14.2 1D motion with 2D and 3D forces

Even if all the motion is in a single direction, an engineer may still have to consider two- or three-dimensional forces.

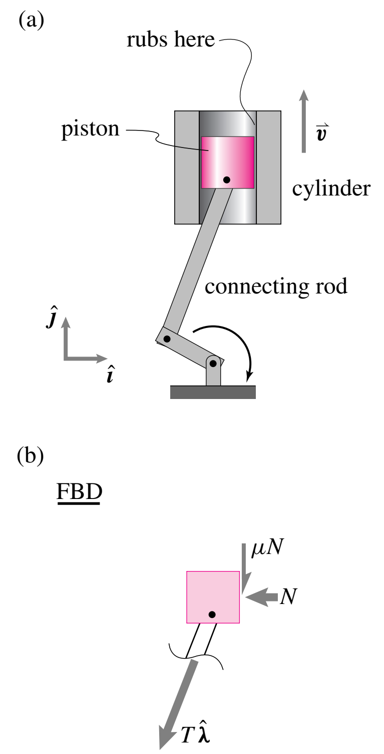


Figure 14.21: (a) shows a piston in a cylinder. (b) shows a free-body diagram of the piston. To draw this FBD, we have assumed: (1) a coefficient of friction μ between the piston and cylinder wall, and (2) negligible mass for the connecting rod, and (3) ignored the spatial extent of the cylinder.

Example: Piston in a cylinder.
A piston slides vertically in a cylinder with coefficient of friction μ between the piston and the cylinder wall. Assume the connecting rod has negligible mass so it can be treated as a two-force member as discussed in section 5.2b. The free-body diagram of the piston (with a bit of the connecting rod) is shown in fig. 14.21. The piston is moving up, so the friction force resists the motion and points down. Linear momentum balance for this system is:

$$\begin{aligned} \sum \vec{F}_i &= \dot{\vec{L}} \\ -N\hat{\imath} - \mu N\hat{\jmath} + T\hat{\lambda}_{rod} &= m_{piston}\, a\hat{\jmath}. \end{aligned}$$

If we assume that the acceleration $a\hat{\jmath}$ of the piston is known, as is its mass m_{piston}, the coefficient of friction μ, and the orientation of the connecting rod $\hat{\lambda}_{rod}$, then we can solve for the rod tension T and the normal reaction N.

Note: *even though the piston moves in one direction, the momentum balance equation is a two-dimensional vector equation.*

Unlike the 1D mechanics of the previous chapter, in this section on 1D motion, the momentum balance equations are 2D and 3D vector equations. Compared to more general 2D and 3D motion, the 1-D motions we assume in this chapter allow an easy introduction to 2D and 3D dynamics calculations.

Highly constrained bodies

This chapter is about rigid objects that move in straight lines. Most objects will not agree to be the topic of such discussion without being forced into doing so. Without being held in place they would rotate and move in a curvy way. To keep an object that is subject to various forces from rotating or curving takes some constraint by wires, rods, rails, hinges, welds, *etc.*. Of course the presence of constraint is not always associated with the disallowance of rotation — constraints could even cause rotation. But in this chapter, constraints keep a rigid object in straight-line motion.

Constraint forces are of interest

Of common interest is making sure that static and dynamic loads do not cause failure of parts that enforce constraints. For example, suppose a truck hauls a very heavy load that is held down by chains or straps. When the truck accelerates, what is the tension in the chains, and will it exceed their strength limits?

1D mechanics with 1D forces, moving on

This all is in contrast with the situation in 1D "unconstrained" dynamics of the previous chapter. For one-dimensional mechanics we assumed that everything of interest mechanically happened in, say, the $\hat{\imath}$ (x) direction.

That is, we ignored *all* torques and angular momenta, and only considered the $\hat{\imath}$ components of the forces (*i.e.*, $\vec{F}\cdot\hat{\imath}$) and linear momentum ($\vec{L}\cdot\hat{\imath}$), namely F_x and L_x.

Kinematics of parallel motion and straight line motion

Let's consider a set of points in the system of interest. Let's call them A to G, or generically, P. For convenience we distinguish a reference point O'. O' may be the center-of-mass, the origin of a local coordinate system, or a fleck of dirt that serves as a marker. By *parallel motion*, we mean that the system happens to move in such a way that $\vec{a}_P = \vec{a}_{O'}$, and $\vec{v}_P = \vec{v}_{O'}$ (fig. 14.22). That is,

$$\vec{a}_A = \vec{a}_B = \vec{a}_C = \vec{a}_D = \vec{a}_E = \vec{a}_F = \vec{a}_G = \vec{a}_P = \vec{a}_{O'}$$

at every instant in time. We also assume that $\vec{v}_A = ... = \vec{v}_P = \vec{v}_{O'}$.

A special case of parallel motion is straight-line motion.

> *a system moves with straight-line motion if it moves like a non-rotating rigid body, in a straight line.*

For straight-line motion, the velocity of the body is in a fixed unchanging direction. If we call a unit vector in that direction $\hat{\lambda}$, then we have

$$\vec{v}(t) = v(t)\hat{\lambda}, \qquad \vec{a}(t) = a(t)\hat{\lambda} \qquad \text{and} \qquad \vec{r}(t) = \vec{r}_0 + s(t)\hat{\lambda}$$

for every point in the system. $\vec{r}_0$ is the position of a point at time 0 and s is the distance the point moves in the $\hat{\lambda}$ direction. Every point in the system has the same s, v, a, and $\hat{\lambda}$ as the other points. There are a variety of problems of practical interest that can be idealized as fitting into this class, notably, the motions of things constrained to move on belts, roads, and rails, like the train on a straight track.

Example: Parallel swing is not straight-line motion

The swing shown does not rotate — all points on the swing have the same velocity. The velocities of all particles are parallel but, since paths are curved, this motion is not straight-line motion. Such curvilinear parallel motion will be discussed later in the book.

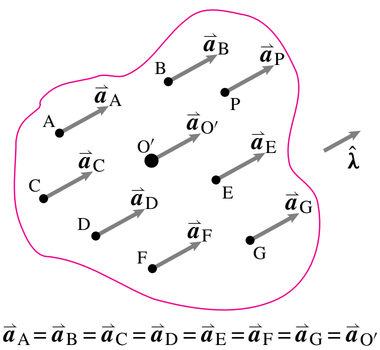


Figure 14.22: Parallel motion: all points on the body have the same acceleration $\vec{a} = a\hat{\lambda}$. For straight-line motion: $\hat{\lambda}(t)$=constant in time and $\vec{v} = v\hat{\lambda}$.

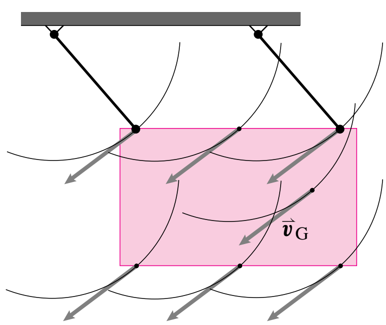


Figure 14.23: A swing showing instantaneous parallel motion which is *curvilinear*. At every instant, each point has the same velocity as the others, but the motion is not in a straight line.

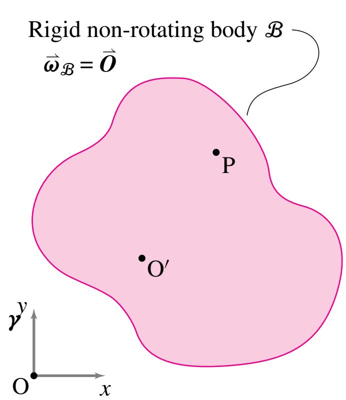


Figure 14.24: A non-rotating body $\mathcal{B}$ with points O' and P.

Velocity of a point

The velocity of any point P on a non-rotating rigid body (such as for straight-line motion) is the same as that of any reference point on the body (see Fig. 14.24).

$$\vec{v}_{\mathrm{P}} = \vec{v}_{\mathrm{O'}}$$

A more general case, which you will learn in later chapters, is shown as 5b in Table II at the back of the book. This formula concerns rotational rate which we will measure with the vector $\vec{\omega}$. For now all you need to know is that $\vec{\omega} = \vec{0}$ when something is not rotating. In 5b in Table II, if you set $\vec{\omega}_{\mathcal{B}} = 0$ and $\vec{v}_{\mathrm{P}/\mathcal{B}} = \vec{0}$ it says that $\vec{v}_{\mathrm{P}} = \dot{\vec{r}}_{\mathrm{O'/O}}$ or in shorthand, $\vec{v}_{\mathrm{P}} = \vec{v}_{\mathrm{O'}}$, as we have written above.

Acceleration of a point

Similarly, the acceleration of every point on a non-rotating rigid body is the same as every other point. The more general case, not needed in this chapter, is shown as entry 5c in Table II at the back of the book.

General results

Before we proceed with discussion of the details of the mechanics of straight-line motion we present some ideas that are also more generally applicable. That is, the concept of the center-of-mass allows some useful simplifications of the general expressions for $\vec{\boldsymbol{L}}$, $\dot{\vec{\boldsymbol{L}}}$, $\vec{\boldsymbol{H}}_{/\mathrm{C}}$, $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}}$ and E_{K}.

Linear momentum $\vec{L}$ and its rate of change $\dot{\vec{L}}$

Although we are dealing with zillions of atoms in a given object, the linear momentum and angular momentum are simple to evaluate:

$$\vec{\boldsymbol{L}} = m_{\mathrm{tot}}\vec{\boldsymbol{v}}_{\mathrm{cm}} \qquad \text{and} \qquad \dot{\vec{\boldsymbol{L}}} = m_{\mathrm{tot}}\vec{\boldsymbol{a}}_{\mathrm{cm}}.$$

Actually, as the front inside cover states, these formulas are good for any motion of any system. The nice simplification for the straight-line motion of this chapter is that all points on a given object have the same velocity and acceleration. So we don't need to find or track the center of mass, but can track the motion of any point on the object.

Angular momentum $\vec{H}_{/\mathrm{C}}$ and its rate of change, $\dot{\vec{H}}_C$ for straight-line motion

For the motions in this chapter, where $\vec{\boldsymbol{a}}_i = \vec{\boldsymbol{a}}_{cm}$ and thus $\vec{\boldsymbol{a}}_{i/cm} = \vec{\boldsymbol{0}}$, angular momentum considerations are simplified, as explained in Box 14.1 on page 3①.

$$\vec{\boldsymbol{H}}_{/\mathrm{C}} = \vec{\boldsymbol{r}}_{\mathrm{cm}/\mathrm{C}} \times \vec{\boldsymbol{v}}_{\mathrm{cm}} m_{\mathrm{tot}} \qquad \text{and} \qquad \dot{\vec{\boldsymbol{H}}}_{\mathrm{C}} = \vec{\boldsymbol{r}}_{\mathrm{cm}/\mathrm{C}} \times \vec{\boldsymbol{a}}_{\mathrm{cm}} m_{\mathrm{tot}}$$

① Calculating rate of change of angular momentum will get more difficult as the book progresses. For a rigid body $\mathcal{B}$ in more general motion, the calculation of rate of change of angular momentum involves the angular velocity $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$, its rate of change $\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}}$, and the moment of inertia matrix $[\boldsymbol{I}^{\mathrm{cm}}]$. If you look in the back of the book at Table I, entries 6c and 6d, you will see formulas that reduce to the formulas below if you assume no rotation and thus use $\vec{\omega} = \vec{\boldsymbol{0}}$ and $\dot{\vec{\omega}} = \vec{\boldsymbol{0}}$. On the other hand, rate of change of linear momentum is simple, at least in concept, in this chapter, as well as in the rest of this book. For all motions of all systems we have

$$\dot{\vec{\boldsymbol{L}}} = m_{tot}\vec{\boldsymbol{a}}_{cm}.$$

14.1 Angular momentum for straight-line motion

For straight-line motion, and parallel motion in general, we can derive the simplification in the calculation of $\vec{\boldsymbol{H}}_{/\mathrm{C}}$ as follows:

$$\begin{aligned}
\vec{\boldsymbol{H}}_{/\mathrm{C}} &\equiv \sum \vec{\boldsymbol{r}}_{i/\mathrm{C}} \times \mathrm{m}_i \vec{\boldsymbol{v}}_i (\text{ definition}) \\
&= \sum \vec{\boldsymbol{r}}_{i/\mathrm{C}} \times \mathrm{m}_i \vec{\boldsymbol{v}}_{\mathrm{cm}} \quad (\text{since, } \vec{\boldsymbol{v}}_i = \vec{\boldsymbol{v}}_{\mathrm{cm}}) \\
&= \left(\sum \vec{\boldsymbol{r}}_{i/\mathrm{C}}\, \mathrm{m}_i\right) \times \vec{\boldsymbol{v}}_{\mathrm{cm}}, \\
&= \vec{\boldsymbol{r}}_{\mathrm{cm}/\mathrm{C}} \times (m_{\mathrm{tot}} \vec{\boldsymbol{v}}_{\mathrm{cm}}), \\
&\quad (\text{ since, } \textstyle\sum \vec{\boldsymbol{r}}_{i/\mathrm{C}}\, \mathrm{m}_i \equiv m_{\mathrm{tot}} \vec{\boldsymbol{r}}_{\mathrm{cm}/\mathrm{C}}).
\end{aligned}$$

The derivation that $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}} = \vec{\boldsymbol{r}}_{\mathrm{cm}/\mathrm{C}} \times (m\vec{\boldsymbol{a}}_{\mathrm{cm}})$ follows from $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}} \equiv \sum \vec{\boldsymbol{r}}_{i/\mathrm{C}} \times \mathrm{m}_i \vec{\boldsymbol{a}}_i$ by the same reasoning.

But for straight-line motion (and, slightly more generally, for any parallel motion), the calculations turn out to be the same as we would get if we put a single point mass at the center-of-mass②:

②**Caution:** The special motions in this chapter are almost the only cases where the angular momentum and its rate of change are so easy to calculate.

$$\begin{aligned}\vec{\boldsymbol{H}}_{/C} &\equiv \sum(\vec{\boldsymbol{r}}_{i/C} \times m_i \vec{\boldsymbol{v}}_i) &= \vec{\boldsymbol{r}}_{cm/C} \times (m_{\text{total}} \vec{\boldsymbol{v}}_{\text{cm}}),\\ \dot{\vec{\boldsymbol{H}}}_{/C} &\equiv \sum(\vec{\boldsymbol{r}}_{i/C} \times m_i \vec{\boldsymbol{a}}_i) &= \vec{\boldsymbol{r}}_{cm/C} \times (m_{\text{total}} \vec{\boldsymbol{a}}_{cm}).\end{aligned}$$

Note, there is some subtlety in the definition of $\vec{\boldsymbol{H}}_{/C}$, as explained in section B.

Kinetic energy

Generally things will not be so simple, but for straight-line motion, or any parallel motion where all points on an object have the same velocity and acceleration, kinetic energy and its rate of change are also easy to calculate:

$$E_K = m_{\text{tot}} v_{\text{cm}}^2/2 \qquad \text{and} \qquad \dot{E}_K = m_{\text{tot}} v_{\text{cm}} a_{\text{cm}}.$$

The kinetic energy works the same as if all the mass was concentrated at the center of mass. This result does not generalize to more complex motions.

Approach

To study systems in straight-line motion (as always) we:

- draw a free-body diagram, showing the appropriate forces and couples at places where connections are 'cut',
- state reasonable kinematic assumptions based on the motions that the constraints allow,
- write linear and/or angular momentum balance equations and/or energy balance, and
- solve for quantities of interest.

Angular momentum balance about a judiciously chosen axis is a particularly useful tool for reducing the number of equations that need to be solved.

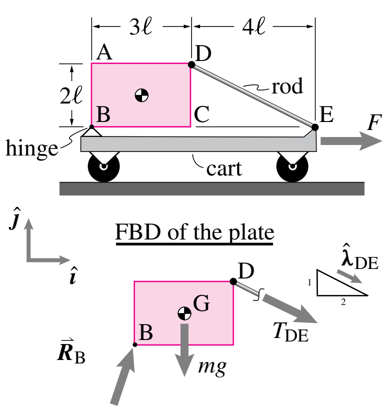


Figure 14.25: Uniform plate supported by a hinge and a rod on an accelerating cart.

Example: Plate on a cart
A uniform rectangular plate $ABCD$ of mass m is supported by a light rigid rod DE and a hinge joint at point B. The dimensions are as shown. The cart has acceleration $a_x \hat{\imath}$ due to a force $F\hat{\imath}$ and the constraints of the wheels. Referring to the free-body diagram in fig. 14.25 and writing angular momentum balance for the plate about point B, we can get an equation for the tension in the rod T_{DE} in terms of m and a_x:

$$\begin{aligned} \sum \vec{M}_{/B} &= \dot{\vec{H}}_{/B} \\ \left\{ \vec{r}_{D/B} \times (T_{DE}\hat{\lambda}_{DE}) + \vec{r}_{G/B} \times (-mg\hat{\jmath}) \right. &= \left. \vec{r}_{G/B} \times (ma_x \hat{\imath}) \right\} \\ \{\} \cdot \hat{k} \Rightarrow \qquad T_{DE} &= \frac{\sqrt{5}}{7} m(a_x - \frac{3}{2} g). \end{aligned}$$

Summarizing note:

angular momentum balance is important even when there is no rotation.

Sliding and pseudo-sliding objects

A car coming to a stop can be roughly modeled as a rigid body that translates and does not rotate. That is, at least for a first approximation, the rotation of the car due to the suspension and tire deformation, can be neglected. The free-body diagram will show various forces with lines of action that do not all act through a single point so that angular momentum balance must be used to analyze the system. Similarly, a bicycle which is braking or a box that is skidding (if not tipping) may be analyzed by assuming straight-line motion.

Example: Car skidding
Consider the accelerating four-wheel drive car in fig. 14.26. The motion quantities for the car are $\dot{\vec{L}} = m_{car}\vec{a}_{car}$ and $\dot{\vec{H}}_{/C} = \vec{r}_{cm/C} \times \vec{a}_{car} m_{car}$. We could calculate angular momentum balance relative to the car's center of mass in which case $\sum \vec{M}_{cm} = \dot{\vec{H}}_{cm} = \vec{0}$ (because the position of the center-of-mass relative to the center-of-mass is $\vec{0}$).

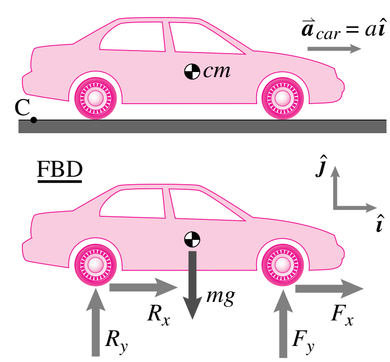


Figure 14.26: A four-wheel drive car accelerating but not tipping. See fig. 2.72 on page 192 for more about FBDs involving wheel contact.

As mentioned, it is often useful to calculate angular momentum balance of sliding objects about points of contact (such as where tires contact the road) or about points that lie on lines of action of applied forces when writing angular momentum balance to solve for forces or accelerations. To do so usually eliminates some unknown reactions from the equations to be solved. For example, the angular momentum balance equation about the rear-wheel contact of a car does not contain the rear-wheel contact forces.

Wheels

The function of wheels is to allow easy sliding-like (pseudo-sliding) motion between objects, at least in the direction they are pointed. On the other hand, wheels do sometimes slip due to:

- being overpowered (as in a screeching accelerating car),
- being braked hard, or
- having very bad bearings (like a rusty toy car).

How wheels are treated when analyzing cars, bikes, and the like depends on both the application and on the level of detail one requires. In *this chapter*, we will always assume that wheels have negligible mass. Thus, when we treat the special case of un-driven and un-braked wheels our free-body diagrams will be as in fig. 2.73a on page 192 and *not* like the one in fig. 2.73b. With the ideal wheel approximation, all of the various cases for a car traveling to the right are shown with partial free-body diagrams of a wheel in fig. 2.72. For the purposes of actually solving problems, we have accepted Coulomb's law of friction as a model for contacting interaction (see page 188 in sec. 2.4).

3-D forces in straight-line motion

The ideas we have discussed apply as well in three dimensions as in two. As you learned from doing statics problems, working out the details in 3D, where vector methods must be used carefully, is more involved than in 2D. As for statics, three-dimensional problems often yield simple results and simple intuitions by considering angular momentum balance about an axis.

Angular momentum balance about an axis

The simplest way to think of angular momentum balance about an axis is to look at angular momentum balance about a point and then take a dot product with a unit vector along an axis:

$$\hat{\boldsymbol{\lambda}} \cdot \left\{ \sum \vec{\boldsymbol{M}}_{/\mathrm{C}} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}} \right\}.$$

Note that the axis need not correspond to any mechanical device in any way resembling an axle. The equation above applies for any point C and any vector $\hat{\boldsymbol{\lambda}}$. If you choose C and $\hat{\boldsymbol{\lambda}}$ judiciously many terms in your equations may drop out.

SAMPLE 14.6 Force in braking. A front-wheel-drive car of mass $m = 1200\,\text{kg}$ is cruising at $v = 60\,\text{mph}$ on a straight road when the driver slams on the brake. The car slows down to 20 mph in 4 s while maintaining its straight path.

1. What is the average force (average in time) applied on the car during braking?
2. What is the average power of braking?

Solution

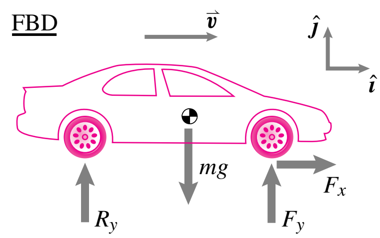


Figure 14.27: Free-body diagram of a front-wheel-drive car during braking. Note that we have (arbitrarily) pointed F_x to the right. The algebra in this problem will tell us that $F_x < 0$.

1. Let us assume that we have an xy coordinate system in which the car is traveling along the x-axis during the entire time under consideration. Then, the velocity of the car before braking, $\vec{v}_1$, and after braking, $\vec{v}_2$, are

$$\vec{v}_1 = v_1\hat{\imath} = 60\,\text{mph}\,\hat{\imath} \quad \text{and} \quad \vec{v}_2 = v_2\hat{\imath} = 20\,\text{mph}\,\hat{\imath}.$$

The linear impulse during braking is $\vec{F}_{\text{ave}}\Delta t$ where $\vec{F} \equiv F_x\hat{\imath}$ (see free-body diagram of the car). Now, from the impulse-momentum relationship,

$$\vec{F}\Delta t = \vec{L}_2 - \vec{L}_1,$$

where $\vec{L}_1$ and $\vec{L}_2$ are linear momenta of the car before and after braking, respectively, and $\vec{F}$ is the average applied force. Therefore,

$$\begin{aligned}
\vec{F} &= \frac{1}{\Delta t}(\vec{L}_2 - \vec{L}_1) = \frac{m}{\Delta t}(\vec{v}_2 - \vec{v}_1) \\
&= \frac{1200\,\text{kg}}{4\,\text{s}}(20 - 60)\,\text{mph}\hat{\imath} \\
&= -12000\frac{\text{kg}}{\text{s}} \cdot \frac{\cancel{\text{mi}}}{\cancel{\text{hr}}} \cdot \frac{1600\ \text{m}}{1\ \cancel{\text{mi}}} \cdot \frac{1\ \cancel{\text{hr}}}{3600\,\text{s}}\hat{\imath} \\
&= -\frac{16,000}{3}\,\text{kg}\cdot\ \text{m/s}^2\hat{\imath} = -5.33\,\text{kN}\hat{\imath}.
\end{aligned}$$

Thus

$$F_x\hat{\imath} = -5.33\,\text{kN}\hat{\imath} \quad \Rightarrow \quad F_x = -5.33\,\text{kN}.$$

$$\boxed{F_x = -5.33\,\text{kN}}$$

2. Let the average power during braking be P_{ave}. Then the work done during braking is $W = \int P_{\text{ave}}dt$. From work-energy principle, we have

$$\begin{aligned}
W &= \Delta E_{\text{K}} \\
\int_{t_1}^{t_2} P_{\text{ave}}dt &= \frac{1}{2}m(v_2^2 - v_1^2) \\
P_{\text{ave}}(t_2 - t_1) &= \frac{1}{2}m(v_2^2 - v_1^2) \\
P_{\text{ave}} &= \frac{m}{2\Delta t}(v_2^2 - v_1^2)
\end{aligned}$$

Substituting $m = 1200\,\text{kg}$, $\Delta t = 4\,\text{s}$, $v_1 = 60\,\text{mph} = 26.67\,\text{m/s}$ and $v_2 = 20\,\text{mph} = 8.89\,\text{m/s}$, we get

$$P_{\text{ave}} = -94815\,\text{N}\cdot\text{m/ s} = -94.815\,\text{kW}.$$

It is easy to check that if we take the average force F_{ave} calculated above and the average speed $v_{\text{ave}} = (v_1 + v_2)/2 = 40\,\text{mph} = 17.77\,\text{m/s}$, then

$$P_{\text{ave}} = F_{\text{ave}}v_{\text{ave}} = -5.33\,\text{kN} \cdot 17.77\,\text{m/s} = -94.815\,\text{kW},$$

as obtained above.

$$\boxed{P_{\text{ave}} = -94.815\,\text{kW}}$$

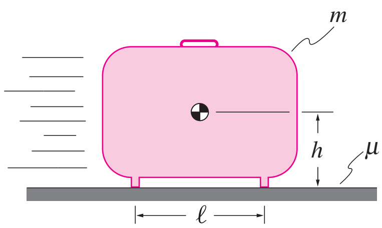


Figure 14.28: A suitcase in motion.

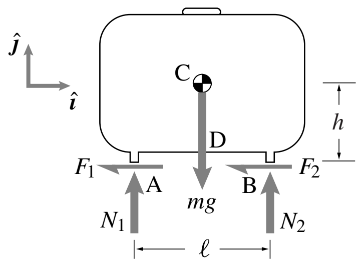


Figure 14.29: FBD of the suitcase.

SAMPLE 14.7 A suitcase skidding on frictional ground. A suitcase of mass m is pushed and sent sliding on a horizontal surface. The suitcase slides without any rotation. A and B are the only contact points of the suitcase with the ground. If the coefficient of friction between the suitcase and the ground is μ, find all the forces applied by the ground on the suitcase. Discuss the results obtained for normal forces.

Solution As usual, we first draw a free-body diagram of the suitcase. The FBD is shown in Fig. 14.29. Assuming Coulomb's law of friction holds, we can write

$$\vec{F}_1 = -\mu N_1 \hat{\imath} \quad \text{and} \quad \vec{F}_2 = -\mu N_2 \hat{\imath}. \tag{14.29}$$

Now we write the balance of linear momentum for the suitcase:

$$\sum \vec{F} = m\vec{a}_{\text{cm}}$$

$$\Rightarrow \qquad -(F_1 + F_2)\hat{\imath} + (N_1 + N_2 - mg)\hat{\jmath} = ma_C\hat{\imath} \tag{14.30}$$

where $\vec{a}_C = a_C\hat{\imath}$ is the unknown acceleration. Dotting eqn. (14.30) with $\hat{\imath}$ and $\hat{\jmath}$ and substituting for F_1 and F_2 from eqn. (14.29) we get

$$-\mu(N_1 + N_2) = ma_C \tag{14.31}$$

$$N_1 + N_2 = mg. \tag{14.32}$$

Equations (14.31) and (14.32) represent 2 scalar equations in three unknowns N_1, N_2 and a. Obviously, we need another equation to solve for these unknowns.

We can write the balance of angular momentum about any point. Points A or B are good choices because they each eliminate some reaction components. Let us write the balance of angular momentum about point A:

$$\sum \vec{M}_{\text{A}} = \dot{\vec{H}}_{\text{A}}$$

$$\begin{aligned}
\sum \vec{M}_{\text{A}} &= \vec{r}_{B/A} \times N_2\hat{\jmath} + \vec{r}_{D/A} \times (-mg)\hat{\jmath} \\
&= \ell\hat{\imath} \times N_2\hat{\jmath} + \frac{\ell}{2}\hat{\imath} \times (-mg)\hat{\jmath} \\
&= (\ell N_2 - mg\frac{\ell}{2})\hat{k}
\end{aligned} \tag{14.33}$$

and

$$\begin{aligned}
\dot{\vec{H}}_{\text{A}} &= \vec{r}_{C/A} \times m\vec{a}_C \\
&= (\frac{\ell}{2}\hat{\imath} + h\hat{\jmath}) \times ma_C\hat{\imath} \\
&= -ma_C h\hat{k}.
\end{aligned} \tag{14.34}$$

Equating (14.33) and (14.34) and dotting both sides with $\hat{k}$ we get the following third scalar equation:

$$\ell N_2 - mg\frac{\ell}{2} = -ma_C h. \tag{14.35}$$

Solving eqns. (14.31) and (14.32) for a we get

$$a_C = -\mu g$$

and substituting this value of a_C in eqn. (14.35) we get

$$\begin{aligned}
N_2 &= \frac{m\mu gh + mg\ell/2}{\ell} \\
&= mg\left(\frac{1}{2} + \frac{h}{\ell}\mu\right).
\end{aligned}$$

Substituting the value of N_2 in either of the equations (14.31) or (14.32) we get

$$N_1 = mg\left(\frac{1}{2} - \frac{h}{\ell}\mu\right).$$

$$N_1 = mg(\tfrac{1}{2} - \tfrac{h}{\ell}\mu),\ N_2 = mg(\tfrac{1}{2} + \tfrac{h}{\ell}\mu),\ f_1 = \mu N_1,\ f_2 = \mu N_2.$$

Discussion: From the expressions for N_1 and N_2 we see that

1. $N_1 = N_2 = \frac{1}{2}mg$ if $\mu = 0$ because without friction there is no deceleration. The problem becomes equivalent to a statics problem.
2. $N_1 = N_2 \approx \frac{1}{2}mg$ if $\ell >> h$. In this case, the moment produced by the friction forces is too small to cause a significant difference in the magnitudes of the normal forces. For example, take $\ell = 20h$ and calculate moment about the center-of-mass to convince yourself.

Graphically, N_1, N_2 and their difference $N_1 - N_2$ are shown in the plot below as a function of h/ℓ for a particular value of μ and mg. As the equations indicate, $N_1 - N_2$ increases steadily as h/ℓ increases, showing how the moment produced by the friction forces makes a bigger and bigger difference between N_1 and N_2 as this moment gets bigger.

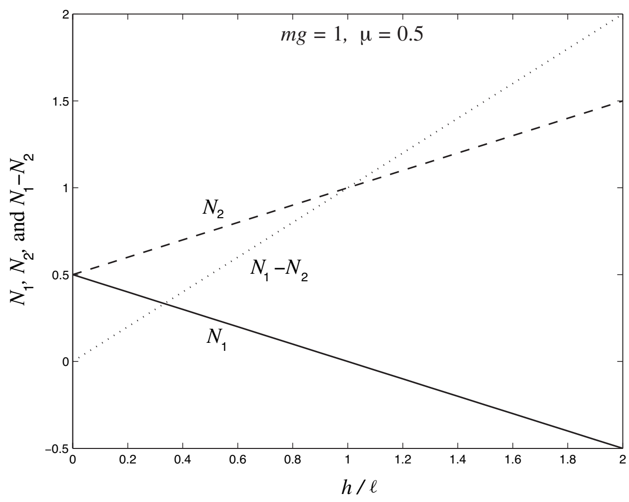


Figure 14.30: The normal forces N_1 and N_2 differ from each other more and more as h/ℓ increases.

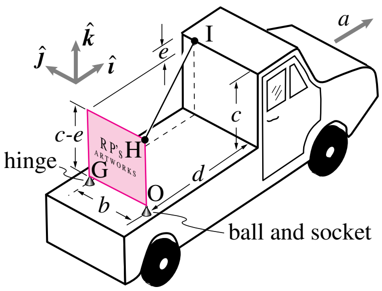


Figure 14.31: An accelerating board in 3-D

SAMPLE 14.8 **Uniform acceleration of a board in 3-D.** A uniform sign-board of mass $m = 20\,\text{kg}$ sits in the back of an accelerating flatbed truck. The board is supported with a ball-and-socket joint at O and a hinge at G. A light rod from H to I keeps the board from falling over. The truck is on level ground and has forward acceleration $\vec{\boldsymbol{a}} = 0.6\,\text{m/s}^2\hat{\boldsymbol{\imath}}$. The relevant dimensions are $b = 1.5\,\text{m}$, $c = 1.5\,\text{m}$, $d = 3\,\text{m}$, $e = 0.5\,\text{m}$. There is gravity ($g = 10\,\text{m/s}^2$).

1. Draw a free-body diagram of the board.
2. Set up equations to solve for all the unknown forces shown on the FBD.
3. Use the balance of angular momentum about an axis to find the tension in the rod.

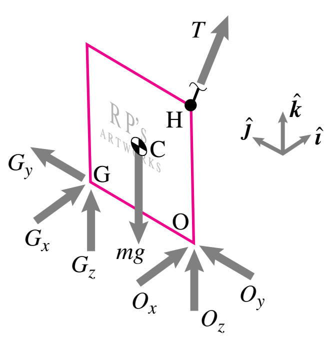


Figure 14.32: FBD of the board

Solution

1. The free-body diagram of the board is shown in Fig. 14.32.
2. Linear momentum balance for the board:

$$\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}, \quad \text{or}$$

$$(G_x + O_x)\hat{\boldsymbol{\imath}} + (G_y + O_y)\hat{\boldsymbol{\jmath}} + (G_z + O_z - mg)\hat{\boldsymbol{k}} + T\hat{\boldsymbol{\lambda}}_{HI} = ma\hat{\boldsymbol{\imath}} \tag{14.36}$$

where

$$\hat{\boldsymbol{\lambda}}_{HI} = \frac{d\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}} + e\hat{\boldsymbol{k}}}{\sqrt{d^2 + b^2 + e^2}} = \frac{d\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}} + e\hat{\boldsymbol{k}}}{\ell},$$

and ℓ is the length of the rod HI.

Dotting eqn. (14.36) with $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ we get the following three scalar equations:

$$G_x + O_x + T\frac{d}{\ell} = ma \tag{14.37}$$

$$G_y + O_y + T\frac{b}{\ell} = 0 \tag{14.38}$$

$$G_z + O_z + T\frac{e}{\ell} = mg \tag{14.39}$$

Angular momentum balance about point G:

$$\sum \vec{\boldsymbol{M}}_G = \dot{\vec{\boldsymbol{H}}}_G$$

$$\begin{aligned}
\sum \vec{\boldsymbol{M}}_G &= \vec{\boldsymbol{r}}_{C/G} \times (-mg\hat{\boldsymbol{k}}) + \vec{\boldsymbol{r}}_{O/G} \times (O_x\hat{\boldsymbol{\imath}} + O_z\hat{\boldsymbol{k}}) + \vec{\boldsymbol{r}}_{H/G} \times T\hat{\boldsymbol{\lambda}}_{HI} \\
&= (-\frac{b}{2}\hat{\boldsymbol{\jmath}} + \frac{c-e}{2}\hat{\boldsymbol{k}}) \times (-mg\hat{\boldsymbol{k}}) - b\hat{\boldsymbol{\jmath}} \times (O_x\hat{\boldsymbol{\imath}} + O_z\hat{\boldsymbol{k}}) \\
&\quad + [-b\hat{\boldsymbol{\jmath}} + (c-e)\hat{\boldsymbol{k}}] \times \frac{T}{\ell}(d\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}} + e\hat{\boldsymbol{k}}) \\
&= \left(\frac{b}{2}mg - bO_z - be\frac{T}{\ell} - (c-e)b\frac{T}{\ell}\right)\hat{\boldsymbol{\imath}} \\
&\quad + (c-e)d\frac{T}{\ell}\hat{\boldsymbol{\jmath}} + \left(bO_x + bd\frac{T}{\ell}\right)\hat{\boldsymbol{k}}
\end{aligned} \tag{14.40}$$

and

$$\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_G &= \vec{\boldsymbol{r}}_{C/G} \times ma\hat{\boldsymbol{\imath}} \\
&= (-\frac{b}{2}\hat{\boldsymbol{\jmath}} + \frac{c-e}{2}\hat{\boldsymbol{k}}) \times ma\hat{\boldsymbol{\imath}} \\
&= \frac{b}{2}ma\hat{\boldsymbol{k}} + \frac{c-e}{2}ma\hat{\boldsymbol{\jmath}}.
\end{aligned} \tag{14.41}$$

Equating (14.40) and (14.41) and dotting both sides with $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ we get the following three additional scalar equations:

$$O_z + \frac{c}{\ell}T = \frac{1}{2}mg \tag{14.42}$$

$$\frac{d}{\ell}T = \frac{1}{2}ma \tag{14.43}$$

$$O_x + \frac{d}{\ell}T = \frac{1}{2}ma \tag{14.44}$$

Now we have six scalar equations in seven unknowns — O_x, O_y, O_z, G_x, G_y, G_z, and T. From basic linear algebra, we know that we cannot find unique solutions for all these unknowns from the given equations. A closer inspection of eqns. (14.37–14.39) and (14.42–14.44) shows that we can easily solve for O_x, O_z, G_x, G_z, and T, but O_y and G_y cannot be determined uniquely because they appear together as the sum $G_y + O_y$. ① Fortunately, we can find the tension in the wire HI without worrying about the values of O_y and G_y as we show below.

① Note that G_y and O_y will always appear together as the sum $G_y + O_y$ even if you took the angular momentum balance about some other point. This is because they have the same line of action. Thus, they cannot be found independently. This mathematical problem corresponds to the physical reality that the supports at points O and G could be squeezing the plate along the line OG with, say, $O_y = 1000\,\mathrm{N}$ and $G_y = -1000\,\mathrm{N}$ even if there were no gravity, and the truck was not accelerating. To make prestress problems like this tractable, people often make assumptions like, 'Assume $G_y = 0$', that is, they try to get rid of the redundancy in supports to make the problem statically determinate.

3. Balance of angular momentum about axis OG gives:

$$\begin{aligned}\hat{\boldsymbol{\lambda}}_{OG} \cdot \sum \vec{\boldsymbol{M}}_G &= \hat{\boldsymbol{\lambda}}_{OG} \cdot \dot{\vec{\boldsymbol{H}}}_G \\ &= \hat{\boldsymbol{\lambda}}_{OG} \cdot (\vec{\boldsymbol{r}}_{C/G} \times ma\hat{\boldsymbol{\imath}}).\end{aligned} \tag{14.45}$$

Since all reaction forces and the weight go through axis OG, they do not produce any moment about this axis (convince yourself that the forces from the reactions have no torque about the axis by calculation or geometry). Therefore,

$$\begin{aligned}\hat{\boldsymbol{\lambda}}_{OG} \cdot \sum \vec{\boldsymbol{M}}_G &= \hat{\boldsymbol{\jmath}} \cdot (\vec{\boldsymbol{r}}_{H/G} \times T\hat{\boldsymbol{\lambda}}_{HI}) \\ &= T\frac{d(c-e)}{\ell}.\end{aligned} \tag{14.46}$$

$$\begin{aligned}\hat{\boldsymbol{\lambda}}_{OG} \cdot (\vec{\boldsymbol{r}}_{C/G} \times ma\hat{\boldsymbol{\imath}}) &= \hat{\boldsymbol{\jmath}} \cdot \left[\left(\frac{b}{2}\hat{\boldsymbol{\jmath}} + \frac{c-e}{2}\hat{\boldsymbol{k}}\right) \times ma\hat{\boldsymbol{\imath}}\right] \\ &= ma\frac{(c-e)}{2}.\end{aligned} \tag{14.47}$$

Equating (14.46) and (14.47), as required by eqn. (14.45), we get

$$\begin{aligned}T &= \frac{ma\ell}{2d} \\ &= \frac{20\,\mathrm{kg} \cdot 0.6\,\mathrm{m/s^2} \cdot 3.39\,\mathrm{m}}{2 \cdot 3\,\mathrm{m}} \\ &= 6.78\,\mathrm{N}.\end{aligned}$$

$$\boxed{T_{HI} = 6.78\,\mathrm{N}}$$

SAMPLE 14.9 Computer solution of algebraic equations. In the previous sample problem (Sample 14.8), six equations were obtained to solve for the six unknown forces (assuming $G_y = 0$). (i) Set up the six equations in matrix form and (ii) solve the matrix equation on a computer. Check the solution by substituting the values obtained in one or two equations.

Solution

1. The six scalar equations — (14.37), (14.38), (14.39), (14.42), (14.43), and (14.44) are amenable to hand calculations. We, however, set up these equations in matrix form and solve the matrix equation on the computer. The matrix form of the equations is:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & \frac{d}{\ell} \\ 0 & 1 & 0 & 0 & 0 & \frac{b}{\ell} \\ 0 & 0 & 1 & 0 & 1 & \frac{e}{\ell} \\ 0 & 0 & 1 & 0 & 0 & \frac{c}{\ell} \\ 0 & 0 & 0 & 0 & 0 & \frac{d}{\ell} \\ 1 & 0 & 0 & 0 & 0 & \frac{d}{\ell} \end{bmatrix} \begin{Bmatrix} O_x \\ O_y \\ O_z \\ G_x \\ G_z \\ T \end{Bmatrix} = \begin{Bmatrix} ma \\ 0 \\ mg \\ mg/2 \\ ma/2 \\ ma/2 \end{Bmatrix}. \tag{14.48}$$

The above equation can be written, in matrix notation, as

$$\mathbf{A\,x} = \mathbf{b}$$

where **A** is the coefficient matrix, **x** is the vector of the unknown forces, and **b** is the vector on the right hand side of the equation. Now we are ready to solve the system of equations on the computer.

2. We use the following pseudo-code to solve the above matrix equation. ②

② Be careful with units. Most computer programs will not take care of your units. They only deal with numerical input and output. You should, therefore, make sure that your variables have proper units for the required calculations. Either do dimensionless calculations or use consistent units for all quantities.

```
m = 20, a = 0.6,
b = 1.5, c = 1.5, d = 3, e = 0.5, g = 10,
l = sqrt(b^2 + d^2 + e^2),

A = [1 0 0 1 0 d/l
     0 1 0 0 0 b/l
     0 0 1 0 1 e/l
     0 0 1 0 0 c/l
     0 0 0 0 0 d/l
     1 0 0 0 0 d/l]
b = [m*a, 0, m*g, m*g/2, m*a/2, m*a/2]'

{Solve A x = b for x}
x =                  % this is the computer output
         0
   -3.0000
   97.0000
    6.0000
  102.0000
    6.7823
```

The solution obtained from the computer means:

$$O_x = 0,\ O_y = -3\,\mathrm{N},\ O_z = 97\,\mathrm{N},\ G_x = 6\,\mathrm{N},\ G_z = 102\,\mathrm{N},\ T = 6.78\,\mathrm{N}.$$

We now hand-check the solution by substituting the values obtained in, say, Eqns. (14.38) and (14.43). Before we substitute the values of forces, we need to calculate the length ℓ.

$$\begin{aligned} \ell &= \sqrt{d^2 + b^2 + e^2} \\ &= 3.3912\,\mathrm{m}. \end{aligned}$$

Therefore,

$$
\begin{aligned}
\text{Eqn. (14.38):} \qquad O_y + T\frac{b}{\ell} &= -3\,\mathrm{N} + 6.78\,\mathrm{N}\cdot\frac{1.5\,\mathrm{m}}{3.3912\,\mathrm{m}} \\
&\stackrel{\surd}{=} 0, \\
\text{Eqn. (14.43):} \qquad \frac{d}{\ell}T - \frac{1}{2}ma &= \frac{3\,\mathrm{m}}{3.3912\,\mathrm{m}}6.78\,\mathrm{N} - \frac{1}{2}20\,\mathrm{kg}\,0.6\,\mathrm{m/s^2} \\
&\stackrel{\surd}{=} 0.
\end{aligned}
$$

Thus, the computer solution agrees with our equations.
Comments: We could have solved the six equations for seven unknowns without assuming $G_y = 0$ if our computer program or package allows us to do so. We will, of course, not get a unique solution. For example, by taking the following **A**, a 6×7 matrix, and solving $\mathbf{A}\,\mathbf{x} = \mathbf{b}$ for $\mathbf{x} = [O_x\ \ O_y\ \ O_z; G_x\ \ G_y\ \ G_z\ \ T]^T$ with the same **b** as input above, we get the solution as shown below.

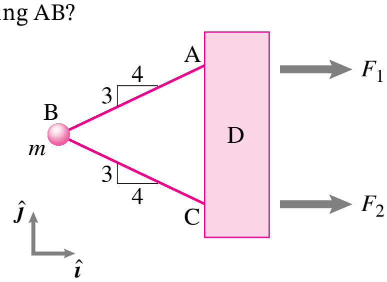
```
 A = [1 0 0 1 0 0 d/l
      0 1 0 0 1 0 b/l
      0 0 1 0 0 1 e/l
      0 0 1 0 0 0 c/l
      0 0 0 0 0 0 d/l
      1 0 0 0 0 0 d/l]
 b = [m*a, 0, m*g, m*g/2, m*a/2, m*a/2]'

{Solve A x = b for x}
x =               % this is the computer output
         0
   -3.0000
   97.0000
    6.0000
         0
  102.0000
    6.7823
```

This is the same solution as we got before except that it includes $G_y = 0$ in the solution. Now, if we add a vector $\Delta\,\mathbf{x} = [0\ \alpha\ 0\ 0\ -\alpha\ 0\ 0]^T$ to $\mathbf{x}$ where α is any number, and compute $\mathbf{A}\,(\mathbf{x}{+}\Delta\mathbf{x})$, we get back **b**. That is, the six equilibrium conditions are satisfied irrespective of the actual values of O_y and G_y as long as the value of $O_y + G_y$ remains the same.

14.2 1D motion with 2D and 3D forces

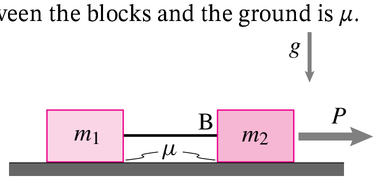
Preparatory Problems

14.2.1 Mass pulled by two strings. F_1 and F_2 are applied so that the system shown accelerates to the right at $5\,\mathrm{m/s^2}$ (*i.e.*, $\boldsymbol{a} = 5\,\mathrm{m/s^2}\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}}$) and has no rotation. The mass of D and forces F_1 and F_2 are unknown. What is the tension in string AB?



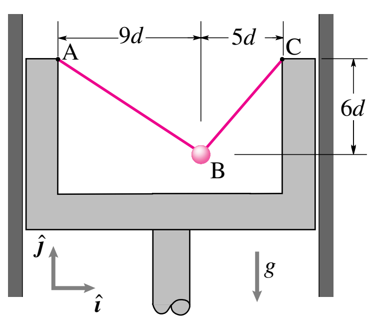
Problem 14.2.1

14.2.2 The two blocks, $m_1 = m_2 = m$, are connected by an inextensible string AB. The string can only withstand a tension T_{cr}. Find the maximum value of the applied force P so that the string does not break. The sliding coefficient of friction between the blocks and the ground is μ.



Problem 14.2.2

14.2.3 A point mass m is attached to a piston by two inextensible cables. The piston has upwards acceleration of $a_y\hat{\boldsymbol{\jmath}}$. There is gravity. In terms of some or all of m, g, d, and a_y find the tension in cable AB. *

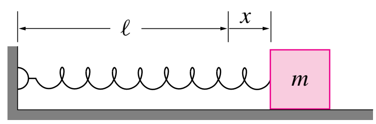


Problem 14.2.3

14.2.4 A point mass of mass m moves on a frictional surface with coefficient of friction μ and is connected to a spring with constant k and unstretched length ℓ. There is gravity. At the instant of interest, the mass is at a distance x to the right from its position where the spring is unstretched and is moving with $\dot{x} > 0$ to the right.

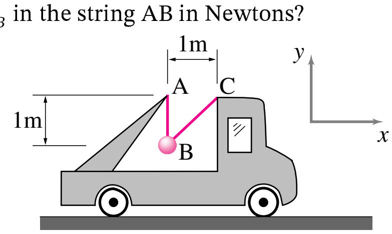
a) Draw a free-body diagram of the mass at the instant of interest.

b) At the instant of interest, write the equation of linear momentum balance for the block evaluating the left hand side as explicitly as possible. Let the acceleration of the block be $\vec{\boldsymbol{a}} = \ddot{x}\hat{\boldsymbol{\imath}}$.



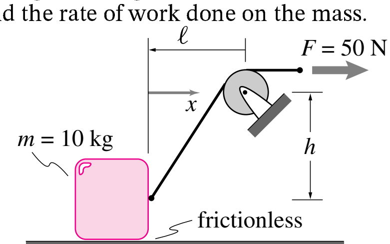
Problem 14.2.4

14.2.5 Consider the mass at B (2 kg) supported by two strings in the back of a truck which has acceleration of $3\,\mathrm{m/s^2}$. Use $g = 10\,\mathrm{m/s^2}$. What is the tension T_{AB} in the string AB in Newtons?



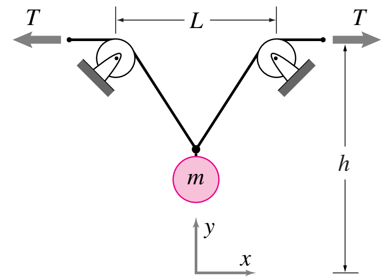
Problem 14.2.5:

14.2.6 At the instant shown, the mass is moving to the right at speed $v = 3\,\mathrm{m/s}$. Find the rate of work done on the mass.



Problem 14.2.6

14.2.7 A point mass 'm' is pulled straight up by two strings. The two strings pull the mass symmetrically about the vertical axis with constant and equal force T. At an instant in time t, the position and the velocity of the mass are $y(t)\hat{\boldsymbol{\jmath}}$ and $\dot{y}(t)\hat{\boldsymbol{\jmath}}$, respectively. Find the power input to the moving mass.



Problem 14.2.7

More-Involved Problems

14.2.8 Two blocks, each of mass m, are connected by a rod of length S. They slide down a slope of angle θ. Do not neglect gravity but do neglect friction.

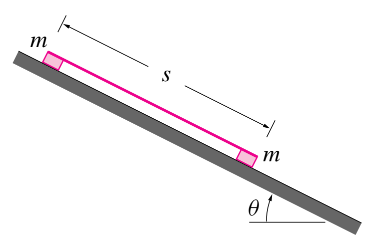
a) Draw separate free-body diagrams of each block, the rod, and the system of the two blocks and rod.

b) Write separate equations for linear momentum balance for each block, the rod, and the system of blocks and rod.

c) What is the acceleration of the center of mass of the two blocks? *

d) What is the force in the rod? *

e) What is the speed of the center of mass for the two blocks after they have traveled a distance d down the slope, having started from rest. [Hint: Dot your momentum balance equations with a unit vector along the ramp in order to reduce this problem to a problem in one dimensional mechanics.] *



Problem 14.2.8

14.2.9 Two blocks, each of mass m, are connected by a massless rod of length S;

the blocks' dimensions are small compared to S. They slide down a slope of angle θ. The coefficient of friction of the top block is μ and of the bottom block is $\mu/2$.

a) Draw separate free-body diagrams of each block, the string, and the system of the two blocks and rod.

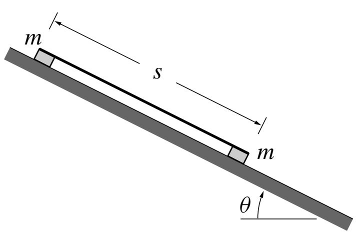
b) Write separate equations for linear momentum balance for each block, the string, and the system of blocks and rod.

c) What is the acceleration of the center of mass of the two blocks? *

d) What is the force in the rod? *

e) What is the speed of the center of mass for the two blocks after they have traveled a distance d down the slope, having started from rest. *

f) How would your solutions to parts (a) and (c) differ if the two blocks were interchanged with the slippery one on top? *



Problem 14.2.9

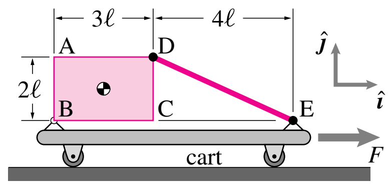
14.2.10 Coin on a car on a ramp. A student engineering design course asked students to build a cart (mass = m_c) that rolls down a ramp with angle θ. A small weight (mass $m_w \ll m_c$) is placed on top of the cart on a surface tipped with respect to the cart (angle ϕ). Assume the small mass does not slide. Assume massless wheels with frictionless bearings. $\hat{\imath}$ is horizontal and $\hat{\jmath}$ is vertical up.

a) Find the acceleration of the cart. Answer in terms of some or all of $m_c, g, \hat{\imath}, \theta$ and $\hat{\jmath}$.

b) What coefficient of friction μ is required (the smallest that will work) to keep the small mass from sliding as the cart rolls down the slope? Answer in terms of some or all of m_c, m_w, g, θ, and ϕ.

c) What angle ϕ will allow a small mass to ride on the cart with the smallest coefficient of friction? Answer in terms of some or all of m_c, m_w, g, and θ.

14.2.11 Guyed plate on a cart A uniform rectangular plate $ABCD$ of mass m is supported by a rod DE and a hinge joint at point B. The dimensions are as shown. There is gravity. What must the acceleration of the cart be in order for massless rod DE to be in tension? *



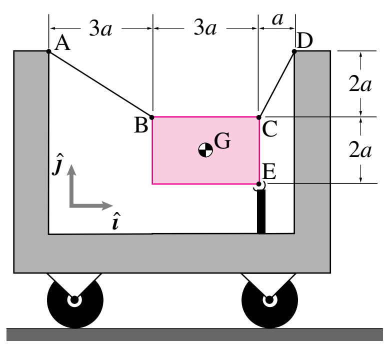
Problem 14.2.11: Uniform plate supported by a hinge and a cable on an accelerating cart.

14.2.12 A uniform rectangular plate of mass m is supported by two inextensible cables AB and CD and by a hinge at point E on the cart as shown. The cart has acceleration $a_x\hat{\imath}$ due to a force not shown. There is gravity.

a) Draw a free-body diagram of the plate.

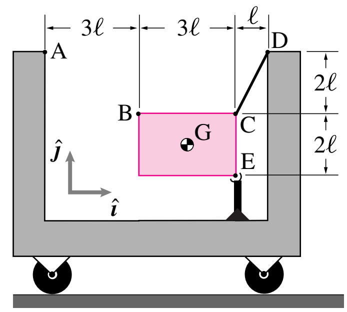
b) Write the equation of linear momentum balance for the plate and evaluate the left hand side as explicitly as possible.

c) Write the equation for angular momentum balance about point E and evaluate the left hand side as explicitly as possible.



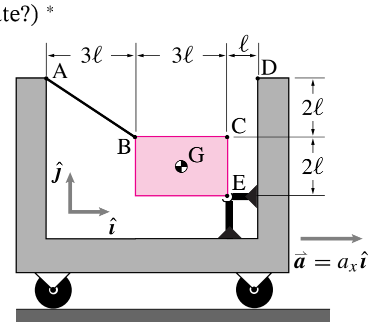
Problem 14.2.12

14.2.13 A uniform rectangular plate of mass m is supported by an inextensible cable CD and a hinge joint at point E on the cart as shown. The hinge joint is attached to a rigid column welded to the floor of the cart. The cart is at rest. There is gravity. Find the tension in cable CD.



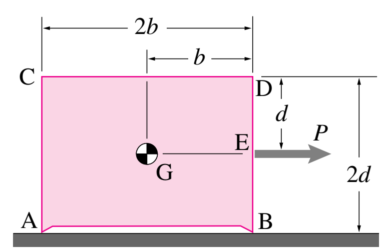
Problem 14.2.13

14.2.14 A uniform rectangular plate of mass m is supported by an inextensible cable AB and a hinge joint at point E on the cart as shown. The hinge joint is attached to a rigid column welded to the floor of the cart. The cart has acceleration $a_x\hat{\imath}$. There is gravity. Find the tension in cable AB. (What's 'wrong' with this problem? What if instead point B were at the bottom left hand corner of the plate?) *



Problem 14.2.14

14.2.15 A block of mass m is sitting on a frictionless surface and acted upon at point E by the horizontal force P through the center of mass. Draw a free-body diagram of the block. There is gravity. Find a) the acceleration of the block and b) reactions on the block at points A and B. *

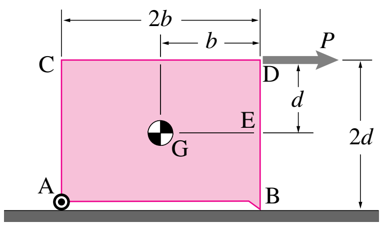


Problem 14.2.15

14.2.16 Reconsider the block in problem 14.2.15. This time, find the acceleration of the block and the reactions at A

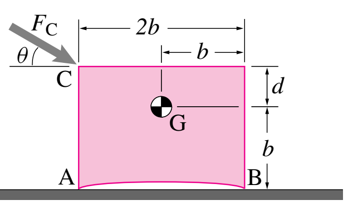
and B if the force P is applied instead at point D. Are the acceleration and the reactions on the block different from those found when P is applied at point E? *

14.2.17 A block of mass m is sitting on a frictional surface and acted upon at point D by the horizontal force P. The block is resting on a sharp edge at point B and is supported by an ideal wheel at point A. There is gravity. Assuming the block is sliding with coefficient of friction μ at point B, find the acceleration of the block and the reactions on the block at points A and B.



Problem 14.2.17

14.2.18 A force F_C is applied to the corner C of a box of weight W with dimensions and center of gravity at G as shown in the figure. The coefficient of sliding friction between the floor and the points of contact A and B is μ. Assuming that the box slides when F_C is applied, find the acceleration of the box and the reactions at A and B in terms of W, F_C, θ, b, and d.

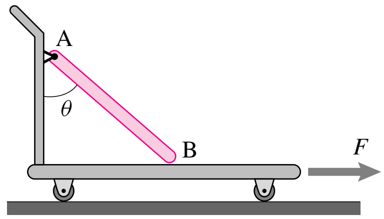


Problem 14.2.18

14.2.19 A uniform rod with mass m_r rests on a cart (mass m_c) which is being pulled to the right. The rod is hinged at one end (with a frictionless hinge) and has no friction at the contact with the cart. The cart is rolling on wheels that are modeled as having no mass and no bearing friction (ideal massless wheels). Answer in terms of g, m_r, m_c, θ and F. Find:

a) The force on the rod from the cart at point B.

b) The force on the rod from the cart at point A.



Problem 14.2.19

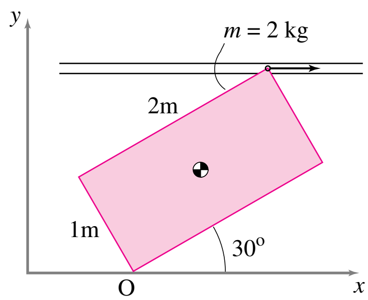
14.2.20 The box shown in the figure is dragged in the x-direction with a constant acceleration $\vec{a} = 0.5\,\mathrm{m/s^2}\hat{\imath}$. At the instant shown, the velocity of (every point on) the box is $\vec{v} = 0.8\,\mathrm{m/s}\hat{\imath}$.

a) Find the linear momentum of the box.

b) Find the rate of change of linear momentum of the box.

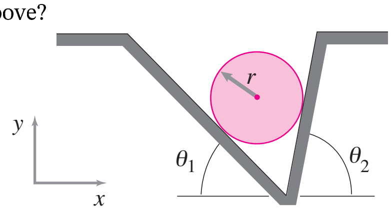
c) Find the angular momentum of the box about the contact point O.

d) Find the rate of change of angular momentum of the box about the contact point O.



Problem 14.2.20

14.2.21 The groove and disk accelerate upwards, $\vec{a} = a\hat{\jmath}$. Neglecting gravity, what are the forces on the disk due to the groove?



Problem 14.2.21

14.2.22 The following problems concern a box that is in the back of a pickup truck. The pickup truck is moving forward with acceleration of a_t. The truck's speed is v_t. The box has sharp feet at the front and back ends so the only place it contacts the truck is at the feet. The center of mass of the box is at the geometric center of the box. The box has height h, length ℓ and depth w (into the paper.) Its mass is m. There is gravity. The friction coefficient between the truck and the box edges is μ.

In the problems below you should express your solutions in terms of the variables given in the figure, ℓ, h, μ, m, g, a_t, and v_t. If any variables do not enter the expressions comment on why they do not. In all cases you may assume that the box does not rotate (though it might be on the verge of doing so).

a) Assuming the box does not slide, what is the total force that the truck exerts on the box (i.e. the sum of the reactions at A and B)?

b) Assuming the box does not slide what are the reactions at A and B? [Note: You cannot find both of them without additional assumptions.]

c) Assuming the box does slide, what is the total force that the truck exerts on the box?

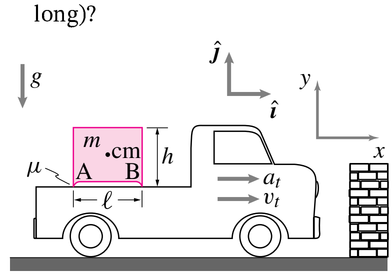
d) Assuming the box does slide, what are the reactions at A and B?

e) Assuming the box does not slide, what is the maximum acceleration of the truck for which the box will not tip over (hint: just at that critical acceleration what is the vertical reaction at B?)?

f) What is the maximum acceleration of the truck for which the block will not slide?

g) The truck hits a brick wall and stops instantly. Does the block tip over?

Assuming the block does not tip over, how far does it slide on the truck before stopping (assume the bed of the truck is sufficiently long)?



Problem 14.2.22

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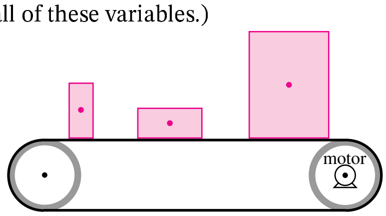
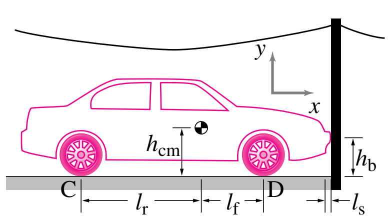
14.2.23 A collection of uniform boxes with various heights h and widths w and masses m sit on a horizontal conveyer belt. The acceleration $a(t)$ of the conveyer belt gets extremely large sometimes due to an erratic over-powered motor. Assume the boxes touch the belt at their left and right edges only and that the coefficient of friction there is μ. It is observed that some boxes never tip over. What is true about μ, g, w, h, and m for the boxes that always maintain contact at both the right and left bottom edges? (Write an inequality that involves some or all of these variables.)



Problem 14.2.23

14.2.24 After failure of her normal brakes, a driver pulls the emergency brake of her old car. This action locks the rear wheels (friction coefficient = μ) but leaves the well lubricated and light front wheels spinning freely. The car, braking inadequately as is the case for rear wheel braking, hits a stiff and slippery phone pole which compresses the car bumper. The car bumper is modeled here as a linear spring (constant = k, rest length = l_0, present length = l_s). The car is still traveling forward at the instant of interest. The bumper is at a height h_b above the ground. Assume that the car, excepting the bumper, is a non-rotating rigid body and that the wheels remain on the ground (that is, the bumper is compliant but the suspension is stiff).

- What is the acceleration of the car in terms of g, m, μ, l_f, l_r, k, h_b, h_{cm}, l_0, and l_s (and any other parameters if needed)?



Problem 14.2.24

14.2.25 Car braking: front brakes versus rear brakes versus all four brakes. What is the peak deceleration of a car when you apply: the front brakes till they skid, the rear brakes till they skid, and all four brakes till they skid? Assume that the coefficient of friction between rubber and road is $\mu = 1$ (about right, the coefficient of friction between rubber and road varies between about .7 and 1.3) and that $g = 10\,\mathrm{m/s^2}$ (2% error). Pick the dimensions and mass of the car, but assume the center of mass height h is greater than zero but is less than half the wheel base w, the distance between the front and rear wheel. Also assume that the CM is halfway between the front and back wheels (*i.e.,* $l_f = l_r = w/2$). The car has a stiff suspension so the car does not move up or down or tip appreciably during braking. Neglect the mass of the rotating wheels in the linear and angular momentum balance equations. Treat this problem as two-dimensional problem; i.e., the car is symmetric left to right, does not turn left or right, and that the left and right wheels carry the same loads. To organize your work, here are some steps to follow.

a) Draw a FBD of the car assuming rear wheel is skidding. The FBD should show the dimensions, the gravity force, what you know *a priori* about the forces on the wheels from the ground (i.e., that the friction force $F_r = \mu N_r$, and that there is no friction at the front wheels), and the coordinate directions. Label points of interest that you will use in your momentum balance equations. (Hint: also draw a free-body diagram of the rear wheel.)

b) Write the equation of linear momentum balance.

c) Write the equation of angular momentum balance relative to a point of your choosing. Some particularly useful points to use are:
   - the point above the front wheel and at the height of the center of mass;
   - the point at the height of the center of mass, behind the rear wheel that makes a 45 degree angle line down to the rear wheel ground contact point; and
   - the point on the ground straight under the front wheel that is as far below ground as the wheel base is long.

d) Solve the momentum balance equations for the wheel contact forces and the deceleration of the car. If you have used any or all of the recommendations from part (c) you will have the pleasure of only solving one equation in one unknown at a time. *

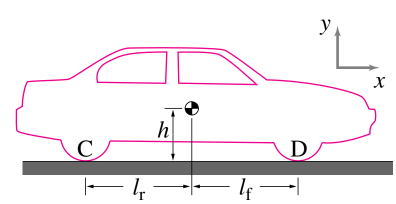
e) Repeat steps (a) to (d) for front-wheel skidding. Note that the advantageous points to use for angular momentum balance are now different. Does a car stop faster or slower or the same by skidding the front instead of the rear wheels? Would your solution to (e) be different if the center of mass of the car were at ground level(h=0)? *

f) Repeat steps (a) to (d) for all-wheel skidding. There are some shortcuts here. You determine the car deceleration without ever knowing the wheel reactions (or using angular momentum balance) if you look at the linear momentum balance equations carefully. *

g) Does the deceleration in (f) equal the sum of the decelerations in (d) and (e)? Why or why not? *

h) What peculiarity occurs in the solution for front-wheel skidding if the wheel base is twice the height of the CM above ground and $\mu = 1$? *

i) What impossibility does the solution predict if the wheel base is shorter than twice the CM height? What wrong assumption gives rise to this impossibility? What would really happen if one tried to skid a car this way? *



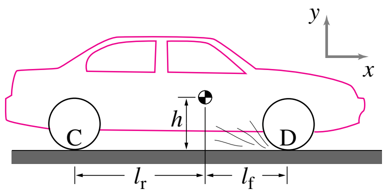
Problem 14.2.25

14.2.26 Assuming massless wheels, an infinitely powerful engine, a stiff suspension (*i.e.,* no rotation of the car) and a coefficient of friction μ between tires and road,

a) what is the maximum forward acceleration of this front wheel drive car? *

b) what is the force of the ground on the rear wheels during this acceleration?

c) what is the force of the ground on the front wheels?



Problem 14.2.26

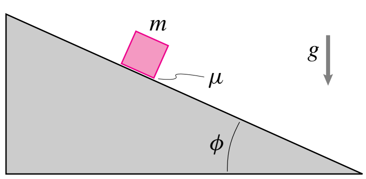
14.2.27 At time $t = 0$, the block of mass m is released from rest on the slope of angle ϕ. The coefficient of friction between the block and slope is μ.

a) What is the acceleration of the block for $\mu > 0$? *

b) What is the acceleration of the block for $\mu = 0$? *

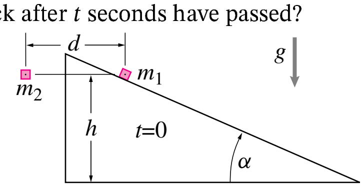
c) Find the position and velocity of the block as a function of time for $\mu > 0$. *

d) Find the position and velocity of the block as a function of time for $\mu = 0$. *



Problem 14.2.27

14.2.28 A small block of mass m_1 is released from rest at altitude h on a frictionless slope of angle α. At the instant of release, another small block of mass m_2 is dropped vertically from rest at the same altitude. The second block does not interact with the ramp. What is the velocity of the first block relative to the second block after t seconds have passed?



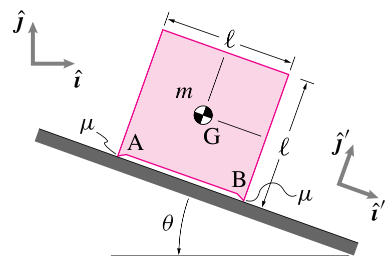
Problem 14.2.28

14.2.29 Block sliding on a ramp with friction. A square box is sliding down a ramp of angle θ with instantaneous velocity $v\hat{\imath}'$. Assume it does not tip over.

a) What is the force on the block from the ramp at point A? Answer in terms of any or all of θ, ℓ, m, g, μ, v, $\hat{\imath}'$, and $\hat{\jmath}'$. As a check, your answer should reduce to $\frac{mg}{2}\hat{\jmath}'$ when $\theta = \mu = 0$. *

b) In addition to solving the problem by hand, see if you can write a set of computer commands that, if θ, μ, ℓ, m, v and g were specified, would give the correct answer.

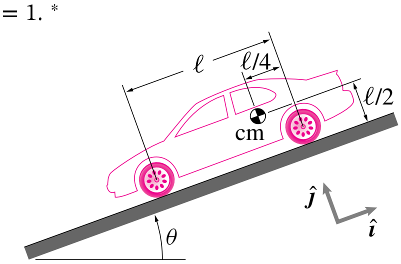
c) Assuming $\theta = 80°$ and $\mu = 0.9$, can the box slide this way or would it tip over? Why? *



Problem 14.2.29

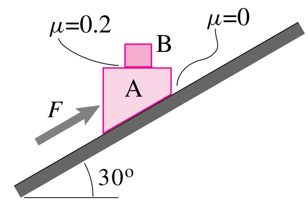
14.2.30 A coin is given a sliding shove up a ramp with angle ϕ with the horizontal. It takes twice as long to slide down as it does to slide up. What is the coefficient of friction μ between the coin and the ramp. Answer in terms of some or all of m, g, ϕ and the initial sliding velocity v.

14.2.31 A skidding car. What is the braking acceleration of the front-wheel braked car as it slides down hill. Express your answer as a function of any or all of the following variables: the slope θ of the hill, the mass of the car m, the wheel base ℓ, and the gravitational constant g. Use $\mu = 1$. *



Problem 14.2.31: A car skidding downhill on a slope of angle θ

14.2.32 Two blocks A and B are pushed up a frictionless inclined plane by an external force F as shown in the figure. The coefficient of friction between the two blocks is $\mu = 0.2$. The masses of the two blocks are $m_A = 5\,\text{kg}$ and $m_B = 2\,\text{kg}$. Find the magnitude of the maximum allowable force such that no relative slip occurs between the two blocks.



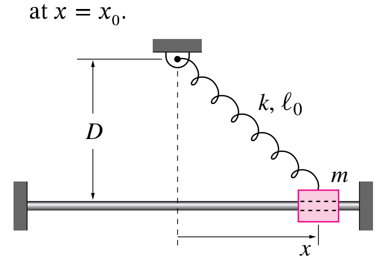
Problem 14.2.32

14.2.33 A bead slides on a frictionless rod. The spring has constant k and rest length ℓ_0. The bead has mass m.

a) Given x and $\dot{x}$ find the acceleration of the bead (in terms of some or all of $D, \ell_0, x, \dot{x}, m, k$ and any base vectors that you define).

b) If the bead is allowed to move, as constrained by the slippery rod and the spring, find a differential equation that must be satisfied by the variable x. (Do not try to solve this somewhat ugly non-linear equation.)

c) In the special case that $\ell_0 = 0$* find how long it takes for the block to return to its starting position after release with no initial velocity at $x = x_0$.



Problem 14.2.33

14.2.34 A bead oscillates on a straight frictionless wire. The spring obeys the equation $F = k(\ell - \ell_0)$, where ℓ = length of the spring and ℓ_0 is the 'rest' length. Assume

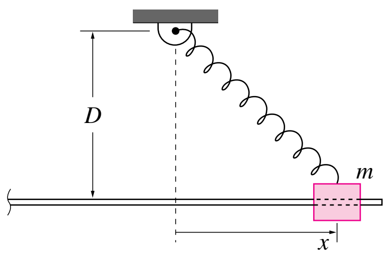
$$x(t = 0) = x_0,\ \dot{x}(t = 0) = 0.$$

a) Write a differential equation satisfied by $x(t)$.

b) What is $\dot{x}$ when $x = 0$? [hint: Don't try to solve the equation in (a)!]

c) What is the simplification in (a) if $\ell_o = 0$ (spring is then a so-called "zero–length" spring).

d) For this special case ($\ell_o = 0$) solve the equation in (a) and show the result agrees with (b) in this special case.

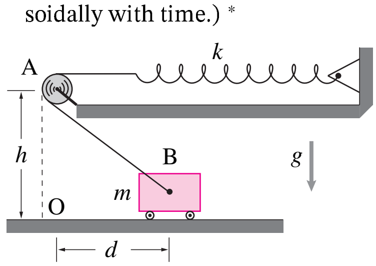


Problem 14.2.34

14.2.35 A cart on an elastic leash. A cart B (mass m) rolls on a frictionless level floor. One end of an inextensible string is attached to the cart. The string wraps around a pulley at point A and the other end is attached to a spring with constant k. When the cart is at point O, it is in static equilibrium. The spring relaxed length, rope length, and room height h are such that the spring would be relaxed if the end of rope at B were disconnected from the cart and brought up to point A. The gravitational constant is g. The cart is pulled a horizontal distance d from the center of the room (at O) and released.

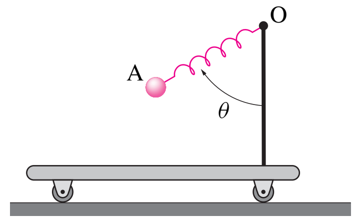
a) Assuming that the cart never leaves the floor, what is the speed of the cart when it passes through the center of the room, in terms of m, h, g and d. *

b) Does the cart undergo simple harmonic motion for small or large oscillations (specify which if either)? (Simple harmonic motion occurs when position varies sinusoidally with time.) *



Problem 14.2.35

14.2.36 The cart moves to the right with constant acceleration a. The ball has mass m. The spring has unstretched length ℓ_0 and spring constant k. Assuming the ball is stationary with respect to the cart find the distance from O to A in terms of k, ℓ_0, and a . [Hint: find θ first.]



Problem 14.2.36

14.2.37 Consider a person, modeled as a rigid body, riding an accelerating motorcycle (2-D). The person is sitting on the seat and cannot slide fore or aft, but is free to rock in the plane of the motorcycle (as if there were a hinge connecting the motorcycle to the rider at the seat). The person's feet are off the pegs and the legs are sticking down and not touching anything. The person's arms are like cables (they are massless and only carry tension). Assume all dimensions and masses are known (you have to define them carefully with a sketch and words). Assume the forward acceleration of the motorcycle is known. You may use numbers and/or variables to describe the quantities of interest.

a) Draw a clear sketch of the problem showing needed dimensional information and the coordinate system you will use.

b) Draw a Free-Body Diagram of the rider.

c) Write the equations of linear and angular momentum balance for the rider.

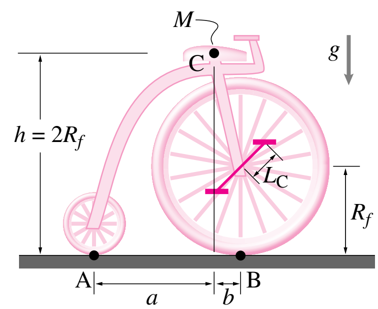
d) Find all forces on the rider from the motorcycle (i.e., at the hands and the seat).

e) What are the forces on the motorcycle from the rider?

14.2.38 Acceleration of a bicycle on level ground. 2-D . A very compact bicycler (modeled as a point mass M at the bicycle seat C with height h, and distance b behind the front wheel contact), rides a very light old-fashioned bicycle (all components have negligible mass) that is well maintained (all bearings have no frictional torque) and streamlined (neglect air resistance). The rider applies a force F_p to the pedal perpendicular to the pedal crank (with length L_c). No force is applied to the other pedal. The radius of the front wheel is R_f.

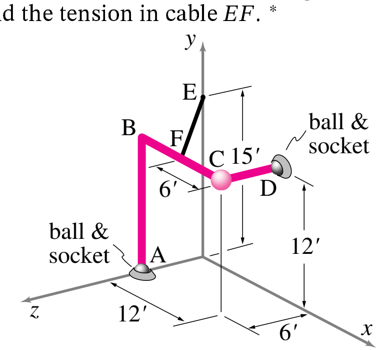
a) Assuming no slip, what is the forward acceleration of the bicycle? [Hint: draw a FBD of the front wheel and crank, and another FBD of the whole bicycle-rider system.] *

b) (Harder) Assuming the rider can push arbitrarily hard but that $\mu = 1$, what is the maximum possible forward acceleration of the bicycle. *



Problem 14.2.38

14.2.39 A 320 lbm mass is attached at the corner C of a light rigid piece of pipe bent as shown. The pipe is supported by ball-and-socket joints at A and D and by cable EF. The points A, D, and E are fastened to the floor and vertical sidewall of a pick-up truck which is accelerating in the z-direction. The acceleration of the truck is $\vec{a} = 5\,\mathrm{ft/s^2}\hat{k}$. There is gravity. Find the tension in cable EF. *

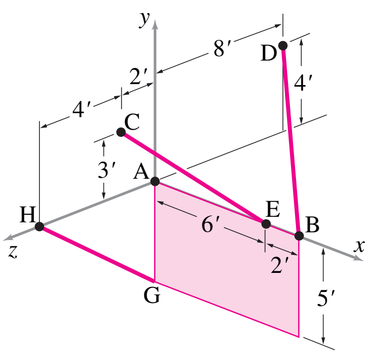


Problem 14.2.39

14.2.40 A 5 ft by 8 ft rectangular plate of uniform density has mass $m = 10\,\mathrm{lbm}$ and is supported by a ball-and-socket joint at point A and the light rods CE,BD, and GH. The entire system is attached to a truck which is moving with acceleration $\vec{\boldsymbol{a}}_T$. The plate is moving without rotation or angular acceleration relative to the truck. Thus, the center of mass acceleration of the plate is the same as the truck's. Dimensions are as shown. Points A, C, and D are fixed to the truck but the truck is not touching the plate at any other points. Find the tension in rod BD.

a) If the truck's acceleration is $\boldsymbol{a}_{cm} = (5\,\mathrm{ft/s^2})\boldsymbol{k}$, what is the tension or compression in rod BD? *

b) If the truck's acceleration is $\vec{\boldsymbol{a}}_{\mathrm{cm}} = (5\,\mathrm{ft/s^2})\hat{\boldsymbol{\jmath}} + (6\,\mathrm{ft/s^2})\hat{\boldsymbol{k}}$, what is the tension or compression in rod GH? *



Problem 14.2.40

14.2.41 Hanging a shelf. A shelf with negligible mass supports a 0.5 kg mass at its center. The shelf is supported at one corner with a ball and socket joint and at the other three corners with strings. At the instant of interest the shelf is in a rocket in outer space and accelerating at $10\,\mathrm{m/s^2}$ in the $\boldsymbol{k}$ direction. The shelf is in the xy plane.

a) Draw a FBD of the shelf.

b) Challenge: without doing any calculations on paper can you find one of the reaction force components or the tension in any of the cables? Give yourself a few minutes of staring to try this approach. If you can't, then come back to this question after you have done all the calculations. *

c) Write down the linear momentum balance equation (a vector equation). *

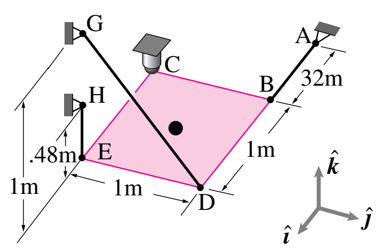
d) Write down the angular momentum balance equation using the center of mass as a reference point. *

e) By taking components, turn (b) and (c) into six scalar equations in six unknowns. *

f) Solve these equations by hand or on the computer. *

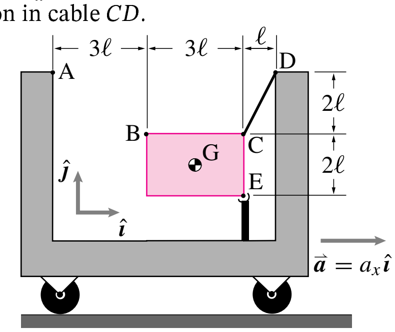
g) Instead of using a system of equations try to find a single equation which can be solved for T_{EH}. Solve it and compare to your result from before. *

h) Challenge: For how many of the reactions can you find one equation which will tell you that particular reaction without knowing any of the other reactions? [Hint, try angular momentum balance about various axes as well as linear momentum balance in an appropriate direction. It is possible to find five of the six unknown reaction components this way.] Must these solutions agree with (d)? Do they?



Problem 14.2.41

14.2.42 A uniform rectangular plate of mass m is supported by an inextensible cable CD and a hinge joint at point E on the cart as shown. The hinge joint is attached to a rigid column welded to the floor of the cart. The cart has acceleration $a_x\hat{\boldsymbol{\imath}}$. There is gravity. Find the tension in cable CD.

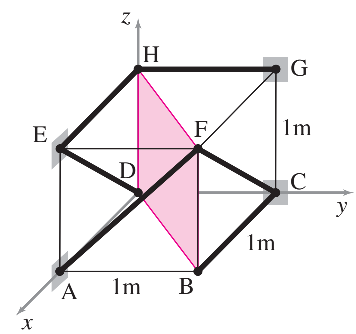


Problem 14.2.42

14.2.43 The uniform 2 kg plate DBFH is held by six massless rods (AF, CB, CF, GH, ED, and EH) which are hinged at their ends. The support points A, C, G, and E are all accelerating in the x-direction with acceleration $\boldsymbol{a} = 3\,\mathrm{m/s^2}\boldsymbol{\imath}$. There is no gravity.

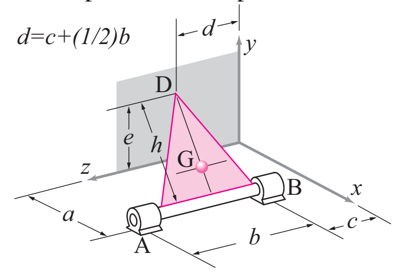
a) What is $\{\sum \vec{\boldsymbol{F}}\} \cdot \hat{\boldsymbol{\imath}}$ for the forces acting on the plate?

b) What is the tension in bar CB?



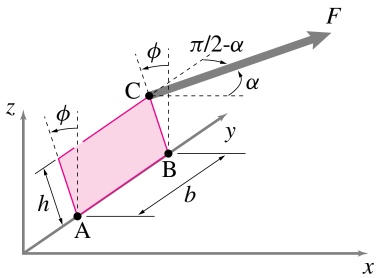
Problem 14.2.43

14.2.44 A massless triangular plate rests against a frictionless wall of a pick-up truck at point D and is rigidly attached to a massless rod supported by two ideal bearings fixed to the floor of the pick-up truck. A ball of mass m is fixed to the centroid of the plate. There is gravity. The pick-up truck skids across a road with acceleration $\vec{\boldsymbol{a}} = a_x\hat{\boldsymbol{\imath}} + a_z\hat{\boldsymbol{k}}$. What is the reaction at point D on the plate?



Problem 14.2.44

14.2.45 Towing a bicycle. A bicycle on the level xy plane is steered straight ahead and is being towed by a rope. The bicycle and rider are modeled as a uniform plate with mass m (for the convenience of the artist). The tow force F applied at C has no z component and makes an angle α with the x axis. The rolling wheel contacts are at A and B. The bike is tipped an angle ϕ from the vertical. The towing force F is the magnitude needed to keep the bike accelerating in a straight line (along the y axis) without tipping any more or less than the angle ϕ. What is the acceleration of the bicycle? Answer in terms of some or all of b, h, α, ϕ, m, g and $\hat{\jmath}$ (Note: F should *not* appear in your final answer.)



Problem 14.2.45

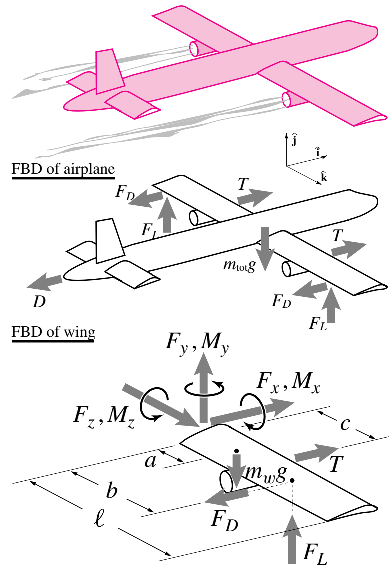
14.2.46 An airplane is in straight level flight but is accelerating in the forward direction. In terms of some or all of the following parameters,

- $m_{tot} \equiv$ the total mass of the plane (including the wings),
- D= the drag force on the fuselage,
- F_D= the drag force on each wing,
- g= gravitational constant, and,
- T = the thrust of one engine.

a) What is the lift on each wing F_L? *

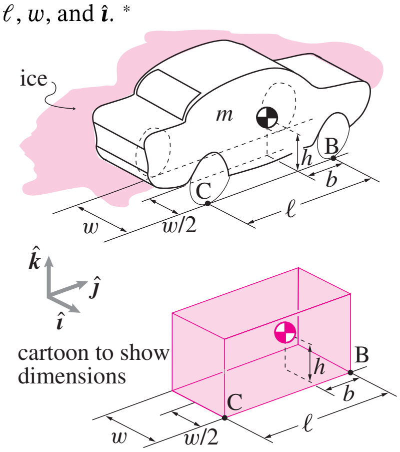
b) What is the acceleration of the plane $\vec{a}_P$? *

c) A free-body diagram of one wing is shown. The mass of one wing is m_w. What, in terms of m_{tot}, m_w, F_L, F_D, g, a,b, c, and ℓ are the reactions at the base of the wing (where it is attached to the plane), $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$ and $\vec{M} = M_x\hat{\imath} + M_y\hat{\jmath} + M_z\hat{k}$? *



Problem 14.2.46

14.2.47 A rear-wheel drive car on level ground. The two left wheels are on perfectly slippery ice. The right wheels are on dry pavement. The negligible-mass front right wheel at B is steered straight ahead and rolls without slip. The right rear wheel at C also rolls without slip and drives the car forward with velocity $\vec{v} = v\hat{\jmath}$ and acceleration $\vec{a} = a\hat{\jmath}$. Dimensions are as shown and the car has mass m . What is the sideways force from the ground on the right front wheel at B? Answer in terms of any or all of m, g, a, b, ℓ, w, and $\hat{\imath}$. *

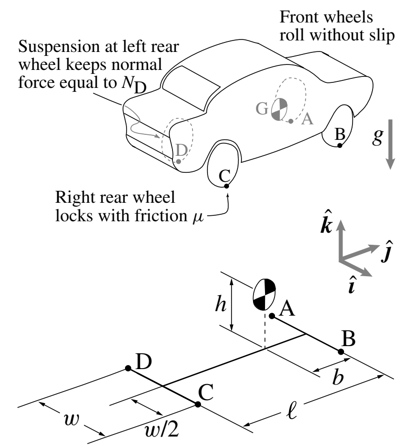


Problem 14.2.47: The left wheels of this car are on ice.

14.2.48 A somewhat crippled car slams on the brakes. The suspension springs at A, B, and C are frozen and keep the car level and at constant height. The normal force at D is kept equal to N_D by the only working suspension spring which is on the left rear wheel at D. The only brake which is working is that of the right rear wheel at C which slides on the ground with friction coefficient μ. Wheels A, B, and D roll freely without slip. Dimensions are as shown.

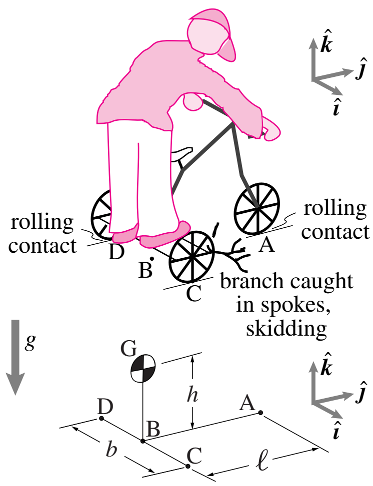
a) Find the acceleration of the car in terms of some or all of $m, w, \ell, b, h, g, \mu, \hat{\jmath}$, and N_D.

b) From the information given could you also find all of the reaction forces at all of the wheels? If so, why? If not, what can't you find and why? (No credit for correct answer. Credit depends on clear explanation.)



Problem 14.2.48: Given normal force at D, skidding at C.

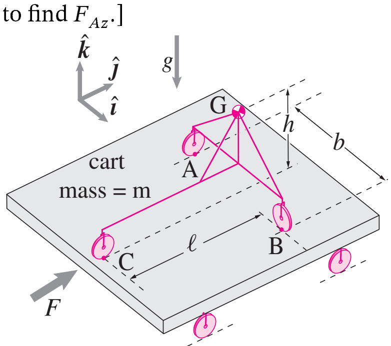
14.2.49 Speeding tricycle gets a branch caught in the right rear wheel. A scared-stiff tricyclist riding on level ground gets a branch stuck in the right rear wheel so the wheel skids with friction coefficient μ. Assume that the center of mass of the tricycle-person system is directly above the rear axle. Assume that the left rear wheel and the front wheel have negligible mass, good bearings, and have sufficient friction that they roll in the $\hat{\jmath}$ direction without slip, thus constraining the overall motion of the tricycle. Dimensions are shown in the lower sketch. *Find the acceleration of the tricycle* (in terms of some or all of ℓ, h, b, m, $[I^{cm}]$, μ, g, $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$). [Hint: check your answer against special cases for which you might guess the answer, such as when $\mu = 0$ or when $h = 0$.]



Problem 14.2.49: Right rear wheel skids.

14.2.50 A 3-wheeled robot. A 3-wheeled robot with mass m is being transported on a level flatbed trailer also with mass m. The trailer is being pushed with a force $F\hat{\jmath}$. The ideal massless trailer wheels roll without slip. The ideal massless robot wheels also roll without slip. The robot steering mechanism has turned the wheels so that wheels at A and C are free to roll in the $\hat{\jmath}$ direction and the wheel at B is free to roll in the $\hat{\imath}$ direction. The center of mass of the robot at G is h above the trailer bed and symmetrically above the axle connecting wheels A and B. The wheels A and B are a distance b apart. The length of the robot is ℓ.

Find the force vector $\vec{F}_A$ of the trailer on the robot at A in terms of some or all of $m, g, \ell, F, b, h, \hat{\imath}, \hat{\jmath}$, and $\hat{k}$. [Hints: Use a free-body diagram of the cart with robot to find their acceleration. With reference to a free-body diagram of the robot, use angular momentum balance about axis BC to find F_{Az}.]



Problem 14.2.50