SAMPLE 13.3 Projectile hits a slanted floor: A ball of mass $m = 0.2\,\mathrm{kg}$ is thrown in the air at an angle $\theta = 60°$ with initial speed $v_0 = 10\,\mathrm{m/s}$. The ball lands on a hard, frictionless floor that is tilted at angle $\alpha = 20°$ with the horizontal. The coefficient of restitution between the floor and the ball is $e = 0.85$. Ignore air resistance. Find the height of the ball after the rebound from the floor.

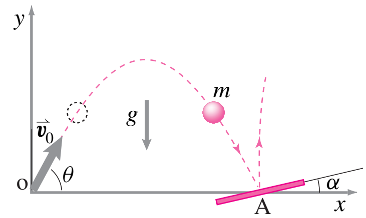

Figure 13.13

Solution This problem has two parts to it. In order to figure out the height after rebound, we need to find the rebound velocity. But to find the rebound velocity, we need to know the velocity of the ball before impact with the floor. Let the velocity just before the impact be $\vec{v}^-$ and the velocity of rebound (just after impact) be $\vec{v}^+$. Let us first find $\vec{v}^-$.

The ball undergoes projectile motion before it lands at A. Its initial (launch) velocity is $\vec{v}_0 = v_0(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath})$. From energy conservation, we know that the kinetic energy just before impact at A, $m|\vec{v}^-|^2/2$, must be the same as kinetic energy at launch, $mv_0^2/2$. Thus $|\vec{v}^-| = v_0$. And, from the symmetry of the flight, we can conclude that $\vec{v}^-$ must make the same angle θ with the horizontal that $\vec{v}_0$ does. Thus, using fig. 13.14, we have

$$\vec{v}^- = v_0(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}).$$

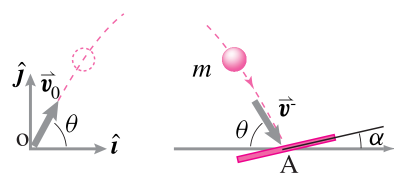

Figure 13.14: Trajectory of the ball through air: From the symmetry of the trajectory, we can conclude that the angle the velocity vector makes with the horizontal at point A is the same as the angle of launch, θ.

Now we are ready to do collision mechanics at point A. We need to determine $\vec{v}^+$ given $\vec{v}^-$ and the coefficient of restitution for the collision at A. From collision law, we know that the velocity component normal to the floor changes because of the normal impulse during collision, while the tangential velocity remains the same because there is no force or impulse parallel to the floor. Thus,

$$\begin{aligned} \vec{v}^+ \cdot \hat{\boldsymbol{\lambda}} &= \vec{v}^- \cdot \hat{\boldsymbol{\lambda}} \\ \vec{v}^+ \cdot \hat{\boldsymbol{n}} &= -e(\vec{v}^- \cdot \hat{\boldsymbol{n}}). \end{aligned}$$

Writing out $\vec{v}^+ = v_x^+\hat{\imath} + v_y^+\hat{\jmath}$, and noting that $\hat{\boldsymbol{\lambda}} = \cos\alpha\hat{\imath} + \sin\alpha\hat{\jmath}$ and $\hat{\boldsymbol{n}} = -\sin\alpha\hat{\imath} + \cos\alpha\hat{\jmath}$, we get, from the equations above,

$$\begin{aligned} \cos\alpha v_x^+ + \sin\alpha v_y^+ &= v_0\cos\theta\cos\alpha - v_0\sin\theta\sin\alpha = v_0\cos(\theta+\alpha) \\ -\sin\alpha v_x^+ + \cos\alpha v_y^+ &= -e\,v_0(-\cos\theta\sin\alpha - \sin\theta\cos\alpha) = e\,v_0\sin(\theta+\alpha). \end{aligned}$$

These are two equations in two unknowns, v_x^+ and v_y^+. Writing them in a matrix form and solving the matrix equation, we get

$$\begin{pmatrix} v_x^+ \\ v_y^+ \end{pmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{pmatrix} v_0\cos(\theta+\alpha) \\ e\,v_0\sin(\theta+\alpha) \end{pmatrix} = \begin{pmatrix} v_0\cos(\theta+2\alpha) \\ e\,v_0\sin(\theta+2\alpha) \end{pmatrix}.$$

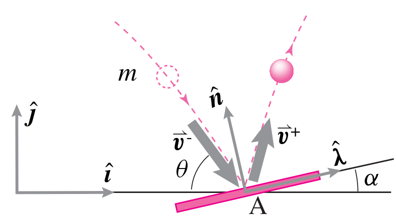

Figure 13.15: Collision of the ball with the tilted floor: The ball strikes the floor with velocity $\vec{v}^-$. It rebounds with velocity $\vec{v}^+$. The impulse during the collision acts in the normal direction to the floor at A. The unit normal and the unit tangent vectors at A are $\hat{\boldsymbol{n}} = -\sin\alpha\hat{\imath} + \cos\alpha\hat{\jmath}$, and $\hat{\boldsymbol{\lambda}} = \cos\alpha\hat{\imath} + \sin\alpha\hat{\jmath}$ respectively.

Thus, we know the rebound velocity $\vec{v}+ = v_0[\cos(\theta+2\alpha)\hat{\imath} + e\sin(\theta+2\alpha)\hat{\jmath}]$.

To find the maximum height reached by the ball on the rebound, we only need the vertical component of the rebound velocity. Since the ball has a constant deceleration g, we can use the formula $(v_y)^2 = (v_{y_0})^2 - 2gh$ with $v_y = 0$ at the maximum height $h_{\max}$ to get,

$$h_{\max} = \frac{(v_y^+)^2}{2g} = \frac{e^2\,v_0^2\,\sin^2(\theta+2\alpha)}{2g}$$

Substituting the given values of v_0, e, θ, and α, and using $g = 9.81\,\mathrm{m/s^2}$, we get,

$$h_{\max} = 3.57\,\mathrm{m}.$$

$$\boxed{h_{\max} = 3.57\,\mathrm{m}}$$

You can see that the inclined plane helps in getting the ball to reach higher on the bounce. If the floor were flat ($\alpha = 0$), we would get $h_{\max} = 2.76\,\mathrm{m}$. It should be obvious that for maximum height, we should have $\sin(\theta + 2\alpha) = 1$ which gives $\alpha = \frac{1}{2}(\frac{\pi}{2} - \theta)$.

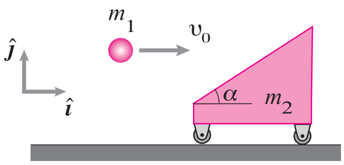

Figure 13.16

SAMPLE 13.4 Simultaneous collisions: This problem involves two simultaneous collisions. In general, such problems are hard to solve. We are going to show one way of solving such problems by treating the collisions successively. However, this leads to nonuniqueness of solution. Here we solve the problem in one way and in the next sample, we solve the same problem in another way.

A 12 kg cart with an inclined face rests on a frictionless floor. A ball of mass 3 kg is shot horizontally with speed 30 m/s at the inclined face of the cart. The coefficient of restitution between the cart and the ball is 0.9. The cart subsequently moves horizontally on the floor. Find the velocity of the ball and that of the cart after the collision.

Solution There are two simultaneous collisions in this problem. One collision is between the ball and the cart and the other is between the cart and the ground. Here, we will treat the two collisions one after the other, the one between the ball and the cart preceding the one between the cart and the ground. In Sample 13.5, we treat the ground collision first.

Collision between the ball and the cart: Here we assume that the ball hits the cart and both are free to move in any direction immediately after the collision. Let the mass of the ball be m_1 and that of the cart be m_2. Let their after collision velocities be $\vec{\boldsymbol{v}}_1^+$ and $\vec{\boldsymbol{v}}_2^+$, respectively. Let the impulse during this collision be P_1.

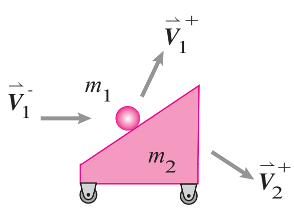

Figure 13.17: The ball and the cart as a system during the collision. There are no external forces or impulses acting on the system. Therefore, the linear momentum must be conserved during the collision; *i.e.,* $\vec{\boldsymbol{L}}^- = \vec{\boldsymbol{L}}^+$.

Let us consider the cart and the ball as a single system during the collision. Then, the impulse becomes internal to this system and there is no net impulse on this system. Therefore, the linear momentum is conserved; that is, $\vec{\boldsymbol{L}}^+ = \vec{\boldsymbol{L}}^-$. From this relationship, we have,

$$\mathrm{m}_1\vec{\boldsymbol{v}}_1^+ + \mathrm{m}_2\vec{\boldsymbol{v}}_2^+ = \mathrm{m}_1\vec{\boldsymbol{v}}_1^- + \mathrm{m}_2\vec{\boldsymbol{v}}_2^- = m_1 v_0\hat{\boldsymbol{\imath}}.$$

Writing out the unknown velocities in terms of their x and y components and dotting the resulting equation with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ separately, we get the following two scalar equations:

$$\begin{aligned} m_1 v_{1x}^+ + m_2 v_{2x}^+ &= m_1 v_0 && (13.7)\\ m_1 v_{1y}^+ + m_2 v_{2y}^+ &= 0. && (13.8)\end{aligned}$$

We have four unknowns here, v_{1x}^+, v_{1y}^+, v_{2x}^+, and v_{2y}^+. So far, we have just two equations. We need more equations. We can write restitution equation relating the relative velocities of the ball and the cart in the normal direction before and after the collision:

$$(\vec{\boldsymbol{v}}_2^+ - \vec{\boldsymbol{v}}_1^+)\cdot\hat{\boldsymbol{n}} = -e(\vec{\boldsymbol{v}}_2^- - \vec{\boldsymbol{v}}_1^-)\cdot\hat{\boldsymbol{n}} = e(v_0\hat{\boldsymbol{\imath}})\cdot\hat{\boldsymbol{n}}.$$

Now, writing $\hat{\boldsymbol{n}} = n_x\hat{\boldsymbol{\imath}} + n_y\hat{\boldsymbol{\jmath}}$ and carrying out the dot products (after writing $\vec{\boldsymbol{v}}_1^+$ and $\vec{\boldsymbol{v}}_2^+$ in terms of their components), we get,

$$(v_{2x}^+ - v_{1x}^+)n_x + (v_{2y}^+ - v_{1y}^+)n_y = -e\,v_0\,n_x. \qquad (13.9)$$

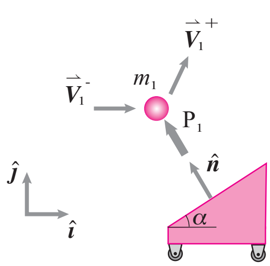

Figure 13.18: Free-body diagram of the ball during the collision. The impulse P_1 acts normal to the plane of collision which is the inclined surface of the cart.

We still need another equation. Let us now consider the impulse acting on the ball during the collision. From the free-body diagram shown in fig. 13.2, we can write the change in momentum of the ball as,

$$\begin{aligned} P_1\hat{\boldsymbol{n}} &= m_1\vec{\boldsymbol{v}}_1^+ - m_1\vec{\boldsymbol{v}}_1^-\\ \text{or}\qquad P_1(n_x\hat{\boldsymbol{\imath}} + n_y\hat{\boldsymbol{\jmath}}) &= m_1(v_{1x}^+\hat{\boldsymbol{\imath}} + v_{1y}^+\hat{\boldsymbol{\jmath}} - v_0\hat{\boldsymbol{\imath}}).\end{aligned}$$

Again, separating out this equation in scalar equations (by dotting the equation with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ separately), we get,

$$\begin{aligned} P_1 n_x - m_1 v_{1x}^+ &= -m_1 v_0 && (13.10)\\ P_1 n_y - m_1 v_{1y}^+ &= 0. && (13.11)\end{aligned}$$

Now, we have added another unknown P_1, but fortunately, we have got an extra equation too. We now have five unknowns and five independent equations. So we should be able to solve for all the unknowns.

For solving these equations, we first write them in matrix form and then use a computer to solve them. We write eqn. (13.7)–eqn. (13.11) as,

$$\begin{bmatrix} m_1 & 0 & m_2 & 0 & 0 \\ 0 & m_1 & 0 & m_2 & 0 \\ -n_x & -n_y & n_x & n_y & 0 \\ -m_1 & 0 & 0 & 0 & n_x \\ 0 & -m_1 & 0 & 0 & n_y \end{bmatrix} \begin{pmatrix} v_{1x}^+ \\ v_{1y}^+ \\ v_{2x}^+ \\ v_{2y}^+ \\ P_1 \end{pmatrix} = \begin{pmatrix} m_1 v_0 \\ 0 \\ -e v_0 n_x \\ -m_1 v_0 \\ 0 \end{pmatrix}.$$

Here is the pseudo computer code to solve this matrix equation:

```
m1 = 3, m2 = 12
theta = pi/6                  % angle in radians
nx = -sin(theta), ny = cos(theta) % components of the normal
v0 = 30
e =  0.9

A = [ m1   0   m2  0    0    % x comp of lin mom bal
      0    m1  0   m2   0    % y comp of lin mom bal
     -nx  -ny  nx  ny   0    % restitution equation
     -m1   0   0   0   -nx   % impulse-momentum for m1, x comp
      0   -m1  0   0   -ny]  % impulse-momentum for m1, y comp
b = [m1*v0  0  -e*v0*nx  -m1*v0  0]' % the known right hand side
solve A*x = b for x
```

The solution thus computed gives us

$$\begin{aligned} v_{1x}^+ &= 18.60\,\mathrm{m/s}, & v_{1y}^+ &= 19.74\,\mathrm{m/s}, \\ v_{2x}^+ &= 2.85\,\mathrm{m/s}, & v_{2y}^+ &= -4.94\,\mathrm{m/s}, \\ P_1 &= -68.40\,\mathrm{N\cdot s}. \end{aligned}$$

$$\begin{aligned} \vec{\boldsymbol{v}}_1^+ &= (18.60\,\mathrm{m/s})\hat{\boldsymbol{\imath}} + (19.74\,\mathrm{m/s})\hat{\boldsymbol{\jmath}} \\ \vec{\boldsymbol{v}}_2^+ &= (2.85\,\mathrm{m/s})\hat{\boldsymbol{\imath}} + (-4.94\,\mathrm{m/s})\hat{\boldsymbol{\jmath}} \end{aligned}$$

Collision between the cart and the ground: Now, we consider the collision between the cart and the ground, taking $\vec{\boldsymbol{v}}_2^+$ as the velocity of the cart just before the collision. Figure 13.19 shows the impulse from the ground acting on the cart. We know the final velocity of the cart has to be in the $\hat{\boldsymbol{\imath}}$ direction. Just to keep our notations straight, let us denote the velocity of the cart after collision as $\vec{\boldsymbol{v}}_2^{++}$ (after the second collision) and keep the incoming velocity as $\vec{\boldsymbol{v}}_2^+$. Then, from impulse momentum, we have,

$$P_2\hat{\boldsymbol{\jmath}} = m_2\vec{\boldsymbol{v}}_2^{++} - m_2\vec{\boldsymbol{v}}_2^+.$$

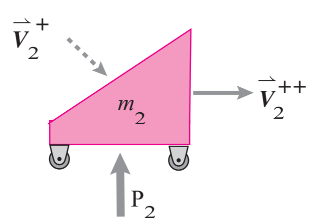

Figure 13.19

This is a vector equation which we can write as two scalar equations in the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ directions. Note that $\vec{\boldsymbol{v}}_2^{++} = v_2^{++}\hat{\boldsymbol{\imath}}$ and we already know $\vec{\boldsymbol{v}}_2^+ = (2.85\,\mathrm{m/s})\hat{\boldsymbol{\imath}} + (-4.94\,\mathrm{m/s})\hat{\boldsymbol{\jmath}}$ as found before. Thus,

$$\begin{aligned} v_2^{++} &= \vec{\boldsymbol{v}}_2^+ \cdot \hat{\boldsymbol{\imath}} = 2.85\,\mathrm{m/s} \\ P_2 &= -m_2\vec{\boldsymbol{v}}_2^+ \cdot \hat{\boldsymbol{\jmath}} = -(12\,\mathrm{kg})(-4.94\,\mathrm{m/s}) = 59.24\,\mathrm{kg\cdot m/s} = 59.24\,\mathrm{N\cdot s}. \end{aligned}$$

$$\vec{\boldsymbol{v}}_2^{++} = 2.85\,\mathrm{m/s}\hat{\boldsymbol{\imath}}$$

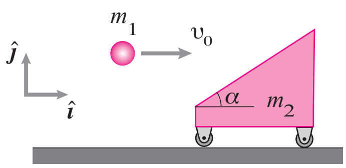

Figure 13.20

SAMPLE 13.5 Simultaneous collisions again: Consider the ball and the cart collision problem of Sample 13.4 again. This time, consider the ball and the cart together to have a collision with the ground first. Then consider the collision between the cart and the ball. Once again, you are to find the final horizontal velocity of the cart. The problem parameters are the same — mass of the ball $m_1 = 3\,\mathrm{kg}$, mass of the cart $m_2 = 12\,\mathrm{kg}$, $e = 0.9$ between the ball and the cart, and the velocity of the ball before impact, $v_0 = 30\,\mathrm{m/s}$.

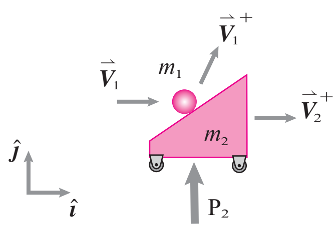

Figure 13.21: Free-body diagram of the ball-cart system during the collision with the ground. The ground impulse is P, the velocities of the ball and the cart before the collision are given — $\vec{\boldsymbol{v}}_1^- = v_0\hat{\boldsymbol{\imath}}$, $\vec{\boldsymbol{v}}_2^- = \vec{\boldsymbol{0}}$, and the velocities after the collision $\vec{\boldsymbol{v}}_2^+ = v_2\hat{\boldsymbol{\imath}}$ and $\vec{\boldsymbol{v}}_1^+ = v_{1x}^+\hat{\boldsymbol{\imath}} + v_{1y}^+\hat{\boldsymbol{\jmath}}$ are unknown.

Solution Let us consider the ball and the cart as a system colliding with the ground as shown in fig. 13.21. There is an unknown external impulse P_2 from the ground acting on this system in the $\hat{\boldsymbol{\jmath}}$ direction. Using this information, we now write impulse-momentum equation for this system:

$$P_2\hat{\boldsymbol{\jmath}} = \vec{\boldsymbol{L}}_2 - \vec{\boldsymbol{L}}_1 = m_1\vec{\boldsymbol{v}}_1^+ + m_2\vec{\boldsymbol{v}}_2^+ - m_1\vec{\boldsymbol{v}}_1^-$$

Assuming that $\vec{\boldsymbol{v}}_1^+ = v_{1x}^+\hat{\boldsymbol{\imath}} + v_{1y}^+\hat{\boldsymbol{\jmath}}$ and $\vec{\boldsymbol{v}}_2^+ = v_2^+\hat{\boldsymbol{\imath}}$, and using the given information $\vec{\boldsymbol{v}}_1^- = v_0\hat{\boldsymbol{\imath}}$, we obtain the following two scalar equations from the vector impulse-momentum equation:

$$\begin{aligned} m_1 v_{1x}^+ + m_2 v_2^+ &= m_1 v_0 && (13.12)\\ m_1 v_{1y}^+ - P_2 &= 0 && (13.13)\end{aligned}$$

So far, we have two equations and four unknowns — v_{1x}^+, v_{1y}^+, v_2^+ and P_2. Obviously, we need more equations. Now, let us consider the collision between the cart and the ball. Let the impulse of this collision be P_1. Then the impulse-momentum equation for the ball gives us,

$$P_1\hat{\boldsymbol{n}} = m_1(v_{1x}^+\hat{\boldsymbol{\imath}} + v_{1y}^+\hat{\boldsymbol{\jmath}}) - m_1 v_0\hat{\boldsymbol{\imath}}.$$

Once again, we separate out the scalar equations from this vector equation, using the information $\hat{\boldsymbol{n}} = n_x\hat{\boldsymbol{\imath}} + n_y\hat{\boldsymbol{\jmath}}$:

$$\begin{aligned} m_1 v_{1x}^+ - P n_x &= m_1 v_0 && (13.14)\\ m_1 v_{1y}^+ - P n_y &= 0 && (13.15)\end{aligned}$$

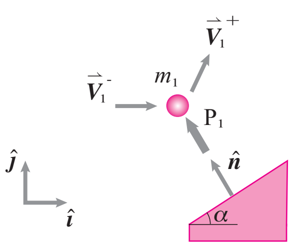

Figure 13.22: Free-body diagram of the ball during the collision with the cart. The cart impulse P_1 acts along the normal $\hat{\boldsymbol{n}}$. From geometry, we find that $\hat{\boldsymbol{n}} \equiv n_x\hat{\boldsymbol{\imath}} + n_y\hat{\boldsymbol{\jmath}} = -\sin\alpha\,\hat{\boldsymbol{\imath}} + \cos\alpha\,\hat{\boldsymbol{\jmath}}$.

Thus, we have now four equations; we still need one more. We now use the restitution equation to relate the normal components of the relative velocities of approach and departure of the ball and the cart:

$$\begin{aligned} (\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+)\cdot\hat{\boldsymbol{n}} &= -e(v_1^- - v_2^-)\cdot\hat{\boldsymbol{n}} \\ \Rightarrow\quad v_{1x}^+ n_x + v_{1y}^+ n_y - v_2 n_x &= -e\,v_0 n_x. && (13.16)\end{aligned}$$

Now we have five equations in five unknowns. All we need to do now is to solve these linear equations for all the unknowns. We do so by first writing the five equations (eqn. (13.12) to eqn. (13.16) in matrix form and then solving the matrix equation on a computer. The matrix equation is:

$$\begin{bmatrix} m_1 & 0 & m_2 & 0 & 0\\ 0 & m_1 & 0 & 0 & -1\\ m_1 & 0 & 0 & -n_x & 0\\ 0 & m_1 & 0 & -n_y & 0\\ n_x & n_y & -n_x & 0 & 0 \end{bmatrix} \begin{Bmatrix} v_{1x}^+\\ v_{1y}^+\\ v_2^+\\ P_1\\ P_2 \end{Bmatrix} = \begin{Bmatrix} m_1 v_0\\ 0\\ m_1 v_0\\ 0\\ -e\,v_0 n_x. \end{Bmatrix}.$$

Solving this equation as in the previous sample, we get,

$$v_{1x}^+ = 16.59\,\mathrm{m/s},\ v_{1y}^+ = 23.23\,\mathrm{m/s},\ v_2^+ = 3.35\,\mathrm{m/s},\ P_1 = 80.47\,\mathrm{N\cdot s},\ P_2 = 69.69\,\mathrm{N\cdot s}.$$

$$\vec{\boldsymbol{v}}_2^+ = (3.35\,\mathrm{m/s})\hat{\boldsymbol{\imath}}$$

Note that the answer obtained here is not the same as that found in Sample 13.4; the cart moves a bit faster to the right in this answer. Depending on the mass ratios and the angle of impact, the two methods can give very different answers or very close answers. Welcome to the world of modeling!