SAMPLE 14.1 Find the motion of two cars. One car is towing another of equal mass on level ground. The thrust of the wheels of the first car is F. The second car rolls frictionlessly. Find the acceleration of the system two ways:

1. using separate free-body diagrams,
2. using a system free-body diagram.

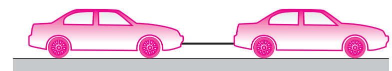

Figure 14.7

Solution

1. The free-body diagram of each car is shown below, in fig. 14.8.

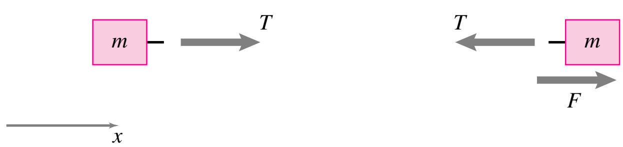

Figure 14.8: Partial free-body diagrams of the two cars (the vertical ground reactions are not shown as they are of no interest to us for the horizontal motion.

From the linear momentum balance of each car, we get

$$\begin{aligned} m\ddot{x}_1 &= T && (14.9)\\ F-T &= m\ddot{x}_2 && (14.10)\end{aligned}$$

The kinematic constraint of towing (the cars move together, *i.e.*, no relative displacement between the cars) gives

$$\ddot{x}_1 - \ddot{x}_2 = 0 \qquad (14.11)$$

Solving eqns. (14.9), (14.10), and (14.11) simultaneously, we get

$$\ddot{x}_1 = \ddot{x}_2 = \frac{F}{2m} \quad (T = \frac{F}{2})$$

2. The free-body diagram of the two cars together is shown below, in fig. 14.9.

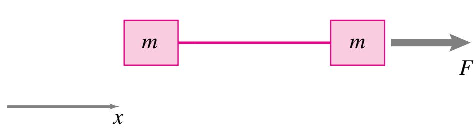

Figure 14.9:

From the linear momentum balance of the two cars as one system, we get

$$\begin{aligned} m\ddot{x} + m\ddot{x} &= F\\ \ddot{x} &= F/2m\end{aligned}$$

$$\boxed{\ddot{x} = \ddot{x}_1 = \ddot{x}_2 = F/2m}$$

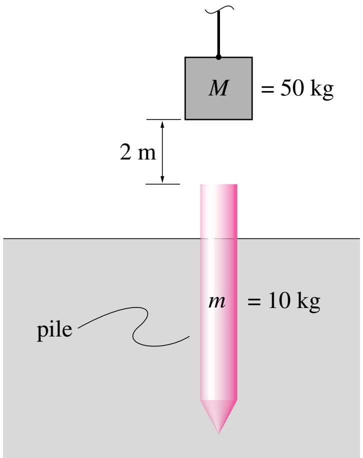

Figure 14.10

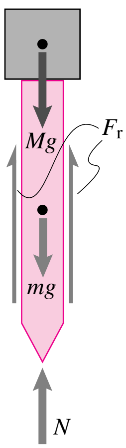

Figure 14.11: Free-body diagram of the hammer and pile system. F_r is the total resistance of the ground.

SAMPLE 14.2 **Driving a pile into the ground.** A cylindrical wooden pile of mass 10 kg and cross-sectional diameter 20 cm is driven into the ground with the blows of a hammer. The hammer is a block of steel with mass 50 kg which is dropped from a height of 2 m to deliver the blow. At the nth blow the pile is driven into the ground by an additional 5 cm. Assuming the impact between the hammer and the pile to be totally inelastic (*i.e.,* the two stick together), find the average resistance of the soil to penetration of the pile.

Solution Let F_r be the average (constant over the period of driving the pile by 5 cm) resistance of the soil. From the free-body diagram of the pile and hammer system, we have

$$\sum \vec{F} = -mg\hat{\boldsymbol{\jmath}} - Mg\hat{\boldsymbol{\jmath}} + N\hat{\boldsymbol{\jmath}} + F_r\hat{\boldsymbol{\jmath}}.$$

But N is the normal reaction of the ground, which from static equilibrium, must be equal to $mg + Mg$. Thus,

$$\sum \vec{F} = F_r\hat{\boldsymbol{\jmath}}.$$

Therefore, from linear momentum balance ($\sum \vec{F} = m\vec{a}$),

$$\vec{a} = \frac{F_r}{M+m}\hat{\boldsymbol{\jmath}}.$$

Now we need to find the acceleration from given conditions. Let v be the speed of the hammer just before impact and V be the combined speed of the hammer and the pile immediately after impact. Then, treating the hammer and the pile as one system, we can ignore all other forces *during* the impact (none of the external forces: gravity, soil resistance, ground reaction, is comparable to the impulsive impact force, see page 904). The impact force is internal to the system. Therefore, during impact, $\sum \vec{F} = \vec{0}$ which implies that linear momentum is conserved. Thus

$$\begin{aligned} -Mv\hat{\boldsymbol{\jmath}} &= -(m+M)V\hat{\boldsymbol{\jmath}} \\ \Rightarrow \quad V &= \left(\frac{M}{m+M}\right)v = \frac{50\,\text{kg}}{60\,\text{kg}}v = \frac{5}{6}v. \end{aligned}$$

The hammer speed v can be easily calculated, since it is the free fall speed from a height of 2 m:

$$v = \sqrt{2gh} = \sqrt{2\cdot(9.81\,\text{m/s}^2)\cdot(2\,\text{m})} \quad = 6.26\,\text{m/s} \quad \Rightarrow \quad V = \frac{5}{6}v = 5.22\,\text{m/s}.$$

The pile and the hammer travel a distance of $s = 5\,\text{cm}$ under the deceleration a. The initial speed $V = 5.22\,\text{m/s}$ and the final speed $= 0$. Plugging these quantities into the one-dimensional kinematic formula

$$v^2 = v_0^2 + 2as,$$

we get,

$$\begin{aligned} 0 &= V^2 - 2as \quad \text{(Note that } a \text{ is negative)} \\ \Rightarrow \quad a &= \frac{V^2}{2s} = \frac{(5.22\,\text{m/s})^2}{2\times 0.05\,\text{m}} = 272.48\,\text{m/s}^2. \end{aligned}$$

Thus $\vec{a} = 272.48\,\text{m/s}^2\,\hat{\boldsymbol{\jmath}}$. Therefore,

$$F_r = (m+M)a = (60\,\text{kg})\cdot(272.48\,\text{m/s}^2) = 1.635\times 10^4\,\text{N}$$

$$\boxed{F_r \approx 16.35\,\text{kN}}$$

SAMPLE 14.3 **Pulley kinematics.** For the masses and ideal-massless pulleys shown in figure 14.12, find the acceleration of mass A in terms of the acceleration of mass B. Pulley C is fixed to the ceiling and pulley D is free to move vertically. All strings are inextensible.

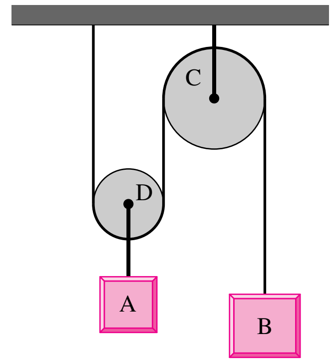

Figure 14.12

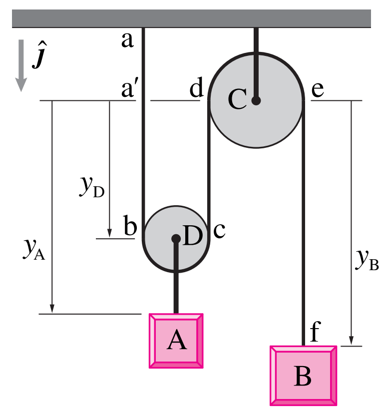

Figure 14.13

Solution Let us measure the position of the two masses from a fixed point, say the center of pulley C. (Since C is fixed, its center is fixed too.) Let y_A and y_B be the vertical distances of masses A and B, respectively, from the chosen reference (C). Then the position vectors of A and B are:

$$\vec{r}_A = y_A\hat{\jmath} \quad \text{and} \quad \vec{r}_B = y_B\hat{\jmath}.$$

Therefore, the velocities and accelerations of the two masses are

$$\begin{aligned} \vec{v}_A &= \dot{y}_A\hat{\jmath}, \qquad \vec{v}_B = \dot{y}_B\hat{\jmath}, \\ \vec{a}_A &= \ddot{y}_A\hat{\jmath}, \qquad \vec{a}_B = \ddot{y}_B\hat{\jmath}. \end{aligned}$$

Since all quantities are in the same direction ($\hat{\jmath}$), we can drop $\hat{\jmath}$ from our calculations and just do scalar calculations. We are asked to relate $\ddot{y}_A$ to $\ddot{y}_B$.

In all pulley problems, the trick in doing kinematic calculations is to relate the variable positions to the fixed length of the string. Here, the length of the string ℓ_{tot} is: (1)

$$\begin{aligned} \ell_{tot} &= \text{ab} + \text{bc} + \text{cd} + \text{de} + \text{ef} = \text{constant} \\ \text{where} \quad \text{ab} &= \underbrace{\text{aa}'}_{\text{constant}} + \underbrace{\text{a}'\text{b}}_{(=\text{cd}=y_D)} \\ \text{bc} &= \text{string over the pulley D} = \text{constant} \\ \text{de} &= \text{string over the pulley C} = \text{constant} \\ \text{ef} &= y_B \\ \text{thus} \quad \ell_{tot} &= 2y_D + y_B + \overbrace{\text{constant}}^{(\text{aa}'+\text{bc}+\text{de})}. \end{aligned}$$

Taking the time derivative on both sides, we get

$$\overbrace{\frac{d}{dt}(\ell_{tot})}^{\boxed{=\ 0,\text{ because } \ell_{tot} \text{ does not change with time}}} = 2\dot{y}_D + \dot{y}_B \quad \Rightarrow \quad \dot{y}_D = -\frac{1}{2}\dot{y}_B \tag{14.12}$$

$$\Rightarrow \quad \ddot{y}_D = -\frac{1}{2}\ddot{y}_B. \tag{14.13}$$

$$\begin{aligned} \text{But} \quad & y_D = y_A - \text{AD and AD} = \text{constant} \\ \Rightarrow \quad & \dot{y}_D = \dot{y}_A \quad \text{and} \quad \ddot{y}_D = \ddot{y}_A. \end{aligned}$$

Thus, substituting $\dot{y}_A$ and $\ddot{y}_A$ for $\dot{y}_D$ and $\ddot{y}_D$ in (14.12) and (14.13) we get

$$\dot{y}_A = -\frac{1}{2}\dot{y}_B \quad \text{and} \quad \ddot{y}_A = -\frac{1}{2}\ddot{y}_B$$

$$\ddot{y}_A = -\tfrac{1}{2}\ddot{y}_B$$

(1) We have done an elaborate calculation of ℓ_{tot} here. Usually, the constant lengths over the pulleys and some constant segments such as aa′ are ignored in calculating ℓ_{tot}. These constant length segments can be ignored because they drop out of the equation when we take time derivatives to relate velocities and accelerations of different points, such as B and D here.

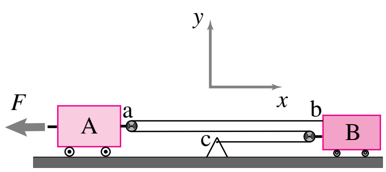

Figure 14.14: A two-mass pulley system.

SAMPLE 14.4 A two-mass pulley system. The two masses shown in Fig. 14.14 have frictionless bases and round frictionless pulleys. The inextensible cord connecting them is always taut. Given that $F = 130\,\text{N}$, $m_A = m_B = m = 40\,\text{kg}$, find the acceleration of the two blocks using:

1. linear momentum balance and
2. energy balance.

Solution

1. **Using Linear Momentum Balance:** The free-body diagrams of the two masses A and B are shown in Fig. 14.15.

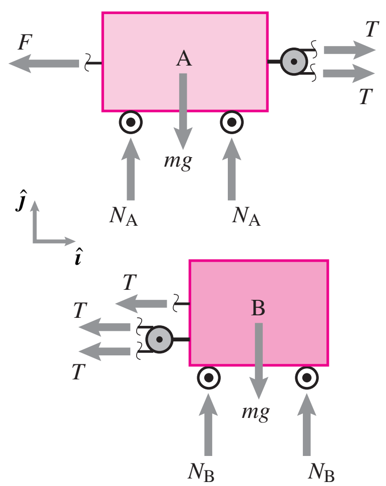

Figure 14.15: Free-body diagrams of the two masses.

Linear momentum balance for mass A gives (assuming $\vec{\boldsymbol{a}}_A = a_A\hat{\boldsymbol{\imath}}$ and $\vec{\boldsymbol{a}}_B = a_B\hat{\boldsymbol{\imath}}$):

$$
\begin{array}{rrcl}
 & (2T-F)\hat{\boldsymbol{\imath}} + (2N_A - mg)\hat{\boldsymbol{\jmath}} & = & m\vec{\boldsymbol{a}}_A = ma_A\hat{\boldsymbol{\imath}} \\
(\text{dotting with } \hat{\boldsymbol{\jmath}}) & \Rightarrow \quad 2N_A & = & mg \\
(\text{dotting with } \hat{\boldsymbol{\imath}}) & \Rightarrow \quad 2T - F & = & ma_A
\end{array} \tag{14.14}
$$

Similarly, linear momentum balance for mass B gives:

$$
\begin{array}{rcl}
-3T\hat{\boldsymbol{\imath}} + (2N_B - mg)\hat{\boldsymbol{\jmath}} & = & m\vec{\boldsymbol{a}}_B = ma_B\hat{\boldsymbol{\imath}} \\
\Rightarrow \quad 2N_B & = & mg \\
\text{and} \quad -3T & = & ma_B.
\end{array} \tag{14.15}
$$

From (14.14) and (14.15) we have three unknowns: T, a_A, a_B, but only 2 equations!. We need an extra equation to solve for the three unknowns. 2

2 You may be tempted to use angular momentum balance (AMB) to get an extra equation. In this case AMB could help determine the vertical reactions, but offers no help in finding the rope tension or the accelerations.

We can get the extra equation from kinematics. Since A and B are connected by a string of fixed length, their accelerations must be related. For simplicity, and since these terms drop out anyway, we neglect the radius of the pulleys and the lengths of the little connecting cords. We use the left wall as the reference position to get

$$
\begin{array}{rcl}
\ell_{tot} & \equiv & \text{length of the string connecting A and B} \\
 & = & 3(x_B - X_C) + 2(X_C - x_A) \\
 & = & 3x_B - X_C - 2x_A
\end{array}
$$

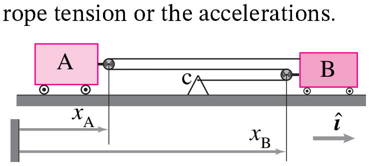

Figure 14.16: Pulley kinematics. The left wall is added as a reference point, off to the side of all pulleys and masses, to help prevent sign errors.

Now, differentiating the expression for ℓ_{tot} with time and noting that the total length of the string remains constant and that X_C is a fixed location in space, we get

$$
\begin{array}{rrcl}
 & \overbrace{\frac{d}{dt}(\ell_{tot})}^{0} & = & 3\dot{x}_B - \overset{0}{\cancel{\dot{X}_C}} - 2\dot{x}_A \\
\Rightarrow & \ddot{x}_B & = & \frac{2}{3}\ddot{x}_A
\end{array} \tag{14.16}
$$

$$\tag{14.17}$$

Since

$$
\begin{array}{rcl}
\vec{\boldsymbol{v}}_A & = & v_A\hat{\boldsymbol{\imath}} = \dot{x}_A\hat{\boldsymbol{\imath}}, \\
\vec{\boldsymbol{a}}_A & = & a_A\hat{\boldsymbol{\imath}} = \ddot{x}_A\hat{\boldsymbol{\imath}}, \\
\vec{\boldsymbol{v}}_B & = & v_B\hat{\boldsymbol{\imath}} = \dot{x}_B\hat{\boldsymbol{\imath}}, \text{ and} \\
\vec{\boldsymbol{a}}_B & = & a_B\hat{\boldsymbol{\imath}} = \ddot{x}_B\hat{\boldsymbol{\imath}},
\end{array}
$$

we get

$$a_B = \frac{2}{3} a_A. \tag{14.18}$$

Substituting (14.18) into (14.15), we get

$$9T = -2ma_A. \tag{14.19}$$

Now solving (14.14) and (14.19) for T, we get

$$T = \frac{2F}{13} = \frac{2 \cdot 130\,\mathrm{N}}{13} = 20\,\mathrm{N}.$$

Therefore,

$$\begin{aligned} a_A &= -\frac{9T}{2m} = -\frac{9 \cdot 20\,\mathrm{N}}{2 \cdot 40\,\mathrm{kg}} = -2.25\,\mathrm{m/s^2} \\ a_B &= \frac{2}{3} a_A = -1.5\,\mathrm{m/s^2} \end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}}_A = -2.25\,\mathrm{m/s^2}\hat{\boldsymbol{\imath}},\ \vec{\boldsymbol{a}}_B = -1.5\,\mathrm{m/s^2}\hat{\boldsymbol{\imath}}.}$$

2. **Using Power Balance (III):** We have,

$$P = \dot{E}_{\mathrm{K}}.$$

The power balance equation becomes

$$\sum \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}} = m\, a_A\, v_A + m_B\, a_B\, v_B.$$

Because the force at A is the only force that does work on the system, 3 when we apply power balance to the whole system (see the FBD in fig. 14.17), we get,

$$\begin{aligned} -Fv_A - T \overbrace{v_q}^{=v_c=0} &= m_A v_A a_A + m_B v_B a_B \\ \text{or} \qquad F &= -ma_A - m\frac{v_B}{v_A} a_B \\ &= -a_A\left(m + m\frac{v_B}{v_A}\frac{a_B}{a_A}\right). \end{aligned}$$

Substituting $a_B = 2/3 a_A$ and $v_B = 2/3 v_A$ from eqn. (14.18),

$$\begin{aligned} a_A &= \frac{-F}{m + \frac{4}{9}m} \\ &= \frac{-130\,\mathrm{N}}{40\,\mathrm{kg}(1 + \frac{4}{9})} \\ &= -2.25\,\mathrm{m/s^2}, \end{aligned}$$

and since $a_B = 2/3 a_A$,

$$a_B = -1.5\,\mathrm{m/s^2},$$

which are the same accelerations as found before.

$$\boxed{a_A = -2.25\,\mathrm{m/s^2}\hat{\boldsymbol{\imath}},\ a_B = -1.5\,\mathrm{m/s^2}\hat{\boldsymbol{\imath}}}$$

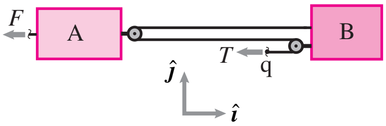

Figure 14.17: 1D free-body diagram of the whole system. Note that except F, no other forces do any work.

3 It may not be obvious why T shown in the FBD in fig. 14.17 does no work. T acts on a material point q' somewhere on the inextensible string from the fixed point 'c'. Even though mass B moves, point 'q' on the string has no displacement. Because the material point q on which it acts does not move, the work done by T is zero.

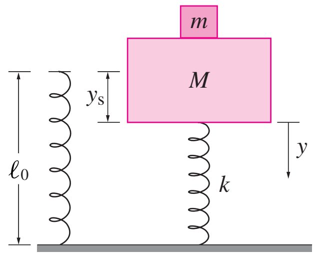

Figure 14.18

SAMPLE 14.5 In static equilibrium the spring in fig. 14.18 is compressed by y_s from its unstretched length ℓ_0. Now, the spring is compressed by an additional amount y_0 and released with no initial velocity.

1. Find the force on the top mass m exerted by the lower mass M.
2. When does this force become minimum? Can this force become zero?
3. Can the force on m due to M ever be negative?

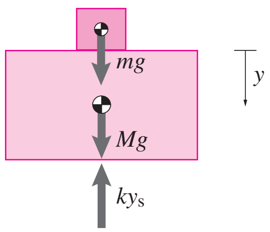

Figure 14.19: Free-body diagram of the two masses as one system in static equilibrium (this special case could be skipped as it follows from the free-body diagram below).

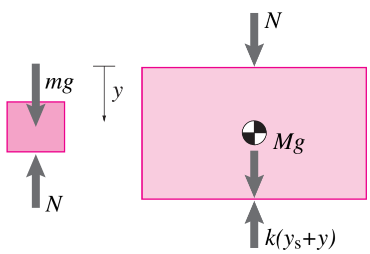

Figure 14.20: Free-body diagrams of the individual masses.

Solution

1. The free-body diagram of the two masses is shown in Figure 14.19 when the system is in static equilibrium. From linear momentum balance we have

$$\sum \vec{F} = \vec{\mathbf{0}} \quad \Rightarrow \quad ky_s = (m+M)g. \tag{14.20}$$

The free-body diagrams of the two masses at an arbitrary position y during motion are shown in Figure 14.20. Since the two masses oscillate together, they have the same acceleration. From linear momentum balance for mass m we get (note that we have chosen y to be positive downwards),

$$mg - N = m\ddot{y}. \tag{14.21}$$

We are interested in finding the normal force N. Clearly, we need to find $\ddot{y}$ to calculate N. Now, from linear momentum balance for mass M we get

$$Mg + N - k(y + y_s) = M\ddot{y}. \tag{14.22}$$

Adding eqn. (14.21) with eqn. (14.22) we get

$$(m+M)g - ky - ky_s = (m+M)\ddot{y}.$$

But $ky_s = (m+M)g$ from eqn. (14.20). Therefore, the equation of motion of the system is

$$\begin{aligned} -ky &= (m+M)\ddot{y} \\ \text{or} \quad \ddot{y} + \frac{k}{(m+M)}y &= 0. \end{aligned} \tag{14.23}$$

As you recall from your study of the harmonic oscillator, the general solution of this differential equation is

$$y(t) = A\sin\lambda t + B\cos\lambda t \tag{14.24}$$

$$\text{where} \quad \lambda = \sqrt{\frac{k}{m+M}}. \tag{14.25}$$

The constants A and B are to be determined from the initial conditions. From eqn. (14.24) we obtain

$$\dot{y}(t) = A\lambda\cos\lambda t - B\lambda\sin\lambda t. \tag{14.26}$$

Substituting the given initial conditions $y(0) = y_0$ and $\dot{y}(0) = 0$ in eqns. (14.24) and (14.26), respectively, we get

$$\begin{aligned} \overbrace{y(0)}^{y_0} &= A\sin(\lambda\cdot 0) + B\cos(\lambda\cdot 0) \quad \Rightarrow \quad B = y_0 \\ \underbrace{\dot{y}(0)}_{0} &= A\lambda\cos(\lambda\cdot 0) - B\lambda\sin(\lambda\cdot 0) \quad \Rightarrow \quad A = 0. \end{aligned}$$

Thus,

$$y(t) = y_0 \cos \lambda t. \tag{14.27}$$

Now we can find the acceleration by differentiating eqn. (14.27) twice :

$$\ddot{y} = -y_0 \lambda^2 \cos \lambda t.$$

Substituting this expression in eqn. (14.21) we get the force applied by mass M on the smaller mass m:

$$\begin{aligned} mg - N &= m \overbrace{(-y_0 \lambda^2 \cos \lambda t)}^{\ddot{y}} \\ \Rightarrow \quad N &= mg + m y_0 \lambda^2 \cos \lambda t \\ &= mg \left(1 + \frac{y_0 \lambda^2}{g} \cos \lambda t\right) \qquad (14.28) \end{aligned}$$

$$\boxed{N = mg \left(1 + \frac{y_0 \lambda^2}{g} \cos \lambda t\right)}$$

2. Since $\cos \lambda t$ varies between ± 1, the value of the force N varies between $mg \pm y_0 \lambda^2$. Clearly, N attains its minimum value when $\cos \lambda t = -1$, *i.e.,* when $\lambda t = \pi$. This condition is met when the spring is fully stretched and the mass is at its highest vertical position. At this point,

$$N \equiv N_{\text{min}} = mg \left(1 - \frac{y_0 \lambda^2}{g}\right).$$

If y_0, the initial displacement from the static equilibrium position, is chosen such that $y_0 \lambda^2 = g$ (that is, the amplitude of the harmonically varying acceleration equals g), then $N = 0$ when $\cos \lambda t = -1$, *i.e.,* at the topmost point in the vertical motion. This condition, $N = 0$, means that the two masses momentarily lose contact with each other; and it happens precisely when they are about to begin their downward motion.

3. From eqn. (14.28) we can get a negative value of N when $\cos \lambda t = -1$ and $y_0 \lambda^2 > g$. However, a negative value for N is nonsense unless the blocks are glued. Without glue the bigger mass M cannot apply a negative force (or a compression) on m, *i.e.,* it cannot "suck" m. When $y_0 \lambda^2 > g$ then N becomes zero before $\cos \lambda t$ decreases to -1. That is, assuming no bonding, the two masses lose contact on their way to the highest vertical position but before reaching the highest point. Beyond that point, the equations of motion derived above are no longer valid for unglued blocks because the equations assume contact between m and M. Equation (14.28) is inapplicable when $N \leq 0$.