13.1 Coupled motions of particles in space

Assume you know enough about a system so that you know the forces on each particle if someone tells you the time and the positions and velocities of all the particles. This means you can write the governing equations for the system of particles like this:

$$
\begin{aligned}
\vec{\boldsymbol{a}}_1 &= \frac{1}{m_1}\vec{\boldsymbol{F}}_1 \\
\vec{\boldsymbol{a}}_2 &= \frac{1}{m_2}\vec{\boldsymbol{F}}_2 \\
\vec{\boldsymbol{a}}_3 &= \frac{1}{m_3}\vec{\boldsymbol{F}}_3 \\
&\text{etc.}
\end{aligned}
\tag{13.1}
$$

where $\vec{\boldsymbol{F}}_1$, $\vec{\boldsymbol{F}}_2$ *etc.* are the total of the forces on the corresponding particles. The force on each particle may come from air-friction, from springs or dashpots connected here and there, or from gravity interactions with other particles, from known applied loads, *etc.*. One way or another, all the forces on all the particles are known given the time, the positions and velocities of the particles. Thus eqn. (13.1) can be written as a system of first order differential equations in standard form, ready for computer simulation. Given accurate initial conditions and a good computer then the motions of all the particles can be found accurately.

Example: Coupled motion of the earth and moon in three dimensions. Let's neglect the sun and just look at the coupled motions of the earth and moon. They attract each other by the same law of gravity that we used for the sun and earth. The difference between this problem and a "central-force" problem is that we now need to look at the 'absolute' positions of the earth and the moon ($\vec{\boldsymbol{r}}_e$ and $\vec{\boldsymbol{r}}_m$), as well as the 'relative' position $\vec{\boldsymbol{r}}_{m/e} \equiv \vec{\boldsymbol{r}}_m - \vec{\boldsymbol{r}}_e$ (fig. 13.1).

The linear momentum balance equations are now

$$
m_e \ddot{\vec{\boldsymbol{r}}}_e = \frac{+G m_e m_m \vec{\boldsymbol{r}}_{m/e}}{|\vec{\boldsymbol{r}}_{m/e}|^3} \quad \text{and} \tag{13.2}
$$

$$
m_m \ddot{\vec{\boldsymbol{r}}}_m = \frac{-G m_e m_m \vec{\boldsymbol{r}}_{m/e}}{|\vec{\boldsymbol{r}}_{m/e}|^3}, \tag{13.3}
$$

which, when broken into x, y, and z components give 6 second order ordinary differential equations. These equations can be written as 12 first order equations by defining a list of 12 z variables: $z_1 = x_e$, $z_2 = \dot{x}_e$, $z_3 = y_e$, $z_4 = \dot{y}_e$, *etc.*

After you find solutions, using various initial conditions you can check if the computer finds such truths (that is, features of the exact solution of the differential equations) as:

1. that the line between the earth and moon always lies on one fixed plane,
2. the center-of-mass moves at constant speed on a straight line,
3. relative to the center-of-mass both the earth and moon travel on paths that are conic sections (circle, ellipse, parabola, hyperbola or a straight line).
4. the total energy ($E_K + E_P$) of the system is constant,

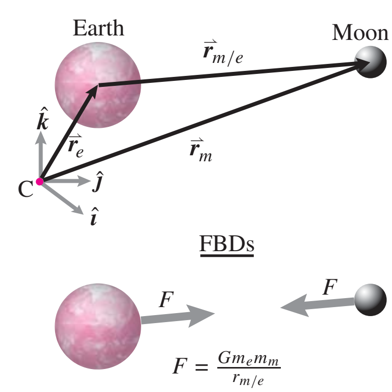

Figure 13.1: The earth and moon. Position is measured relative to some "fixed" point C.

5. and that the angular momentum of the system about the center-of-mass is a constant.

These facts are discussed further below in the subsection on 'Two-particle central force motion'.

Momentum and energy of systems

There are a plethora of theorems about the momentum and energy of systems of particles. These are discussed in section B. The simplest of these are just the ones that you get from adding up the results for a single particle from section 12.2:

Linear momentum balance. $\sum_{\text{all forces}} \vec{\boldsymbol{F}}_j = \sum m_i \vec{\boldsymbol{a}}_i.$

Either because the forces between particles in a system are usually assumed to come in equal and opposite pairs or because it is an independent postulate of mechanics for general systems, the force sum can be replaced with a sum over all the external forces.

Angular momentum balance. $\sum_{\text{all forces}} \vec{\boldsymbol{r}}_{j/C} \times \vec{\boldsymbol{F}}_j = \sum \vec{\boldsymbol{r}}_{i/C} \times (m_i \vec{\boldsymbol{a}}_i).$

As for linear momentum, the force sum can be replaced with only the forces that act externally on the system.

Power balance. $\sum_{\text{all forces}} \vec{\boldsymbol{F}}_j \cdot \vec{\boldsymbol{v}}_j = \frac{d}{dt} \sum m_i v_i^2 / 2.$

In this case the sum is over *all* the forces, internal and external. The simplification to just external forces doesn't apply to system kinetic energy like it does for momentum and angular momentum.

The one-body problem

Let's review one special problem from the previous section. The 'one-body' problem should properly be about the mechanics of a single particle interacting with nothing else. Such a particle moves at constant velocity and is too boring to get a name. Instead, when people refer to the 'one-body' problem they are talking about a particle flying around a stationary point mass to which it is attracted. That stationary point mass is held in place by, well, who knows what. It's just an idealized thing anchored by a massless structure. The 'one-body' problem is to find the motion of the particle flying around.

As we discussed in the previous sections, if the gravitational attraction follows an inverse square law then the particle moves on a plane on a curve which is either an ellipse, a circle, a parabola or a hyperbola. These are, quite accurately, the trajectories of the planets and comets around the sun.

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The two-body problem: two mutually attracting particles

If two particles are attracted equally to each other by mutually central forces, and no other forces act, this is called 'the two-body problem'①. Assume the two particles are m_1 and m_2 with positions $\vec{\boldsymbol{r}}_1$ and $\vec{\boldsymbol{r}}_2$ (relative to the origin of a coordinate system fixed in a Newtonian frame). The force on particle 1 from particle 2 is

①This is such a famous problem in the history of science that people use it for word play to describe certain social situations. For example if two people in a couple are having trouble finding jobs in the same city they are said to have a 'two-body problem'.

$$\vec{\boldsymbol{F}}_{12} = F(r_{12})\frac{\vec{\boldsymbol{r}}_{12}}{r_{12}}$$

where $\vec{\boldsymbol{r}}_{12} = \vec{\boldsymbol{r}}_{2/1}$ is the position of particle 2 relative to particle 1, r_{12} is the distance $|\vec{\boldsymbol{r}}_{12}|$ between the particles and F is the magnitude of the attractive force. We assume the force on particle 2 is the opposite of this

$$\vec{\boldsymbol{F}}_{21} = -\vec{\boldsymbol{F}}_{12}.$$

The instantaneous velocities are $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$. We can find the center of mass G of the pair of particles as

$$m_{tot}\vec{\boldsymbol{r}}_G = m_1\vec{\boldsymbol{r}}_1 + m_2\vec{\boldsymbol{r}}_2$$

with $m_{tot} = m_1 + m_2$.

Either by system linear momentum balance or by adding up $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ for each of the particles it is easy to see that

$$\vec{\boldsymbol{a}}_G = \vec{\boldsymbol{0}} \qquad \text{and} \qquad \vec{\boldsymbol{v}}_G = \text{constant}.$$

Thus we could put the origin of a good Newtonian reference frame at the center of mass $\vec{\boldsymbol{r}}_G$. The positions, velocities and accelerations relative to G, indicated with a prime ('), are

$$\begin{aligned}
\vec{\boldsymbol{r}}'_1 &= \vec{\boldsymbol{r}}_1 - \vec{\boldsymbol{r}}_G \\
\vec{\boldsymbol{r}}'_2 &= \vec{\boldsymbol{r}}_2 - \vec{\boldsymbol{r}}_G \\
\vec{\boldsymbol{v}}'_1 &= \vec{\boldsymbol{v}}_1 - \vec{\boldsymbol{v}}_G \\
\vec{\boldsymbol{v}}'_2 &= \vec{\boldsymbol{v}}_2 - \vec{\boldsymbol{v}}_G \\
\vec{\boldsymbol{a}}'_1 &= \vec{\boldsymbol{a}}_1 \\
\vec{\boldsymbol{a}}'_2 &= \vec{\boldsymbol{a}}_2
\end{aligned}$$

where we can skip use of the prime for the acceleration. Now some facts.

- $m_1\vec{\boldsymbol{r}}'_1 + m_2\vec{\boldsymbol{r}}'_2 = \vec{\boldsymbol{0}}$ so $\vec{\boldsymbol{r}}'_1 = -(m_2/m_1)\vec{\boldsymbol{r}}'_2$. For all time the two positions (relative to the center of mass) are in the opposite direction and proportional. Similarly $\vec{\boldsymbol{v}}'_1 = -(m_2/m_1)\vec{\boldsymbol{v}}'_2$ and $\vec{\boldsymbol{a}}'_1 = -(m_2/m_1)\vec{\boldsymbol{a}}'_2$.
- At a given instant there is a single plane defined by $\vec{\boldsymbol{r}}'_1, \vec{\boldsymbol{r}}'_2, \vec{\boldsymbol{v}}'_1, \vec{\boldsymbol{v}}'_2, \vec{\boldsymbol{a}}'_1$ and $\vec{\boldsymbol{a}}'_2$ because the positions and accelerations are all parallel (or antiparallel) and the two velocities are (anti) parallel.

- The plane above is constant in time. This is because neither the velocity nor the acceleration has a component orthogonal to the plane, thus there is no tendency to leave the plane.
- Each particle moves as if it was a single particle attracted to a central force at G. Why? Let's look at the force on mass 1

$$\vec{\boldsymbol{F}}_{12} = F(r_{12})\frac{\vec{\boldsymbol{r}}_{12}}{r_{12}} = -F(r_{12})\frac{\vec{\boldsymbol{r}}'_1}{r'_1}$$

  because the relative position of the masses passes through the origin G.
- In the special case of inverse-square gravitational attraction

$$F = \frac{Gm_1m_2}{r_{12}^2} = \frac{Gm_1m_2}{(r'_1 + r'_2)^2} = \frac{Gm_1M}{{r'_1}^2}$$

  where $M = m_2/(1 + 2(m_1/m_2) + (m_1/m_2)^2)$ is a fictitious mass at G we find using the substitution $r'_2 = (m_1/m_2)r'_1$.

What we have found here is somewhat remarkable. Two particles are flying around in space attracted to each other by inverse-square gravitational attraction. Instead of doing something wild, they each move, relative to their joint center of mass, as if they were in central force motion with a fixed mass. That is, the 3D two-body problem reduces, exactly, to the 2D one body problem. You just have to use a coordinate system that is on the plane of motion and whose origin is at the center of mass.

Thus, the moon doesn't really go around the earth. Rather the moon and earth go around their common center of mass (a point about 3/4 of the way out towards the earth's surface from its center). And Jupiter doesn't go around the sun, the sun and Jupiter go around their combined center of mass just outside the sun. But both of these examples are, in detail, wrong. Because the earth-moon system is affected by the sun and jupiter. And the Jupiter-sun system is affected by the earth and moon.

The three-body problem

With inverse-square attraction, one body goes around a fixed point on one or another conic section. Two bodies go around each other in exactly the same way as one body about a fixed point. The two-body problem reduced to the one-body problem. What about lots of bodies? Let's start with three. How, in general, do three bodies move that are all mutually attracted with inverse-square gravitation? Great question. Lots of people have asked it. And no-one knows the answer. Given any three masses and their initial conditions we could use a computer program to find out their subsequent positions and velocities. But no-one knows how to categorize all the possible motions of such systems.

Some things are known about 'the three body problem.' One is that it is hard, the best minds haven't been able to solve it in general. Another is

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that the solutions can be pretty wild. For example, three particles might tumble around each other for a long time and, with no change in the equations, all of a sudden one of the particles will be ejected at high speed and never return (as if on a hyperbolic trajectory relative to the other two particles). A few special solutions of the three-body problem are known. For example, with the right initial conditions, three identical particles can move in either a circle or in Montgomery's figure 8.

Despite the difficulty of analytic description, there is no special impediment to finding solutions to any 3-body problem with computer simulation.

The n-body problem

With many particles all manner of complicated motion is possible. And there are few solutions which are known analytically. One solution has the n particles chasing each-other around in a circle, with the particles forming a regular polygon. Another amazing approximate solution, the Buck solution, is that a string of thousands of particles will all chase each-other around an arbitrary curve in 3-dimensional space. At least approximately, for a while.

By applying $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ to 3 or 1000 interacting particles you can see all manner of n-body solutions on your computer.

SAMPLE 13.1 **Location of the center-of-mass.** A structure is made up of three point masses, $m_1 = 1\,\mathrm{kg}$, $m_2 = 2\,\mathrm{kg}$ and $m_3 = 3\,\mathrm{kg}$. At the instant of interest, the coordinates of the three masses are (1.25 m, 3 m), (2 m, 2 m), and (0.75 m, 0.5 m), respectively. At the same instant, the velocities of the three masses are $2\,\mathrm{m/s}\hat{\imath}$, $2\,\mathrm{m/s}(\hat{\imath} - 1.5\hat{\jmath})$ and $1\,\mathrm{m/s}\hat{\jmath}$, respectively.

1. Find the coordinates of the center-of-mass of the structure.
2. Find the velocity of the center-of-mass.

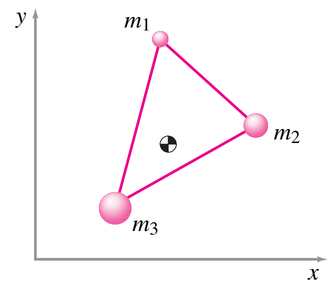

Figure 13.2

Solution

1. Let $(\bar{x}, \bar{y})$ be the coordinates of the mass-center. Then from the definition of mass-center

$$\begin{aligned}\bar{x} &= \frac{\sum m_i x_i}{\sum m_i} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}\\ &= \frac{1\,\mathrm{kg}\cdot 1.25\,\mathrm{m} + 2\,\mathrm{kg}\cdot 2\,\mathrm{m} + 3\,\mathrm{kg}\cdot 0.75\,\mathrm{m}}{1\,\mathrm{kg} + 2\,\mathrm{kg} + 3\,\mathrm{kg}}\\ &= \frac{7.5\,\cancel{\mathrm{kg}}\cdot\,\mathrm{m}}{6\,\cancel{\mathrm{kg}}} = 1.25\,\mathrm{m}.\end{aligned}$$

Similarly,

$$\begin{aligned}\bar{y} &= \frac{\sum m_i y_i}{\sum m_i}\\ &= \frac{1\,\mathrm{kg}\cdot 3\,\mathrm{m} + 2\,\mathrm{kg}\cdot 2\,\mathrm{m} + 3\,\mathrm{kg}\cdot 0.5\,\mathrm{m}}{1\,\mathrm{kg} + 2\,\mathrm{kg} + 3\,\mathrm{kg}}\\ &= \frac{8.5\,\cancel{\mathrm{kg}}\cdot\,\mathrm{m}}{6\,\cancel{\mathrm{kg}}} = 1.42\,\mathrm{m}.\end{aligned}$$

Thus the center-of-mass is located at the coordinates (1.25 m, 1.42 m).

$$\boxed{(1.25\,\mathrm{m},\ 1.42\,\mathrm{m})}$$

2. For a system of particles, the linear momentum

$$\begin{aligned}\vec{\boldsymbol{L}} &= \sum m_i\vec{\boldsymbol{v}}_i = m_{\mathrm{tot}}\vec{\boldsymbol{v}}_{cm}\\ \Rightarrow\quad \vec{\boldsymbol{v}}_{cm} &= \frac{\sum m_i\vec{\boldsymbol{v}}_i}{m_{\mathrm{tot}}}\\ &= \frac{1\,\mathrm{kg}\cdot(2\,\mathrm{m/s}\hat{\imath}) + 2\,\mathrm{kg}\cdot(2\hat{\imath} - 3\hat{\jmath})\,\mathrm{m/s} + 3\,\mathrm{kg}\cdot(1\,\mathrm{m/s}\hat{\jmath})}{6\,\mathrm{kg}}\\ &= \frac{(6\hat{\imath} - 3\hat{\jmath})\,\cancel{\mathrm{kg}}\cdot\,\mathrm{m/s}}{6\,\cancel{\mathrm{kg}}}\\ &= 1\,\mathrm{m/s}\hat{\imath} - 0.5\,\mathrm{m/s}\hat{\jmath}.\end{aligned}$$

$$\boxed{\vec{\boldsymbol{v}}_{cm} = 1\,\mathrm{m/s}\hat{\imath} - 0.5\,\mathrm{m/s}\hat{\jmath}}$$

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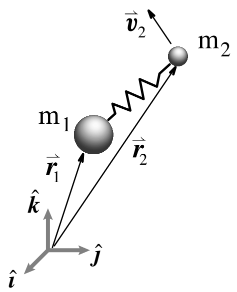


Figure 13.3

SAMPLE 13.2 **A spring-mass system in space.** A spring-mass system consists of two masses, $m_1 = 10\,\mathrm{kg}$ and $m_2 = 1\,\mathrm{kg}$, and a weak spring with stiffness $k = 1\,\mathrm{N/m}$. The spring has zero relaxed length. The system is in 3-D space where there is no gravity. At the instant of observation, *i.e.,* at $t = 0, \vec{r}_1 = \vec{0}, \vec{r}_2 = 1\,\mathrm{m}(\hat{\imath} + \hat{\jmath} + \hat{k}), \dot{\vec{r}}_1 = \vec{0}$, and $\dot{\vec{r}}_2 = \sqrt{6}\,\mathrm{m/s}(-\hat{\imath} + \hat{\jmath})$. Track the motion of the system for the next 20 seconds. In particular,

1. Plot the trajectory of the two masses in space.
2. Plot the trajectory of the center-of-mass of the system.
3. Plot the trajectory of the two masses as seen by an observer sitting at the center-of-mass.
4. Compute and plot the total energy of the system and show that it remains constant during the entire motion.

Solution The free-body diagrams of the two masses are shown in fig. 13.4. The only force acting on each mass is the force due to the spring which is directed along the line joining the two masses. Thus, the system represents a central force problem. From the linear momentum balance of the two masses, we can write the equations of motion as follows.

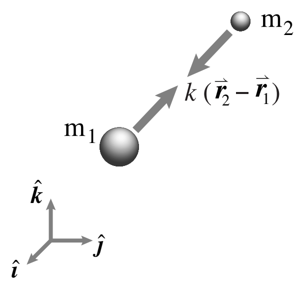


Figure 13.4: Free-body diagram of the two masses.

$$
\begin{aligned}
m_1\ddot{\vec{r}}_1 &= k(\vec{r}_2 - \vec{r}_1)\\
m_2\ddot{\vec{r}}_2 &= -k(\vec{r}_2 - \vec{r}_1)
\end{aligned}
$$

Let $\vec{r}_1 = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$ and $\vec{r}_2 = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$. Substituting above and dotting the two equations with $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, we get

$$
\begin{aligned}
\ddot{x}_1 &= \frac{k}{m_1}(x_2 - x_1); & \ddot{x}_2 &= -\frac{k}{m_2}(x_2 - x_1)\\
\ddot{y}_1 &= \frac{k}{m_1}(y_2 - y_1); & \ddot{y}_2 &= -\frac{k}{m_2}(y_2 - y_1)\\
\ddot{z}_1 &= \frac{k}{m_1}(z_2 - z_1); & \ddot{z}_2 &= -\frac{k}{m_2}(z_2 - z_1)
\end{aligned}
$$

Thus we get six second order coupled linear ODEs as equations of motion.

1. To plot the trajectory of the two masses, we need to solve for $\vec{r}_1(t)$ and $\vec{r}_2(t)$, *i.e.,* for $x_1(t), y_1(t), z_1(t)$, and $x_2(t), y_2(t), z_2(t)$. We can do this by first writing the six second order equations as a set of 12 first order equations and then solving them using a numerical ODE solver. Here is a pseudocode to accomplish this task.

```
ODEs = {x1dot = u1,
        u1dot = k/m1*(x2-x1),
        y1dot = v1,
        v1dot = k/m1*(y2-y1),
        z1dot = w1,
        w1dot = k/m1*(z2-z1),
        x2dot = u2,
        u2dot = -k/m2*(x2-x1),
        y2dot = v2,
        v2dot = -k/m2*(y2-y1),
        z2dot = w2,
        w2dot = -k/m2*(z2-z1) }
IC   = {x1(0)=0, y1(0)=0, z1(0)=0,
        u1(0)=0, v1(0)=0, w1(0)=0,
        x2(0)=1, y2(0)=1, z2(0)=1,
        u2(0)=-sqrt(6), v2(0)=sqrt(6), w2(0)=0}
Set  k=1, m1=10, m2=1
Solve ODEs with IC for t=0 to t=20
Plot {x1,y1,z1} and {x2,y2,z2}
```

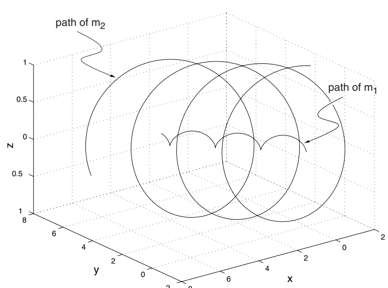


Figure 13.5: 3-D trajectory of m_1 and m_2 plotted from numerical solution of the equations of motion.

The 3-D plot showing the trajectory of the two masses obtained from the numerical solution is shown in fig. 13.5. From the plot, it seems like the smaller mass goes around the bigger mass as the bigger mass moves on its trajectory.

2. We can find the trajectory of the center-of-mass using the following relationships.

$$x_{\mathrm{cm}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, \quad y_{\mathrm{cm}} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}, \quad z_{\mathrm{cm}} = \frac{m_1 z_1 + m_2 z_2}{m_1 + m_2}.$$

Since there is no external force on the system if we consider the two masses and the spring together, the center-of-mass of the system has zero acceleration. Therefore, we expect the center-of-mass to move on a straight path with constant velocity. The center-of-mass coordinates x_{cm}, y_{cm}, and z_{cm} are plotted against time in fig. 13.6 which show that the center-of-mass moves on a straight line in a plane parallel to the xy-plane (z is constant). This is expected since the initial velocity of the center of mass has no z-component:

$$\begin{aligned}
\vec{\boldsymbol{v}}_{\mathrm{cm}} &= \frac{m_1 \vec{\boldsymbol{v}}_1 + m_2 \vec{\boldsymbol{v}}_2}{m_1 + m_2} \\
&= \frac{m_1 \cdot \vec{\mathbf{0}} + 1\,\mathrm{kg} \cdot \sqrt{(6)}\,\mathrm{m/s}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})}{10\,\mathrm{kg} + 1\,\mathrm{kg}} \\
&= 0.22\,\mathrm{m/s}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}).
\end{aligned}$$

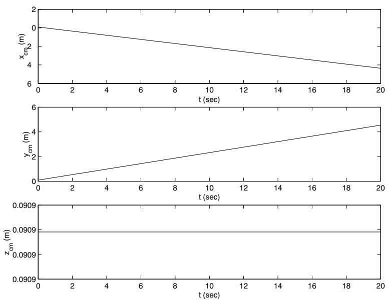


Figure 13.6: The center-of-mass coordinates $x_{\mathrm{cm}}(t)$, $y_{\mathrm{cm}}(t)$, and $z_{\mathrm{cm}}(t)$. The center-of-mass moves on a straight line in a plane parallel to the xy-plane.

3. The trajectory of the two masses with respect to the center-of-mass can be easily obtained by the following relationships.

$$\begin{aligned}
x_{1/cm} &= x_1 - x_{\mathrm{cm}}, & y_{1/cm} &= y_1 - y_{\mathrm{cm}}, & z_{1/cm} &= z_1 - z_{\mathrm{cm}} \\
x_{2/cm} &= x_2 - x_{\mathrm{cm}}, & y_{2/cm} &= y_2 - y_{\mathrm{cm}}, & z_{2/cm} &= z_2 - z_{\mathrm{cm}}
\end{aligned}$$

The trajectories thus obtained are shown in fig. 13.7. It is clear that the two masses have closed orbits with respect to the center-of-mass. These closed orbits are actually conic sections as we would expect in a central force problem.

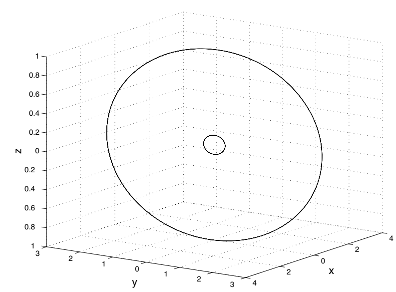


Figure 13.7: The paths of m_1 and m_2 as seen from the center-of-mass. The two masses are on closed orbits with respect to the center-of-mass.

4. We can calculate the kinetic energy of the two masses and the potential energy of the spring at each instant during the motion and add them up to find the total energy.

$$\begin{aligned}
(E_k)_{m_1} &= \frac{1}{2} m_1 (u_1^2 + v_1^2 + w_1^2) \\
(E_k)_{m_2} &= \frac{1}{2} m_2 (u_2^2 + v_2^2 + w_2^2) \\
E_p &= \frac{1}{2} k[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2] \\
E_{\mathrm{total}} &= (E_k)_{m_1} + (E_k)_{m_2} + E_p
\end{aligned}$$

The energies so calculated are plotted in fig. 13.8. It is clear from the plot that the total energy remains constant during the entire motion.

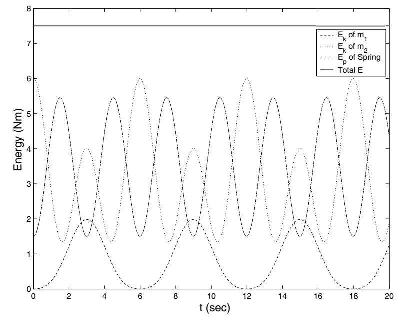


Figure 13.8: The kinetic energy of the two masses and the potential energy of the spring sum up to the constant total energy of the system.

13.1 Coupled motions of particles in space

Preparatory Problems

13.1.1 Linear momentum balance applied to the whole of a system consisting of multiple interacting particles reduces to $\vec{F} = m\vec{a}$ *if you interpret the terms correctly.* What are the correct interpretations of $\vec{F}$, m and $\vec{a}$? *

13.1.2 A particle of mass $m_1 = 6\,\text{kg}$ and a particle of mass $m_2 = 10\,\text{kg}$ are moving in the xy-plane. At a particular instant of interest, particle 1 has position $\vec{r}_1 = 3\,\text{m}\hat{\imath} + 2\,\text{m}\hat{\jmath}$, velocity $\vec{v}_1 = -16\,\text{m/s}\hat{\imath} + 6\,\text{m/s}\hat{\jmath}$, and acceleration $\vec{a}_1 = 10\,\text{m/s}^2\hat{\imath} - 24\,\text{m/s}^2\,\hat{\jmath}$; and particle 2 has position $\vec{r}_2 = -6\,\text{m}\hat{\imath} - 4\,\text{m}\hat{\jmath}$, velocity $\vec{v}_2 = 8\,\text{m/s}\hat{\imath} + 4\,\text{m/s}\hat{\jmath}$, and acceleration $\vec{a}_2 = 5\,\text{m/s}^2\hat{\imath} - 16\,\text{m/s}^2\,\hat{\jmath}$.

a) Find the linear momentum $\vec{L}$ and its rate of change $\dot{\vec{L}}$ of each particle at the instant of interest. *

b) Find the linear momentum $\vec{L}$ and its rate of change $\dot{\vec{L}}$ of the system of the two particles at the instant of interest. *

c) Find the center of mass of the system at the instant of interest. *

d) Find the velocity and acceleration of the center of mass. *

13.1.3 A particle of mass $m_1 = 5\,\text{kg}$ and a particle of mass $m_2 = 10\,\text{kg}$ are moving in space. At a particular instant of interest, particle 1 has position, velocity, and acceleration

$$\begin{aligned}\vec{r}_1 &= 1\,\text{m}\hat{\imath} + 1\,\text{m}\,\hat{\jmath}\\ \vec{v}_1 &= 2\,\text{m/s}\,\hat{\jmath}\\ \vec{a}_1 &= 3\,\text{m/s}^2\hat{k}\end{aligned}$$

respectively, and particle 2 has position, velocity, and acceleration

$$\begin{aligned}\vec{r}_2 &= 2\,\text{m}\hat{\imath}\\ \vec{v}_2 &= 1\,\text{m/s}\hat{k}\\ \vec{a}_2 &= 1\,\text{m/s}^2\,\hat{\jmath}\end{aligned}$$

respectively. For the system of particles at the instant of interest, find its

a) linear momentum $\vec{L}$, *

b) rate of change of linear momentum $\dot{\vec{L}}$, *

c) angular momentum about the origin $\vec{H}_{/O}$, *

d) rate of change of angular momentum about the origin $\dot{\vec{H}}_{/O}$, *

e) kinetic energy E_K, and *

f) rate of change of kinetic energy. *

13.1.4 If you are given the total mass, the position, the velocity, and the acceleration of the center of mass of a system of particles can you find the angular momentum $\vec{H}_{/O}$ of the system, where O is not at the center of mass? If so, how and why? If not, then give a reason and/or a counter example. *

13.1.5 Seventeen particles are interacting with the force on particle i from particle j being $\vec{F}_{ij}$ with all $\vec{F}_{ij}$ known.

a) What is the commonly assumed assumption about the relation between, say, $\vec{F}_{36}$ and $\vec{F}_{63}$? *

b) What is the total force on particle 5? *

More-Involved Problems

13.1.6 Two particles each of mass m are connected by a massless elastic spring of spring constant k and unextended length $2R$. The system slides without friction on a horizontal table, so that no net external forces act.

a) Is the total linear momentum conserved? Justify your answer. *

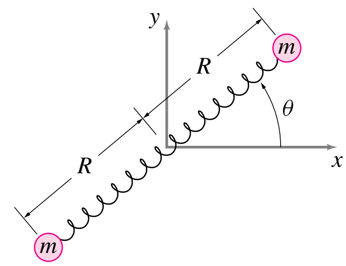
b) Can the center of mass accelerate? Justify your answer. *

c) Draw free-body diagrams for each mass.

d) Derive the equations of motion for each mass in terms of cartesian coordinates.

e) What are the total kinetic and potential energies of the system? *

f) For constant values and initial conditions of your choosing, plot the trajectories of the two particles and of the center of mass (on the same plot).



Problem 13.1.6

13.1.7 Two ice skaters whirl around one another. They are connected by a linear elastic cord whose center is stationary in space. We wish to consider the motion of one of the skaters by modeling her as a mass m held by a cord that exerts k Newtons for each meter it is extended from the central position.

a) Draw a free-body diagram showing the forces that act on the mass at an arbitrary position.

b) Write the differential equations that describe the motion. *

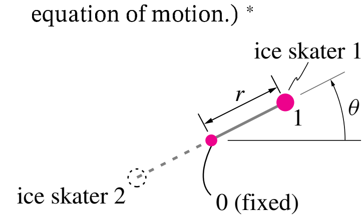
c) Describe in physical and mathematical terms the nature of the motion for the three cases

a) $\omega < \sqrt{k/m}$;

b) $\omega = \sqrt{k/m}$;

c) $\omega > \sqrt{k/m}$.

(You are not asked to solve the equation of motion.) *



Problem 13.1.7

13.1.8 n identical particles with mass m are on the vertices of an n sided regular polygon. Equivalently, n particles are equally spaced on a circle with radius R. At $t = 0$ they all have velocities tangent to the circle and equal in magnitude v_0. All the particles are attracted to each other with an inverse square gravitational attraction. For the numerical simulations below pick values of n, m, G and R any way that pleases you.

a) Find an initial value for v_0 so that all the masses spiral in and then bounce out again. Plot the trajectories of all the masses on one plot for a long-enough time so the plot is pleasing to the eye.

b) Find a value for v_0 so all the particles travel on circular trajectories.

c) Can you find a formula for v_0 above in terms of the other parameters in the problem? *

13.1.9 Two masses, both with $M = 1000m$ travel in circles on the xy plane according to

$$\vec{\boldsymbol{r}}_1 = -\vec{\boldsymbol{r}}_2 = R(\cos\omega t\hat{\boldsymbol{\imath}} + \sin\omega t\,\hat{\boldsymbol{\jmath}})$$

a) Assume inverse square attraction and find a set of values for m, G, R and ω so the assumed circular path is a solution of the equations of motion. *

b) A third mass m is introduced which is gravitationally attracted to the other two. Pick initial conditions for the two big masses that are consistent with their circular motion solution. For the third mass use initial conditions $[\vec{\boldsymbol{r}}_0]_{xyz} = [00z_0]$. Run a simulation for some time.

- Does the third mass stay *exactly* on the z axis for all time in the simulation? Would it if the simulation was exact? If so, why? If not, why not? *
- Is the motion of the third mass exactly periodic in the computer simulation? Would it be if the solution was exact? If so, why? If not, why not? *

13.1.10 The amazing eight. Three equal masses, say $m = 1$, are attracted by an inverse-square gravity law with $G = 1$. That is, each mass is attracted to the other by $F = Gm_1m_2/r^2$ where r is the distance between them. Use these unusual and special initial positions:

$$\begin{aligned}(x1, y1) &= (-0.97000436, 0.24308753)\\ (x2, y2) &= (-x1, -y1)\\ (x3, y3) &= (0, 0)\end{aligned}$$

and initial velocities

$$\begin{aligned}(vx3, vy3) &= (0.93240737, 0.86473146)\\ (vx1, vy1) &= -(vx3, vy3)/2\\ (vx2, vy2) &= -(vx3, vy3)/2.\end{aligned}$$

For each of the problems below show accurate computer plots and explain any curiosities. (Note, this solution was discovered independently, and nearly simultaneously, by Richard Montgomery and Chris Moore.)

a) Use computer integration to find and plot the motions of the particles. Plot each with a different color. Run the program for 2.1 time units. *

b) Same as above, but run for 10 time units. *

c) Same as above, but change the initial conditions slightly. *

d) Same as above, but change the initial conditions more and run for a much longer time. *

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