13.1.1 Linear momentum balance applied to the whole of a system consisting of multiple interacting particles reduces to $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ *if you interpret the terms correctly.* What are the correct interpretations of $\vec{\boldsymbol{F}}$, m and $\vec{\boldsymbol{a}}$?

Answer:
$\vec{\boldsymbol{F}}$ is the sum of all external forces, m is total mass of the system of particles, and $\vec{\boldsymbol{a}}$ is the acceleration of the center of mass of the system of particles.

$\vec{F} = m\vec{a}$

Acceleration of the C.M in the inertial frame of reference

Total mass of the system

External force on the system

13.1.2 A particle of mass $m_1 = 6\,\text{kg}$ and a particle of mass $m_2 = 10\,\text{kg}$ are moving in the xy-plane. At a particular instant of interest, particle 1 has position $\vec{r}_1 = 3\,\text{m}\hat{\imath} + 2\,\text{m}\hat{\jmath}$, velocity $\vec{v}_1 = -16\,\text{m/s}\hat{\imath} + 6\,\text{m/s}\hat{\jmath}$, and acceleration $\vec{a}_1 = 10\,\text{m/s}^2\hat{\imath} - 24\,\text{m/s}^2\hat{\jmath}$; and particle 2 has position $\vec{r}_2 = -6\,\text{m}\hat{\imath} - 4\,\text{m}\hat{\jmath}$, velocity $\vec{v}_2 = 8\,\text{m/s}\hat{\imath} + 4\,\text{m/s}\hat{\jmath}$, and acceleration $\vec{a}_2 = 5\,\text{m/s}^2\hat{\imath} - 16\,\text{m/s}^2\hat{\jmath}$.

a) Find the linear momentum $\vec{L}$ and its rate of change $\dot{\vec{L}}$ of each particle at the instant of interest.

b) Find the linear momentum $\vec{L}$ and its rate of change $\dot{\vec{L}}$ of the system of the two particles at the instant of interest.

c) Find the center of mass of the system at the instant of interest.

d) Find the velocity and acceleration of the center of mass.

Answer:
(a) $\vec{L}_1 = (18\hat{\imath}+12\hat{\jmath})\,\text{kg m/s}$, $\dot{\vec{L}}_1 = (60\hat{\imath} - 144\hat{\jmath})\,\text{N}$, $\vec{L}_2 = (80\hat{\imath} + 40\hat{\jmath})\,\text{kg m/s}$, and $\dot{\vec{L}}_2 = (50\hat{\imath} - 160\hat{\jmath})\,\text{N}$.
(b) $\vec{L} = \vec{L}_1 + \vec{L}_2 = (98\hat{\imath} + 52\hat{\jmath})\,\text{kg m/s}$ and $\dot{\vec{L}} = \dot{\vec{L}}_1 + \dot{\vec{L}}_2 = (110\hat{\imath} - 304\hat{\jmath})\,\text{N}$
(c) $\vec{r}_{cm} = -\frac{7}{8}(3\hat{\imath} + 2\hat{\jmath})\,\text{m}$
(d) $\vec{v}_{cm} = (-1\hat{\imath} + 4.75\hat{\jmath})\,\text{m/s}$ and $\vec{a}_{cm} = (6.875\hat{\imath} - 19\hat{\jmath})\,\text{m/s}^2$

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \end{pmatrix} \text{kg}$$

$$\underline{r}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \text{m}, \quad \underline{r}_2 = \begin{pmatrix} -6 \\ -4 \end{pmatrix} \text{m}$$

$$\dot{\underline{r}}_1 = \begin{pmatrix} -16 \\ 6 \end{pmatrix} \text{m/s}, \quad \dot{\underline{r}}_2 = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \text{m/s}$$

$$\ddot{\underline{r}}_1 = \begin{pmatrix} 10 \\ -24 \end{pmatrix} \text{m/s}^2, \quad \ddot{\underline{r}}_2 = \begin{pmatrix} 5 \\ -16 \end{pmatrix} \text{m/s}^2$$

a) $$\vec{L}_1 = m_1 \dot{\underline{r}}_1 = 6 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \text{kg m/s}$$

$$\dot{\vec{L}}_1 = m_1 \ddot{\underline{r}}_1 = 6 \begin{pmatrix} 10 \\ -24 \end{pmatrix} \text{kg m/s}^2$$

$$\vec{L}_2 = m_2 \dot{\underline{r}}_2 = 10 \begin{pmatrix} 8 \\ 4 \end{pmatrix} \text{kg m/s}$$

$$\dot{\vec{L}}_2 = m_2 \ddot{\underline{r}}_2 = 10 \begin{pmatrix} 5 \\ -16 \end{pmatrix} \text{kg m/s}^2$$

b) $$\vec{L} = \vec{L}_1 + \vec{L}_2 = \begin{pmatrix} 98 \\ 52 \end{pmatrix} \text{kg m/s}$$

$$\dot{\vec{L}} = \dot{\vec{L}}_1 + \dot{\vec{L}}_2 = \begin{pmatrix} 110 \\ -304 \end{pmatrix} \text{kg m/s}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

c) $\vec{r}_c = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$

$$\vec{r}_c = \frac{6\begin{pmatrix}3\\2\end{pmatrix} + 10\begin{pmatrix}-6\\-4\end{pmatrix}}{16} = \frac{1}{16}\begin{pmatrix}-42\\-28\end{pmatrix} = -\frac{7}{8}\begin{pmatrix}3\\2\end{pmatrix} m$$

$$\dot{\vec{r}}_c = \frac{m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2}{m_1+m_2}$$

$$= \frac{6\begin{pmatrix}-16\\6\end{pmatrix} + 10\begin{pmatrix}8\\4\end{pmatrix}}{16} = \frac{1}{16}\begin{pmatrix}-16\\76\end{pmatrix} = \begin{pmatrix}-1\\4.75\end{pmatrix}\frac{m}{s}$$

13.1.3 A particle of mass $m_1 = 5\,\text{kg}$ and a particle of mass $m_2 = 10\,\text{kg}$ are moving in space. At a particular instant of interest, particle 1 has position, velocity, and acceleration

$$\begin{aligned}\vec{r}_1 &= 1\,\text{m}\hat{\imath} + 1\,\text{m}\hat{\jmath}\\ \vec{v}_1 &= 2\,\text{m/s}\hat{\jmath}\\ \vec{a}_1 &= 3\,\text{m/s}^2\hat{k}\end{aligned}$$

respectively, and particle 2 has position, velocity, and acceleration

$$\begin{aligned}\vec{r}_2 &= 2\,\text{m}\hat{\imath}\\ \vec{v}_2 &= 1\,\text{m/s}\hat{k}\\ \vec{a}_2 &= 1\,\text{m/s}^2\hat{\jmath}\end{aligned}$$

respectively. For the system of particles at the instant of interest, find its

a) linear momentum $\vec{L}$,

b) rate of change of linear momentum $\dot{\vec{L}}$,

c) angular momentum about the origin $\vec{H}_{/O}$,

d) rate of change of angular momentum about the origin $\dot{\vec{H}}_{/O}$,

e) kinetic energy E_K, and

f) rate of change of kinetic energy.

Answer:
(a) $(10\hat{\jmath} + 10\hat{k})\,\text{kg m/s}$
(b) $(10\hat{\jmath} + 15\hat{k})\,\text{kg m/s}^2$
(c) $(-20\hat{\jmath} + 10\hat{k})\,\text{kg}\cdot\text{m}^2/\text{s}$
(d) $(15\hat{\imath} - 15\hat{\jmath} + 30\hat{k})\,\text{kg}\cdot\text{m}^2/\text{s}^2$
(e) $15\,\text{kg}\cdot\text{m}^2/\text{s}^2$
(f) $40\,\text{W}$

$m_1 = 5\,\text{kg}$

$\vec{r}_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}\text{m}$

$\vec{v}_1 = \begin{pmatrix}0\\2\\0\end{pmatrix}\text{m/s}$

$\vec{a}_1 = \begin{pmatrix}0\\0\\3\end{pmatrix}\text{m/s}^2$

$m_2 = 10\,\text{kg}$

$\vec{r}_2 = \begin{pmatrix}2\\0\\0\end{pmatrix}\text{m}$

$\vec{v}_2 = \begin{pmatrix}0\\0\\1\end{pmatrix}\text{m/s}$

$\vec{a}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}\text{m/s}^2$

a) $\vec{L} = m_1\vec{v}_1 + m_2\vec{v}_2$

$$= 5\begin{pmatrix}0\\2\\0\end{pmatrix} + 10\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\10\\10\end{pmatrix}\text{kg m/s}$$

b) $$\dot{\vec{L}} = 5\begin{pmatrix}0\\0\\3\end{pmatrix} + 10\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\10\\15\end{pmatrix}\text{kg m/s}^2$$

c) Angular momentum about origin

$$\vec{H}_{/O} = m_1\vec{r}_1\times\vec{v}_1 + m_2\vec{r}_2\times\vec{v}_2$$

$$= 5\begin{pmatrix}0\\0\\2\end{pmatrix} + 10\begin{pmatrix}0\\-2\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\-20\\10\end{pmatrix}\text{kg m}^2/\text{s}$$

d) $\dot{\vec{H}}_{/O} = m_1 \vec{r}_1 \times \vec{a}_1 + m_2 \vec{r}_2 \times \vec{a}_2$

$= 5\begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} + 10\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 15 \\ -15 \\ 20 \end{pmatrix} \text{kg m}^2/\text{s}^2$

e) $E_K = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2$

$= \frac{1}{2} \times 5 \times 4 + \frac{1}{2} \times 10 \times 1$

$= 15 \text{ kg m}^2/\text{s}^2$

f) $\dot{E}_K = m_1 \dot{r}_1 \ddot{r}_1 + m_2 \dot{r}_2 \cdot \ddot{r}_2$

$= 10 \times 3 + 10 \times 1 \times 1$

$= 40 \text{ kg m}^2/\text{s}^3$

13.1.4 If you are given the total mass, the position, the velocity, and the acceleration of the center of mass of a system of particles can you find the angular momentum $\vec{H}_{/O}$ of the system, where O is not at the center of mass? If so, how and why? If not, then give a reason and/or a counter example.

Answer:
No. You need to know the angular momenta of the particles relative to the center of mass to complete the calculation, information which is not given.

No. You need to know the angular momenta of the particles relative to the center of mass to complete the calculation, information which is not given. If the angular momenta of the paticles relative to the center of mass is zero or known, it is possible to estimate the angular momentum of the system with the given information.

13.1.5 Seventeen particles are interacting with the force on particle i from particle j being $\vec{F}_{ij}$ with all $\vec{F}_{ij}$ known.

a) What is the commonly assumed assumption about the relation between, say, $\vec{F}_{36}$ and $\vec{F}_{63}$?

b) What is the total force on particle 5?

Answer:
(a) $\vec{F}_{36} = -\vec{F}_{63}$
(b) Total force on particle 5 $= \sum_{i=1}^{17} \vec{F}_{5i}$

a) $\vec{F}_{ij} = -\vec{F}_{ji}$

b) Total force on 5 $= \sum_{j=1}^{17} \vec{F}_{5j}$

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13.1.6 Two particles each of mass m are connected by a massless elastic spring of spring constant k and unextended length $2R$. The system slides without friction on a horizontal table, so that no net external forces act.

a) Is the total linear momentum conserved? Justify your answer.

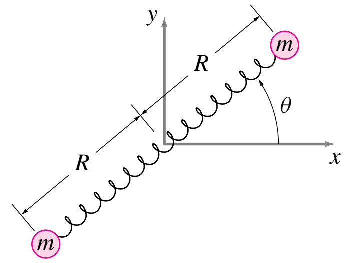
b) Can the center of mass accelerate? Justify your answer.

c) Draw free-body diagrams for each mass.

d) Derive the equations of motion for each mass in terms of cartesian coordinates.

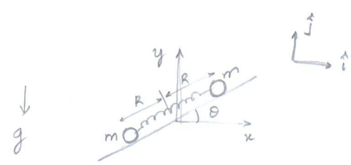
e) What are the total kinetic and potential energies of the system?

f) For constant values and initial conditions of your choosing, plot the trajectories of the two particles and of the center of mass (on the same plot).



Problem 13.6

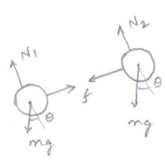
Answer:
(a) Yes, total external force on the system is zero.
(b) No, sum of all external forces on the system is zero.
(e) KE $= \frac{1}{2}(m_1 v_1^2 + m_2 v_2^2)$, PE $= \frac{1}{2}k(|\vec{r}_{12}| - 2R)^2$



a) Total linear momentum is not conserved. Gravity is acting on the system.

b) Yes, due to gravity

c)



$N_1 = mg\cos\theta = N_2$

$\sin\theta\, mg - f = ma$

both masses fall together so $f = 0$

$a = g\sin\theta$ along the plane

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

d)

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = -g\sin\theta \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\ddot{x} = -g\sin\theta\cos\theta \ \hat{i}$$

$$\ddot{y} = -g\sin^2\theta \ \hat{j}$$

e) The total Kinetic Energy + Potential Energy = constant

13.1.7 Two ice skaters whirl around one another. They are connected by a linear elastic cord whose center is stationary in space. We wish to consider the motion of one of the skaters by modeling her as a mass m held by a cord that exerts k Newtons for each meter it is extended from the central position.

a) Draw a free-body diagram showing the forces that act on the mass at an arbitrary position.

b) Write the differential equations that describe the motion.

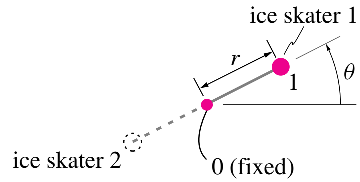
c) Describe in physical and mathematical terms the nature of the motion for the three cases

  a) $\omega < \sqrt{k/m}$;

  b) $\omega = \sqrt{k/m}$;

  c) $\omega > \sqrt{k/m}$.

(You are not asked to solve the equation of motion.)



Problem 13.7

Answer:
(b) $2r\dot{r}\dot{\theta} + r^3\ddot{\theta} = 0$
(c) For $\omega = \sqrt{k/m}$ the skaters will follow an elliptical trajectory with rate of whirling slowing down as the skater goes away from the center similar to planetary motion.

13.1.8 n identical particles with mass m are on the vertices of an n sided regular polygon. Equivalently, n particles are equally spaced on a circle with radius R. At $t = 0$ they all have velocities tangent to the circle and equal in magnitude v_0. All the particles are attracted to each other with an inverse square gravitational attraction. For the numerical simulations below pick values of n, m, G and R any way that pleases you.

a) Find an initial value for v_0 so that all the masses spiral in and then bounce out again. Plot the trajectories of all the masses on one plot for a long-enough time so the plot is pleasing to the eye.

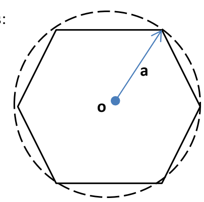
b) Find a value for v_0 so all the particles travel on circular trajectories.

c) Can you find a formula for v_0 above in terms of the other parameters in the problem?

Answer:

(c) $v_0 = \sqrt{\frac{3m}{r}\sum_{i=2}^{n-1}\cot((i-1)\frac{\pi}{n})}$

13.1.8 For 'n' equally spaced masses on a circle force acting on any mass is:



$$\vec{F} = \frac{-GmM}{R^2}\hat{e}_r \quad \text{where,}$$

$$M = (n-1)m \quad \text{and} \quad R = a + x$$

x can be obtained by using the fact that center of mass is at 'o'.

i.e. $ma = (n-1)mx$,

for circular orbit

$$\frac{mv_o^2}{R} = \frac{-GmM}{R^2}\hat{e}_r \quad \Rightarrow v_o = \sqrt{G(n-1)^2 m / nR}$$

For velocities less than $v_o = \sqrt{G(n-1)^2 m / nR}$ the trajectory of masses will look spiral. Each individual mass follows an elliptical path around the center of mass.

The spiral trajectory for case when $v=0.5*vo$ is as shown, for n=20 [ref m-file]

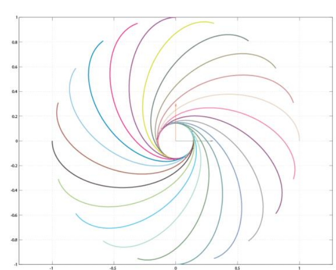
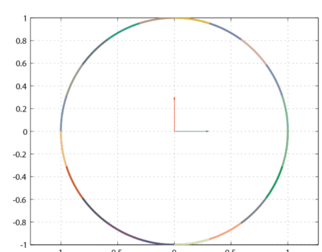


Fig: Spiral orbit obtained for v=0.5*vo



Circular orbit: n=20

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13.1.9 Two masses, both with $M = 1000m$ travel in circles on the xy plane according to

$$\vec{r}_1 = -\vec{r}_2 = R(\cos\omega t\hat{\imath} + \sin\omega t\hat{\jmath})$$

a) Assume inverse square attraction and find a set of values for m, G, R and ω so the assumed circular path is a solution of the equations of motion.

b) A third mass m is introduced which is gravitationally attracted to the other two. Pick initial conditions for the two big masses that are consistent with their circular motion solution. For the third mass use initial conditions $[\vec{r}_0]_{xyz} = [00z_0]$. Run a simulation for some time.

- Does the third mass stay *exactly* on the z axis for all time in the simulation? Would it if the simulation was exact? If so, why? If not, why not?
- Is the motion of the third mass exactly periodic in the computer simulation? Would it be if the solution was exact? If so, why? If not, why not?

Answer:
(a) $\omega = \sqrt{\frac{3M}{2R}}$
(b) The mass would stay on the z axis if the solution was exact.
(b) The solution would be exactly periodic if the ratio of the masses was infinite rather than just 1000. There are special initial conditions for which the motion is periodic for any mass ratio, the oscillations of the light mass need to be synchronous with the in-and-out oscillations of the heavier nearly-circular-motion masses.

13.1.10 The amazing eight. Three equal masses, say $m = 1$, are attracted by an inverse-square gravity law with $G = 1$. That is, each mass is attracted to the other by $F = Gm_1m_2/r^2$ where r is the distance between them. Use these unusual and special initial positions:

$$\begin{aligned}(x1, y1) &= (-0.97000436, 0.24308753)\\ (x2, y2) &= (-x1, -y1)\\ (x3, y3) &= (0, 0)\end{aligned}$$

and initial velocities

$$\begin{aligned}(vx3, vy3) &= (0.93240737, 0.86473146)\\ (vx1, vy1) &= -(vx3, vy3)/2\\ (vx2, vy2) &= -(vx3, vy3)/2.\end{aligned}$$

For each of the problems below show accurate computer plots and explain any curiosities. (Note, this solution was discovered independently, and nearly simultaneously, by Richard Montgomery and Chris Moore.)

a) Use computer integration to find and plot the motions of the particles. Plot each with a different color. Run the program for 2.1 time units.
b) Same as above, but run for 10 time units.
c) Same as above, but change the initial conditions slightly.
d) Same as above, but change the initial conditions more and run for a much longer time.

Answer:
(a) These three trajectories are all parts of the same figure 8.
(b) The trajectories trace and retrace the same figure 8. If your integration is not accurate, the curves will not exactly retrace.
(c) The trajectories make a beautiful swirl resembling a figure 8.
(d) The trajectories get wild, possibly ejecting one or more masses off to infinity.

Duan Li

13.1.10 Montgomery's eight. Three equal masses, say $m = 1$, are attracted by an inverse-square gravity law with $G = 1$. That is, each mass is attracted to the other by $F = Gm_1m_2/r^2$ where r is the distance between them. Use these unusual and special initial positions:

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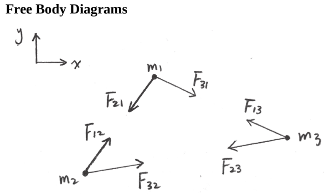
b) Same as above, but run for 10 time units. *
c) Same as above, but change the initial conditions slightly. *
d) Same as above, but change the initial conditions more and run for a much longer time. *

Given:
Three masses: $m_1 = m_2 = m_3 = m$
Inverse-square gravity law: $F = Gm_1m_2/r^2$, $G = 1$
Initial conditions:

$$\begin{aligned}(x_1, y_1)' &= (-0.97000436, 0.24308753)'\\ (x_2, y_2)' &= (-x_1, -y_1)'\\ (x_3, y_3)' &= (0, 0)'\\ (v_{x3}, v_{y3})' &= (0.93240737, 0.86473146)'\\ (v_{x1}, v_{y1})' &= -(v_{x3}, v_{y3})'/2\\ (v_{x2}, v_{y2})' &= -(v_{x3}, v_{y3})'/2\end{aligned}$$

Find: Three body motions

Free Body Diagrams



Define variables:
$\vec{F}_{ij}$: the gravitational force m_i imposes on m_j
$\vec{r}_{ji}$: the position vector pointing from m_j to m_i
$\vec{r}_i$: the position of m_i, $\vec{r}_i = (x_i, y_i)'$
$\vec{v}_i$: the velocity of m_i, $\vec{v}_i = (v_{xi}, v_{yi})'$
Inverse-square gravity law in vector form $\vec{F}_{ij} = \frac{Gm^2\vec{r}_{ji}}{|\vec{r}_{ji}|^3}$

Linear Momentum Balance for each mass
Mass 1

$$m\vec{a}_1 = \sum \vec{F} = \vec{F}_{21} + \vec{F}_{31}$$

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$$m\vec{a}_1 = Gm^2\left[\frac{\vec{r}_{12}}{|\vec{r}_{12}|^3}+\frac{\vec{r}_{13}}{|\vec{r}_{13}|^3}\right] = Gm^2\left[\frac{(\vec{r}_2-\vec{r}_1)}{|\vec{r}_2-\vec{r}_1|^3}+\frac{(\vec{r}_3-\vec{r}_1)}{|\vec{r}_3-\vec{r}_1|^3}\right]$$

$$\vec{a}_1 = Gm\left[\frac{(\vec{r}_2-\vec{r}_1)}{|\vec{r}_2-\vec{r}_1|^3}+\frac{(\vec{r}_3-\vec{r}_1)}{|\vec{r}_3-\vec{r}_1|^3}\right]$$

Mass 2 (by the same logic as mass 1)

$$\vec{a}_2 = Gm\left[\frac{(\vec{r}_1-\vec{r}_2)}{|\vec{r}_1-\vec{r}_2|^3}+\frac{(\vec{r}_3-\vec{r}_2)}{|\vec{r}_3-\vec{r}_2|^3}\right]$$

Mass 3 (by the same logic as mass 1)

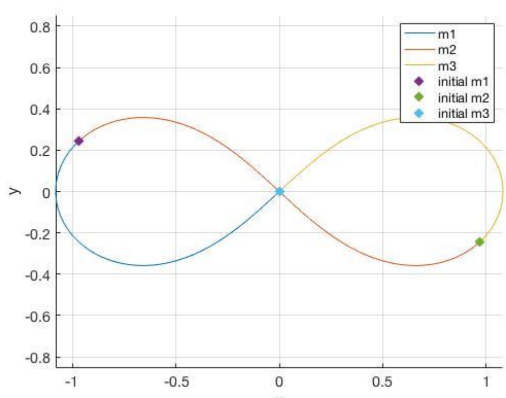
$$\vec{a}_3 = Gm\left[\frac{(\vec{r}_1-\vec{r}_3)}{|\vec{r}_1-\vec{r}_3|^3}+\frac{(\vec{r}_2-\vec{r}_3)}{|\vec{r}_2-\vec{r}_3|^3}\right]$$

Matlab ODE structure

Use the augmented state $\vec{z} = (\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{v}_1, \vec{v}_2, \vec{v}_3)'$, where $\vec{r}_i$ and $\vec{v}_i$ are both 2*1 column vector, so $\vec{z}$ is a 12*1 column vector.

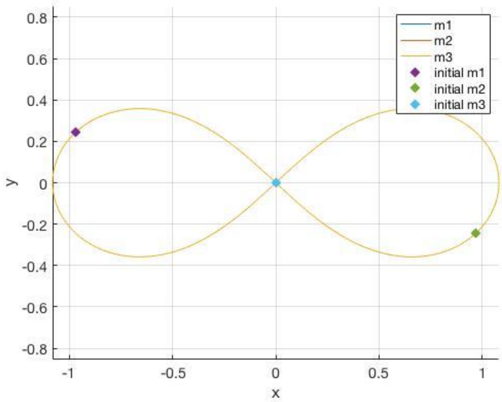
$$\dot{\vec{z}} = \begin{pmatrix}\vec{v}_1\\ \vec{v}_2\\ \vec{v}_3\\ \vec{a}_1\\ \vec{a}_2\\ \vec{a}_3\end{pmatrix} = \begin{pmatrix}\vec{v}_1\\ \vec{v}_2\\ \vec{v}_3\\ Gm\left[\frac{(\vec{r}_2-\vec{r}_1)}{|\vec{r}_2-\vec{r}_1|^3}+\frac{(\vec{r}_3-\vec{r}_1)}{|\vec{r}_3-\vec{r}_1|^3}\right]\\ Gm\left[\frac{(\vec{r}_1-\vec{r}_2)}{|\vec{r}_1-\vec{r}_2|^3}+\frac{(\vec{r}_3-\vec{r}_2)}{|\vec{r}_3-\vec{r}_2|^3}\right]\\ Gm\left[\frac{(\vec{r}_1-\vec{r}_3)}{|\vec{r}_1-\vec{r}_3|^3}+\frac{(\vec{r}_2-\vec{r}_3)}{|\vec{r}_2-\vec{r}_3|^3}\right]\end{pmatrix}$$

(a) Plot: Three body motion for 2.1 time units, with the initial conditions in the problem statement



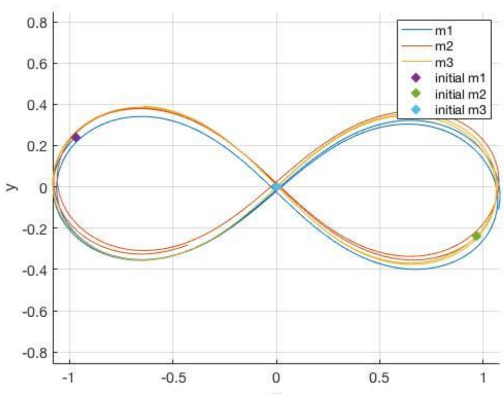
The trajectories of the three masses form a closed 8-shape.

(b) Plot: Three body motion for 10 time units, with the initial conditions in the problem statement



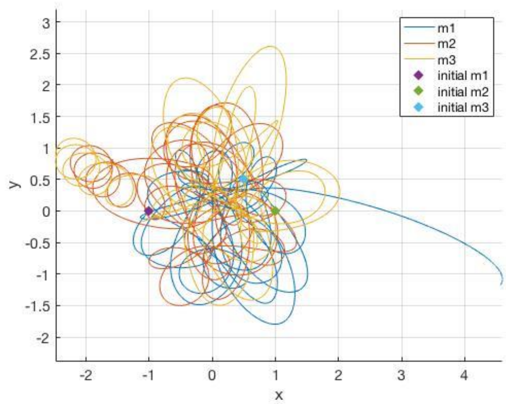
The trajectories of the three masses overlap and the system is stable.

(c) Plot: Three body motion for 10 time units with slightly changed initial conditions:
$\vec{r}_1 = (x_1, y_1)' \ = \ (-0.97, 0.24)'$
$\vec{v}_3 = (v_{x3}, v_{y3})' \ = \ (0.93, 0.86)'$
while the other initial conditions remain the same as in the problem statement.



The system is perturbed. The solution stays close to a period motion and is slowly diverging from the 8-shaped trajectory in part (b).

(d) Plot: Three body motion for 100 time units with significantly changed initial conditions:
$\vec{x}_1 = (x_1, y_1)' \ = \ (-1, 0)'$
$\vec{x}_3 = (x_3, y_3)' \ = \ (0.5, 0.5)'$
$\vec{v}_3 = (v_{x3}, v_{y3})' \ = \ (1, 1)'$
while the other initial conditions remain the same as in the problem statement.



The system becomes more chaotic with fast evolving orbits. In fact, if you run the code for long enough, you can see eventually m3 will fly off to infinity in one direction, while m1 and m2 will stay coupled and fly off to infinity in the other direction.

Montgomery's Eight

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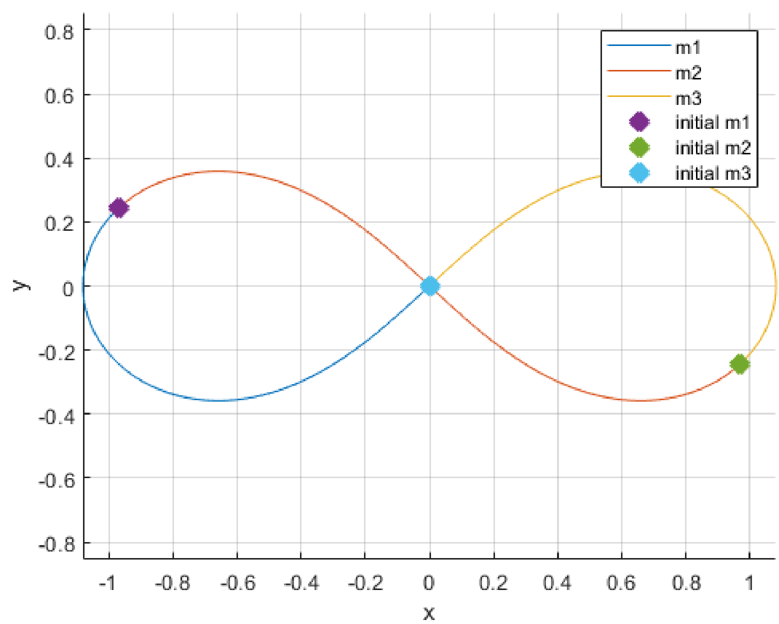
```
% This program simulates a three-body motion with special initial
% conditions.

clear all; close all;

% Parameters
p.G = 1;    % gravitational constant
p.m = 1;    % same mass for all three bodies

% Augmented state z = [r1; r2; r3; v1; v2; v3]
% where r1, r2, r3 are the positions of the three masses and
% v1, v2, v3 are the velocities of the three masses.
% r1, r2, r3, v1, v2, v3 are all column vectors consisting of
% their x and y components.

% Settings for different part

part = 'a'; % ##### USER INPUT #####

if part == 'a'
    % inital conditions for part (a)
    r10 = [-0.97000436; 0.24308753];
    r20 = -r10;
    r30 = [0; 0];
    v30 = [0.93240737; 0.86473146];
    v10 = -v30/2;
    v20 = -v30/2;
    % end time for part (a)
    tend = 2.1;
elseif part == 'b'
    % inital conditions for part (b)
    r10 = [-0.97000436; 0.24308753];
    r20 = -r10;
    r30 = [0; 0];
    v30 = [0.93240737; 0.86473146];
    v10 = -v30/2;
    v20 = -v30/2;
    % end time for part (b)
    tend = 10;
elseif part == 'c'
    % initial conditions for part (c)
    r10 = [-0.97; 0.24];
    r20 = -r10;
    r30 = [0; 0];
    v30 = [0.93; 0.86];
    v10 = -v30/2;
    v20 = -v30/2;
```

```
    % end time for part (c)
    tend = 10;
elseif part == 'd'
    % initial conditions for part (d)
    r10 = [-1; 0];
    r20 = -r10;
    r30 = [0.5; 0.5];
    v30 = [1; 1];
    v10 = -v30/2;
    v20 = -v30/2;
    % end time for part (d)
    tend = 100;
else
    error('Invalid part number.')
end

% Initial conditions
z0 = [r10; r20; r30; v10; v20; v30]; % pack variables

% Time span
tspan = [0, tend];

% Solving ODE
f =  @(t,z) rhs(z,p);                                % anonymous function
opts = odeset('RelTol',1e-8,'AbsTol',1e-8);          % tolerance
[tarray, zarray] = ode45(f, tspan, z0, opts);        % solve ODE

% Plot three masses' trajectories
figure(1);
hold on;
plot(zarray(:,1), zarray(:,2));     % trajectory of m1
plot(zarray(:,3), zarray(:,4));     % trajectory of m2
plot(zarray(:,5), zarray(:,6));     % trajectory of m3
% Plot three masses' initial positions
s1 = scatter(r10(1), r10(2), 'x');  % initial position of m1
s2 = scatter(r20(1), r20(2), 'x');  % initial position of m2
s3 = scatter(r30(1), r30(2), 'x');  % initial position of m3
s1.LineWidth = 8;
s2.LineWidth = 8;
s3.LineWidth = 8;
axis equal;
grid on;
xlabel('x');
ylabel('y');
legend('m1','m2', 'm3', 'initial m1', 'initial m2', 'initial m3');
hold off;

% State z = [r1; r2; r3; v1; v2; v3]
function zdot = rhs(z,p)

    % unpack parameters
    G = p.G;
    m = p.m;
```

```
    % unpack states into variables
    r1 = z(1:2);    % position of m1
    r2 = z(3:4);    % position of m2
    r3 = z(5:6);    % position of m3
    v1 = z(7:8);    % velocity of m1
    v2 = z(9:10);   % velocity of m2
    v3 = z(11:12);  % velocity of m3

    % 1st order ODEs
    r1dot = v1;
    r2dot = v2;
    r3dot = v3;
    v1dot = G*m*( (r3-r1)/norm(r3-r1)^3 + (r2-r1)/norm(r2-r1)^3 );
    v2dot = G*m*( (r1-r2)/norm(r1-r2)^3 + (r3-r2)/norm(r3-r2)^3 );
    v3dot = G*m*( (r1-r3)/norm(r1-r3)^3 + (r2-r3)/norm(r2-r3)^3 );

    % pack rate of change of the states
    zdot = [r1dot; r2dot; r3dot; v1dot; v2dot; v3dot];
end
```



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Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*