SAMPLE 12.8 **Basic calculations:** Find $\vec{L}, \dot{\vec{L}}, \vec{H}_{/\mathrm{C}}, \dot{\vec{H}}_{/\mathrm{C}}, E_K, \dot{E}_K$ for a given particle P with mass $m_P = 1\,\mathrm{kg}$, given position, velocity, acceleration, and a point C. Specifically, we are given $\vec{r}_P = (\hat{\imath} + \hat{\jmath} + \hat{k})\,\mathrm{m}$, $\vec{v}_P = 3\,\mathrm{m/s}(\hat{\imath} + \hat{\jmath})$, $\vec{a}_P = 2\,\mathrm{m/s^2}(\hat{\imath} - \hat{\jmath} - \hat{k})$, and $\vec{r}_C = (2\hat{\imath} + \hat{k})\,\mathrm{m}$.

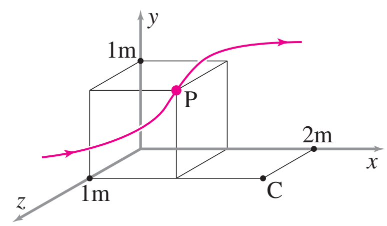

Figure 12.26

Solution Since $\vec{r}_P = (\hat{\imath} + \hat{\jmath} + \hat{k})\,\mathrm{m}$ and $\vec{r}_C = (2\hat{\imath} + \hat{k})\,\mathrm{m}$,

$$\vec{r}_{P/C} = \vec{r}_P - \vec{r}_C = (-\hat{\imath} + \hat{\jmath})\,\mathrm{m}.$$

So we have the motion quantities

$$\begin{aligned}
\vec{L} &= m\vec{v}_P \\
&= (1\,\mathrm{kg})\cdot[(3\,\mathrm{m/s})(\hat{\imath} + \hat{\jmath})] \\
&= 3(\hat{\imath} + \hat{\jmath})\frac{\mathrm{kg\cdot m}}{\mathrm{s}} \\
&= 3\,\mathrm{N\cdot s}(\hat{\imath} + \hat{\jmath})
\end{aligned}$$

$$\begin{aligned}
\dot{\vec{L}} &= m\vec{a}_P \\
&= (1\,\mathrm{kg})[(2\,\mathrm{m/s^2})(\hat{\imath} - \hat{\jmath} - \hat{k})] \\
&= 2(\hat{\imath} - \hat{\jmath} - \hat{k})\frac{\mathrm{kg\cdot m}}{\mathrm{s^2}} \\
&= 2\,\mathrm{N}(\hat{\imath} - \hat{\jmath} - \hat{k})
\end{aligned}$$

$$\begin{aligned}
\vec{H}_{/\mathrm{C}} &= \vec{r}_{P/C} \times m\vec{v}_P \\
&= [(-\hat{\imath} + \hat{\jmath})\,\mathrm{m}] \times [(1\,\mathrm{kg})3\,\mathrm{m/s}(\hat{\imath} + \hat{\jmath})] \\
&= -(6\,\mathrm{kg\cdot m^2/s})\hat{k} \qquad (12.11)
\end{aligned}$$

$$\begin{aligned}
\dot{\vec{H}}_{/\mathrm{C}} &= \vec{r}_{P/C} \times m\vec{a} \\
&= [(-\hat{\imath} + \hat{\jmath})m] \times [(1\,\mathrm{kg})2\,\mathrm{m/s^2}(\hat{\imath} - \hat{\jmath} - \hat{k})] \\
&= -(2\,\mathrm{kg\cdot m^2/s^2})(\hat{\imath} + \hat{\jmath})
\end{aligned}$$

$$\begin{aligned}
E_{\mathrm{K}} &= \frac{1}{2}m|\vec{v}_P|^2 \\
&= \frac{1}{2}(1\,\mathrm{kg})(3\sqrt{2}\,\mathrm{m/s})^2 \\
&= 9\frac{\mathrm{kg\cdot m^2}}{\mathrm{s^2}} \\
&= 9\,\mathrm{N\cdot m}
\end{aligned}$$

$$\begin{aligned}
\dot{E}_{\mathrm{K}} &= \frac{d}{dt}(\frac{m}{2}\vec{v}_P\cdot\vec{v}_P) \\
&= \frac{m}{2}[\vec{v}_P\cdot\dot{\vec{v}}_P + \dot{\vec{v}}_P\cdot\vec{v}_P] \\
&= m\vec{v}_P\cdot\vec{a}_P \\
&= 1\,\mathrm{kg}[(3\,\mathrm{m/s})(\hat{\imath} + \hat{\jmath})]\cdot[(2\,\mathrm{m/s^2})(\hat{\imath} - \hat{\jmath} - \hat{k})] \\
&= 0.
\end{aligned}$$

Note: $\frac{d}{dt}(\frac{1}{2}v^2) \neq |\vec{v}||\vec{a}|$.

SAMPLE 12.9 **Direct application of the formulas:** A 2 kg block is moving with a velocity $\vec{v}(t) = u_0 e^{-ct}\hat{\imath} + v_0\hat{\jmath}$, where $u_0 = 5\,\mathrm{m/s}$, $v_0 = 10\,\mathrm{m/s}$, and $c = 0.5/\,\mathrm{s}$. Consider the time interval between $t_1 = 1\,\mathrm{s}$ to $t_2 = 3\,\mathrm{s}$.

1. Find the net change in the linear momentum of the block, $\Delta\vec{L} = \vec{L}(t_2) - \vec{L}(t_1)$.
2. Find the force $\vec{F}(t)$ on the block and compute the impulse $\int_{t_1}^{t_2} \vec{F}dt$ and show that it is the same as $\Delta\vec{L}$ computed above.
3. Find the change in kinetic energy from direct computation of energy and compare with work done by computing $\int_{t_1}^{t_2} Pdt$.

Solution

1. For the given block we have, $\vec{L} = m\vec{v} = m(u_0 e^{-ct}\hat{\imath} + v_0\hat{\jmath})$. Therefore,

$$\Delta L = \vec{L}(t_2) - \vec{L}(t_1) = mu_0\left(e^{-ct_2} - e^{-ct_1}\right)\hat{\imath}.$$

Substituting the given values, $m = 2\,\mathrm{kg}$, $u_0 = 5\,\mathrm{m/s}$, $v_0 = 10\,\mathrm{m/s}$, $c = 0.5/\,\mathrm{s}$, $t_1 = 1\,\mathrm{s}$, and $t_2 = 3\,\mathrm{s}$ we get

$$\Delta\vec{L} = 2\,\mathrm{kg}\cdot 5\,\mathrm{m/s}(e^{-0.5/\,\mathrm{s}\cdot 3\,\mathrm{s}} - e^{-0.5/\,\mathrm{s}\cdot 1\,\mathrm{s}})\hat{\imath} = -(3.83\,\mathrm{kg}\cdot\,\mathrm{m/s})\hat{\imath}.$$

$$\boxed{\Delta\vec{L} = -(3.83\,\mathrm{kg}\cdot\,\mathrm{m/s})\hat{\imath}}$$

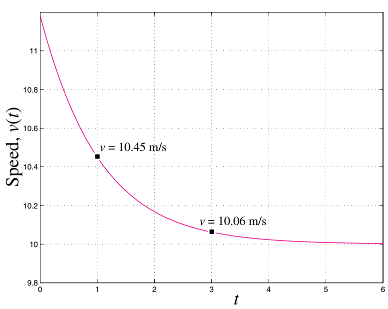

Figure 12.27: The plot of speed $v = |\vec{v}| = \sqrt{v_x^2 + v_y^2}$ vs time. The speeds at t_1 and t_2 are of interest for computing the kinetic energy at the two instants.

2. To calculate the impulse, $\int \vec{F}dt$, we need to find the force first. Since $\vec{F} = m\vec{a} = m\dot{\vec{v}}$, we get

$$\vec{F}(t) = m\frac{d}{dt}(u_0 e^{-ct}\hat{\imath} + v_0\hat{\jmath}) = -mcu_0 e^{-ct}\hat{\imath}.$$

Hence, the impulse is

$$\begin{aligned}\int_{t_1}^{t_2}\vec{F}dt &= -\int_{t_1}^{t_2} mcu_0 e^{-ct}dt\hat{\imath} = mu_0(e^{-ct_2} - e^{-ct_1}) \\ &= 2\,\mathrm{kg}\cdot 5\,\mathrm{m/s}(e^{-0.5/\,\mathrm{s}\cdot 3\,\mathrm{s}} - e^{-0.5/\,\mathrm{s}\cdot 1\,\mathrm{s}})\hat{\imath} \\ &= -3.83\ \mathrm{kg}\cdot\,\mathrm{m/s}\hat{\imath}\end{aligned}$$

which is, expectedly, the same answer as obtained above for ΔL.

3. To find the kinetic energy, we need the speed of the particle, $v = |\vec{v}| = \sqrt{v_x^2 + v_y^2}$. Now, the change in kinetic energy is

$$\begin{aligned}\Delta E_\mathrm{K} &= E_{\mathrm{K}_2} - E_{\mathrm{K}_1} = \frac{1}{2}m\left(v_2^2 - v_1^2\right) \\ &= \frac{1}{2}m\left(u_0^2 e^{-2ct_2} + v_0^2 - u_0^2 e^{-2ct_1} - v_0^2\right) \\ &= \frac{1}{2}mu_0^2\left(e^{-2ct_2} - e^{-2ct_1}\right) = -7.95\,\mathrm{N{\cdot}m}.\end{aligned}$$

Now, we can compare this value by computing the work done $\int Pdt$, since $\Delta E_\mathrm{K} = \int Pdt$. To compute the power $P = \vec{F}\cdot\vec{v}$, we need to find the dot product between the force and the velocity. Since $\vec{F} = -mcu_0 e^{-ct}\hat{\imath}$, and $\vec{v} = u_0 e^{-ct}\hat{\imath} + v_0\hat{\jmath}$, we get, $\vec{F}\cdot\vec{v} = -mcu_0^2 e^{-2ct}$. Therefore, the work done is,

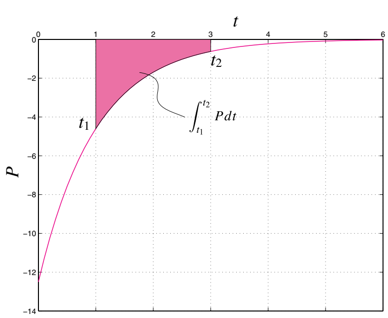

Figure 12.28: The area under the $P(t)$ curve between t_1 and t_2 is the work done $W = \int_{t_1}^{t_2} Pdt$.

$$\begin{aligned}W &= \int_{t_1}^{t_2} Pdt = \int_{t_1}^{t_2} -mcu_0^2 e^{-2ct}dt \\ &= -mcu_0^2(e^{-2ct_2} - e^{-2ct_1}) = -7.95\,\mathrm{N{\cdot}m}.\end{aligned}$$

$$\boxed{\Delta E_\mathrm{K} = -7.95\,\mathrm{N{\cdot}m}, \quad W = -7.95\,\mathrm{N{\cdot}m}}$$

SAMPLE 12.10 Angular momentum: direct application of the formula. The position of a particle of mass $m = 0.5\,\text{kg}$ is $\vec{r}(t) = \ell\ \sin(\lambda t)\hat{\imath} + h\hat{\jmath}$; where $\lambda = \pi/2\,\text{rad/s}$, $h = 2\,\text{m}$, $\ell = 2\,\text{m}$, and $\vec{r}$ is measured from the origin.

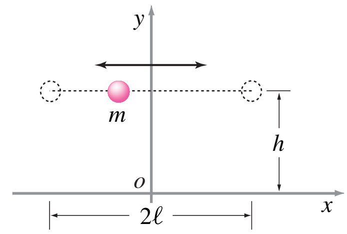

Figure 12.29

1. Find the net change in linear momentum $\Delta\vec{L}$ of the particle between $t = 1\,\text{s}$ and $t = 3\,\text{s}$.
2. Find the net change in angular momentum $\Delta\vec{H}_{/O}$ of the particle about the origin between $t = 1\,\text{s}$ and $t = 3\,\text{s}$.
3. Find the angular impulse $\int_{1\,\text{s}}^{3\,\text{s}} M dt$ about the origin and compare the result with $\Delta\vec{H}_{/O}$ found above.

Solution

1. Linear momentum: Let the two instants of interest be $t_1(= 1\,\text{s})$ and $t_2(= 3\,\text{s})$. The net change in linear momentum, $\Delta\vec{L} = \vec{L}_2 - \vec{L}_1 = m(\vec{v}_2 - \vec{v}_1)$. Since $\vec{v} = \dot{\vec{r}} == \ell\,\lambda\,\cos(\lambda t)\hat{\imath}$, we get

$$\begin{aligned}\Delta\vec{L} &= m(\vec{v}_2 - \vec{v}_1) = m\,\ell\,\lambda(\cos\lambda t_2 - \cos\lambda t_1)\hat{\imath}\\ &= (0.5\,\text{kg})(2\,\text{m})\left(\frac{\pi}{2}\,\text{rad/s}\right)\left(\cos\frac{\pi}{2} - \cos\frac{3\pi}{2}\right)\hat{\imath}\\ &= \vec{0}.\end{aligned}$$

The answer makes sense because both $\vec{v}_1 = \vec{0}$ and $\vec{v}_2 = \vec{0}$. In fact, finding the velocity at $t_1 = 1\,\text{s}$ and $t_2 = 3\,\text{s}$ would have made the calculation much simpler.

$$\boxed{\Delta\vec{L} = \vec{0}}$$

2. Angular momentum: The net change in angular momentum between t_1 and t_2 is,

$$\Delta\vec{H}_{/O} = (\vec{H}_{/O})_2 - (\vec{H}_{/O})_1 = \vec{r}_{/O} \times m\underbrace{\vec{v}_2}_{\vec{0}} - \vec{r}_{/O} \times m\underbrace{\vec{v}_1}_{\vec{0}} = \vec{0}.$$

$$\boxed{\Delta\vec{H}_{/O} = \vec{0}}$$

Note that it so happens that velocities at the two instants are zero and hence, both $(\vec{H}_{/O})_1$ and $(\vec{H}_{/O})_2$ are zero, making $\Delta\vec{H}_{/O}$ also zero. It is, however, possible that we could get $\Delta\vec{H}_{/O}$ to be zero even if the $(\vec{H}_{/O})_1$ and $(\vec{H}_{/O})_2$ were non-zero (when they are equal).

3. Moment impulse: Now, let us find the impulse due to the moment, $\int M dt$ between the two given time instants and see if that matches with the net zero change in angular momentum. We first need to compute the moment $\vec{M}_O = \vec{r}_{/O} \times \vec{F} = \vec{r}_{/O} \times m\vec{a}$:

$$\vec{M}_{/O}(t) = \vec{r}_{/O} \times m\vec{a} = (\ell\ \sin(\lambda t)\hat{\imath} + h\hat{\jmath}) \times m(-\ell\,\lambda^2 \sin\lambda t\hat{\imath}) = m\,\ell\,h\,\lambda^2 \sin\lambda t\hat{k}.$$

Therefore, the impulse due to this moment is

$$\begin{aligned}\int_{t_1}^{t_2} \vec{M}_{/O} dt &= \int_{1\,\text{s}}^{3\,\text{s}} (m\,\ell\,h\,\lambda^2 \sin\lambda t\hat{k})dt = m\,\ell\,h\,\lambda^2\hat{k}\int_{1\,\text{s}}^{3\,\text{s}} \sin(\lambda t)dt\\ &= m\,\ell\,h\,\lambda^2\hat{k}\left[-\frac{\cos\lambda t}{\lambda}\right]_{1\,\text{s}}^{3\,\text{s}} = m\,\ell\,h\,\lambda\hat{k}\left[\cos\frac{3\pi}{2} - \cos\frac{\pi}{2}\right]\\ &= \vec{0}\end{aligned}$$

as expected. It can also be seen from a plot of $|\vec{M}_{/O}|$ vs t, as shown in fig. 12.30, that the net area under the moment between t_1 and t_2 is zero, giving a zero moment impulse.

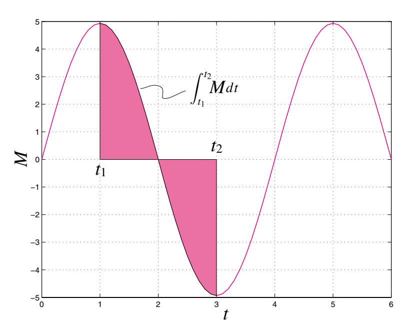

Figure 12.30: Plot of $M(t)$ where $\vec{M}_{/O}(t) = M(t)\hat{k}$. The area under the moment curve between t_1 and t_2 is the magnitude of the moment impulse.