12.1 Dynamics of a particle in space

Time derivative of a vector: position, velocity and acceleration

From here to the end of the book most of our calculations will involve vector-valued functions of time. For example, the vectors linear momentum $\vec{\boldsymbol{L}}$ and angular momentum $\vec{\boldsymbol{H}}$ have a central place in mechanics. Evaluating them depends, in turn, on understanding the relation between position $\vec{\boldsymbol{r}}$, and its rate of change, called velocity $\vec{\boldsymbol{v}}$. We also need to know the relation between velocity $\vec{\boldsymbol{v}}$ and its rate of change, the acceleration $\vec{\boldsymbol{a}}$.

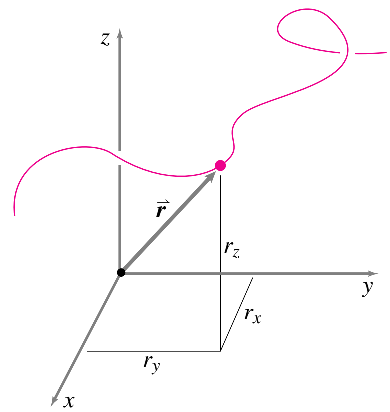


Figure 12.5: The path of a particle and its position at one time projected onto Cartesian coordinates. The path, or trajectory, of the particle is the sequence of locations of the tip of the $\vec{\boldsymbol{r}}$ vector, if the vector is drawn with its tail at the origin.

What do we mean by the rate of change of a vector? The rate of change of any quantity, including a vector, is the ratio of the change of that quantity to the amount of time that passes, for very small amounts of time. ①

$$\text{rate of change of any } (\textit{thing}) \equiv \frac{\text{amount } \textit{thing} \text{ changes}}{\text{amount of time for that change}}$$

①As you should remember from Calculus, these words really describe the average rate of change over the time interval. Only in the mathematical limit, as the time interval approaches zero, is the ratio of "amount of change over the time interval" not just approximately, but exactly, the instantaneous rate of change.

The notation for the rate of change of a vector $\vec{\boldsymbol{r}}$ is

$$\frac{d\vec{\boldsymbol{r}}}{dt}.$$

Or, in the short hand 'dot' notation invented by Newton for just this purpose, $\vec{\boldsymbol{v}} = \dot{\vec{\boldsymbol{r}}}$. The definition of the derivative $\frac{d\vec{\boldsymbol{r}}}{dt}$ or $\dot{\vec{\boldsymbol{r}}}$ is the same as for any*thing* else,

$$\frac{d\vec{\boldsymbol{r}}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{\boldsymbol{r}}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{\boldsymbol{r}}(t + \Delta t) - \vec{\boldsymbol{r}}(t)}{\Delta t}$$

where the top of the fraction is the change in $\vec{\boldsymbol{r}}$ and the bottom is the change in t. Underlying most dynamics calculations are derivatives of $\vec{\boldsymbol{r}}(t)$. But we also sometimes need to take derivatives of linear momentum $\vec{\boldsymbol{L}}$, angular momentum $\vec{\boldsymbol{H}}$ and some other quantities (e.g., the angular velocity $\vec{\boldsymbol{\omega}}$ of a rigid object). But all of these quantities somehow depend on the derivatives of $\vec{\boldsymbol{r}}(t)$.

Cartesian coordinates

A simple way to think about vector derivatives is with cartesian coordinates. A moving point has a location $\vec{\boldsymbol{r}}$, relative to the origin of a 'good' (i.e., $\mathcal{N}$ewtonian) reference frame as shown in fig. 12.5, which can be written as:

$$\vec{\boldsymbol{r}} = r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}} + r_z\hat{\boldsymbol{k}} \qquad \text{or} \qquad \vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}.$$

So velocity is the derivative of $\vec{\boldsymbol{r}}$. Since the base vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ are constant, differentiation to get velocity and acceleration is simple:

$$\vec{\boldsymbol{v}} = \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}} + \dot{z}\hat{\boldsymbol{k}} \qquad \text{and} \qquad \vec{\boldsymbol{a}} = \ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}} + \ddot{z}\hat{\boldsymbol{k}}.$$

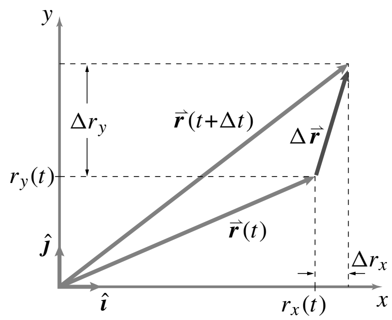


Figure 12.6: Change of position in Δt broken into components in 2-D . $\Delta\vec{\boldsymbol{r}}$ is $\vec{\boldsymbol{r}}(t{+}\Delta t){-}\vec{\boldsymbol{r}}(t)$. $\Delta\vec{\boldsymbol{r}}$ has components Δr_x and Δr_y. So $\Delta\vec{\boldsymbol{r}} = \Delta r_x\hat{\boldsymbol{\imath}} + \Delta r_y\hat{\boldsymbol{\jmath}}$. In the limit as Δt goes to zero, $\dot{\vec{\boldsymbol{r}}}$ is the ratio of $\Delta\vec{\boldsymbol{r}}$ to Δt.

The idea is illustrated in fig. 12.6. Let's take

$$\vec{\boldsymbol{r}} = r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}} \quad \text{or} \quad \vec{\boldsymbol{r}}(t) = r_x(t)\hat{\boldsymbol{\imath}} + r_y(t)\hat{\boldsymbol{\jmath}}.$$

We can apply the definition of derivative and find

$$\begin{aligned}
\dot{\vec{\boldsymbol{r}}}(t) &= \lim_{\Delta t\to 0} \frac{\vec{\boldsymbol{r}}(t+\Delta t) - \vec{\boldsymbol{r}}(t)}{\Delta t} \\
&= \frac{\overbrace{\Big(r_x(t+\Delta t)\hat{\boldsymbol{\imath}} + r_y(t+\Delta t)\hat{\boldsymbol{\jmath}}\Big)}^{\vec{\boldsymbol{r}}(t+\Delta t)} - \overbrace{\Big(r_x(t)\hat{\boldsymbol{\imath}} + r_y(t)\hat{\boldsymbol{\jmath}}\Big)}^{\vec{\boldsymbol{r}}(t)}}{\Delta t} \\
&= \frac{r_x(t+\Delta t) - r_x(t)}{\Delta t}\hat{\boldsymbol{\imath}} + \frac{r_y(t+\Delta t) - r_y(t)}{\Delta t}\hat{\boldsymbol{\jmath}} \\
&= \dot{r}_x(t)\hat{\boldsymbol{\imath}} + \dot{r}_y(t)\hat{\boldsymbol{\jmath}}
\end{aligned}$$

thus showing the palatable result that

> the components of the velocity vector are the time derivatives of the components of the position vector.

(**Beware.** Later in the book we will use base vectors that change in time, such as polar coordinate base vectors, path basis vectors, or basis vectors attached to a rotating frame. For these vectors the components of the vector's derivative will *not* be the derivatives of its components. See box 12.2.)

Figure 12.7 shows a particle P's path, its position at a sequence of times. The position vector $\vec{\boldsymbol{r}}_{P/O}$ is the arrow from the origin to a point on the curve, a different point on the curve at each instant of time. The velocity $\vec{\boldsymbol{v}}$ at time t is the rate of change of position at that time, $\vec{\boldsymbol{v}} \equiv \dot{\vec{\boldsymbol{r}}}$.

Example: Given position as a function of time, find the velocity.
Given that the position of a point is:

$$\vec{\boldsymbol{r}}(t) = C_1\cos(\omega\, t)\hat{\boldsymbol{\imath}} + C_2\sin(\omega\, t)\hat{\boldsymbol{\jmath}}$$

with C_1, C_2 and ω given constants what is the velocity (a vector) at a given time t?

First we note that the components of $\vec{\boldsymbol{r}}(t)$ have been given implicitly as

$$r_x(t) = C_1\cos(\omega\, t) \qquad \text{and} \qquad r_y(t) = C_2\sin(\omega\, t).$$

Then we find the velocity by differentiating each of the components with respect to time and re-assembling as a vector to get

$$\vec{\boldsymbol{v}}(t) = \dot{\vec{\boldsymbol{r}}} = -C_1\omega\sin(\omega\, t)\hat{\boldsymbol{\imath}} + C_2\omega\cos(\omega\, t)\hat{\boldsymbol{\jmath}}$$

Now we evaluate this expression with the given values of C_1, C_2, ω and t .

Are position, velocity and acceleration all parallel? Sometimes this is a right intuition. For example, after some time has passed the change in position is exactly the average velocity times the elapsed time. And the change in velocity is exactly the average acceleration times the elapsed time. So in the long run, if something accelerates in some more-or-less constant direction then the position will change in that same direction. But actually, at any instant in time, position, velocity and acceleration are basically unrelated.

Example: Position, velocity and acceleration can be mutually orthogonal.
Here is a motion where, at least at one instant in time, the position, velocity, and acceleration are mutually orthogonal as in fig. 12.8. For example, look at the path in fig. 12.9. At the point where the path intersects the y axis the position relative to the origin is in the $\hat{\boldsymbol{\jmath}}$ direction, the velocity is tangent to the path in the $\hat{\boldsymbol{\imath}}$ direction and the acceleration is at least partially up, in the $\hat{\boldsymbol{k}}$ direction. Working this out with equations, if we take the position as a function of time to be

$$\vec{\boldsymbol{r}}(t) = A\hat{\boldsymbol{\jmath}} - Bt\hat{\boldsymbol{\imath}} + Ct^2\hat{\boldsymbol{k}}$$

we can calculate the velocity and acceleration by differentiation as

$$\vec{\boldsymbol{v}} = \frac{d\vec{\boldsymbol{r}}}{dt} = -B\hat{\boldsymbol{\imath}} + 2Ct\hat{\boldsymbol{k}}, \quad \vec{\boldsymbol{a}} = \frac{d\vec{\boldsymbol{v}}}{dt} = 2C\hat{\boldsymbol{k}}.$$

So, at $t = 0$,

$$\vec{\boldsymbol{r}} = A\hat{\boldsymbol{\jmath}}, \quad \vec{\boldsymbol{v}} = -B\hat{\boldsymbol{\imath}}, \quad \text{and} \quad \vec{\boldsymbol{a}} = 2C\hat{\boldsymbol{k}}.$$

The dot products between $\vec{\boldsymbol{r}}$, $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{a}}$ are: $\vec{\boldsymbol{r}}\cdot\vec{\boldsymbol{v}} = 0$, $\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}} = 0$, and $\vec{\boldsymbol{r}}\cdot\vec{\boldsymbol{a}} = 0$, so these vectors are mutually orthogonal at the instant marked. (**Aside:** Why is there a $-B$ in this example? Answer: no reason, we could have used $+B$ just as well.)

In constant rate circular motion position (relative to the circle's center) and velocity remain perpendicular for all time, and so do velocity and acceleration. However, the directions of position, velocity and acceleration are not arbitrary. For example, there is no motion where position, velocity and acceleration are exactly mutually orthogonal for an extended time. Imagine a slender circular cone. If position is measured relative to the apex of the cone then constant-rate circular motion about the base of the cone *almost* has position, velocity and acceleration mutually orthogonal for all time. But position and acceleration are only exactly orthogonal in the limit as the cone becomes infinitely slender.

The product rule of differentiation

We know three ways to multiply vectors: multiplying a vector by a scalar, taking the dot product of two vectors, and taking the cross product of two vectors (please review Chapter 2). Because all of these quantities might be functions of time we need to know how to differentiate products. It's simple. All three kinds of vector multiplication obey 'the product rule'

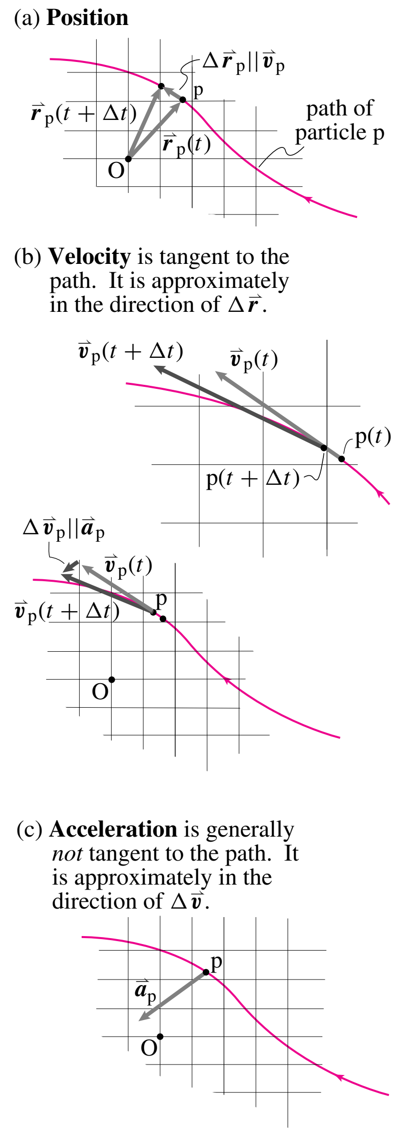


Figure 12.7: A particle moving on a curve. (a) shows the position vector is an arrow from the origin to the point on the curve. On the position curve the particle is shown at two times: t and $t + \Delta t$. The velocity at time t is roughly parallel to the difference between these two positions. The velocity is then shown at these two times in (b). The acceleration is roughly parallel to the difference between these two velocities. In (c) the acceleration is drawn on the path roughly parallel to the difference in velocities.

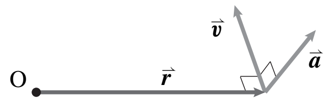


Figure 12.8: It is possible for position, velocity and acceleration to be mutually orthogonal.

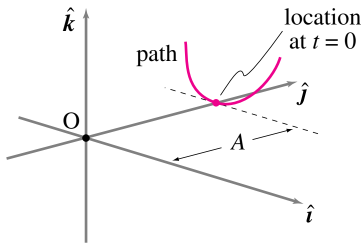


Figure 12.9: A path where, at the marked dot, the position (relative to the origin), velocity and acceleration can be mutually orthogonal: Position is in the $\hat{j}$ direction, velocity is in the $\hat{\imath}$ direction and (if the particle moves at constant speed), the acceleration is in the $\hat{k}$ direction.

① Eventually you may gain the math skills to shortcut this brute-force numerical approach, at least for some simple problems. But for most problems, even math geniuses use the numerical approach here.

that you learned in freshman calculus.

$$\begin{aligned}
\frac{d}{dt}(a\vec{A}) &= \dot{a}\vec{A} + a\dot{\vec{A}} \\
\frac{d}{dt}(\vec{A}\cdot\vec{B}) &= \dot{\vec{A}}\cdot\vec{B} + \vec{A}\cdot\dot{\vec{B}} \\
\frac{d}{dt}(\vec{A}\times\vec{B}) &= \dot{\vec{A}}\times\vec{B} + \vec{A}\times\dot{\vec{B}}.
\end{aligned}$$

The proofs of these identities are the same as the proof used for scalar multiplication, it follows from the definition of derivative (above) and the elimination of terms with Δ^2 as negligible compared to terms with just Δ in the limit $\Delta \to 0$.

Example: Derivative of a vector of constant length.
Assume a vector $\vec{C}$ has constant length so

$$|\vec{C}| = \text{constant} \qquad \text{and} \qquad |\vec{C}|^2 = \text{constant}$$

so, differentiating and using the product rule, working from left to right:

$$\frac{d}{dt}|\vec{C}|^2 = \frac{d}{dt}(\vec{C}\cdot\vec{C}) = \vec{C}\cdot\dot{\vec{C}} + \dot{\vec{C}}\cdot\vec{C} = 2\vec{C}\cdot\dot{\vec{C}} = 0.$$

Because $\vec{C}\cdot\dot{\vec{C}} = 0$ we then know that $\vec{C} \perp \dot{\vec{C}}$. That is, for any vector $\vec{C}$ that has constant length, its rate of change is perpendicular to itself. This is a useful fact to remember about time-varying constant-magnitude vectors, especially time-varying unit vectors.

To make this more intuitive, imagine a dog on a taut fixed-length leash anchored to the ground. The length of the leash is the magnitude $|\vec{C}|$ of the position vector $\vec{C}$, from ground-to-neck, and is constant. So our result is obvious, the neck can only move with a velocity $\dot{\vec{C}}$ that is tangent to the circle that the neck moves on because the tangent of a circle is orthogonal to the radius.

In 3D, space-dogs on taut leashes can only move tangent to the sphere they are stuck on $|\vec{C}| = \text{constant} \quad \Rightarrow \quad \vec{C}\cdot\dot{\vec{C}} = 0 \quad \Rightarrow \quad \vec{C} \perp \dot{\vec{C}}$. And, intuitively again, all tangents to the surface of a sphere are orthogonal to the radius of the sphere at that point.

Dynamics in space

Isaac Newton wondered how the planets move around the sun. By applying his equation $\vec{F} = m\vec{a}$, his law of gravitation, his calculus, and his inimitable geometric reasoning, he learned a lot about the moon and the planets. After you learn the material in this section you will know enough to reproduce many of Newton's calculations. You won't need to be a Newton-like genius to solve Newton's differential equations. You can solve them on a computer. And you can use the same computer approach to find motions that Newton could never find, say the trajectory of a projectile with a realistic model of air friction. In this chapter, the basic recipe is this: ①

Write $\vec{F} = m\vec{a}$ and solve the equations.

In some sense it's that simple.

A sure-fire recipe. Here's how to find the motion of a particle:

1. Draw a free-body diagram of the particle,
2. Find the forces on the particle in terms of its position, velocity and time. External forces (external forces might come, for example, from a spring, dashpot, gravity, or air friction),
3. Write the linear momentum balance equation for the particle (translation: write $\vec{F} = m\vec{a}$).
4. Break the vector equation into components to make 2 or 3 2nd order scalar ODEs, in 2 or 3 dimensions, respectively.
5. Write the 2 or 3 2nd order ODEs in first order form. You now have 4 or 6 first order ordinary differential equations (for a 2 or 3 dimensional problem, respectively).
6. Write these first order equations in standard form, with all the time derivatives on the left hand side.
7. Feed these equations to the computer, substituting values for the various parameters and appropriate initial conditions.
8. Plot some aspect(s) of the solution and
   a) Use the solution to help you find errors in your formulation, and
   b) Interpret the solution so that it makes sense to you and increases your understanding of the system of study.

Instantaneous dynamics. Some problems are even easier, problems of the *instantaneous dynamics* type. They use the equations of dynamics but do not track the motion over time.

Example: Knowing the forces find the acceleration.
Say you know the forces on a particle at some instant in time, say $\vec{F}_1$ and $\vec{F}_2$, and you just want to know the acceleration at that instant. The answer is given directly by linear momentum balance as

$$\sum \vec{F}_i = m\vec{a} \quad \Rightarrow \quad \vec{a} = \frac{\vec{F}_1 + \vec{F}_2}{m}$$

Sometimes this 'instantaneous' dynamics, with the motion given and the forces to be determined, is called '*inverse dynamics*'. The inside back cover of the book compares the solution methods for instantaneous dynamics to those where differential equations need be solved.

Analytic solution. Some problems involving motion are simple and you can determine almost all you want to know with pencil and paper. You can bypass the whole computer recipe above.

Example: Parabolic trajectory of a projectile
If we assume a constant gravitational field, neglect air drag, and take the y direction as up the only force acting on a projectile is $\vec{\boldsymbol{F}} = -mg\hat{\boldsymbol{\jmath}}$. Thus the "equations of motion" (linear momentum balance) are

$$-mg\hat{\boldsymbol{\jmath}} = m\vec{\boldsymbol{a}}.$$

Taking the dot product of this equation with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ (equivalent to taking the x and y components) we get the following two differential equations,

$$\ddot{x} = 0 \quad \text{and} \quad \ddot{y} = -g$$

which are decoupled and have the general solution

$$\vec{\boldsymbol{r}} = (A + Bt)\hat{\boldsymbol{\imath}} + (C + Dt - gt^2/2)\hat{\boldsymbol{\jmath}}$$

which is a parametric description of all possible trajectories. By making plots or by using simple algebra you could convince yourself that these trajectories are parabolas for all possible A, B, C, and D. That is, neglecting air drag, the predicted trajectory of a thrown ball is a parabola.

Some other special problems turn out to be easy, although you might not recognize such problems at first glance.

Example: Mass tethered by a zero-length spring
Imagine a massless spring whose un-stretched length is zero (See page 382 in section 7.1 for a discussion of zero length springs). Assume one end is connected to a pivot at the origin and the other to a particle. Neglect gravity and air drag. The force on the mass is thus proportional to its distance from the pivot and the spring constant and pointed towards the origin: $\vec{\boldsymbol{F}} = -k\vec{\boldsymbol{r}}$. Thus linear momentum balance yields

$$-k\vec{\boldsymbol{r}} = m\vec{\boldsymbol{a}}.$$

Breaking into components we get

$$\ddot{x} = (-k/m)x \quad \text{and} \quad \ddot{y} = (-k/m)y.$$

Thus the motion can be thought of as two independent harmonic oscillators, one in the x direction and one in the y direction. The general solution is

$$\vec{\boldsymbol{r}} = \left(A\cos\sqrt{\frac{k}{m}}\,t + B\sin\sqrt{\frac{k}{m}}\,t\right)\hat{\boldsymbol{\imath}} + \left(C\cos\sqrt{\frac{k}{m}}\,t + D\sin\sqrt{\frac{k}{m}}\,t\right)\hat{\boldsymbol{\jmath}}$$

which is always an ellipse (special cases of which are a circle and a straight line).

Even with gravity and springs together some (only some) problems are easy.

Example: Mass hanging from a zero-length spring
If gravity in the $-\hat{\boldsymbol{\jmath}}$ direction is included in the above problem the solution is only changed by the addition of a constant to be:

$$\vec{\boldsymbol{r}}_{\text{with gravity}} = \vec{\boldsymbol{r}}_{\text{previous example}} - (mg/k)\hat{\boldsymbol{\jmath}}$$

Analytic methods sometimes just can't do the job. Some problems are hard and can't be solved without a computer.

Example: Trajectory with quadratic air drag.
For motions of things you can see with your bare eyes moving in air, the drag force is roughly proportional to the speed squared and opposes the motion. Thus the total force on a particle is $\vec{F} = -mg\hat{\jmath} - Cv^2(\vec{v}/v)$, where $\vec{v}/v$ is a unit vector in the direction of motion. So linear momentum balance gives

$$-mg\hat{\jmath} - Cv\vec{v} = m\vec{a}.$$

If we dot this equation with $\hat{\imath}$ and $\hat{\jmath}$ we get

$$\ddot{x} = -(C/m)\left(\sqrt{\dot{x}^2 + \dot{y}^2}\right)\dot{x} \quad \text{and} \quad \ddot{y} = -(C/m)\left(\sqrt{\dot{x}^2 + \dot{y}^2}\right)\dot{y} - g.$$

These are two coupled second order equations that are probably not solvable with pencil and paper. But they are easily put in the form of a set of four first order equations for direct numerical solution.

On the edge. Some problems are within the reach of advanced analytic methods, but might be easier to solve with a computer.

Example: Path of the earth around the sun.
Assume the sun is big and unmovable with mass M and the earth has mass m. Take the origin to be at the sun. The force on the earth is $\vec{F} = -(mMG/r^2)(\vec{r}/r)$ where $\vec{r}/r$ is a unit vector pointing from the sun to the earth. So linear momentum balance gives

$$-\frac{mMG\vec{r}}{r^3} = m\vec{a}.$$

This equation *can* be solved with pencil and paper, Newton did it but many of us find it too tricky. On the other hand the equations of motion for planetary trajectories are easily broken into components and then into a set of 4 ODEs which can be easily solved on the computer. Either by pencil and paper, or by investigation of numerical solutions, you will find that all solutions are conic sections (straight lines, parabolas, hyperbolas, and ellipses). The special case of circular motion is not far from what the earth does around the sun, what the moon does around the earth, and what most artificial satellites do around the earth.

Summary

If, given the time, the particle's position and the particle's velocity, you know the force on a particle, then you know $\vec{F}(t, \vec{r}, \vec{v})$[2]. That means you can write $\vec{F} = m\vec{a}$ as

$$\vec{a} = \vec{F}(t, \vec{r}, \vec{v})/m$$

where $\vec{F}$ is known. This can, in turn, be written as two vector first order equations

$$\begin{aligned} \dot{\vec{r}} &= \vec{v} \\ \dot{\vec{v}} &= \vec{F}(t, \vec{r}, \vec{v})/m. \end{aligned} \tag{12.2}$$

[2] Typically you would know this because an applied force would vary in time in a known way (the dependence on t), gravity and spring forces would vary with position in a known way (dependence on r), and you would know forces due to various friction (dependence on $\vec{v}$).

which are equivalent, written out long hand, to the 6 first order equations

$$\begin{aligned}
\dot{x} &= v_x \\
\dot{y} &= v_y \\
\dot{z} &= v_z \\
\dot{v}_x &= F_x(t, x, y, z, v_x, v_y, v_z)/m \\
\dot{v}_y &= F_y(t, x, y, z, v_x, v_y, v_z)/m \\
\dot{v}_z &= F_z(t, x, y, z, v_x, v_y, v_z)/m.
\end{aligned} \tag{12.3}$$

Given the position and velocity at some starting time, these equations can be integrated, sometimes by hand but generally on the computer, to give position and velocity as a function of time.

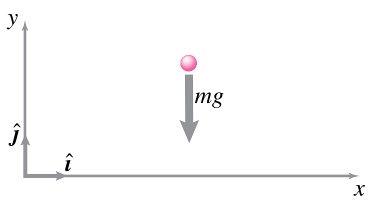


Figure 12.10: Free-body diagram of a particle in flight, neglecting air friction.

Example: Simple ballistics.
This is an example that can be solved with pencil and paper. A computer is not needed. It is the classic from high school and freshman physics. A particle has only one force on it, gravity. A free-body diagram is shown in fig. 12.10. Linear momentum balance gives

$$\begin{aligned}
\vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \\
-mg\hat{\boldsymbol{\jmath}} &= m\vec{\boldsymbol{a}} \\
\vec{\boldsymbol{a}} &= -g\hat{\boldsymbol{\jmath}}
\end{aligned}$$

So

$$\begin{aligned}
\dot{x} &= v_x \\
\dot{y} &= v_y \\
\dot{v}_x &= 0 \\
\dot{v}_y &= -g
\end{aligned}$$

Integrating the last two of these equations and plugging the result into the first two we get:

$$\begin{aligned}
\vec{\boldsymbol{v}} &= v_{0x}\hat{\boldsymbol{\imath}} + (v_{0y} - gt)\hat{\boldsymbol{\jmath}} \\
\vec{\boldsymbol{r}} &= (x_0 + v_{0x}t)\hat{\boldsymbol{\imath}} + (y_0 + v_{0y}t - gt^2/2)\hat{\boldsymbol{\jmath}}
\end{aligned}$$

This solution is plotted various ways in fig. 12.22 on page 626.

More complicated examples are given in the samples on the following pages.

12.2 The rate of change of a vector depends on reference frame

The time derivative of a vector can be found by differentiating each of its components. This calculation depended on having a reference frame, an imaginary piece of big graph paper, and a corresponding set of base (or basis) vectors, say $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$. But there can be more than one piece of imaginary graph paper. You could be holding one, Jo another, and Tanya a third. Each could be moving their graph paper around and on each paper the same given vector would change in a different way.

As noted earlier, but for the special case of one frame moving at constant velocity (without rotation) with respect to another, the rate of change of a given vector is different if calculated in different reference frames.

For mechanics we have to differentiate vectors with respect to a Newtonian frame.

Because most often we use the "fixed" ground under us as a practical approximation of a Newtonian frame, we label a Newtonian frame with a curly script $\mathcal{F}$, for fixed. So, when being fussy about notation we will sometimes write

$$\overset{\mathcal{F}}{\dot{\boldsymbol{r}}}_{\mathrm{B/O}} = \text{The velocity of point B as calculated in frame } \mathcal{F}.$$

Non-Newtonian frames

Even though the laws of mechanics are not valid in non-Newtonian frames, non-Newtonian frames are a useful aid with the understanding of the motion of and forces on systems composed of objects with complex relative motion. So eventually we need to understand frames that accelerate and rotate with respect to each other and with reference to Newtonian frames. Such non-Newtonian frames will be discussed in later chapters.

SAMPLE 12.1 Velocity and acceleration from derivative of position:
The position vector of a particle is given as a function of time:

$$\vec{r}(t) = (C_1 + C_2 t + C_3 t^2)\hat{\imath} + C_4 t\,\hat{\jmath}$$

where $C_1 = 1\,\text{m}, C_2 = 3\,\text{m/s}, C_3 = 1\,\text{m/s}^2$, and $C_4 = 2\,\text{m/s}$.

1. Find the position, velocity, and acceleration of the particle at $t = 2\,\text{s}$.
2. Find the change in the position of the particle between $t = 2\,\text{s}$ and $t = 3\,\text{s}$.

Solution We are given,

$$\vec{r} = (C_1 + C_2 t + C_3 t^2)\hat{\imath} + C_4 t\,\hat{\jmath}.$$

Therefore,

$$\begin{aligned}
\vec{v} &\equiv \dot{\vec{r}} = \frac{d\vec{r}}{dt} = (C_2 + 2C_3 t)\hat{\imath} + C_4\hat{\jmath} \\
\vec{a} &\equiv \ddot{\vec{r}} = \frac{d^2\vec{r}}{dt^2} = 2C_3\hat{\imath}.
\end{aligned}$$

1. Substituting the given values of the constants and $t = 2\,\text{s}$ in the equations above we get,

$$\begin{aligned}
\vec{r}(t = 2\,\text{s}) &= (1\,\text{m} + 3\frac{\text{m}}{\text{s}}\cdot 2\,\text{s} + 1\frac{\text{m}}{\text{s}^2}\cdot 4\,\text{s}^2)\hat{\imath} + (2\frac{\text{m}}{\text{s}}\cdot 2\,\text{s})\hat{\jmath} \\
&= 11\,\text{m}\hat{\imath} + 4\,\text{m}\hat{\jmath} \\
\vec{v}(t = 2\,\text{s}) &= (3\frac{\text{m}}{\text{s}} + 2\cdot 1\frac{\text{m}}{\text{s}^2}\cdot 2\,\text{s})\hat{\imath} + (2\frac{\text{m}}{\text{s}})\hat{\jmath} \\
&= 7\,\text{m/s}\hat{\imath} + 2\,\text{m/s}\hat{\jmath} \\
\vec{a}(t = 2\,\text{s}) &= (2\cdot 1\frac{\text{m}}{\text{s}^2})\hat{\imath} = 2\,\text{m/s}^2\hat{\imath}.
\end{aligned}$$

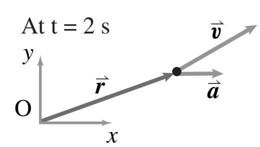


Figure 12.11

$$\boxed{\vec{r} = (11\hat{\imath} + 4\hat{\jmath})\,\text{m}, \quad \vec{v} = (7\hat{\imath} + 2\hat{\jmath})\,\text{m/s}, \quad \vec{a} = 2\,\text{m/s}^2\hat{\imath}}$$

2. The change in the position of the particle between the two time instants is,

$$\Delta\vec{r} = \vec{r}(t = 3\,\text{s}) - \vec{r}(t = 2\,\text{s}).$$

We already have $\vec{r}$ at $t = 2\,\text{s}$. We need to calculate $\vec{r}$ at $t = 3\,\text{s}$.

$$\begin{aligned}
\vec{r}(t = 3\,\text{s}) &= (1\,\text{m} + 3\frac{\text{m}}{\text{s}}\cdot 3\,\text{s} + 1\frac{\text{m}}{\text{s}^2}\cdot 9\,\text{s}^2)\hat{\imath} + (2\frac{\text{m}}{\text{s}}\cdot 3\,\text{s})\hat{\jmath} \\
&= 19\,\text{m}\hat{\imath} + 6\,\text{m}\hat{\jmath}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Delta\vec{r} &= (19\,\text{m}\hat{\imath} + 6\,\text{m}\hat{\jmath}) - (11\,\text{m}\hat{\imath} + 4\,\text{m}\hat{\jmath}) \\
&= 8\,\text{m}\hat{\imath} + 2\,\text{m}\hat{\jmath}.
\end{aligned}$$

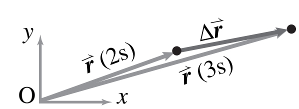


Figure 12.12

$$\boxed{\Delta\vec{r} = 8\,\text{m}\hat{\imath} + 2\,\text{m}\hat{\jmath}}$$

SAMPLE 12.2 Velocity and acceleration from position on a helix. Given that the position of a particle is

$$\vec{r} = A\cos(\lambda t)\hat{\imath} + B\sin(\lambda t)\hat{\jmath} + Ct\hat{k},$$

where A, B, C, and λ are constants, find

1. the velocity as a function of time,
2. the acceleration as a function of time,
3. a condition under which the acceleration vector is normal to the velocity vector.

Solution

1. The velocity:

$$\begin{aligned}\vec{v} &= \frac{d\vec{r}}{dt} \\ &= \frac{d}{dt}[A\cos(\lambda t)\hat{\imath} + B\sin(\lambda t)\hat{\jmath} + Ct\hat{k}] \\ &= -A\lambda\sin(\lambda t)\hat{\imath} + B\lambda\cos(\lambda t)\hat{\jmath} + C\hat{k}.\end{aligned}$$

$$\boxed{\vec{v} = -A\lambda\sin(\lambda t)\hat{\imath} + B\lambda\cos(\lambda t)\hat{\jmath} + C\hat{k}}$$

2. The acceleration:

$$\begin{aligned}\vec{a} &= \frac{d\vec{v}}{dt} \\ &= \frac{d}{dt}[-A\lambda\sin(\lambda t)\hat{\imath} + B\lambda\cos(\lambda t)\hat{\jmath} \\ &= -A\lambda^2\cos(\lambda t)\hat{\imath} - B\lambda^2\sin(\lambda t)\hat{\jmath}\end{aligned}$$

$$\boxed{\vec{a} = -A\lambda^2\cos(\lambda t)\hat{\imath} - B\lambda^2\sin(\lambda t)\hat{\jmath}}$$

3. The velocity vector and the acceleration vector will be orthogonal to each other if $\vec{v}\cdot\vec{a} = 0$. Taking the dot product of the two vectors, we find,

$$\begin{aligned}\vec{v}\cdot\vec{a} &= (-A\lambda\sin(\lambda t)\hat{\imath} + B\lambda\cos(\lambda t)\hat{\jmath} + C\hat{k})\cdot(-A\lambda^2\cos(\lambda t)\hat{\imath} - B\lambda^2\sin(\lambda t)\hat{\jmath}) \\ &= A^2\lambda^3\sin(\lambda t)\cos(\lambda t) - B^2\lambda^3\sin(\lambda t)\cos(\lambda t) \\ &= (A^2 - B^2)\lambda^3\sin(\lambda t)\cos(\lambda t).\end{aligned}$$

Now, this dot product must be zero for all t if $\vec{a}$ is normal to $\vec{v}$. This is indeed the case if $A = B$. Thus, the condition for orthogonality of $\vec{v}$ and $\vec{a}$ is $A = B$.

$$\boxed{A = B \quad \Rightarrow \quad \vec{v}\cdot\vec{a} = 0}$$

Note: The path is an elliptical helix with axis in the z direction. The z-component of velocity is constant so the acceleration is entirely in the xy plane. In fact, the acceleration vector points from the particle towards the axis of the helix.

SAMPLE 12.3 **Position from velocity.** Assume the expression for velocity $\vec{v}$ of a particle is given: $\vec{v} = v_0\hat{\imath} - gt\,\hat{\jmath}$. Find the expressions for the x and y coordinates of the particle at a general time t, if the initial coordinates at $t = 0$ are (x_0, y_0). Plot the path of the particle taking $x_0 = 0$, $y_0 = 80\,\mathrm{m}$, $v_0 = 2\,\mathrm{m/s}$, $g = 10\,\mathrm{m/s^2}$, and $t = 1 \ldots 4\,\mathrm{s}$.

Solution The position vector of the particle at any time t is

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}.$$

We are given that

$$\vec{r}(t=0) = x_0\hat{\imath} + y_0\hat{\jmath}.$$

Now

$$\begin{aligned} \vec{v} \equiv \frac{d\vec{r}}{dt} &= v_0\hat{\imath} - gt\,\hat{\jmath} \\ \text{or} \qquad dx\hat{\imath} + dy\,\hat{\jmath} &= (v_0\hat{\imath} - gt\,\hat{\jmath})\,dt. \end{aligned}$$

Integrating both sides of the equation with appropriate limits, we get

$$\begin{aligned} \int_{x_0\hat{\imath}+y_0\hat{\jmath}}^{x\hat{\imath}+y\hat{\jmath}} (dx\hat{\imath} + dy\,\hat{\jmath}) &= \int_0^t (v_0\hat{\imath} - gt\,\hat{\jmath})dt \\ \int_{x_0}^{x} dx\hat{\imath} + \int_{y_0}^{y} dy\,\hat{\jmath} &= v_0\hat{\imath}\int_0^t dt - g\hat{\jmath}\int_0^t t\,dt \\ (x - x_0)\hat{\imath} + (y - y_0)\hat{\jmath} &= v_0 t\hat{\imath} - \frac{1}{2}gt^2\,\hat{\jmath} \\ x\hat{\imath} + y\,\hat{\jmath} &= (x_0 + v_0 t)\hat{\imath} + (y_0 - \frac{1}{2}gt^2)\hat{\jmath}. \end{aligned}$$

Therefore,

$$\vec{r}(t) = (x_0 + v_0 t)\hat{\imath} + (y_0 - \frac{1}{2}gt^2)\hat{\jmath}$$

and the (x, y) coordinates are

$$\begin{aligned} x(t) &= x_0 + v_0 t \\ y(t) &= y_0 - \frac{1}{2}gt^2. \end{aligned}$$

$$\boxed{(x_0 + v_0 t,\ y_0 - \tfrac{1}{2}gt^2)}$$

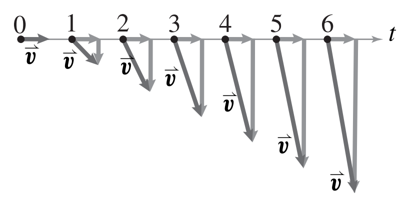


Figure 12.13: The given velocity vector at different time instants ($t = 0$, 1 2, etc.). Note that the x-component of the velocity remains constant (v_0) while the y-component grows linearly with time ($-gt$).

Plugging in $x0 = 0$, $y_0 = 80\,\mathrm{m}$, $v_0 = 2\,\mathrm{m/s}$, $g = 10\,\mathrm{m/s^2}$, and taking 20 points between $t = 0$ to $t = 4$, we compute the values of x and y and plot them to get the path of the particle. The plot is shown in fig. 12.14 with a few intermediate positions marked on the path.

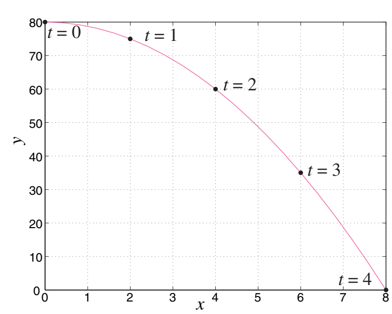


Figure 12.14: Path of the particle plotted by computing the coordinates $x(t)$ and $y(t)$ at 20 equal time intervals between $t = 0$ to 4 seconds. Five positions are marked on the path corresponding to $t = 0$, 1, 2, 3, and 4 seconds.

Comments: From the x and y coordinates, it is possible to get the equation of the path of the particle by eliminating the time from the two equations. From the expression for $x(t)$, we get $t = (x - x_0)/v_0$. Substituting this expression for t in the equation for $y(t)$, we get,

$$y - y_0 = \frac{g}{2v_0^2}(x - x_0)^2$$

which is the equation of the path. From this equation it should be clear that the path is parabolic. It is easier to see this if you shift the origin to (x_0, y_0) and use the new coordinates $\hat{x} = x - x_0$ and $\hat{y} = y - y_0$. Then, in terms of the new coordinates, the path becomes,

$$\hat{y} = \frac{g}{2v_0^2}\hat{x}^2.$$

SAMPLE 12.4 **Acceleration of a point mass in 3-D.** A ball of mass $m = 13\,\text{kg}$ is being pulled by three strings as shown in Fig. 12.15. The tension in each string is $T = 13\,\text{N}$. Find the acceleration of the ball.

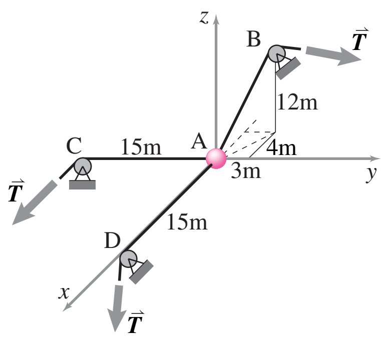


Figure 12.15: A ball in 3-D

Solution The forces acting on the body are shown in the free-body diagram in Fig.12.16. From geometry:

$$\begin{aligned}\hat{\boldsymbol{\lambda}} &= \frac{\vec{\boldsymbol{r}}_{AB}}{|\vec{\boldsymbol{r}}_{AB}|} = \frac{-4\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}}{\sqrt{4^2 + 3^2 + 12^2}} \\ &= \frac{-4\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}}{13}.\end{aligned}$$

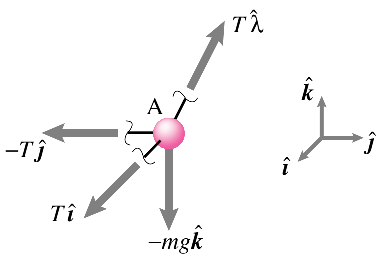


Figure 12.16: FBD of the ball

The balance of linear momentum for the ball gives

$$\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}} \tag{12.4}$$

where

$$\begin{aligned}\sum \vec{\boldsymbol{F}} &= T\hat{\boldsymbol{\imath}} - T\hat{\boldsymbol{\jmath}} + T\hat{\boldsymbol{\lambda}} - mg\hat{\boldsymbol{k}} \\ &= T\left(\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + \frac{-4\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}}{13}\right) - mg\hat{\boldsymbol{k}} \\ &= \frac{T}{13}(9\hat{\boldsymbol{\imath}} - 10\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}) - mg\hat{\boldsymbol{k}}.\end{aligned}$$

Substituting $\sum \vec{\boldsymbol{F}}$ in eqn. (12.4):

$$\vec{\boldsymbol{a}} = \frac{T}{13m}(9\hat{\boldsymbol{\imath}} - 10\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}) - g\hat{\boldsymbol{k}}.$$

Now plugging in the given values: $T = 13\,\text{N}$, $m = 13\,\text{kg}$, and $g = 10\,\text{m/s}^2$, we get

$$\begin{aligned}\vec{\boldsymbol{a}} &= \frac{\cancel{13}\,\text{N}}{\cancel{13}\cdot 13\,\text{kg}}(9\hat{\boldsymbol{\imath}} - 10\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}) - 10\,\text{m/s}^2\hat{\boldsymbol{k}} \\ &= (0.69\hat{\boldsymbol{\imath}} - 0.77\hat{\boldsymbol{\jmath}} - 9.08\hat{\boldsymbol{k}})\,\text{m/s}^2.\end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}} = (0.69\hat{\boldsymbol{\imath}} - 0.77\hat{\boldsymbol{\jmath}} - 9.08\hat{\boldsymbol{k}})\,\text{m/s}^2}$$

SAMPLE 12.5 Projectile motion with air drag. A projectile is fired into the air at an initial angle θ_0 and with initial speed v_0. The air resistance to the motion is proportional to the square of the speed of the projectile. Take the constant of proportionality to be k. Find the equations of motion of the projectile in the horizontal and vertical directions assuming the air resistance to be in the opposite direction of the velocity.

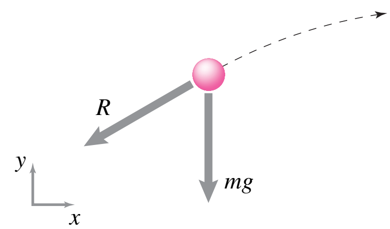


Figure 12.17: FBD of the projectile.

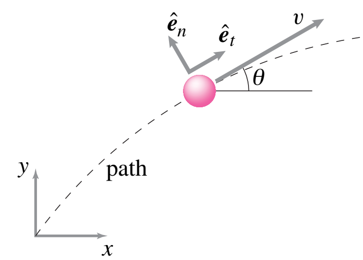


Figure 12.18

Solution The free-body diagram of the projectile is shown in the figure at some constant t during motion. At the instant shown, let the velocity of the projectile be $\vec{v} = v\hat{e}_t$ where

$$\hat{e}_t = \vec{v}/|\vec{v}| = (v_x/|\vec{v}|)\hat{\imath} + (v_y/|\vec{v}|)\hat{\jmath}.$$

Then the force due to air resistance is

$$\vec{R} = -kv^2\hat{e}_t.$$

Now applying the linear momentum balance on the projectile, we get

$$\begin{aligned} \vec{R} + m\vec{g} &= m\vec{a} \\ \text{or} \qquad -kv^2\hat{e}_t - mg\hat{\jmath} &= m\overbrace{(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath})}^{\vec{a}}. \end{aligned} \tag{12.5}$$

Noting that $v = |\vec{v}| = |\dot{x}\hat{\imath} + \dot{y}\hat{\jmath}| = \sqrt{\dot{x}^2 + \dot{y}^2}$, and dotting both sides of eqn. (12.5) with $\hat{\imath}$ and $\hat{\jmath}$ we get

$$\begin{aligned} -k(\dot{x}^2 + \dot{y}^2)\cdot(\hat{e}_t\cdot\hat{\imath}) &= m\ddot{x} \\ -k(\dot{x}^2 + \dot{y}^2)\cdot(\hat{e}_t\cdot\hat{\jmath}) - mg &= m\ddot{y}. \end{aligned}$$

Rearranging terms and carrying out the dot products, we get

$$\begin{aligned} \ddot{x} &= -\frac{k}{m}(\dot{x}^2 + \dot{y}^2)(v_x/|\vec{v}|) \\ \ddot{y} &= -g - \frac{k}{m}(\dot{x}^2 + \dot{y}^2)(v_y/|\vec{v}|). \end{aligned}$$

Substituting these expressions in to the equations for $\ddot{x}$ and $\ddot{y}$ we get

$$\boxed{\ddot{x} = -\tfrac{k}{m}\dot{x}\sqrt{\dot{x}^2 + \dot{y}^2}, \quad \ddot{y} = -\tfrac{k}{m}\dot{y}\sqrt{\dot{x}^2 + \dot{y}^2} - g}$$

For initial conditions we could use

$$x_0 = 0, y_0 = 0 \qquad \text{and} \qquad \dot{x}_0 = v_{x0} = v_0\cos\theta_0, \qquad \text{and} \qquad \dot{y}_0 = v_{y0} = v_0\sin\theta_0.$$

Note that θ changes with time. We can express θ in terms of $\dot{x}$ and $\dot{y}$. We can calculate the slope of the trajectory from

$$\tan\theta = \frac{\dot{y}}{\dot{x}}.$$

SAMPLE 12.6 Trajectory of a food-bag. In a flood hit area relief supplies are dropped in a 20 kg bag from a helicopter. The helicopter is flying parallel to the ground at 200 km/h and is 80 m above the ground when the package is dropped. How much horizontal distance does the bag travel before it hits the ground? Take the value of g, the gravitational acceleration, to be $10\,\mathrm{m/s^2}$. Ignore air drag.

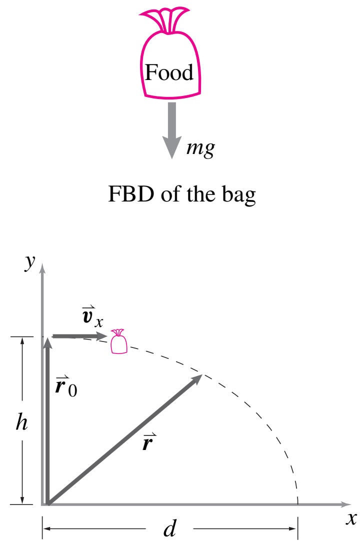


Figure 12.19: Free-body diagram of the bag and the geometry of its motion.

Solution You must have solved such problems in elementary physics courses. Usually, in all projectile motion problems the equations of motion are written separately in the x and y directions, realizing that there is no force in the x direction, and then the equations are solved. Here we show you how to write and keep the equations in vector form all the way through.

The free-body diagram of the bag during its free flight is shown in Fig. 12.19. The only force acting on the bag is its weight. Therefore, from the linear momentum balance for the bag we get

$$m\vec{a} = -mg\hat{\jmath}.$$

Let us choose the origin of our coordinate system on the ground exactly below the point at which the bag is dropped from the helicopter. Then, the initial position of the bag $\vec{r}(0) = h\hat{\jmath} = 80\,\mathrm{m}\,\hat{\jmath}$. The fact that the bag is dropped from a helicopter flying horizontally gives us the initial velocity of the bag:

$$\vec{v}(0) \equiv \dot{\vec{r}}(0) = v_x\hat{\imath} = 200\,\mathrm{km/h}\hat{\imath}.$$

So now we have a 2nd order differential equation (from linear momentum balance):

$$\ddot{\vec{r}} = -g\hat{\jmath}$$

with two initial conditions:

$$\vec{r}(0) = h\hat{\jmath} \quad \text{and} \quad \dot{\vec{r}}(0) = v_x\hat{\imath}$$

which we can solve to get the position vector of the bag at any time. Since the basis vectors $\hat{\imath}$ and $\hat{\jmath}$ do not change with time, solving the differential equation is a matter of simple integration:

$$\begin{aligned} \ddot{\vec{r}} &\equiv \frac{d\dot{\vec{r}}}{dt} = -g\hat{\jmath} \\ \int d\dot{\vec{r}} &= -\hat{\jmath}\int g\,dt \\ \text{or} \qquad \dot{\vec{r}} &= -gt\hat{\jmath} + \vec{c}_1 \end{aligned} \tag{12.6}$$

and integrating once again, we get

$$\begin{aligned} \vec{r} &= \int(-gt\hat{\jmath} + \vec{c}_1)\,dt \\ &= -\frac{1}{2}gt^2\hat{\jmath} + \vec{c}_1 t + \vec{c}_2 \end{aligned} \tag{12.7}$$

where $\vec{c}_1$ and $\vec{c}_2$ are constants of integration and are vector quantities. Now substituting the initial conditions in eqn. (12.6) and eqn. (12.7) we get

$$\begin{aligned} \dot{\vec{r}}(0) &= v_x\hat{\imath} = \vec{c}_1, \quad \text{and} \\ \vec{r}(0) &= h\hat{\jmath} = \vec{c}_2. \end{aligned}$$

Therefore, the solution is

$$\begin{aligned} \vec{r}(t) &= -\frac{1}{2}gt^2\hat{\jmath} + v_x t\hat{\imath} + h\hat{\jmath} \\ &= v_x t\hat{\imath} + (h - \frac{1}{2}gt^2)\hat{\jmath}. \end{aligned}$$

So how do we find the horizontal distance traveled by the bag from our solution? The distance we are interested in is the x-component of $\vec{\boldsymbol{r}}$, *i.e.,* $v_x t$. But we do not know t. However, when the bag hits the ground, its position vector has no y-component, *i.e.,* we can write $\vec{\boldsymbol{r}} = d\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}}$ where d is the distance we are interested in. Now equating the components of $\vec{\boldsymbol{r}}$ with the obtained solution, we get

$$d = v_x t \quad \text{and} \quad 0 = h - \frac{1}{2}gt^2.$$

Solving for t from the second equation and substituting in the first equation we get

$$d = v_x\sqrt{\frac{2h}{g}} = \frac{200\,\text{km}}{3600\,\text{s}} \cdot \sqrt{\frac{2 \cdot 80\,\text{m}}{10\,\text{m/s}^2}} = \frac{2}{9}\,\text{km} \approx 222\,\text{m}.$$

$$\boxed{d = 222\,\text{m}}$$

Comments: Here we have tried to show you that solving for position from the given acceleration in vector form is not really any different than solving in scalar form provided the unit vectors involved are fixed in time. As long as the right hand side of the differential equation is integrable, the solution can be obtained. If the method shown above seems too “*mathy*” or intimidating to you then follow the usual scalar way of doing this problem.

The scalar method:

From the linear momentum balance, $-mg\hat{\boldsymbol{\jmath}} = m\vec{\boldsymbol{a}}$, writing the acceleration as $\vec{\boldsymbol{a}} = a_x\hat{\boldsymbol{\imath}} + a_y\hat{\boldsymbol{\jmath}}$ and equating the x and y components from both sides, we get

$$a_x = 0 \quad \text{and} \quad a_y = -g.$$

Now using the formula for distance under uniform acceleration from Chapter 3, $x = x_0 + v_0 t + \frac{1}{2}at^2$, in both x and y directions, we get

$$\begin{aligned}
d &= \overbrace{x_0}^{0} + v_x t + \frac{1}{2}\overbrace{a_x}^{0}t^2 \\
&= v_x t \\
0 &= \overbrace{y_0}^{h} + \overbrace{v_y}^{0}t + \frac{1}{2}\overbrace{a_y}^{-g}t^2 \\
&= h - \frac{1}{2}gt^2 \\
\Rightarrow \quad t &= \sqrt{\frac{2h}{g}}.
\end{aligned}$$

Substituting for t in the equation for d we get

$$d = v_x\sqrt{\frac{2h}{g}} = \frac{200\,\text{km}}{3600\,\text{s}} \cdot \sqrt{\frac{2 \cdot 80\,\text{m}}{10\,\text{m/s}^2}} = \frac{2}{9}\,\text{km} \approx 222\,\text{m}.$$

as above.

SAMPLE 12.7 Cartoon mechanics: The cannon. It is sometimes claimed that students have trouble with dynamics because they built their intuition by watching cartoons. This claim could be rebutted on many grounds.

1) Students don't have trouble with dynamics! They love the subject.
2) Nowadays many cartoons are made using 'correct' mechanics, and
3) the cartoons are sometimes more accurate than the pedagogues anyway.

Problem: What is the path of a cannon ball? In the cartoon world the cannon ball goes in a straight line out the cannon then comes to a stop and then starts falling. Of course a good physicist knows the path is a parabola. Or is it?

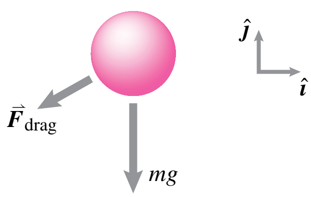


Figure 12.20

Solution The drag force on a cannon ball moving through air is approximately proportional to the speed squared and resists motion. Gravity is approximately constant. Then

$$
\begin{aligned}
\vec{F}_{\text{drag}} &= cv^2 \cdot (\text{unit vector opposing motion}) \\
&= cv^2 \cdot \left(\frac{-\vec{v}}{|\vec{v}|}\right) \\
&= -c|\vec{v}|\vec{v} \\
&= -c\sqrt{\dot{x}^2+\dot{y}^2}\,(\dot{x}\hat{\imath}+\dot{y}\hat{\jmath})
\end{aligned}
$$

So the linear momentum balance gives

$$
\sum \vec{F} = \dot{\vec{L}}
$$

$$
\left\{-mg\hat{\jmath} - c\sqrt{\dot{x}^2+\dot{y}^2}(\dot{x}\hat{\imath}+\dot{y}\hat{\jmath}) = m(\ddot{x}\hat{\imath}+\ddot{y}\hat{\jmath})\right\}
$$

$$
\begin{aligned}
\{\}\cdot\hat{\imath} \quad &\Rightarrow \quad \ddot{x} = \left[-c\sqrt{\dot{x}^2+\dot{y}^2}\,\dot{x}/m\right] \\
\{\}\cdot\hat{\jmath} \quad &\Rightarrow \quad \ddot{y} = -c\sqrt{\dot{x}^2+\dot{y}^2}\,\dot{y}/m - g
\end{aligned}
$$

Solving these equations numerically with reasonable values ① of $\dot{x}_0$, $\dot{y}_0$, m and c gives which is closer to a cartoon's triangle than to a naive physicist's parabola.

① To be precise, if the launch speed is much faster than the 'terminal velocity' of the falling ball.

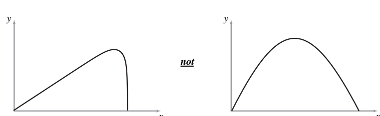


Figure 12.21

12.1 Dynamics of a particle in space

Preparatory Problems

12.1.1 Given $\vec{r}(t) = A\sin(\omega t)\hat{\imath} + Bt\hat{\jmath} + C\hat{k}$, find

1. $\vec{v}(t)$
2. $\vec{a}(t)$
3. $\vec{r}(t) \times \vec{a}(t)$.

12.1.2 A particle of mass =3 kg travels in space with its position known as a function of time, $\vec{r} = (\sin\frac{t}{\mathrm{s}})\,\mathrm{m}\hat{\imath} + (\cos\frac{t}{\mathrm{s}})\,\mathrm{m}\hat{\jmath} + te^{\frac{t}{\mathrm{s}}}\,\mathrm{m/s}\hat{k}$. At $t = 3\,\mathrm{s}$, find the particle's

a) velocity and
b) acceleration.

12.1.3 A particle of mass $m = 2\,\mathrm{kg}$ travels in the xy-plane with its position known as a function of time, $\vec{r} = 3t^2\,\mathrm{m/s^2}\hat{\imath} + 4t^3\frac{\mathrm{m}}{\mathrm{s}^3}\hat{\jmath}$. At $t = 5\,\mathrm{s}$, find the particle's

a) velocity and *
b) acceleration, and *
c) draw the three vectors.

12.1.4 The velocity of a particle of mass m on a frictionless surface is given as $\vec{v} = (0.5\,\mathrm{m/s})\hat{\imath} - (1.5\,\mathrm{m/s})\hat{\jmath}$. If the displacement is given by $\Delta\vec{r} = \vec{v}t$, find (a) the distance traveled by the mass in 2 seconds and (b) a unit vector along the displacement.

12.1.5 If $\dot{\vec{r}} = (u_0\sin\Omega t)\hat{\imath} + v_0\hat{\jmath}$ and $\vec{r}(0) = x_0\hat{\imath} + y_0\hat{\jmath}$, with u_0, v_0, and Ω as constants, find $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$. *

12.1.6 For $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$ and $\vec{a} = 2\,\mathrm{m/s^2}\hat{\imath} - (3\,\mathrm{m/s^2} - 1\,\mathrm{m/s^3}\,t)\hat{\jmath} - 5\,\mathrm{m/s^4}\,t^2\hat{k}$, write the vector equation $\vec{v} = \int \vec{a}\,dt$ as three scalar equations (*i.e.*, find $v_x(t)$, $v_y(t)$, and $v_z(t)$).

12.1.7 Find $\vec{r}(5\,\mathrm{s})$ given that $\dot{\vec{r}} = v_1\sin(ct)\hat{\imath} + v_2\hat{\jmath}$ and $\vec{r}(0) = 2\,\mathrm{m}\hat{\imath} + 3\,\mathrm{m}\hat{\jmath}$, and that v_1 is a constant $4\,\mathrm{m/s}$, v_2 is a constant $5\,\mathrm{m/s}$, and c is a constant $4\,\mathrm{s}^{-1}$.

12.1.8 Let $\dot{\vec{r}} = v_0\cos\alpha\hat{\imath} + v_0\sin\alpha\hat{\jmath} + (v_0\tan\theta - gt)\hat{k}$, where v_0, α, θ, and g are constants. If $\vec{r}(0) = \vec{0}$, find $\vec{r}(t)$.

12.1.9 On a smooth circular helical path the velocity of a particle is $\dot{\vec{r}} = -R\sin t\hat{\imath} + R\cos t\hat{\jmath} + gt\hat{k}$. If $\vec{r}(0) = R\hat{\imath}$, find $\vec{r}((\pi/3)\,\mathrm{s})$.

More-Involved Problems

12.1.10 A particle travels on a path in the xy-plane given by $y(x) = \sin^2(\frac{x}{\mathrm{m}})\,\mathrm{m}$. where $x(t) = t^3(\frac{\mathrm{m}}{\mathrm{s}^3})$. What are the velocity and acceleration of the particle in cartesian coordinates when $t = (\pi)^{\frac{1}{3}}\,\mathrm{s}$?

12.1.11 The position of a particle is given by $\vec{r}(t) = (t^2\,\mathrm{m/s^2}\hat{\imath} + e^{\frac{t}{\mathrm{s}}}\,\mathrm{m}\hat{\jmath})$. What are the velocity and acceleration of the particle as functions of time? Draw the path of the particle and show the vectors $\vec{v}$ and $\vec{a}$ at $t = 1\,\mathrm{s}$. *

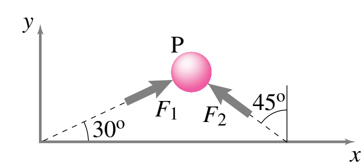
12.1.12 A particle travels on an elliptical path given by $y^2 = b^2(1 - \frac{x^2}{a^2})$ with constant speed v. Find the velocity of the particle when $x = a/2$ and $y > 0$ in terms of a, b, and v.

12.1.13 A particle travels on a path in the xy-plane given by $y(x) = (1 - e^{-\frac{x}{\mathrm{m}}})\,\mathrm{m}$. Make a plot of the path. It is known that the x coordinate of the particle is given by $x(t) = t^2\,\mathrm{m/s^2}$. What is the rate of change of speed of the particle? What angle does the velocity vector make with the positive x axis when $t = 3\,\mathrm{s}$?

12.1.14 A particle starts at the origin in the xy-plane, $(x_0 = 0, y_0 = 0)$ and travels only in the positive xy quadrant. Its speed and x coordinate are known to be $v(t) = \sqrt{1 + (\frac{4}{\mathrm{s}^2})t^2}\,\mathrm{m/s}$ and $x(t) = t\,\mathrm{m/s}$, respectively. What is $\vec{r}(t)$ in cartesian coordinates? What are the velocity, acceleration, and rate of change of speed of the particle as functions of time? What kind of path is the particle on? What is the distance of the particle from the origin and its velocity and acceleration when $x = 3\,\mathrm{m}$?

12.1.15 For a particle, $\sum\vec{F} = m\vec{a}$. Two forces $\vec{F}_1$ and $\vec{F}_2$ act on a mass P as shown in the figure. P has mass 2 lbm. The acceleration of the mass is somehow measured to be $\vec{a} = -2\,\mathrm{ft/s^2}\hat{\imath} + 5\,\mathrm{ft/s^2}\hat{\jmath}$.

a) Write the equation
$$\sum\vec{F} = m\vec{a}$$
in vector form (evaluating each side as much as possible).
b) Write the equation in scalar form (use any method you like to get two scalar equations in the two unknowns F_1 and F_2).
c) Write the equation in matrix form.
d) Find $F_1 = |\vec{F}_1|$ and $F_2 = |\vec{F}_2|$ by the following methods:
   1. from the scalar equations using hand algebra,
   2. from the matrix equation using a computer, and
   3. from the vector equation using a cross product.



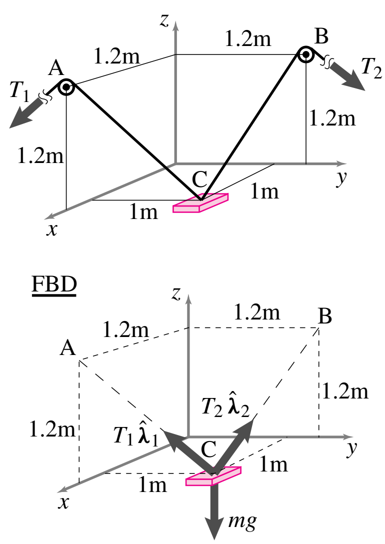
Problem 12.1.15

12.1.16 Three forces, $\vec{F}_1 = 20\,\mathrm{N}\hat{\imath} - 5\,\mathrm{N}\hat{\jmath}$, $\vec{F}_2 = F_{2x}\hat{\imath} + F_{2z}\hat{k}$, and $\vec{F}_3 = F_3\hat{\lambda}$, where $\hat{\lambda} = \frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath}$, act on a body with mass 2 kg. The acceleration of the body is $\vec{a} = -0.2\,\mathrm{m/s^2}\hat{\imath} + 2.2\,\mathrm{m/s^2}\hat{\jmath} + 1.7\,\mathrm{m/s^2}\hat{k}$. Write the equation $\sum\vec{F} = m\vec{a}$ as scalar equations and solve them (most conveniently on a computer) for F_{2x}, F_{2z}, and F_3.

12.1.17 In three-dimensional space with no gravity a particle with $m = 3\,\mathrm{kg}$ at A is pulled by three strings which pass through points B, C, and D respectively. The acceleration is known to be $\vec{a} = (1\hat{\imath} + 2\hat{\jmath} + 3\hat{k})\,\mathrm{m/s^2}$. The position vectors of B, C, and D relative to A are given in the first few lines of code below. Complete the pseudo-code to find the three tensions. The last line should read `T = ...` with `T` being assigned to be a 3-element column vector with the three tensions in Newtons. [Hint: If `x`, `y`, and `z` are three column vectors then `A=[x y z]` is a matrix with `x`, `y`, and `z` as columns.]

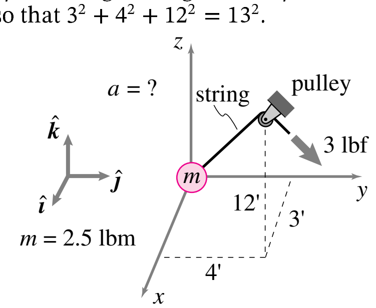
```
% Incomplete PSEUDO-CODE file
m   = 3;
a   = [ 1 2 3]';
~
rAB = [ 2 3 5]';
rAC = [-3 4 2]';
rAD = [ 1 1 1]';
uAB = rAB/(magnitude of rAB);
     .
     .
     .
T   =
```

12.1.18 A block of mass 100 kg is pulled with two strings AC and BC. Given that the tensions $T_1 = 1200\,\mathrm{N}$ and $T_2 = 1500\,\mathrm{N}$, find the magnitude and direction of the acceleration of the block. [$\sum \vec{F} = m\vec{a}$]



Problem 12.1.18

12.1.19 Neglecting gravity, the only force acting on the mass shown in the figure is from the string. Find the acceleration of the mass. Use the dimensions and quantities given. Recall that lbf is a pound force, lbm is a pound mass, and $\mathrm{lbf}/\mathrm{lbm} = g$. Use $g = 32\,\mathrm{ft/s^2}$. Note also that $3^2 + 4^2 + 12^2 = 13^2$.

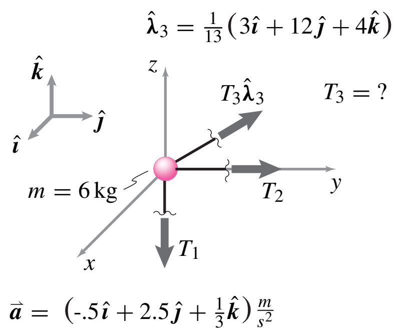


Problem 12.1.19

12.1.20 Three strings are tied to the mass shown with the directions indicated in the figure. They have unknown tensions T_1, T_2, and T_3. There is no gravity. The acceleration of the mass is given as $\vec{a} = (-0.5\hat{\imath} + 2.5\hat{\jmath} + \frac{1}{3}\hat{k})\,\mathrm{m/s^2}$.

a) Given the free-body diagram in the figure, write the equations of linear momentum balance for the mass.

b) Find the tension T_3. *



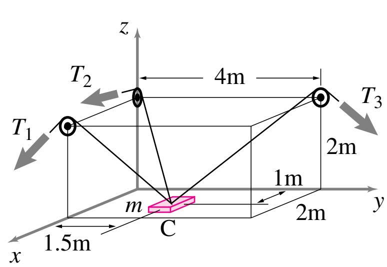
Problem 12.1.20

12.1.21 An object C of mass 2 kg is pulled by three strings as shown. The acceleration of the object at the position shown is $\boldsymbol{a} = \left(-0.6\hat{\imath} - 0.2\hat{\jmath} + 2.0\hat{k}\right)\mathrm{m/s^2}$.

a) Draw a free-body diagram of the mass.

b) Write the equation of linear momentum balance for the mass. Use $\boldsymbol{\lambda}$'s as unit vectors along the strings.

c) Find the three tensions T_1, T_2, and T_3 at the instant shown. You may find these tensions by using hand algebra with the scalar equations, using a computer with the matrix equation, or by using a cross product on the vector equation.



Problem 12.1.21

12.1.22 Particle moves on a strange path. Given that a particle moves in the xy plane for 1.77 s obeying

$$\vec{r} = (5\,\mathrm{m})\cos^2(t^2/\mathrm{s}^2)\hat{\imath} + (5\,\mathrm{m})\sin(t^2/\mathrm{s}^2)\cos(t^2/\mathrm{s}^2)\hat{\jmath}$$

where x and y are the horizontal distance in meters and t is measured in seconds.

a) Accurately plot the trajectory of the particle.

b) Mark on your plot where the particle is going fast and where it is going slow. Explain how you know these points are the fast and slow places.

12.1.23 *Computer question:* **What's the plot? What's the mechanics question?** Shown are shown some pseudo computer commands that are not commented adequately, unfortunately, and no computer is available at the moment.

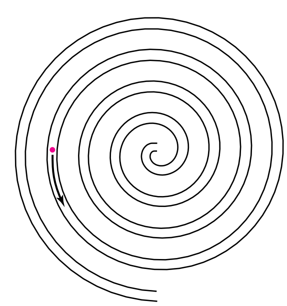
a) Draw as accurately as you can, assigning numbers etc, the plot that results from running these commands.

b) See if you can guess a mechanical situation that is described by this program. Sketch the system and define the variables to make the script file agree with the problem stated.

```
ODEs = {z1dot = z2
        z2dot = 0}
ICs = {z1 =1, z2=1}

Solve ODEs with ICs from t=0 to t=5

plot z2 and z1 vs t on the same plot
```

Problem 12.1.23

12.1.24 A particle is blown out through the uniform spiral tube shown, which lies flat on a horizontal frictionless table. Draw the particle's path after it is expelled from the tube. Defend your answer.



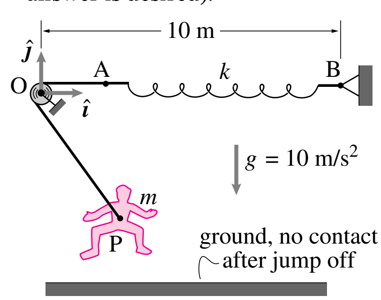
Problem 12.1.24

12.1.25 Bungy Jumping. In a relatively safe bungy jumping system, people jump up from the ground while being pulled up by a rope that runs over a pulley at O and is connected to a stretched spring anchored at B. The ideal pulley has negligible size, mass, and friction. For the situation shown the spring AB has rest length $\ell_0 = 2\,\mathrm{m}$ and a stiffness of $k = 200\,\mathrm{N/m}$. The inextensible massless rope from A to P has length $\ell_r = 8\,\mathrm{m}$, the person has a mass of 100 kg. Take O to be the origin of an xy coordinate system aligned with the unit vectors $\hat{\imath}$ and $\hat{\jmath}$

a) Assume you are given the position of the person $\vec{r} = x\hat{\imath} + y\hat{\jmath}$ and the velocity of the person $\boldsymbol{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$. Find her acceleration in terms of some or all of her position, her velocity, and the other parameters given. Then use the numbers given, where supplied, in your final answer.

b) Given that bungy jumper's initial position and velocity are $\vec{r}_0 = 1\,\mathrm{m}\hat{\imath} - 5\,\mathrm{m}\hat{\jmath}$ and $\boldsymbol{v}_0 = \boldsymbol{0}$ write computer commands to find her position at $t = \pi/\sqrt{2}\,\mathrm{s}$.

c) Find the answer to part (b) with pencil and paper (that is, find an analytic solution to the differential equations, a final numerical answer is desired).



Problem 12.1.25: Conceptual setup for a bungy jumping system.

12.1.26 A softball pitcher releases a ball of mass m upwards from her hand with speed v_0 and angle θ_0 from the horizontal. The only external force acting on the ball after its release is gravity.

a) What is the equation of motion for the ball after its release?

b) What are the position, velocity, and acceleration of the ball?

c) What is its maximum height?

d) At what distance does the ball return to the elevation of release?

e) What kind of path does the ball follow and what is its equation y as a function of x?

12.1.27 Find the trajectory of a not-vertically-fired cannon ball assuming the air drag is proportional to the speed. Assume the mass is 10 kg, $g = 10\,\mathrm{m/s}$, the drag proportionality constant is $C = 5\,\mathrm{N/(m/s)}$. The cannon ball is launched at 100 m/s at a 45 degree angle.

- Draw a free-body diagram of the mass.
- Write linear momentum balance in vector form.
- Solve the equations on the computer and plot the trajectory.
- Solve the equations by hand and then use the computer to plot your solution.
- Compare the two plots and comment on the differences, if any.

12.1.28 A baseball pitching machine releases a baseball of mass m from its barrel with speed v_0 and angle θ_0 from the horizontal. The only external forces acting on the ball after its release are gravity and air resistance. The speed of the ball is given by $v^2 = \dot{x}^2 + \dot{y}^2$. Taking into account air resistance on the ball proportional to its speed squared, $\vec{F}_d = -bv^2\hat{\boldsymbol{e}}_t$, find the equation of motion for the ball, after its release, in cartesian coordinates. *

12.1.29 The equations of motion from problem 12.1.28 are nonlinear and cannot be solved in closed form for the position of the baseball. Instead, solve the equations numerically. Make a computer simulation of the flight of the baseball, as follows.

a) Convert the equation of motion into a system of first order differential equations. *

b) Pick values for the gravitational constant g, the coefficient of resistance b, and initial speed v_0, solve for the x and y coordinates of the ball and make a plot of its trajectory for various initial angles θ_0.

c) Use Euler's, Runge-Kutta, or other suitable method to numerically integrate the system of equations.

d) Use your simulation to find the initial angle that maximizes the distance of travel for the ball, with and without air resistance.

e) If the air resistance is very high, what is a qualitative description for the curve described by the path of the ball? Show this with an accurate plot of the trajectory. (Make sure to integrate long enough for the ball to get back to the ground.)

12.1.30 A particle of mass m moves in a viscous fluid which resists motion with a force of magnitude $F = c|\vec{v}|$, where $\vec{v}$ is the velocity. Do not neglect gravity.

a) (easy) In terms of some or all of g, m, and c, what is the particle's terminal (steady-state) falling speed?

b) Starting with a free-body diagram and linear momentum balance, find two second order scalar differential equations that describe the two-dimensional motion of the particle.

c) (Challenge, long calculation) Assume the particle is thrown from $\vec{r} = \vec{0}$ with $\vec{v} = v_{x0}\hat{\imath} + v_{y0}\hat{\jmath}$ at a vertical wall a distance d away. Find the height h along the wall where the particle hits. (Answer in terms of some or all of v_{x0}, v_{y0}, m, g, c, and d.) [Hint: i) find $x(t)$ and $y(t)$, ii) eliminate t, iii) substitute $x = d$. The answer is not tidy. In the limit $d \to 0$ the answer reduces to a sensible dependence on d (The limit $c \to 0$ is also sensible.).]

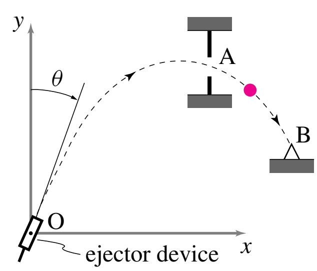
d) (Challenge, computer simulation). Do a computer simulation of the problem and find the solution in your simulation. Choose non-trivial numbers for all constants. To get an accurate solution you need an accurate interpolation to find at what time the particle hits the wall.

12.1.31 Someone in a violent part of the world shot a projectile at someone else. The basic facts:
Launched from the origin.
Projectile mass = 1 kg.
Launch angle 30° above horizontal.
Launch speed 172 m/s.
Drag force $\propto cv^2$ with $c = .01\,\mathrm{kg/m}$.
Gravity $g = 10\,\mathrm{m/s}$.

a) Write and execute computer code to find the height at $t = 1\,\mathrm{s}$. [Hints: sketch of problem, FBD, write drag force in vector form, LMB, 1st order equations, numerical setup, find height at 1 s].

b) Estimate the height at $t = 1\,\mathrm{s}$ using pencil and paper. An answer in meters is desired. [Hints: Assume g is negligible. Good calculus skills are needed but no involved arithmetic is needed. $1 + 1.72 = 2.72 \approx e$. After you have found a solution check that the force of gravity is a small fraction of the drag force throughout the duration of one second of your solution.]

12.1.32 In the arcade game shown, the object of the game is to propel the small ball from the ejector device at O in such a way that it passes through the small aperture at A and strikes the contact point at B. The player controls the angle θ at which the ball is ejected and the initial velocity v_o. The trajectory is confined to the frictionless xy-plane, which may or may not be vertical. Find the value of θ that gives success. The coordinates of A and B are $(2\ell, 2\ell)$ and $(3\ell, \ell)$, respectively, where ℓ is your favorite length unit.



Problem 12.1.32

12.1.1 Given $\vec{\boldsymbol{r}}(t) = A\sin(\omega t)\hat{\boldsymbol{\imath}} + Bt\hat{\boldsymbol{\jmath}} + C\hat{\boldsymbol{k}}$, find

1. $\vec{\boldsymbol{v}}(t)$
2. $\vec{\boldsymbol{a}}(t)$
3. $\vec{\boldsymbol{r}}(t) \times \vec{\boldsymbol{a}}(t)$.

$$\vec{r}(t) = A\sin\omega t\,\hat{i} + Bt\,\hat{j} + C\,\hat{k}$$

1) $\vec{V}(t) = \dot{\vec{r}}(t) = A\omega\cos\omega t\,\hat{i} + B\,\hat{j}$

2) $\vec{a}(t) = \ddot{\vec{r}}(t) = -A\omega^2\sin\omega t\,\hat{i}$

3) $\vec{r}(t)\times\vec{a}(t) = AB\omega^2 t\sin\omega t\,\hat{k} - AC\omega^2\sin\omega t\,\hat{j}$

$$= A\omega^2\sin\omega t\,[\,Bt\,\hat{k} - C\,\hat{j}\,]$$

12.1.2 A particle of mass $=3\,\text{kg}$ travels in space with its position known as a function of time, $\vec{r} = (\sin\frac{t}{\text{s}})\,\text{m}\hat{\imath} + (\cos\frac{t}{\text{s}})\,\text{m}\,\hat{\jmath} + te^{\frac{t}{\text{s}}}\,\text{m/s}\hat{k}$. At $t = 3\,\text{s}$, find the particle's

a) velocity and

b) acceleration.

$$\vec{r} = \sin\left(\frac{t}{s}\right) m\hat{i} + \cos\left(\frac{t}{s}\right) m\hat{j} + te^{t/s}\frac{m}{s}\hat{k}$$

a) $$\vec{v} = \dot{\vec{r}}\Big|_{t=3s} = \cos\left(\frac{t}{s}\right)\frac{m}{s}\hat{i} - \sin\left(\frac{t}{s}\right)\frac{m}{s}\hat{j} + \left[e^{t/s}\frac{m}{s} + te^{t/s}\frac{m}{s^2}\right]\hat{k}$$

$$= \left[-0.989\,\hat{i} - 0.141\,\hat{j} + 80.342\,\hat{k}\right]$$

b) $$\vec{a} = \ddot{\vec{r}}\Big|_{t=3s} = -\sin\left(\frac{t}{s}\right)\frac{m}{s^2}\hat{i} - \cos\left(\frac{t}{s}\right)\frac{m}{s^2}\hat{j} + \left[2e^{t/s}\frac{m}{s^2} + te^{t/s}\frac{m}{s^3}\right]\hat{k}$$

$$= \left[-0.141\,\hat{i} + 0.989\,\hat{j} + 100.42\,\hat{k}\right]$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

12.1.3 A particle of mass $m = 2\,\text{kg}$ travels in the xy-plane with its position known as a function of time, $\vec{r} = 3t^2\,\text{m/s}^2\hat{\imath} + 4t^3\,\frac{\text{m}}{s^3}\hat{\jmath}$. At $t = 5\,\text{s}$, find the particle's

a) velocity and
b) acceleration, and
c) draw the three vectors.

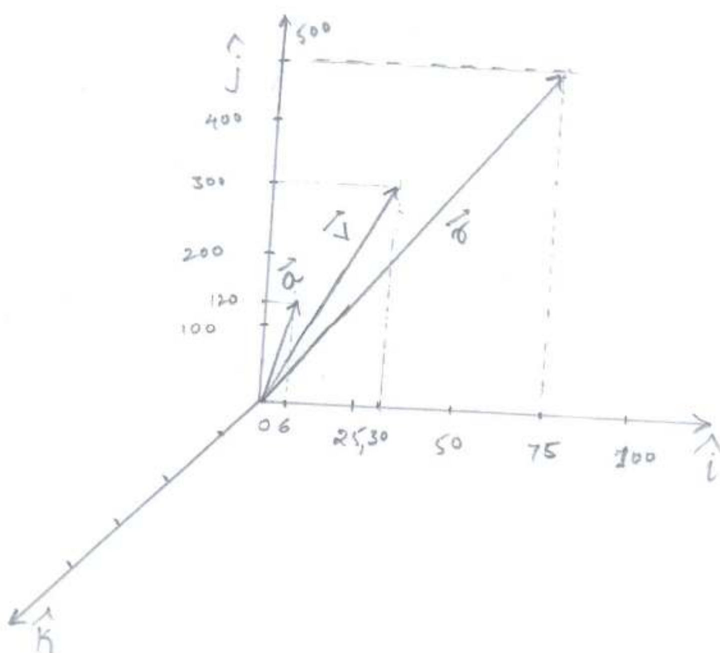
Answer:
(a) $\vec{v}(5\,\text{s}) = (30\hat{\imath} + 300\hat{\jmath})\,\text{m/s}$.
(b) $\vec{a}(5\,\text{s}) = (6\hat{\imath} + 120\hat{\jmath})\,\text{m/s}^2$.

$$\vec{r} = 3t^2\hat{i}\,\frac{m}{s^2} + 4t^3\hat{j}\,\frac{m}{s^3}$$

a) $$\vec{V} = \dot{\vec{r}}\Big|_{t=5} = 6t\,\hat{i}\,\frac{m}{s} + 12t^2\hat{j}\,\frac{m}{s}$$
$$= \left[30\hat{i} + 300\hat{j}\right]\frac{m}{s}$$

b) $$\vec{a} = \ddot{\vec{r}}\Big|_{t=5} = \left[6\hat{i} + 24t\,\hat{j}\right]\frac{m}{s^2}$$
$$= \left[6\hat{i} + 120\hat{j}\right]\frac{m}{s^2}$$

c)



Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

12.1.4 The velocity of a particle of mass m on a frictionless surface is given as $\vec{v} = (0.5\,\mathrm{m/s})\hat{\imath} - (1.5\,\mathrm{m/s})\hat{\jmath}$. If the displacement is given by $\Delta\vec{r} = \vec{v}t$, find (a) the distance traveled by the mass in 2 seconds and (b) a unit vector along the displacement.

11.1.4

a) $|\Delta\vec{r}| = |2(0.5\hat{i} - 1.5\hat{j})| = |\hat{i} - 3\hat{j}| = \sqrt{10}\,\mathrm{m}$

b) Direction vector $= \dfrac{\hat{i}}{\sqrt{10}} - \dfrac{3\hat{j}}{\sqrt{10}}$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

12.1.5 If $\dot{\vec{r}} = (u_0 \sin \Omega t)\hat{\imath} + v_0\hat{\jmath}$ and $\vec{r}(0) = x_0\hat{\imath} + y_0\hat{\jmath}$, with u_0, v_0, and Ω as constants, find $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$.

Answer:
$\vec{r}(t) = \left(x_0 + \frac{u_0}{\Omega} - \frac{u_0}{\Omega}\cos(\Omega t)\right)\hat{\imath} + \left(y_0 + v_0 t\right)\hat{\jmath}$.

11.1.5

$$\dot{\vec{r}} = (u_0 \sin\Omega t)\hat{i} + v_0\hat{j}$$

$$\vec{r}(0) = x_0\hat{i} + y_0\hat{j}$$

integrate $\dot{\vec{r}}$

$$\int\dot{\vec{r}} = \vec{r} = \int(u_0\sin\Omega t)\hat{i} + v_0\hat{j} + C_1$$

$$\vec{r} = -\frac{u_0}{\Omega}\cos\Omega t\,\hat{i} + v_0 t\,\hat{j} + C_1$$

$$\vec{r}(0) = x_0\hat{i} + y_0\hat{j} = -\frac{u_0}{\Omega}\hat{i} + C_1$$

$$C_1 = \left(x_0 + \frac{u_0}{\Omega}\right)\hat{i} + y_0\hat{j}$$

$$\vec{r} = \left[x_0 + \frac{u_0}{\Omega}(1-\cos\Omega t)\right]\hat{i} + (y_0 + v_0 t)\hat{j}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

12.1.6 For $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$ and $\vec{a} = 2\,\text{m/s}^2\hat{\imath} - (3\,\text{m/s}^2 - 1\,\text{m/s}^3\,t)\hat{\jmath} - 5\,\text{m/s}^4\,t^2\hat{k}$, write the vector equation $\vec{v} = \int \vec{a}\,dt$ as three scalar equations (*i.e.,* find $v_x(t)$, $v_y(t)$, and $v_z(t)$).

11.1.6

$$\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$$

$$\vec{a} = 2\hat{i} - (3-t)\hat{j} - 5t^2\hat{k}$$

$$\vec{V}(t) = \int \vec{a}\,dt + c$$

$$\vec{V}(t) = \left[2t\,\hat{i} - \left(3t - \frac{t^2}{2}\right)\hat{j} - \frac{5t^3}{3}\hat{k}\right]$$

The scalar equations are.

$$V_x = 2t$$

$$V_y = -\left(3t - \frac{t^2}{2}\right)$$

$$V_z = -\frac{5t^3}{3}$$

12.1.7 Find $\vec{r}(5\,\mathrm{s})$ given that $\dot{\vec{r}} = v_1 \sin(ct)\hat{\imath} + v_2\hat{\jmath}$ and $\vec{r}(0) = 2\,\mathrm{m}\hat{\imath} + 3\,\mathrm{m}\hat{\jmath}$, and that v_1 is a constant $4\,\mathrm{m/s}$, v_2 is a constant $5\,\mathrm{m/s}$, and c is a constant $4\,\mathrm{s}^{-1}$.

<u>11.1.7</u>

$$\vec{r}(t) = \int \dot{\vec{r}}(t)\,dt + C_1$$

$$= \int (V_1 \sin ct\ \hat{i} + V_2 \hat{j})\,dt + C_1$$

$$\vec{r}(t) = -\frac{V_1}{c}\cos ct\ \hat{i} + V_2 t\ \hat{j} + c$$

$$\vec{r}(t)\Big|_{t=0} = -\frac{V_1}{c}\hat{i} + C_1 = 2\hat{i} + 3\hat{j}$$

$$C_1 = \left(2 + \frac{V_1}{c}\right)\hat{i} + 3\hat{j}$$

12.1.8 Let $\dot{\vec{r}} = v_0 \cos\alpha \hat{\imath} + v_0 \sin\alpha \hat{\jmath} + (v_0 \tan\theta - gt)\hat{k}$, where v_0, α, θ, and g are constants. If $\vec{r}(0) = \vec{0}$, find $\vec{r}(t)$.

11.1.8

$$\vec{r}(t) = \int \dot{\vec{r}}(t)\, dt + C$$

$$= \int \left[V_0 \cos\alpha\, \hat{i} + V_0 \sin\alpha\, \hat{j} + (V_0 \tan\theta - gt)\hat{k} \right] dt + C_1$$

$$= t(V_0 \cos\alpha\, \hat{i} + V_0 \sin\alpha\, \hat{j} + V_0 \tan\theta\, \hat{k}) - \frac{gt^2}{2}\hat{k} + C_1$$

$$\vec{r}(t)\Big|_{t=0} = 0 \quad \text{gives } C = 0$$

$$\vec{r}(t) = t(V_0 \cos\alpha\, \hat{i} + V_0 \sin\alpha\, \hat{j} + V_0 \tan\theta\, \hat{k}) - \frac{gt^2}{2}\hat{k}$$

12.1.9 On a smooth circular helical path the velocity of a particle is $\dot{\vec{r}} = -R\sin t\hat{\imath} + R\cos t\hat{\jmath} + gt\hat{k}$. If $\vec{r}(0) = R\hat{\imath}$, find $\vec{r}((\pi/3)\,\mathrm{s})$.

11.1.9

$$\vec{r}(t) = \int(-R\sin t\,\hat{i} + R\cos t\,\hat{j} + gt\,\hat{k})\,dt + C_1$$

$$= R\cos t\,\hat{i} + R\sin t\,\hat{j} + \frac{gt^2}{2}\hat{k} + C_1$$

$$\vec{r}(0) = R\hat{i} + C_1 = R\hat{i} \implies C_1 = 0$$

$$\vec{r}\left(\frac{\pi}{3}\right) = R\cos\frac{\pi}{3}\,\hat{i} + R\sin\frac{\pi}{3}\,\hat{j} + \frac{g\left(\frac{\pi}{3}\right)^2}{2}\hat{k}$$

$$= \frac{R}{2}\hat{i} + \frac{\sqrt{3}R}{2}\hat{j} + \frac{g\pi^2}{18}\hat{k}$$

12.1.10 A particle travels on a path in the xy-plane given by $y(x) = \sin^2(\frac{x}{\text{m}})$ m. where $x(t) = t^3(\frac{\text{m}}{\text{s}^3})$. What are the velocity and acceleration of the particle in cartesian coordinates when $t = (\pi)^{\frac{1}{3}}$ s?

11.1.10 ⑥

$y(x) = \sin^2(x)$

$x(t) = t^3$

$\vec{V}(t) = \frac{d}{dt}\left[t^3\,\hat{i} + \sin^2(t^3)\,\hat{j}\right]$

$\vec{V}(t) = \left[3t^2\,\hat{i} + 2\sin(t^3)\cos(t^3)\cdot 3\cdot t^2\,\hat{j}\right]$

$\vec{a}(t) = \left[6t\,\hat{i} + \left((3t^2)(6t^2\cos 2t^3) + 6t\sin 2t^3\right)\hat{j}\right]$

$\cancel{\vec{V}(t)\Big|_{t=(\pi)^{1/3}} = 3\pi^{2/3}\,\hat{i}\quad 6\pi^{1/3}(\hat{i} + 3\pi\,\hat{j})}$

$\vec{V}(t)\Big|_{t=(\pi)^{1/3}} = 3\pi^{2/3}\,\hat{i}$

$\vec{a}(\pi^{1/3}) = 6\pi^{1/3}\left[\hat{i} + 3\pi\,\hat{j}\right]$

12.1.11 The position of a particle is given by $\vec{r}(t) = (t^2\,\mathrm{m/s^2}\hat{\imath} + e^{\frac{t}{\mathrm{s}}}\,\mathrm{m}\,\hat{\jmath})$. What are the velocity and acceleration of the particle as functions of time? Draw the path of the particle and show the vectors $\vec{v}$ and $\vec{a}$ at $t = 1\,\mathrm{s}$.

Answer:
$\vec{v} = 2t\,\mathrm{m/s^2}\hat{\imath} + e^{\frac{t}{\mathrm{s}}}\,\mathrm{m/s}\,\hat{\jmath}$, $\vec{a} = 2\,\mathrm{m/s^2}\hat{\imath} + e^{\frac{t}{\mathrm{s}}}\,\mathrm{m/s^2}\,\hat{\jmath}$.

6.1

$$\underline{r}(t) = t^2\left(\frac{m}{s^2}\right)\hat{\imath} + e^{t(1/s)}\,\hat{\jmath}$$

$$\underline{v}(t) = \frac{d\,\underline{r}(t)}{dt} = 2t\left(\frac{m}{s^2}\right)\hat{\imath} + e^{t(1/s)}\,\hat{\jmath}$$

$$\underline{a}(t) = \frac{d\,\underline{v}(t)}{dt} = 2\left(\frac{m}{s^2}\right)\hat{\imath} + e^{t(1/s)}\,\hat{\jmath}$$

12.1.12 A particle travels on an elliptical path given by $y^2 = b^2(1 - \frac{x^2}{a^2})$ with constant speed v. Find the velocity of the particle when $x = a/2$ and $y > 0$ in terms of a, b, and v.

(8)

11.1.12

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$x = \frac{a}{2} \Rightarrow y^2 = b^2\left(1 - \frac{1}{4}\right) = \frac{3b^2}{4}$$

$$y = \pm\frac{\sqrt{3}}{2} b \Rightarrow y = +\frac{\sqrt{3}}{2} b$$

$$\text{Speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = V \longrightarrow (1)$$

$$2y\frac{dy}{dt} = -b^2\frac{2x}{a^2}\frac{dx}{dt}$$

$$\frac{dy}{dt} = -\left(\frac{b^2}{a^2}\frac{x}{y}\right)\frac{dx}{dt} \longrightarrow (2)$$

Putting in (1) we get

$$\frac{dx}{dt}\sqrt{1 + \left(\frac{b^2 x}{a^2 y}\right)^2} = V$$

$$V_x = \frac{dx}{dt} = \frac{V}{\sqrt{1 + \left(\frac{b^2}{a^2}\frac{x}{y}\right)^2}} = \frac{V}{\sqrt{1 + \frac{b^2}{3a^2}}}$$

similarly $V_y = \dfrac{dy}{dt} = \dfrac{V}{\sqrt{1 + \frac{3a^2}{b^2}}}$

$$\vec{V} = \frac{V}{\sqrt{1 + b^2/3a^2}}\,\hat{i} + \frac{V}{\sqrt{1 + 3a^2/b^2}}\,\hat{j}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

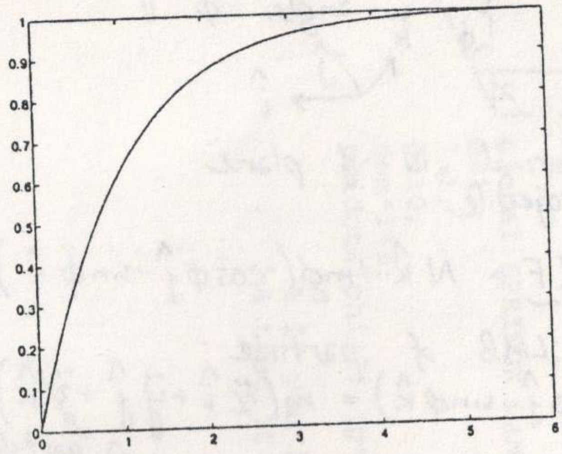
12.1.13 A particle travels on a path in the xy-plane given by $y(x) = (1 - e^{-\frac{x}{\mathrm{m}}})\,\mathrm{m}$. Make a plot of the path. It is known that the x coordinate of the particle is given by $x(t) = t^2\,\mathrm{m/s^2}$. What is the rate of change of speed of the particle? What angle does the velocity vector make with the positive x axis when $t = 3\,\mathrm{s}$?

6.5 p. 8

A particle travels on a path:

$y(x) = (1 - e^{-x/m})\ m.$

Plot:



x-coordinate: $x(t) = t^2\ [m/s^2]$

then y-coord.: $y(t) = (1 - e^{-t^2/[m]})\ [m]$ the m is a unit, not a variable

so $\underset{\sim}{r}(t) = t^2\,\hat{i} + (1 - e^{-t^2})\,\hat{j}$

then $\dfrac{d\underset{\sim}{r}(t)}{dt} = 2t\,\hat{i} + 2t\,e^{-t^2}\,\hat{j} = \underset{\sim}{v}(t)$ Drop units to avoid confusion

so $|\underset{\sim}{v}| = \sqrt{4t^2 + 4t^2 e^{-2t^2}} =$

$$\boxed{|\underset{\sim}{v}| = 2t\sqrt{1 + e^{-2t^2}}\ [m/s^2]}$$

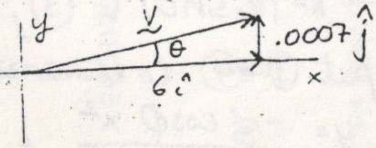
oops this is speed differentiate this wrt t to get its rate of change

b) recall $\underset{\sim}{v}(t) = 2t\,\hat{i} + 2t e^{-t^2}\,\hat{j}$

put in $t = 3$ sec:

$\underset{\sim}{v}(3\,s.) = (6\,\hat{i} + .0007\,\hat{j})\ m/s$

$\cos\theta = \dfrac{6}{\sqrt{6^2 + .0007^2}} \Rightarrow$



$$\boxed{\theta = .007^\circ}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

12.1.14 A particle starts at the origin in the xy-plane, ($x_0 = 0, y_0 = 0$) and travels only in the positive xy quadrant. Its speed and x coordinate are known to be $v(t) = \sqrt{1 + (\frac{4}{s^2})t^2}$ m/s and $x(t) = t$ m/s, respectively. What is $\vec{r}(t)$ in cartesian coordinates? What are the velocity, acceleration, and rate of change of speed of the particle as functions of time? What kind of path is the particle on? What is the distance of the particle from the origin and its velocity and acceleration when $x = 3$ m?

11.1.14

9

$V(t) = \sqrt{1+4t^2}$ $\quad x > 0$

$x(t) = t$ $\quad y > 0$

$V_x = \dot{x}(t) = 1$

$V(t) = \sqrt{V_x^2 + V_y^2} = \sqrt{1 + V_y^2} = \sqrt{1+4t^2}$

$V_y = \pm 2t \quad \Rightarrow \quad V_y = 2t$

$\vec{r}(t) = \int \vec{V}(t)\, dt = \int (\hat{i} + 2t\,\hat{j})\, dt + C$

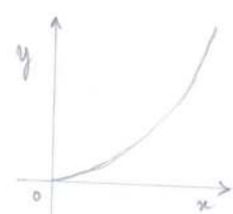
$\vec{r}(t) = t\,\hat{i} + \frac{2t^2}{2}\,\hat{j} + C$

$C = 0$

$\vec{r}(t) = t\,\hat{i} + t^2\,\hat{j}$

$\vec{V}(t) = \hat{i} + 2t\,\hat{j}$

$\vec{a}(t) = 2\,\hat{j}$



Rate of change of speed

$\dot{V} = \frac{1}{2} \frac{8t}{\sqrt{1+4t^2}} = \frac{4t}{\sqrt{1+4t^2}}$

Path of the particle is parabolic upwards constant acceleration.

At $x = 3$, $t = 3$;

$y = t^2 = 9$

distance from origin $= \sqrt{9^2 + 3^2} = 3\sqrt{10}$

Velocity $= (\hat{i} + 6\hat{j})$ m/s

acceleration $= 2\hat{j}$ m/s^2

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

12.1.15 For a particle, $\sum \vec{F} = m\vec{a}$. Two forces $\vec{F}_1$ and $\vec{F}_2$ act on a mass P as shown in the figure. P has mass 2 lbm. The acceleration of the mass is somehow measured to be $\vec{a} = -2\,\text{ft/s}^2\hat{\imath} + 5\,\text{ft/s}^2\hat{\jmath}$.

a) Write the equation

$$\sum \vec{F} = m\vec{a}$$

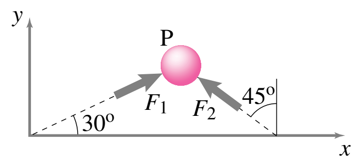
in vector form (evaluating each side as much as possible).

b) Write the equation in scalar form (use any method you like to get two scalar equations in the two unknowns F_1 and F_2).

c) Write the equation in matrix form.

d) Find $F_1 = |\vec{F}_1|$ and $F_2 = |\vec{F}_2|$ by the following methods:

1. from the scalar equations using hand algebra,
2. from the matrix equation using a computer, and
3. from the vector equation using a cross product.



Problem 12.15

10

11.1.15

$\vec{a} = -2\hat{i} + 5\hat{j}$

$m = 2$

a) $\sum \vec{F} = m\vec{a}$

$\vec{F}_1 + \vec{F}_2 = m(-2\hat{i} + 5\hat{j})$

$= -2m\hat{i} + 5m\hat{j}$

$\vec{F}_2 = |\vec{F}_2| \left(-\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right)$,

$\vec{F}_1 = |\vec{F}_1| \left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right)$

$$\left(\frac{\sqrt{3}}{2}|\vec{F}_1| - \frac{|\vec{F}_2|}{\sqrt{2}}\right)\hat{i} + \left(\frac{|\vec{F}_1|}{2} + \frac{|\vec{F}_2|}{\sqrt{2}}\right)\hat{j} = -2m\hat{i} + 5m\hat{j}$$

b) Scalar form

$$\frac{\sqrt{3}}{2}|\vec{F}_1| - \frac{|\vec{F}_2|}{\sqrt{2}} = -2m \quad \text{— (i)}$$

$$\frac{|\vec{F}_1|}{2} + \frac{|\vec{F}_2|}{\sqrt{2}} = 5m \quad \text{— (ii)}$$

c) Matrix form

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} |\vec{F}_1| \\ |\vec{F}_2| \end{bmatrix} = \begin{bmatrix} -2m \\ 5m \end{bmatrix}$$

d) Solution : Hand Algebra (11)

$$\frac{\sqrt{3}+1}{2}|\vec{F}_1| = 3m \Rightarrow |\vec{F}_1| = \frac{6m}{\sqrt{3}+1}$$

$$|\vec{F}_2| = \frac{\sqrt{2}(5\sqrt{3}+2)m}{\sqrt{3}+1}$$

Matrix Method

$$|\vec{F}_1| = \frac{\begin{vmatrix} -2m & -\frac{1}{\sqrt{2}} \\ 5m & \frac{1}{\sqrt{2}} \end{vmatrix}}{\begin{vmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} \end{vmatrix}}$$

$$|\vec{F}_2| = \frac{\begin{vmatrix} \sqrt{3}/2 & -2m \\ 1/2 & 5m \end{vmatrix}}{\begin{vmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} \end{vmatrix}}$$

Solution from vector method.

unit vector along $\vec{F}_1 \Rightarrow \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j} = \hat{e}_1$

Along $\vec{F}_2 = -\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} = \hat{e}_2$

$$\vec{F}_1 + \vec{F}_2 = m(-2\hat{i} + 5\hat{j})$$

12

$$\hat{e}_2 \times (\vec{F}_1 + \vec{F}_2) = \hat{e}_2 \times [m(-2\hat{i} + 5\hat{j})]$$

$$|\vec{F}_1| \left(\frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}}\right)\hat{k} = m\left(\frac{-5}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\hat{k}$$

$$|\vec{F}_1| = \frac{6m}{1+\sqrt{3}}$$

lly

$$\hat{e}_1 \times (\vec{F}_1 + \vec{F}_2) = \hat{e}_1 \times m(-2\hat{i} + 5\hat{j})$$

$$|\vec{F}_2| \left(\frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\right) = \left(\frac{5\sqrt{3}}{2} + 1\right)m$$

$$|\vec{F}_2| = \frac{\sqrt{2}\,(5\sqrt{3}+2)\,m}{1+\sqrt{3}}$$

12.1.16 Three forces, $\vec{F}_1 = 20\,\text{N}\hat{\imath} - 5\,\text{N}\hat{\jmath}$, $\vec{F}_2 = F_{2x}\hat{\imath} + F_{2z}\hat{k}$, and $\vec{F}_3 = F_3\hat{\lambda}$, where $\hat{\lambda} = \frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath}$, act on a body with mass 2 kg. The acceleration of the body is $\vec{a} = -0.2\,\text{m/s}^2\hat{\imath} + 2.2\,\text{m/s}^2\hat{\jmath} + 1.7\,\text{m/s}^2\hat{k}$. Write the equation $\sum \vec{F} = m\vec{a}$ as scalar equations and solve them (most conveniently on a computer) for F_{2x}, F_{2z}, and F_3.

13

11. 1.16

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = m\vec{a}$$

$$\Rightarrow [20\hat{i} - 5\hat{j}] + [F_{2x}\hat{i} + F_{2z}\hat{k}] + [\frac{F_3}{2}\hat{i} + F_3\frac{\sqrt{3}}{2}\hat{j}] = 2[-0.2\hat{i} + 2.2\hat{j} + 1.7\hat{k}]$$

$$[20 + F_{2x} + \frac{F_3}{2}]\hat{i} + [-5 + \frac{F_3\sqrt{2}}{2}]\hat{j} + F_{2z}\hat{k} = -0.4\hat{i} + 4.4\hat{j} + 3.4\hat{k}$$

$$20 + F_{2x} + \frac{F_3}{2} = -0.4 \longrightarrow (i)$$

$$-5 + \frac{F_3\sqrt{3}}{2} = 4.4 \longrightarrow (ii)$$

$$F_{2z} = 3.4 \longrightarrow (iii)$$

$F_{2z} = 3.4$

$F_3 = 10.85$

$F_{2x} = -25.827$

12.1.17 In three-dimensional space with no gravity a particle with $m = 3\,\text{kg}$ at A is pulled by three strings which pass through points B, C, and D respectively. The acceleration is known to be $\vec{a} = (1\hat{\imath} + 2\hat{\jmath} + 3\hat{k})\ \text{m/s}^2$. The position vectors of B, C, and D relative to A are given in the first few lines of code below. Complete the pseudo-code to find the three tensions. The last line should read `T = ...` with `T` being assigned to be a 3-element column vector with the three tensions in Newtons. [Hint: If `x`, `y`, and `z` are three column vectors then `A=[x y z]` is a matrix with `x`, `y`, and `z` as columns.]

```
% Incomplete PSEUDO-CODE file
m   = 3;
a   = [ 1 2 3]';
~
rAB = [ 2 3 5]';
rAC = [-3 4 2]';
rAD = [ 1 1 1]';
uAB = rAB/(magnitude of rAB);
    .
    .
    .
T   =
```

14

11.1.17

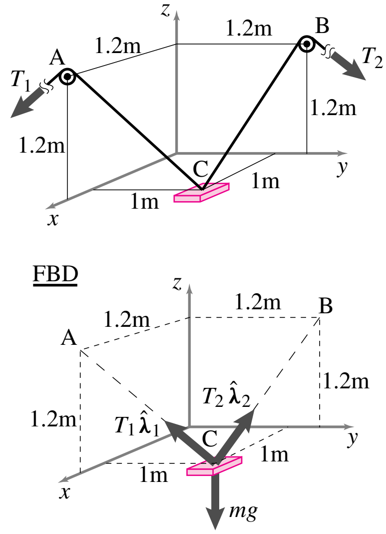
Say the tension be T_1, T_2, T_3

then

$$T_1 \frac{\vec{r}_{AB}}{|\vec{r}_{AB}|} + T_2 \frac{\vec{r}_{AC}}{|\vec{r}_{AC}|} + T_3 \frac{\vec{r}_{AD}}{|\vec{r}_{AD}|} = m\vec{a}$$

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 3.4675 \\ 0.3366 \\ 0.1083 \end{bmatrix}$$

12.1.18 A block of mass 100 kg is pulled with two strings AC and BC. Given that the tensions $T_1 = 1200\,\text{N}$ and $T_2 = 1500\,\text{N}$, find the magnitude and direction of the acceleration of the block. [$\sum \vec{F} = m\vec{a}$]

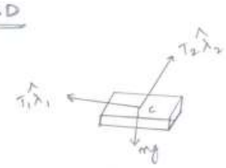


Problem 12.18

(16)

11.1.19

FBD



$$\hat{\lambda}_1 = \frac{\vec{r}_{CA}}{|\vec{r}_{CA}|}$$

$$\hat{\lambda}_1 = \frac{(\hat{i}+\hat{j}) - (1.2\hat{i} + 1.2\hat{k})}{|\vec{r}_{CA}|}$$

$$= -\frac{1}{\sqrt{1.08}}\left[-0.2\hat{i} + \hat{j} - 0.2\hat{k}\right]$$

Illy.

$$\hat{\lambda}_2 = -\frac{\left[\hat{i} - 0.2\hat{j} - 0.2\hat{k}\right]}{\sqrt{1.08}}$$

using $\sum\vec{F} = m\vec{a}$

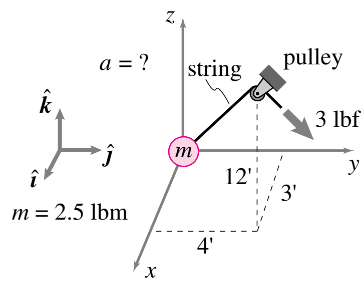
we get $m\vec{a} = T_1\hat{\lambda}_1 + T_2\hat{\lambda}_2 - mg\hat{k}$

$$m\vec{a} = \frac{T_1}{\sqrt{1.08}}\begin{bmatrix}0.2\\-1\\0.2\end{bmatrix} + \frac{T_2}{\sqrt{1.08}}\begin{bmatrix}-1\\0.2\\0.2\end{bmatrix} - mg\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\vec{a} = 6.2\hat{i} - 25.98\hat{j} - 4.8038\hat{k}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

12.1.19 Neglecting gravity, the only force acting on the mass shown in the figure is from the string. Find the acceleration of the mass. Use the dimensions and quantities given. Recall that lbf is a pound force, lbm is a pound mass, and lbf/lbm = g. Use $g = 32\,\mathrm{ft/s^2}$. Note also that $3^2 + 4^2 + 12^2 = 13^2$.



Problem 12.19

11.1.20

$$\vec{F} = m\vec{a}$$

$$3g\,\frac{[3\hat{i} + 4\hat{j} + 12\hat{k}]}{13} = m\vec{a}$$

$$\cancel{\vec{a} = \frac{3}{32} \times (32)}$$

$$\vec{a} = \frac{3}{2.5} \times 32 \left[\frac{3\hat{i} + 4\hat{j} + 12\hat{k}}{13}\right]$$

$$\vec{a} = 5.9\hat{i} + 7.87\hat{j} + 23.63\hat{k}$$

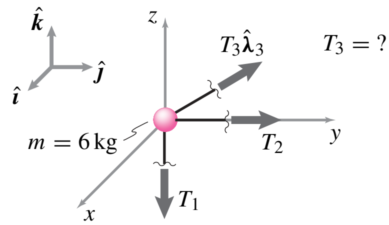
Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

12.1.20 Three strings are tied to the mass shown with the directions indicated in the figure. They have unknown tensions T_1, T_2, and T_3. There is no gravity. The acceleration of the mass is given as $\vec{a} = (-0.5\hat{\imath} + 2.5\hat{\jmath} + \frac{1}{3}\hat{k})\,\mathrm{m/s^2}$.

a) Given the free-body diagram in the figure, write the equations of linear momentum balance for the mass.

b) Find the tension T_3.

$\hat{\lambda}_3 = \frac{1}{13}(3\hat{\imath} + 12\hat{\jmath} + 4\hat{k})$



$\vec{a} = (\text{-}.5\hat{\imath} + 2.5\hat{\jmath} + \frac{1}{3}\hat{k})\frac{m}{s^2}$

Problem 12.20

Answer:
$T_3 = 13\,\mathrm{N}$

11.1.21

$\sum_i \vec{F} = m\vec{a}$

a) $-T_1\hat{k} + T_2\hat{\jmath} + T_3\frac{1}{13}(3\hat{\imath} + 12\hat{\jmath} + 4\hat{k}) = [-0.5\hat{\imath} + 2.5\hat{\jmath} + \frac{1}{3}\hat{k}]\,6$

$\Rightarrow \quad \frac{3T_3\hat{\imath}}{13} = -3\hat{\imath}$

$T_3 = -13N$

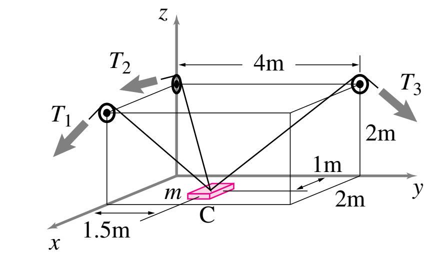
12.1.21

An object C of mass 2 kg is pulled by three strings as shown. The acceleration of the object at the position shown is $\boldsymbol{a} = (-0.6\hat{\imath} - 0.2\hat{\jmath} + 2.0\hat{\boldsymbol{k}})\,\mathrm{m/s^2}$.

a) Draw a free-body diagram of the mass.

b) Write the equation of linear momentum balance for the mass. Use $\boldsymbol{\lambda}$'s as unit vectors along the strings.

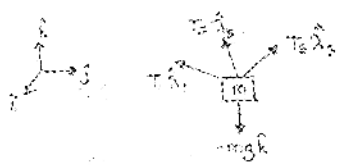
c) Find the three tensions T_1, T_2, and T_3 at the instant shown. You may find these tensions by using hand algebra with the scalar equations, using a computer with the matrix equation, or by using a cross product on the vector equation.



Problem 12.21

$\vec{a} = -0.6\hat{\imath} - 0.2\hat{\jmath} + 2.0\hat{k}\ [\mathrm{m/s^2}]$, $m = 2\ \mathrm{kg}$

a) Free body diagram:



$\vec{r}_m = 1.0\hat{\imath} + 1.5\hat{\jmath}\ [\mathrm{m}]$

$\vec{r}_1 = 2.0\hat{\imath} + 2.0\hat{k}\ [\mathrm{m}]$

$\vec{r}_2 = 2.0\hat{k}\ [\mathrm{m}]$

$\vec{r}_3 = 4.0\hat{\jmath} + 2.0\hat{k}\ [\mathrm{m}]$

b) $\vec{r}_{m1} = \vec{r}_1 - \vec{r}_m = 1.0\hat{\imath} - 1.5\hat{\jmath} + 2.0\hat{k}$ $\quad |\vec{r}_{m1}| = \frac{1}{2}\sqrt{29}$

$\vec{r}_{m2} = \vec{r}_2 - \vec{r}_m = -1.0\hat{\imath} - 1.5\hat{\jmath} + 2.0\hat{k}$ $\quad |\vec{r}_{m2}| = \frac{1}{2}\sqrt{29}$

$\vec{r}_{m3} = \vec{r}_3 - \vec{r}_m = -1.0\hat{\imath} + 2.5\hat{\jmath} + 2.0\hat{k}$ $\quad |\vec{r}_{m3}| = \frac{3}{2}\sqrt{5}$

$$\sum \vec{F} = m\vec{a} = T_1\hat{\lambda}_1 + T_2\hat{\lambda}_2 + T_3\hat{\lambda}_3 - mg\hat{k}$$

or $2\,\mathrm{kg}(-0.6\hat{\imath} - 0.2\hat{\jmath} + 2.0\hat{k}) = T_1\left(\frac{2\sqrt{29}}{29}\right)(\hat{\imath} - 1.5\hat{\jmath} + 2.0\hat{k})$

$+ T_2\left(\frac{2\sqrt{29}}{29}\right)(-\hat{\imath} - 1.5\hat{\jmath} + 2.0\hat{k}) + T_3\left(\frac{2\sqrt{5}}{15}\right)(-\hat{\imath} + 2.5\hat{\jmath} + 2.0\hat{k}) - mg\hat{k}$

In component form:

$\hat{\imath}\,(-1.2) = \hat{\imath}\,(0.3714T_1 - 0.3714T_2 - 0.2981T_3)$

$\hat{\jmath}\,(-0.4) = \hat{\jmath}\,(-0.5571T_1 - 0.5571T_2 + 0.7454T_3)$

$\hat{k}\,(4 + 19.62) = \hat{k}\,(0.7428T_1 + 0.7428T_2 + 0.5963T_3)$

Continued on page 4

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

10.22 continued.

In matrix form, we have:

$$\begin{bmatrix} -1.2 \\ -0.4 \\ 23.62 \end{bmatrix} = \begin{bmatrix} 0.3714 & -0.3714 & -0.2981 \\ -0.5571 & -0.5571 & 0.7454 \\ 0.7428 & 0.7428 & 0.5963 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}$$

Solving in Matlab yields:

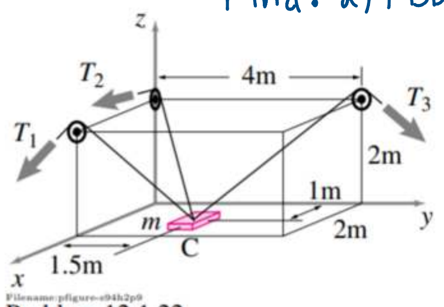
$$\boxed{\begin{aligned} T_1 &= 14.28 \text{ N} \\ T_2 &= 5.86 \text{ N} \\ T_3 &= 14.52 \text{ N} \end{aligned}}$$

Alternately, see Matlab code (modified from problem 10.17) on previous page.

#21 Solution Michelle deSouza

Monday, March 8, 2021 2:54 PM

$\boxed{12.1.22}$ Given: accelerating mass pulled by 3 strings
Find: a) FBD b) LMB c) T_1, T_2, T_3

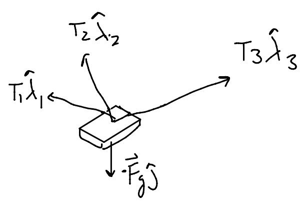


Problem 12.1.22

$m = 2\ kg$

$\vec{a} = -0.6\hat{\imath} - 0.2\hat{\jmath} + 2.0\hat{k}\ m/s^2$

a) **FBD:**



$\hat{\lambda}$s defined as unit vector in direction of string

Assumption: ideal pulley, tension is equal on either side

b) **LMB:** $\sum \vec{F} = T_1\hat{\lambda}_1 + T_2\hat{\lambda}_2 + T_3\hat{\lambda}_3 - mg\hat{k} = m\vec{a}$

What are $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3$?

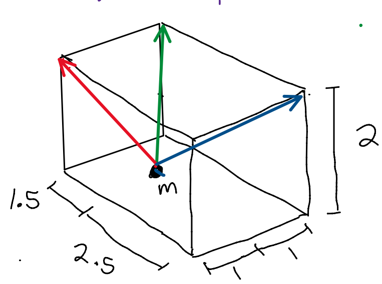
- Find vector $\vec{v}$ in direction of $\hat{\lambda}$, use components of string lengths

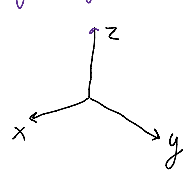
$\vec{v} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$

$\vec{v}_1 = (1\hat{\imath} - 1.5\hat{\jmath} + 2\hat{k})\ m$

$\vec{v}_2 = (-1\hat{\imath} - 1.5\hat{\jmath} + 2\hat{k})\ m$

$\vec{v}_3 = (-1\hat{\imath} + 2.5\hat{\jmath} + 2\hat{k})\ m$





pick mass as origin

- Convert to unit vector

$v = \sqrt{a^2+b^2+c^2}$ (magnitude!)

$\hat{\lambda} = \frac{\vec{v}}{v}$ → total vector has magnitude 1 ✓

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

$$V_1 = \sqrt{(1m)^2 + (-1.5m)^2 + (2m)^2} = 2.69\ m \qquad \hat{\lambda}_1 = \frac{\vec{V}_1}{V_1} = \frac{1}{2.69}(1\hat{\imath} - 1.5\hat{\jmath} + 2\hat{k})\ m$$

$$V_2 = \sqrt{(-1m)^2 + (-1.5m)^2 + (2m)^2} = 2.69\ m \qquad \hat{\lambda}_2 = \frac{\vec{V}_2}{V_2} = \frac{1}{2.69}(-1\hat{\imath} - 1.5\hat{\jmath} + 2\hat{k})\ m$$

$$V_3 = \sqrt{(-1m)^2 + (2.5m)^2 + (2m)^2} = 3.35\ m \qquad \hat{\lambda}_3 = \frac{\vec{V}_3}{V_3} = \frac{1}{3.35}(-1\hat{\imath} + 2.5\hat{\jmath} + 2\hat{k})\ m$$

c) Solve for T_1, T_2, T_3

$$T_1\hat{\lambda}_1 + T_2\hat{\lambda}_2 + T_3\hat{\lambda}_3 = m\vec{a} + mg\hat{k}$$

split into $\hat{\imath}, \hat{\jmath}, \hat{k}$ by dotting

$$T_1(\hat{\lambda}_1\cdot\hat{\imath}) + T_2(\hat{\lambda}_2\cdot\hat{\imath}) + T_3(\hat{\lambda}_3\cdot\hat{\imath}) = m(\vec{a}\cdot\hat{\imath})$$

$$T_1(\hat{\lambda}_1\cdot\hat{\jmath}) + T_2(\hat{\lambda}_2\cdot\hat{\jmath}) + T_3(\hat{\lambda}_3\cdot\hat{\jmath}) = m(\vec{a}\cdot\hat{\jmath})$$

$$T_1(\hat{\lambda}_1\cdot\hat{k}) + T_2(\hat{\lambda}_2\cdot\hat{k}) + T_3(\hat{\lambda}_3\cdot\hat{k}) = m(\vec{a}\cdot\hat{k}) + mg$$

put into matrix form

$$\underbrace{\begin{bmatrix} \hat{\lambda}_1\cdot\hat{\imath} & \hat{\lambda}_2\cdot\hat{\imath} & \hat{\lambda}_3\cdot\hat{\imath} \\ \hat{\lambda}_1\cdot\hat{\jmath} & \hat{\lambda}_2\cdot\hat{\jmath} & \hat{\lambda}_3\cdot\hat{\jmath} \\ \hat{\lambda}_1\cdot\hat{k} & \hat{\lambda}_2\cdot\hat{k} & \hat{\lambda}_3\cdot\hat{k} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}}_{T} = m \underbrace{\begin{bmatrix} \vec{a}\cdot\hat{\imath} \\ \vec{a}\cdot\hat{\jmath} \\ \vec{a}\cdot\hat{k} + g \end{bmatrix}}_{b}$$

3 equations, 3 variables – can solve using substitution, dot + cross products (shown in Matlab), etc

$$A \cdot T = b$$

$$T = A^{-1}b$$

solve in Matlab

↓

$T_1 = 14.3\ N$

$T_2 = 5.9\ N$

$T_3 = 14.5\ N$

my code shown below, for another example, refer to "Matlab Code for Three Strings" on Ed

```
%MAE 2030 HW 4 SOLUTION
%Problem #21 12.1.22
%Michelle deSouza
%For another example, see 'Matlab Code for Three Strings' on Ed

%Define unit vectors
i = [1;0;0];
j = [0;1;0];
k = [0;0;1];

%Define m, a, g
m = 2; %kg
a = -0.6*i -0.2*j + 2.0*k;%m/s2
g = 9.8; %m/s^2

%%FIND LAMBDAS%%
%Define v1,2,3 vectors
v1 =  1*i - 1.5*j + 2*k;
v2 = -1*i - 1.5*j + 2*k;
v3 = -1*i + 2.5*j + 2*k;
%Find v1,2,3 magnitude
v1mag = norm(v1);
v2mag = norm(v2);
v3mag = norm(v3);
%Convert to unit vectors
lam1 = v1/v1mag;
lam2 = v2/v2mag;
lam3 = v3/v3mag;

%%%%%%%%Method 1: invert a matrix%%%%%%%%
%Create A and b
A = [lam1, lam2, lam3];
b = m*a+m*g*k;
%Invert to solve for T1,2,3
T = A\b


%%%%%%%%Method 2: choose good cross product%%%%%%%%
% T1*lam1 + T2*lam2 + T3*lam3 = b
% dot both sides by a vector perpendicular to two of the terms e.g.
 lam2 and lam3
% those two terms go to 0, so -> dot(perp_vec1,lam1)*T1 =
 dot(perp_vec1,b)
% T = dot(perp_vec1,b)/dot(perp_vec1,lam1)

perp_vec1 = cross(lam2,lam3); %perpendicular to T2 and T3
perp_vec2 = cross(lam1,lam3); %perpendicular to T1 and T3
perp_vec3 = cross(lam1,lam2); %perpendicular to T1 and T2

T1 = dot(perp_vec1, b)/dot(perp_vec1, lam1);
T2 = dot(perp_vec2, b)/dot(perp_vec2, lam2);
T3 = dot(perp_vec3, b)/dot(perp_vec3, lam3);
```


Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

```
T = [T1 T2 T3]' %identical to Method 1 :)
```

```
T =

   14.2707
    5.8564
   14.5065


T =

   14.2707
    5.8564
   14.5065
```

Published with MATLAB® R2020b

12.1.22 Particle moves on a strange path. Given that a particle moves in the xy plane for 1.77 s obeying

$$\vec{r} = (5\,\text{m})\cos^2(t^2/\text{s}^2)\hat{\imath} + (5\,\text{m})\sin(t^2/\text{s}^2)\cos(t^2/\text{s}^2)\hat{\jmath}$$

where x and y are the horizontal distance in meters and t is measured in seconds.

a) Accurately plot the trajectory of the particle.

b) Mark on your plot where the particle is going fast and where it is going slow. Explain how you know these points are the fast and slow places.

18

11.1.23

$\vec{r} = 5\cos^2(t^2)\hat{i} + 5\sin(t^2)\cos(t^2)\hat{j}$

1. Plot trajectory

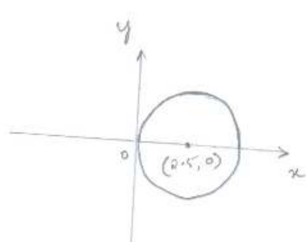
$x = 5\cos^2(t^2)$, $y = 5\sin(t^2)\cdot\cos(t^2)$

$x = 2.5(\cos 2t^2 + 1)$, $y = 2.5\sin(2t^2)$

$\frac{x-2.5}{2.5} = \cos 2t^2$, $\frac{y}{2.5} = \sin 2t^2$

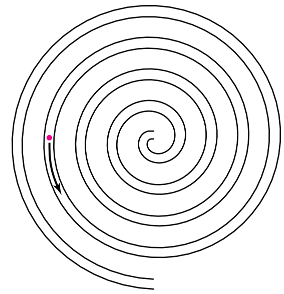
$$(x-2.5)^2 + y^2 = (2.5)^2$$

Trajectory is a circle with radius 2.5 and the centre $x = 2.5$, $y = 0$



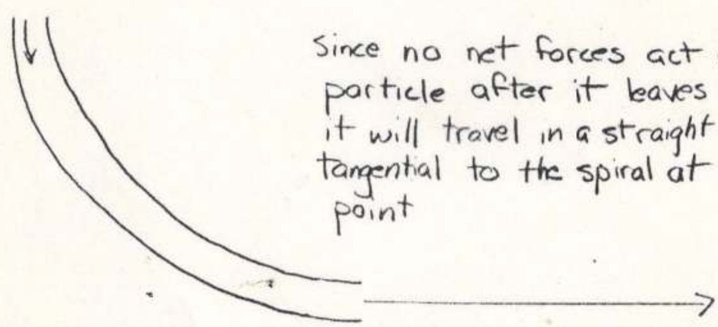
Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

12.1.24 A particle is blown out through the uniform spiral tube shown, which lies flat on a horizontal frictionless table. Draw the particle's path after it is expelled from the tube. Defend your answer.



Problem 12.24

6.1 Determine the path of a particle after it is blown out of a spiral tube which rests on a frictionless table.



Since no net forces act on the particle after it leaves the tube, it will travel in a straight line tangential to the spiral at its end point

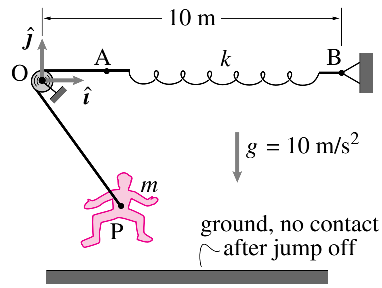
Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

12.1.25 Bungy Jumping. In a relatively safe bungy jumping system, people jump up from the ground while being pulled up by a rope that runs over a pulley at O and is connected to a stretched spring anchored at B. The ideal pulley has negligible size, mass, and friction. For the situation shown the spring AB has rest length $\ell_0 = 2\,\mathrm{m}$ and a stiffness of $k = 200\,\mathrm{N/m}$. The inextensible massless rope from A to P has length $\ell_r = 8\,\mathrm{m}$, the person has a mass of 100 kg. Take O to be the origin of an xy coordinate system aligned with the unit vectors $\hat{\imath}$ and $\hat{\jmath}$

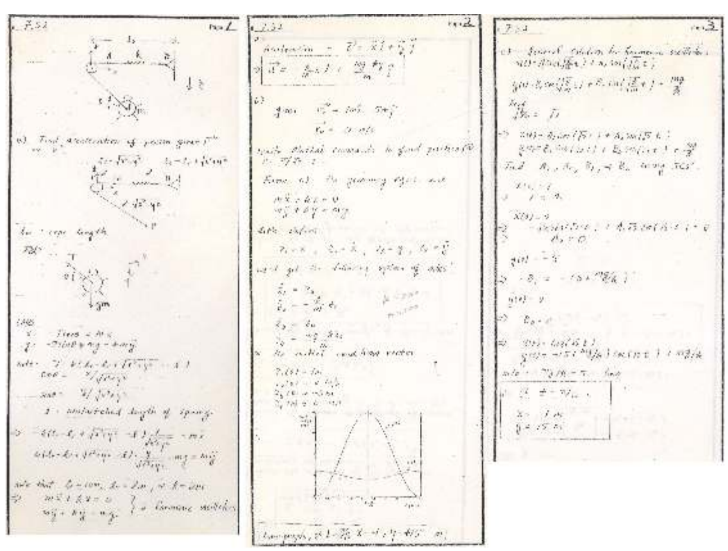
a) Assume you are given the position of the person $\vec{r} = x\hat{\imath} + y\hat{\jmath}$ and the velocity of the person $\boldsymbol{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$. Find her acceleration in terms of some or all of her position, her velocity, and the other parameters given. Then use the numbers given, where supplied, in your final answer.

b) Given that bungy jumper's initial position and velocity are $\vec{r}_0 = 1\,\mathrm{m}\hat{\imath} - 5\,\mathrm{m}\hat{\jmath}$ and $\boldsymbol{v}_0 = \mathbf{0}$ write computer commands to find her position at $t = \pi/\sqrt{2}\,\mathrm{s}$.

c) Find the answer to part (b) with pencil and paper (that is, find an analytic solution to the differential equations, a final numerical answer is desired).



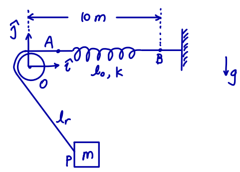
Problem 12.25: Conceptual setup for a bungy jumping system.



Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

HW5 P22 12.1.26

Teaya Yang



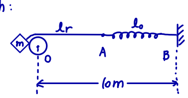
Assume: - pulley has negligible size, mass and friction
- rope is massless and inextensible

Given: $l_o = 2m$ $k = 200 N/m$
$l_r = 8m$ $m = 100\ kg$
$g = 10 m/s^2$

a) Given $\vec{r} = x\hat{\imath} + y\hat{\jmath}$ and $\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$, find $\vec{a}$.

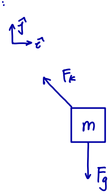
Note that $l_o + l_r = 10m$.

At rest length:



* This means the person is at origin when the spring is at rest length so the spring force is proportional to the position vector.

FBD:



$\vec{F}_g = -mg\,\hat{\jmath}$

$\vec{F}_k = kr\left(-\frac{\vec{r}}{r}\right) = -k\vec{r}$

LMB: $\Sigma\vec{F} = m\vec{a}$

$-mg\hat{\jmath} - k(x\hat{\imath} + y\hat{\jmath}) = m\vec{a}$

$m\vec{a} = -mg\hat{\jmath} - kx\hat{\imath} - ky\hat{\jmath}$

$\vec{a} = -\frac{k}{m}x\hat{\imath} - \left(\frac{k}{m}y + g\right)\hat{\jmath}$

$\vec{a} = -\frac{200}{100}\left(\frac{N/m}{kg}\right)x\hat{\imath} - \left[\frac{200\ N/m}{100\ kg}\,y + 10\ m/s^2\right]\hat{\jmath}$

$= -2\left(\frac{\cancel{N}}{\cancel{kg}m}\right)\left(\frac{\cancel{kg}m/s^2}{\cancel{N}}\right) - \left[2\left(\frac{\cancel{N}}{\cancel{kgm}}\right)\left(\frac{\cancel{kg\,m}/s^2}{\cancel{N}}\right) + 10\ m/s^2\right]\hat{\jmath}$

$$\boxed{\vec{a} = -(2\,s^{-2})\,x\hat{\imath} - \left[(2\,s^{-2})y + 10\,ms^{-2}\right]\hat{\jmath}}$$

b) Given ICs, find $\vec{r}\left(t = \frac{\pi}{\sqrt{2}}s\right)$ numerically.

ICs: $\vec{r}_o = 1m\,\hat{\imath} - 5m\,\hat{\jmath}$

$\vec{v}_o = \vec{0}$

$z = \begin{bmatrix} r_x \\ r_y \\ v_x \\ v_y \end{bmatrix}$ $\dot{z} = \begin{bmatrix} v_x \\ v_y \\ a_x \\ a_y \end{bmatrix}$ $z_o = \begin{bmatrix} 1 \\ -5 \\ 0 \\ 0 \end{bmatrix}$

$[r] = \begin{bmatrix} r_x \\ r_y \end{bmatrix} = z(1:2)$

$[v] = \begin{bmatrix} v_x \\ v_y \end{bmatrix} = z(3:4)$

In rhs:

$[F_{tot}] = -mg[\hat{\jmath}] - k[r]$ where $[\hat{\jmath}] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$[a] = [F_{tot}]/m$

$\dot{z} = \begin{bmatrix} [v] \\ [a] \end{bmatrix}$

This is integrated with ode45 until $t_{end} = \frac{\pi}{\sqrt{2}}$ s.

Then the last data point is printed.

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

b) cont.

solution : $\boxed{\vec{r}(t = \frac{\pi}{\sqrt{2}} s) = -1 m \hat{\imath} - 5 m \hat{\jmath}}$

what the trajectory looks like :

See next page for Matlab file, part c) comes after.

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

Contents


```
% MAE2030 HW5 P22 12.1.26
% Finding the position of bungy jumper using ODE45
```

Part b

Fining position at t = pi/sqrt(2)

```
% Parameters
p.m = 100; % [kg]
p.k = 200; % [N/m]
p.g = 10; % [kg/m^2]

% IC
r0 = [1;-5];
v0 = [0;0];
z0 = [r0;v0];

% Time
tend = pi/sqrt(2);
tspan = [0 tend];

% Equation
eq = @(t,z) rhs(t,z,p);

% Options
smallnumber = 1e-6;
options = odeset('AbsTol',smallnumber,'RelTol',smallnumber);

% Solve
[tarray, zarray] = ode45(eq,tspan,z0,options);

% Display Solution
fprintf('Position at t = pi/sqrt(2) is %.2f m i %.2f m j\n',zarray(end,1),zarray(end,2))


% rhs function
function zdot = rhs(t,z,p)
% Unit Vectors
i = [1;0]; j = [0;1];

% Unpack Variables
r = z(1:2);
v = z(3:4);
% Unpack Parameters
m = p.m; k = p.k; g = p.g;

% Equation
Ftot = -m*g*j-k*r;
vdot = Ftot/m;

% Return zdot
zdot = [v;vdot];
```

```
end
```

```
Position at t = pi/sqrt(2) is -1.00 m i -5.00 m j
```

Published with MATLAB® R2020b

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

c) Solve part b) analytically.

$\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$

$\{\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} = -(2s^{-2})x\hat{\imath} - [(2s^{-2})y + 10\,ms^{-2}]\hat{\jmath}\}$

$\{\ \}\cdot\hat{\imath}\ :\ \ddot{x} = -(2s^{-2})x$ ①

$\{\ \}\cdot\hat{\jmath}\ :\ \ddot{y} = -(2s^{-2})y - 10\,ms^{-2}$ ②

① $\ddot{x} + (2s^{-2})x = 0$

Sol : $x = A\cos(\sqrt{2}s^{-1}t) + B\sin(\sqrt{2}s^{-1}t)$

$\dot{x} = -\sqrt{2}s^{-1}A\sin(\sqrt{2}s^{-1}t) + \sqrt{2}s^{-1}B\cos(\sqrt{2}s^{-1}t)$

IC: $x(0) = 1\,m = A$

$V_x(0) = 0\,ms^{-1} = \sqrt{2}s^{-1}B \quad \Rightarrow \quad x(t) = \cos(\sqrt{2}s^{-1}t)\ m$

$B = 0\,m$

② $\ddot{y} + (2s^{-2})y = -10\,ms^{-2}$

Sol: $y_h = C\cos(\sqrt{2}s^{-1}t) + D\sin(\sqrt{2}s^{-1}t)$

$y_p = E$

$\ddot{y}_p + (2s^{-2})y_p = -10\,ms^{-2}$

$y_p = -5\,m$

IC: $y(0) = -5\,m = C - 5\,m \rightarrow C = 0\,m$

$V_y(0) = 0\,ms^{-1} = \sqrt{2}s^{-1}D \longrightarrow D = 0\,m$

$\hookrightarrow y(t) = -5\,m$

$y = y_h + y_p = C\cos(\sqrt{2}s^{-1}t) + D\sin(\sqrt{2}s^{-1}t) - 5\,m$

$\dot{y} = -\sqrt{2}s^{-1}C\sin(\sqrt{2}s^{-1}t) + \sqrt{2}s^{-1}D\cos(\sqrt{2}s^{-1}t)$

Finding $\vec{r}(t = \frac{\pi}{\sqrt{2}}s)$: $x(t = \frac{\pi}{\sqrt{2}}s) = \cos(\pi s^{-1}t)\,m = -1\,m$

$y(t = \frac{\pi}{\sqrt{2}}s) = -5\,m$

$$\boxed{\vec{r}(t = \tfrac{\pi}{\sqrt{2}}s) = -1\,m\,\hat{\imath} - 5\,m\,\hat{\jmath}}$$

d) General solution.

$$\boxed{\begin{aligned} x(t) &= A\cos(\sqrt{2}s^{-1}t) + B\sin(\sqrt{2}s^{-1}t) \\ y(t) &= C\cos(\sqrt{2}s^{-1}t) + D\sin(\sqrt{2}s^{-1}t) - 5\,m \end{aligned}}$$

x and y are sine waves with same frequency but different phase shift.

The general result is elliptical, but solution becomes a straight line when the phase shift is the same or if one of them is constant. (When both are constant, system is at rest at one point in space.)

See next page for examples.

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

d) cont.

Examples where solution is an ellipse :

i) IC : $z_0 = [1, -5, 3, -3]'$

Initial condition given in part b) with added initial velocity.

Since velocity components are not zero at equilibrium position , there are oscillations in both directions.

ii) $z_0 = [0, 0, 1, -1]'$

Starting at origin with initial velocity

Similar system with nonzero velocity components at equlibrium position.

Different initial conditions for this case change the shape and orientation of the ellipse due to the change in phase shift and amplitude of the sinusoidal waves.

d) cont.

Examples of straight-line solutions :

iii) $z_0 = [1, -6, 0, 0]'$

Starting at a non-equilibrium position, with zero initial velocity

With zero initial velocity, the sine terms in both $x(t)$ and $y(t)$ disappear.

$x(t) = A\cos(\sqrt{3}\,s^{-1}t)$

$y(t) = C\cos(\sqrt{3}\,s^{-1}t) - 5\,m$

Therefore $y = Fx + G$ → linear relationship !

iv) $z_0 = [1, -5, 0, 0]'$

Conditions given in part b)

This condition given in part b) is special because the y component of position is at equilibrium with no initial velocity, while oscillation exists in x.

v) z0 = [0, -5, 0, 1]'

Jumping straight up and down

This is the case where x component of position remains constant while y component oscillates, meaning if the person jumps up and down right under the pulley they wouldn't be swinging left or right (intuitive).

Solution at one point:

vi) z0 = [0, -5, 0, 0]'

Equilibrium position.

If starting at the equilibrium position with no initial velocity, the result is static.

Contents


```
% MAE2030 HW5 P22 12.1.26
% Finding the position of bungy jumper using ODE45
```

Part b

Fining position at t = pi/sqrt(2)

```
% Parameters
p.m = 100; % [kg]
p.k = 200; % [N/m]
p.g = 10; % [kg/m^2]

% IC
r0 = [1;-5];
v0 = [0;0];
z0 = [r0;v0];

% Time
tend = pi/sqrt(2);
tspan = [0 tend];

% Equation
eq = @(t,z) rhs(t,z,p);

% Options
smallnumber = 1e-6;
options = odeset('AbsTol',smallnumber,'RelTol',smallnumber);

% Solve
[tarray, zarray] = ode45(eq,tspan,z0,options);

% Display Solution
fprintf('Position at t = pi/sqrt(2) is %.2f m i %.2f m j\n',zarray(end,1),zarray(end,2))
```

Part d

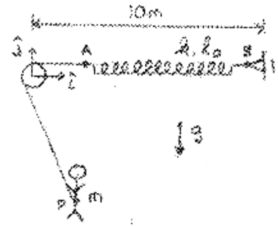
Finding general motion Parameters

```
p.m = 100; % [kg]
p.k = 200; % [N/m]
p.g = 10; % [kg/m^2]

% IC
% z0 = [1,-5,3,-3]';
% z0 = [1,-6,0,0]';
% z0 = [0,0,1,-1]';
% z0 = [1,-5,0,0]';
z0 = [0,-5,0,0]';
% z0 = [0,-5,0,1]';
```

```
% Time
tend = 10;
tspan = [0 tend];

% Equation
eq = @(t,z) rhs(t,z,p);

% Options
smallnumber = 1e-6;
options = odeset('AbsTol',smallnumber,'RelTol',smallnumber);

% Solve
[tarray, zarray] = ode45(eq,tspan,z0,options);

% Plot
x = zarray(:,1);
y = zarray(:,2);
figure(1)
hold on
box on
grid minor
axis equal
plot(x,y,'bo')
xlabel('x(m)')
ylabel('y(m)')
title('Bungy Jumper Trajectory')

% rhs function
function zdot = rhs(t,z,p)
% Unit Vectors
i = [1;0]; j = [0;1];

% Unpack Variables
r = z(1:2);
v = z(3:4);
% Unpack Parameters
m = p.m; k = p.k; g = p.g;

% Equation
Ftot = -m*g*j-k*r;
vdot = Ftot/m;

% Return zdot
zdot = [v;vdot];

end
```

Published with MATLAB® R2020b

10.26



Given: $k = 200$ N/m
$\ell_0 = 2$ m
$\ell_{AP} = 8\text{ m} = \ell_r$
$m = 100$ kg
Assume no air drag.

a) $\vec{r} = x\hat{i} + y\hat{j}$, $\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j}$, Find $\vec{a}$

PERSON: (1) PULLEY: (2)

$$\hat{\lambda} = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2+y^2}}$$

δ = extension of spring = $\ell - \ell_0$, where $\ell = 10 - (\ell_r - \sqrt{x^2+y^2})$

$\therefore \ell = 10 - 8 + \sqrt{x^2+y^2} = 2 + \sqrt{x^2+y^2}$

$\therefore \delta = \ell - \ell_0 = \sqrt{x^2+y^2}$

From FBD 2: $T = k\delta$ (tension on both sides of pulley must be equal) $\therefore$ $T = k\sqrt{x^2+y^2}$

From FBD 1: $\sum \vec{F} = m\vec{a} = -T\hat{\lambda} - mg\hat{j}$

$$\therefore -k\sqrt{x^2+y^2}\, \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2+y^2}} - mg\hat{j} = m\vec{a}$$

or $-kx\hat{i} - (ky + mg)\hat{j} = m\vec{a}$ – plug in values

$$\vec{a} = \frac{-200\,\text{N/m}\; x}{100\,\text{kg}}\hat{i} - \frac{200\,\text{N/m}\; y + 100\,\text{kg}\,(10\,\text{m/s}^2)}{100\,\text{kg}}\hat{j} = \boxed{2x\,[\text{s}^{-2}]\,\hat{i} - (2y + 10\,\text{m})\,[\text{s}^{-2}]\,\hat{j}}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

Problem 10.26 (b).

```
function Prob1026()
% Problem 10.26 Solution
% March 11, [illegible]

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%  VARIABLES (Assume consistent units)
%   r  = displacement (vector with x and y components)
%   v  = dr/dt

% INITIAL CONDITIONS
r0= [1 -5]'; % initial position
v0= [0 0]'; % initial velocity
z0= [r0;v0]; % pack variables

tspan =[0 pi/sqrt(2)]; % time interval of integration

[t zarray] = ode45(@rhs,tspan, z0);

% Unpack Variables
r= zarray(:,1:2);

disp(r(end,:));

% ANSWER:
% ans =
%
%     -1.0000    -5.0000    (meters)
%

end

% THE DIFFERENTIAL EQUATION 'The Right Hand Side'
function zdot = rhs(t,z)

%Unpack variables
r= z(1:2);
v= z(3:4);

%The equations
rdot= v;
vdot= [-2*r(1) -2*r(2)-10]';

% Pack the rate of change of r and v
zdot= [rdot; vdot];

end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

b) See Matlab code on previous page

c) From part (a),

$$\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} = -2x\hat{\imath} - (2y+10)\hat{\jmath}$$

$\therefore$ $(\ddot{x} = -2x)\hat{\imath}\cdot\hat{\imath} \Rightarrow \ddot{x} = -2x$ (1)

$(\ddot{y} = -2y-10)\hat{\jmath}\cdot\hat{\jmath} \Rightarrow \ddot{y} = -2y-10$ (2)

Solve (1) and (2) with $\vec{r}(0) = \hat{\imath} - 5\hat{\jmath}$, $\vec{v}(0) = \vec{0}$

(1) $\ddot{x} = -2x$, so $x(t) = A\sin(\sqrt{2}t) + B\cos(\sqrt{2}t)$

$\dot{x}(t) = \sqrt{2}A\cos(\sqrt{2}t) - \sqrt{2}B\sin(\sqrt{2}t)$

$\dot{x}(0) = 0 = \sqrt{2}A \quad \therefore A = 0$

$x(0) = 1 = B\cos(0) \quad \therefore B = 1$

$\therefore \quad x(t) = \cos(\sqrt{2}t)$

(2) $\ddot{y} = -2y-10$, so $y(t) = C\sin(\sqrt{2}t) + D\cos(\sqrt{2}t) - 5$ y_p

$\dot{y}(0) = 0 = \sqrt{2}C \quad \therefore C = 0$

$y(0) = -5 = D\cos(0) - 5 \quad \therefore D = 0$

$\therefore \quad y(t) = -5$

$\vec{r}(t) = \cos\sqrt{2}t\,\hat{\imath} - 5\hat{\jmath} = x(t)\hat{\imath} + y(t)\hat{\jmath}$

$\therefore \quad \vec{r}\left(\frac{\pi}{\sqrt{2}}\right) = \cos\left(\sqrt{2}\frac{\pi}{\sqrt{2}}\right)\hat{\imath} - 5\hat{\jmath} = \boxed{-\hat{\imath} - 5\hat{\jmath}\ [\mathrm{m}]}$

12.1.26 A softball pitcher releases a ball of mass m upwards from her hand with speed v_0 and angle θ_0 from the horizontal. The only external force acting on the ball after its release is gravity.

a) What is the equation of motion for the ball after its release?

b) What are the position, velocity, and acceleration of the ball?

c) What is its maximum height?

d) At what distance does the ball return to the elevation of release?

e) What kind of path does the ball follow and what is its equation y as a function of x?

19

11.1.27

a) Equation of motion

$$m\ddot{a} = -mg\hat{j}$$

$$\ddot{y} = -g\hat{j}$$

$$\ddot{x} = 0$$

$$\vec{V}_y = \dot{y} = (V_0 \sin\theta_0 - gt)\hat{j}$$

$$\vec{V}_x = V_0\cos\theta_0\,\hat{i}$$

$$\vec{V} = V_0\cos\theta_0\,\hat{i} + (V_0\sin\theta_0 - gt)\hat{j}$$

b) acceleration

$$\vec{a} = -g\hat{j}$$

Velocity

$\cancel{\vec{V} = V_0 t \cos}$

$$\vec{V} = V_0\cos\theta_0\,\hat{i} + (V_0\sin\theta_0 - gt)\hat{j}$$

Position

$$\vec{r} = V_0 t\cos\theta_0\,\hat{i} + (V_0 t\sin\theta_0 - \tfrac{1}{2}gt^2)\hat{j}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

ec

c) maximum height-

$$\frac{d}{dt}(v_y) = 0 \Rightarrow v_0 \sin\theta_0 - gt = 0 \Rightarrow t = \frac{v_0 \sin\theta_0}{g}$$

$$h_{max} = \left(\frac{v_0 \sin\theta_0}{g}\right) v_0 \sin\theta_0 - \frac{1}{2} g \left(\frac{v_0 \sin\theta_0}{g}\right)^2$$

$$h_{max} = \frac{v_0^2 \sin^2\theta_0}{2g}$$

d) $y = 0 \Rightarrow v_0 t \sin\theta_0 - \frac{1}{2} g t^2 = 0$

$$t = \frac{2v_0 \sin\theta_0}{g} = t_0$$

$$r_x \Big|_{t=t_0} = v_0 \cos\theta_0 \left(\frac{2v_0 \sin\theta_0}{g}\right) = \frac{v_0^2 \cos 2\theta_0}{g}$$

e) $y = v_0 t \sin\theta_0 - \frac{1}{2} g t^2$

$x = v_0 t \cos\theta_0$

eleminating 't' we get-

$$y = v_0 \sin\theta_0 \left(\frac{x}{v_0 \cos\theta_0}\right) - \frac{1}{2} g \left(\frac{x}{v_0 \cos\theta_0}\right)^2$$

21

$$y = \frac{x}{V_0 \cos\theta_0}\left[V_0 \sin\theta_0 - \frac{gx}{2V_0\cos\theta_0}\right]$$

$$-y = \frac{gx^2}{2V_0^2\cos^2\theta_0} - x\tan\theta_0$$

$$-\frac{2V_0^2\cos^2\theta_0}{g}\,y = x^2 - 2x\left(\frac{V_0^2\cos^2\theta_0\tan\theta_0}{g}\right)$$

$$\left(x - \frac{V_0^2\sin 2\theta}{2g}\right)^2 = -\frac{2V_0^2\cos^2\theta}{g}\,y + \left(\frac{V_0^2\sin^2\theta}{2g}\right)^2$$

$$\left(x - \frac{V_0^2\sin 2\theta}{2g}\right)^2 = -\frac{2V_0^2\cos^2\theta}{g}\left[y - \frac{V_0^2\sin^2\theta}{2g}\right]$$

This equation represents a parabola

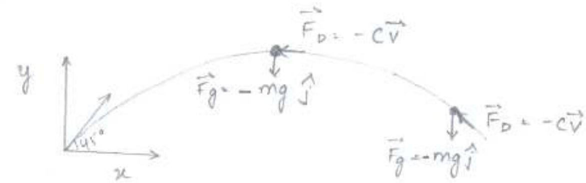
12.1.27 Find the trajectory of a not-vertically-fired cannon ball assuming the air drag is proportional to the speed. Assume the mass is 10 kg, $g = 10\,\mathrm{m/s}$, the drag proportionality constant is $C = 5\,\mathrm{N/(m/s)}$. The cannon ball is launched at 100 m/s at a 45 degree angle.

- Draw a free-body diagram of the mass.
- Write linear momentum balance in vector form.
- Solve the equations on the computer and plot the trajectory.
- Solve the equations by hand and then use the computer to plot your solution.
- Compare the two plots and comment on the differences, if any.

22

11.1.28

a)



b) Linear momentum balance equation

$$m a_x = -c V_x \qquad m\ddot{x} = -c\dot{x}$$

$$m a_y = -mg - cV_y \qquad m\ddot{y} = -mg - c\dot{y}$$

Where $\vec{V} = v_x\hat{i} + v_y\hat{j}$

c) We can put in the state space form as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{c}{m} & 0 \\ 0 & 0 & 0 & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -g \end{bmatrix}$$

d) Matlab

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

23

e) $m\ddot{x} = -c\dot{x} \Rightarrow \dfrac{m\,dv_x}{dt} = -cv_x$

$$\int \frac{dv_x}{v_x} = -\frac{c}{m}\int dt$$

$$\Rightarrow \log v_x = -\frac{ct}{m} + c_1$$

$$v_x = Ae^{-ct/m}$$

at $t=0$, $v_x\big|_{t=0} = u\cos\theta \Rightarrow v_x = u\cos\theta\, e^{-\frac{ct}{m}} \longrightarrow (1)$

Similarly

$$\int \frac{dv_y}{v_y + \frac{mg}{c}} = -\frac{c}{m}\int dt$$

$$\log\left(v_y + \frac{mg}{c}\right) = -\frac{ct}{m} + c_1$$

$$v_y = Ae^{-\frac{ct}{m}} - \frac{mg}{c}$$

$$A - \frac{mg}{c} = u\sin\theta$$

$$A = u\sin\theta + \frac{mg}{c}$$

$$v_y = u\sin\theta\, e^{-ct/m} + \frac{mg}{c}\left(e^{-ct/m} - 1\right)$$

24

Further Integration

$$\frac{dx}{dt} = u\cos\theta\, e^{-ct/m}$$

$$x = u\cos\theta\left(-\frac{m}{c}\right)e^{-ct/m} + B$$

$$x\big|_{t=0} = 0 \Rightarrow B = u\cos\theta\left(\frac{m}{c}\right)$$

$$x = u\cos\theta\left(\frac{m}{c}\right)\left[1 - e^{-ct/m}\right]$$

Similarly

$$\dot{y} = (u\sin\theta + mg/c)\,e^{-ct/m} - \frac{mg}{c}$$

$$y = (u\sin\theta + mg/c)\left(-\frac{m}{c}\right)e^{-ct/m} - \frac{mgt}{c} + B$$

$$y\big|_{t=0} = 0 \Rightarrow B = \left(u\sin\theta + \frac{mg}{c}\right)\left(\frac{m}{c}\right)$$

$$y = \frac{m}{c}\left(u\sin\theta + \frac{mg}{c}\right)\left(1 - e^{-ct/m}\right) - \frac{mgt}{c}$$

12.1.28 A baseball pitching machine releases a baseball of mass m from its barrel with speed v_0 and angle θ_0 from the horizontal. The only external forces acting on the ball after its release are gravity and air resistance. The speed of the ball is given by $v^2 = \dot{x}^2 + \dot{y}^2$. Taking into account air resistance on the ball proportional to its speed squared, $\vec{F}_d = -bv^2\hat{e}_t$, find the equation of motion for the ball, after its release, in cartesian coordinates.

Answer:
Equation of motion: $-mg\hat{\jmath} - b(\dot{x}^2 + \dot{y}^2)\left(\frac{\dot{x}\hat{\imath}+\dot{y}\hat{\jmath}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath})$.

25

11.1.29

Equation of motion

$$m\ddot{x} = \frac{-b(\dot{x}^2+\dot{y}^2)\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} = -b\dot{x}\sqrt{\dot{x}^2+\dot{y}^2}$$

$$m\ddot{y} = -mg - \frac{b(\dot{x}^2+\dot{y}^2)\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} = -mg - b\dot{y}\sqrt{\dot{x}^2+\dot{y}^2}$$

$$\ddot{x} = -\frac{b\dot{x}}{m}\sqrt{\dot{x}^2+\dot{y}^2}$$

$$\ddot{y} = -g - \frac{b\dot{y}}{m}\sqrt{\dot{x}^2+\dot{y}^2}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

12.1.29 The equations of motion from problem 12.1.28 are nonlinear and cannot be solved in closed form for the position of the baseball. Instead, solve the equations numerically. Make a computer simulation of the flight of the baseball, as follows.

a) Convert the equation of motion into a system of first order differential equations.

b) Pick values for the gravitational constant g, the coefficient of resistance b, and initial speed v_0, solve for the x and y coordinates of the ball and make a plot of its trajectory for various initial angles θ_0.

c) Use Euler's, Runge-Kutta, or other suitable method to numerically integrate the system of equations.

d) Use your simulation to find the initial angle that maximizes the distance of travel for the ball, with and without air resistance.

e) If the air resistance is very high, what is a qualitative description for the curve described by the path of the ball? Show this with an accurate plot of the trajectory. (Make sure to integrate long enough for the ball to get back to the ground.)

Answer:
(a) System of equations:

$$\begin{aligned}\dot{x} &= v_x\\ \dot{y} &= v_y\\ \dot{v}_x &= -\frac{b}{m}v_x\sqrt{v_x^2+v_y^2}\\ \dot{v}_y &= -g-\frac{b}{m}v_y\sqrt{v_x^2+v_y^2}\end{aligned}$$

26

11.1.30

a) System of first order differential Equation

$$\dot{x} = u$$
$$\dot{y} = v$$
$$\dot{u} = -\frac{b}{m}u\sqrt{u^2+v^2}$$
$$\dot{v} = -g-\frac{b}{m}v\sqrt{u^2+v^2}$$

b) using $g = 10\,m/sec^2$, $m = 10\,kg$

$b = 5N/(m/sec)^2$, $V_0 = 100\,m/sec$

c) Matlab

d) From Matlab we get initial angle that maximizes range is

Theta = $15.2432°$

without Air resistance $(b=0)$

Theta = $45°$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

12.1.30 A particle of mass m moves in a viscous fluid which resists motion with a force of magnitude $F = c|\vec{\boldsymbol{v}}|$, where $\vec{\boldsymbol{v}}$ is the velocity. Do not neglect gravity.

a) (easy) In terms of some or all of g, m, and c, what is the particle's terminal (steady-state) falling speed?

b) Starting with a free-body diagram and linear momentum balance, find two second order scalar differential equations that describe the two-dimensional motion of the particle.

c) (Challenge, long calculation) Assume the particle is thrown from $\vec{\boldsymbol{r}} = \vec{\boldsymbol{0}}$ with $\vec{\boldsymbol{v}} = v_{x0}\hat{\boldsymbol{\imath}} + v_{y0}\hat{\boldsymbol{\jmath}}$ at a vertical wall a distance d away. Find the height h along the wall where the particle hits. (Answer in terms of some or all of v_{x0}, v_{y0}, m, g, c, and d.) [Hint: i) find $x(t)$ and $y(t)$, ii) eliminate t, iii) substitute $x = d$. The answer is not tidy. In the limit $d \to 0$ the answer reduces to a sensible dependence on d (The limit $c \to 0$ is also sensible.).]

d) (Challenge, computer simulation). Do a computer simulation of the problem and find the solution in your simulation. Choose non-trivial numbers for all constants. To get an accurate solution you need an accurate interpolation to find at what time the particle hits the wall.

HW6-P24-12.1.31

Tuesday, March 16, 2021 2:22 PM

Solution by Michael Zakoworotny

12.1.31 A particle of mass m moves in a viscous fluid which resists motion with a force of magnitude $F = c|\vec{v}|$, where $\vec{v}$ is the velocity. Do not neglect gravity.

a) (easy) In terms of some or all of g, m, and c, what is the particle's terminal (steady-state) falling speed?

b) Starting with a free-body diagram and linear momentum balance, find two second order scalar differential equations that describe the two-dimensional motion of the particle.

c) (Challenge, long calculation) Assume the particle is thrown from $\vec{r} = \vec{0}$ with $\vec{v} = v_{x0}\hat{\imath} + v_{y0}\hat{\jmath}$ at a vertical wall a distance d away. Find the height h along the wall where the particle hits. (Answer in terms of some or all of v_{x0}, v_{y0}, m, g, c, and d.) [Hint: i) find $x(t)$ and $y(t)$, ii) eliminate t, iii) substitute $x = d$. The answer is not tidy. In the limit $d \to 0$ the answer reduces to a sensible dependence on d (The limit $c \to 0$ is also sensible.).]

d) (Challenge, computer simulation). Do a computer simulation of the problem and find the solution in your simulation. Choose non-trivial numbers for all constants. To get an accurate solution you need an accurate interpolation to find at what time the particle hits the wall.

24| 12.1.31 This problem should expand your understanding of parabolic-flight ballistics to the more realistic ballistics of things where air drag is important. Add part (e): Assume, in some consistent units, that $m = 1$, $g = 10$ and $c = 1$ in some consistent unit system. Assume the initial launch speed is 1000. Forget the wall. Find the best angle to achieve the maximum range. What is that angle and range?

Given: m, c, g, linear drag

Find: a) Terminal speed in freefall

b) Two 2nd order scalar ODE's for 2D motion

c) Given $\vec{r}_0 = \vec{0}$, $\vec{v}_0 = v_{x0}\hat{\imath} + v_{y0}\hat{\jmath}$, and a wall distance d away, find height, h, of collision with wall

d) Computer simulation, including time and height of wall collision

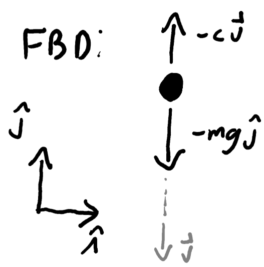
e) Given $m=1$, $g=10$, $c=1$, $|\vec{v}_0| = 1000$, find angle to optimize range

Solution:

a) Find terminal velocity in freefall

Freefall - $\vec{v}\cdot\hat{\imath}=0$

FBD:



In steady state, mass does not accelerate

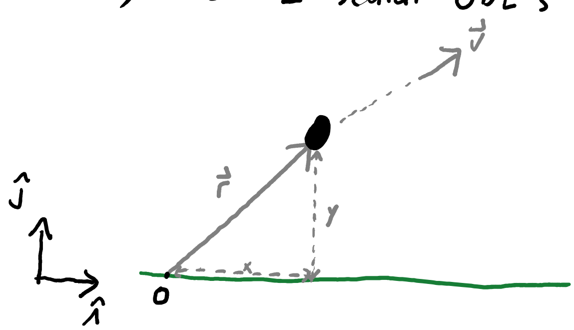
$$\sum \vec{F} = m\vec{a}$$

$$\{-mg\hat{\jmath} - c\vec{v} = m\cancel{\vec{a}}^{\,0}\}$$

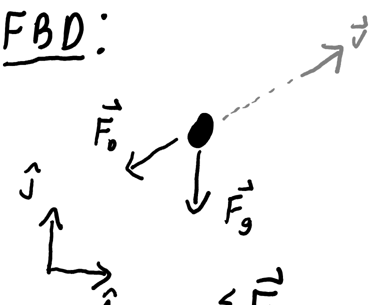
$$\{\}\cdot\hat{\jmath}:\ -mg - cv_t = 0$$

$$\boxed{v_t = \frac{-mg}{c}}$$ ← Terminal speed

b) Get 2 scalar ODE's for particle motion



FBD:



$$\vec{F}_D = \underbrace{c|\vec{v}|}_{\text{scalar magnitude}} \cdot \underbrace{-\frac{\vec{v}}{|\vec{v}|}}_{\text{direction - unit vector}} = -c\vec{v}$$

$$\vec{F}_g = -mg\hat{\jmath}$$

$$\sum \vec{F} = m\vec{a}$$

$$\vec{F}_g + \vec{F}_D = m\vec{a}$$

$$\vec{r} = x\hat{\imath} + y\hat{\jmath}$$

$$\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$$

$$\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$$

$$\{-mg\hat{j} - c(\dot{x}\hat{\imath} + \dot{y}\hat{j}) = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{j})\}$$

$\{\}\cdot\hat{\imath}:\quad -c\dot{x} = m\ddot{x}$

$\{\}\cdot\hat{j}:\quad -mg - c\dot{y} = m\ddot{y}$

$$\boxed{\begin{aligned} m\ddot{x} + c\dot{x} &= 0 \\ m\ddot{y} + c\dot{y} &= -mg \end{aligned}}$$

c) Get height, h, when $x = d$

Solve for $x(t)$ and $y(t)$

$\vec{r}_0 = \vec{0} \Rightarrow x(0) = 0,\ y(0) = 0$

$\vec{v}_0 = v_{x0}\hat{\imath} + v_{y0}\hat{j} \Rightarrow v_x(0) = v_{x0},\ v_y(0) = v_{y0}$

$\underline{x}:\quad m\ddot{x} + c\dot{x} = 0 \Rightarrow m\dot{v}_x + cv_x = 0$

$\frac{dv_x}{dt} + \frac{c}{m}v_x = 0$ ← solve as a 1st order separable ODE

$v_x(t) = Ae^{-c/m \cdot t}$

Solve for A using $v_x(0) = v_{x0}$

$v_{x0} = A\cancel{e^0}^{1} \Rightarrow A = v_{x0}$

$$\boxed{v_x(t) = v_{x0}e^{-c/m \cdot t}}$$

$v_x(t) = \frac{dx}{dt} \Rightarrow x = \int v_x(t)\,dt$

$x(t) = v_{x0}\cdot -\frac{m}{c}e^{-c/m \cdot t} + B$

Solve for B using $x(0) = 0$

$0 = -v_{x0}\frac{m}{c}\cdot\cancel{e^0}^{1} + B \Rightarrow B = v_{x0}\cdot\frac{m}{c}$

$$\boxed{x(t) = \frac{v_{x0} m}{c}\left[1 - e^{-c/m \cdot t}\right]}$$

$\underline{y}$: $m\ddot{y} + c\dot{y} = -mg \Rightarrow m\dot{v}_y + cv_y = -mg$

Homogeneous: $\frac{dv_y}{dt} + \frac{c}{m} v_y = 0$ Particular: $v_y = B$

$v_y(t) = Ae^{-c/m\, t}$ $0 + cB = -mg$

$B = -mg/c$

$$v_y(t) = Ae^{-c/m\, t} - mg/c$$

Solve for A using $v_y(0) = v_{y0}$

$v_{y0} = A \cdot \cancel{e}^{1} - mg/c$

$A = v_{y0} + mg/c$

$$\boxed{v_y(t) = (v_{y0} + mg/c)\, e^{-c/m \cdot t} - mg/c}$$

$v_y(t) = \frac{dy}{dt} \Rightarrow y = \int v_y(t)\, dt$

$y(t) = (v_{y0} + mg/c)(-m/c)\, e^{-c/m \cdot t} - mg/c\, t + B$

Solve for B using $y(0) = 0$

$0 = -\frac{m}{c}(v_{y0} + mg/c)\, \cancel{e}^{1} - 0 + B$

$B = \frac{m}{c}(v_{y0} + mg/c)$

$$\boxed{y(t) = \frac{m}{c}(v_{y0} + mg/c)\left[1 - e^{-c/m \cdot t}\right] - mg/c \cdot t}$$

* To solve for y position as function of x, combine x(t) and y(t)

$$t = \frac{-m}{c} \ln\left[1 - \frac{x(t)\,c}{m\,v_{x0}}\right]$$

$$y(x(t)) = \frac{m}{c}\left(v_{y0} + \frac{mg}{c}\right)\left[1 - \left(1 - \frac{x(t)\,c}{m v_{x0}}\right)\right] - \frac{mg}{c}\cdot\left(\frac{-m}{c}\right)\ln\left(1 - \frac{x(t)\,c}{m v_{x0}}\right)$$

$$\boxed{y(x(t)) = \left(v_{y0} + \frac{mg}{c}\right)\left(\frac{x(t)}{v_{x0}}\right) + \frac{m^2 g}{c^2}\ln\left(1 - \frac{x(t)\,c}{m v_{x0}}\right)}$$

Let $x(t) = d$

$$\boxed{y(x=d) = \left(v_{y0} + \frac{mg}{c}\right)\left(\frac{d}{v_{x0}}\right) + \frac{m^2 g}{c^2}\ln\left(1 - \frac{dc}{m v_{x0}}\right)}$$

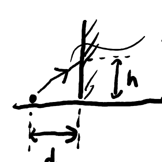
Check: $$\lim_{d \to 0} y(x=d) = \frac{v_{y0} d}{v_{x0}} + \frac{mgd}{c v_{x0}} + \frac{m^2 g}{c^2}\cdot\frac{-dc}{m v_{x0}}$$

$$= \frac{v_{y0} d}{v_{x0}} + \cancel{\frac{mgd}{c v_{x0}}} - \cancel{\frac{mgd}{c v_{x0}}}$$

$$= \boxed{\frac{v_{y0} d}{v_{x0}}}$$

From: $\ln(1-x) \approx -x$ for small x

This represents case where particle is launched very close to wall. Trajectory is approximately a straight line in this limit



Slope: v_{y0}/v_{x0} - tangent to path

$$\frac{v_{y0}}{v_{x0}} = \frac{h}{d} \Rightarrow h = \frac{v_{y0} d}{v_{x0}}$$

$$\lim_{c\to 0} y(x=d) = \frac{v_{y0} d}{v_{x0}} + \lim_{c\to 0}\left[\frac{mgd}{v_{x0}}\cdot\frac{1}{c} + \frac{m^2 g}{c^2}\ln\left(1-\frac{dc}{mv_{x0}}\right)\right]$$

$$\hookrightarrow \frac{1}{v_{x0}}\cdot\frac{mgdc + m^2 g v_{x0}\ln\left(1-\frac{dc}{mv_{x0}}\right)}{c^2} \to \frac{0}{0}$$

L'Hopital's: $\frac{1}{v_{x0}}\cdot\frac{mgd + m^{\cancel{2}} g \cancel{v_{x0}}\cdot\frac{1}{1-\frac{dc}{mv_{x0}}}\cdot -\frac{d}{\cancel{m}\cancel{v_{x0}}}}{2c}$

$$= \frac{mgd}{v_{x0}}\cdot\frac{1-\frac{mv_{x0}}{mv_{x0}-dc}}{2c}$$

$$= \frac{mgd}{v_{x0}}\cdot\frac{\cancel{mv_{x0}} - d\cancel{c} - \cancel{mv_{x0}}}{2\cancel{c}(mv_{x0}-dc)}$$

$$\Rightarrow \lim_{c\to 0}\frac{mgd}{v_{x0}}\cdot\frac{-d}{2(mv_{x0}-dc)}$$

$$= \frac{-\cancel{m}gd^2}{2\cancel{m}v_{x0}^2} = \frac{-gd^2}{2v_{x0}^2}$$

$$= \boxed{\frac{v_{y0} d}{v_{x0}} - \frac{gd^2}{2v_{x0}^2}}$$

This represents the case with no drag

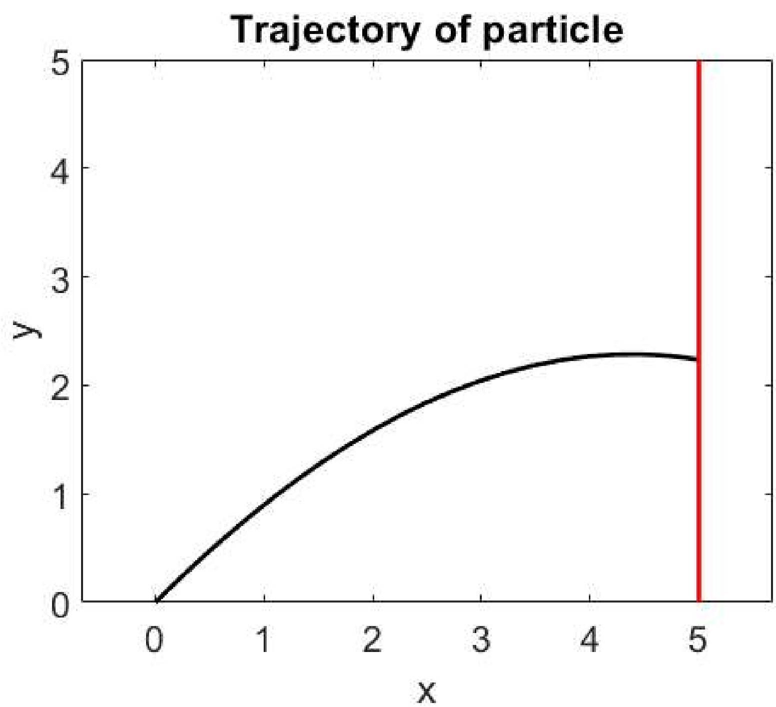
(i.e. a standard ballistic problem, from Kinematics)

d) MATLAB code plotting trajectory

```
% % MAE 2030
% HW 6, Q24 Solution
% Solution by Michael Zakoworotny

clear all; clc; close all;
set(0,'DefaultLineLineWidth',2); % set all plots lines to size of 2
set(0,'DefaultAxesFontSize',16); % set plot font size to 16

% Define parameters - mass, linear drag, gravity and position of wall
m = 5; c = 1; g = 10; d = 5;
p.m = m; p.c = c; p.g = g; p.d = d;

% Time span
t0 = 0; tend = 10;
tspan = [t0 tend];

% Initial conditions
r0 = [0 0];
% v0 = [10 10]; % parameterize by x and y velocity components
v0 = 10*[cosd(45) sind(45)]; % paramaterize by speed and angle
z0 = [r0 v0]';

% Solution step
fun = @(t,z) myRHS(z,t,p);
tol = 1e-8;
wall_event = @(t,z) wallEvent(t,z,p);
opts = odeset('RelTol',tol,'AbsTol',tol,'Events',wall_event);
[t_array, z_array] = ode45(fun, tspan, z0, opts);

% Unpack solution
x_array = z_array(:,1); y_array = z_array(:,2);
y_wall = y_array(end);
fprintf('The y position when the particle collides with the wall is %.4f\n',y_wall);

% Plot solution - x vs y
fig = figure; axis equal; box on; hold on;
plot(x_array, y_array, '-k');
plot([d d],[min(xlim),max(xlim)],'-r');
xlabel('x'); ylabel('y');
title('Trajectory of particle');


function zdot = myRHS(z,t,p)
    % Unpack parameters
    m = p.m; c = p.c; g = p.g;
    % Get position and velocity (column vectors) from current state
    r = z(1:2);
    v = z(3:4);

    % Solve for rdot and vdot
    rdot = v;
    vdot = 1/m*(-c*v - m*g*[0;1]);
    % Store zdot vector from rdot and vdot
    zdot = [rdot; vdot];
```

```
end

function [value,isterminal,direction] = wallEvent(t,z,p)
    % Returns parameters to be used in computing an event in ode45.
    x = z(1);
    d = p.d;
    value = x - d; % event occurs when this is 0
    isterminal = 1; % stop integration at event
    direction = 1; % only meaningful solution is when z increases to d

end
```

```
The y position when the particle collides with the wall is 2.2361
```



Published with MATLAB® R2019b

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

```
% % MAE 2030
% HW 6, Q24 Solution
% Solution by Michael Zakoworotny
%
% Solve for height of wall collision symbolically

clear all; close all; clc;

% Define symbolic variables for position (as function of time), and
% parameters
syms x(t) y(t) m c g d
% Symbolic variables for initial conditions
syms x0 y0 vx0 vy0 real

% Unit vectors
i_ = [1;0]; j_ = [0;1];

% Position, and its derivatives
r = [x; y];
v = diff(r);
a = diff(v);

% Define governing ODE and initial conditions for position and velocity
odes = m*a == -c*v - m*g*j_;
ic_pos = r(0) == [0; 0];
ic_vel = v(0) == [vx0; vy0];

% Solve system for x and y as functions of time
sol = dsolve(odes,ic_pos,ic_vel); % solve a symbolic ODE. sol is a struct
                                  % which contains the x and y scalar
                                  % solutions
x = simplify(sol.x); % extract the x solution
y = simplify(sol.y); % extract the y solution

% Solve for the time, and y position, at which x = d
t_wall = solve(x == d, t); % solve for the t for which x = d
y_wall = simplify(subs(y,t,t_wall)); % substitute t_wall into the equation for y

% Printout of solutions
fprintf('x(t) = \n');
pretty(x);
fprintf('\ny(t) = \n');
pretty(y);
% Printout of time and y position of wall collision
fprintf('\nt at wall collision = \n');
pretty(t_wall);
fprintf('\ny at wall collision = \n');
pretty(y_wall);
% Limit as d->0. MATLAB shows computes this as zero because it does not
% take into account small angle approximation
fprintf('\nThe limit of y as d approaches zero is \n');
pretty(simplify(limit(y_wall,d,0)));
% Limit as c->0. Consistent with our expected kinematic solution
fprintf('\nThe limit of y as c approaches zero is \n');
pretty(simplify(limit(y_wall,c,0)));
```

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

```
x(t) =
                 /   c t \
m vx0 - m vx0 exp| - --- |
                 \    m  /
--------------------------
             c


y(t) =
                    /          /   c t \               \
                    |       exp| - --- | (g m + c vy0) |
                    |          \    m  /               |
                  m | g t + -------------------------- |
m (g m + c vy0)     \                 c                /
--------------- - --------------------------------------
        2                            c
       c


t at wall collision =
       /   c d - m vx0 \
  m log| - ----------- |
       \      m vx0    /
- ----------------------
             c


y at wall collision =
 2            2        /   c d - m vx0 \
c  d vy0 + g m  vx0 log| - ----------- | + c d g m
                       \      m vx0    /
--------------------------------------------------
                        2
                       c  vx0


The limit of y as d approaches zero is
0


The limit of y as c approaches zero is
  d (d g - 2 vx0 vy0)
- -------------------
              2
         2 vx0
```

Published with MATLAB® R2019b

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

e) Get angle of maximum range

$m=1,\ g=10,\ c=1,\ |\vec{v}_0|=1000$

Approximation method:

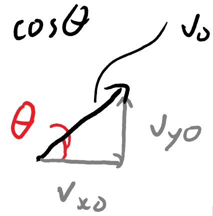
Since initial velocity is so large, drag force is much larger than gravity. So, in limit as $|\vec{v}|\to\infty$, we can ignore gravity.

Consider time to reach ground:

$$y(t) \simeq \frac{v_{y0}\, m}{c}\left[1-e^{-c/m\, t}\right] = 0$$

Set $t\to\infty$ b/c $t>0$ → Under this assumption, particle rapidly flies up, stops due to large drag force, and then drops in freefall when gravity is dominant again

$v_{x0} = |\vec{v}_0|\cos\theta$



$$x(t) = \frac{v_{x0}\, m}{c}[1-0]$$

$$x(t\to\infty) = |\vec{v}_0|\cos\theta \cdot \frac{m}{c}$$ ← maximized when $\boxed{\theta=0}$

Exact method

* For more precise answer solve for t for given θ

---

* For more precise answer solve for t for given θ

$$y(t) = \frac{m}{c}\left(v_{y0} + \frac{mg}{c}\right)\left[1 - e^{-c/m \cdot t}\right] - \frac{mg}{c} \cdot t = 0$$

$\hookrightarrow v_{y0} = |\vec{v}_0| \sin\theta$

Then solve for θ that maximizes x

$v_{x0} = |\vec{v}_0| \cos\theta$ $\qquad$ $t(\theta)$, from eqn. above

$$x(t) = \frac{v_{x0} m}{c}\left[1 - e^{-c/m \cdot t}\right]$$

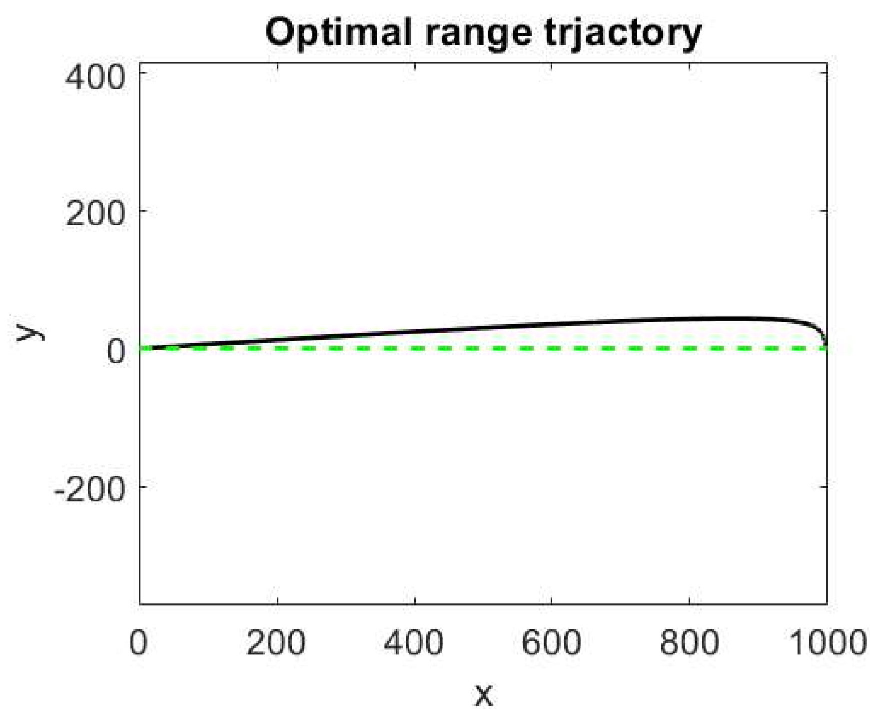
This can be done in MATLAB using the fzero and fminbnd functions.

$$\boxed{\begin{aligned} \theta_{max} &= \operatorname{argmax}(x(\theta)) = 3.649^\circ \\ x_{max} &= 997.34 \end{aligned}}$$

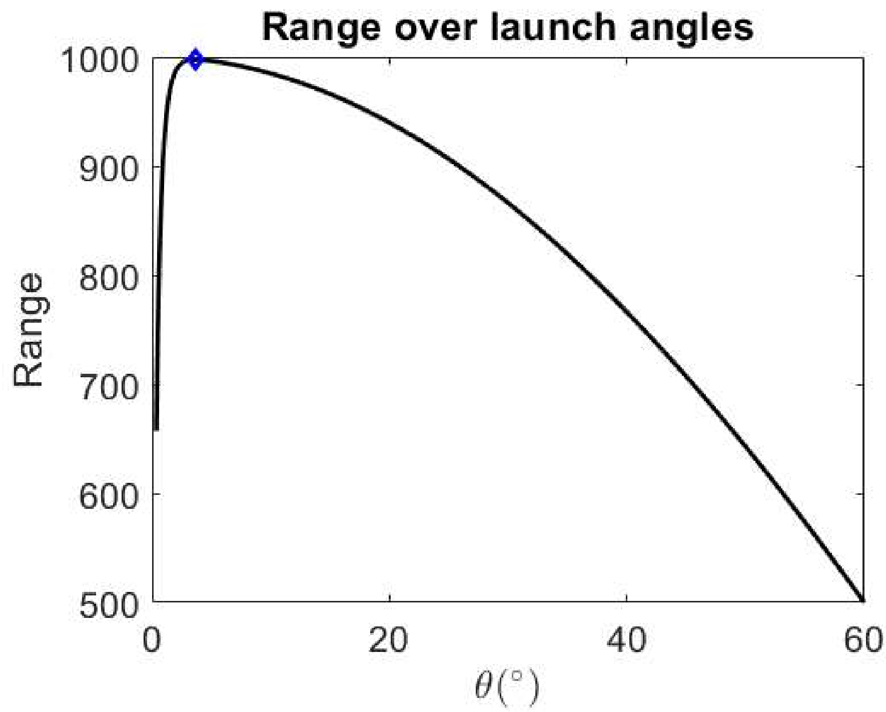
```
% % MAE 2030
% HW 6, Q24 part e Solution
% Solution by Michael Zakoworotny

clear all; close all; clc;
set(0,'DefaultLineLineWidth',2); % set all plots lines to size of 2
set(0,'DefaultAxesFontSize',16); % set plot font size to 16

% Given parameters
m = 1; g = 10; c = 1;
v0 = 1000;

% Anonymous function handle for y(t, th), since y is dependent on launch
% angle as well
y = @(t, th) m/c*(v0*sin(th) + m*g/c) .* (1-exp(-c/m*t)) - m*g/c*t;
% Anonymous function handle for x(t, th)
x = @(t, th) v0*cos(th)*m/c .* (1-exp(-c/m*t));

% Anonymous function which returns the time for the particle to reach y=0,
% given a launch angle, th. fzero finds the roots of a single-variable
% function, which in this case is the function y(t,th), for a given value
% of th. Start search near t = 10 s
t_drag = m/c*log(v0*c/m/g); % appr. time such that drag force = gravity
t_fall = @(th)sqrt(2*v0*t_drag*sin(th)/g); % approximate freefall time from
                           % top of trajectory, as function of launch angle
% Begin search around t = 1.5*(t_drag+t_fall), where 1.5 is a safety factor
% on the apprxomate time of trajectory
t_ = @(th) fzero(@(t) y(t,th), 1.5*(t_drag+t_fall(th)), ...
            optimset('Display','off','TolX',1e-6)); % tolerance on solution
% Anonymous function which returns the range, given a launch angle th. Uses
% the previous function t_ to obtain the launch time
range = @(th) x(t_(th), th);
% Solve for the angle for which the range is maximized. Since fminbnd
% searches for local minima, we are finding the minimum of -range. Search
% between 0 and 10 degrees, since we showed the optimal angle is close to 0
angle_opt = fminbnd(@(th) -range(th), 0, 10*pi/180);
% Get the range using this angle
range_opt = range(angle_opt);
% Flight time
t_opt = t_(angle_opt);

% Printout
fprintf('The optimal angle is %.4f degrees\n',angle_opt*180/pi);
fprintf('The range at this angle is %.4f \n',range_opt);

% Plot flight path
t_plot = linspace(0, t_opt, 1000);
figure; hold on; box on;
plot(x(t_plot,angle_opt),y(t_plot,angle_opt),'-k');
plot(xlim,[0,0],'--g'); % ground
axis equal;
title('Optimal range trjactory');
xlabel('x'); ylabel('y');

% Plot range as a function of launch angle. Ideally, we could use the fplot
```

```
% function to plot an anonymous function handle. However, since x is
% defined as a function of t and th in our code, the fplot function is not
% equipped to deal with this. Instead, we precompute the ranges over a set
% of launch angles
th = linspace(0,pi/3,1000); % list of angles to plot - from 0 to 60 deg.
ranges = zeros(1,length(th)); % empty list of ranges corresponding to th's
for i = 1:length(th) % compute each range and add to list
    ranges(i) = range(th(i));
end
% Plot
figure; hold on; box on;
plot(th*180/pi,ranges,'-k');
plot(angle_opt*180/pi,range_opt,'bd') % show optimal point
xlabel('$\theta ({}^{\circ})$','interpreter','latex');
ylabel('Range');
title('Range over launch angles');
```

```
The optimal angle is 3.6490 degrees
The range at this angle is 997.3376
```



Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*



Published with MATLAB® R2019b

12.1.31 Someone in a violent part of the world shot a projectile at someone else. The basic facts:
Launched from the origin.
Projectile mass = 1 kg.
Launch angle 30° above horizontal.
Launch speed 172 m/s.
Drag force $\propto cv^2$ with $c = .01$ kg / m.
Gravity g = 10 m/s.

a) Write and execute computer code to find the height at $t = 1$ s. [Hints: sketch of problem, FBD, write drag force in vector form, LMB, 1st order equations, numerical setup, find height at 1 s].

b) Estimate the height at $t = 1$ s using pencil and paper. An answer in meters is desired. [Hints: Assume g is negligible. Good calculus skills are needed but no involved arithmetic is needed. $1 + 1.72 = 2.72 \approx e$. After you have found a solution check that the force of gravity is a small fraction of the drag force throughout the duration of one second of your solution.]

11.1.32 (27)

$m = 1$ kg , $\theta = 30°$, $u = 172$ m/sec

$F_D \propto c v^2$, $C = 0.01$ kg/m , $g = 10$ m/sec^2

FBD

Linear momentum balance

$$m\vec{a} = \vec{F}_D + \vec{F}_G$$

$$= -mg\hat{j} - cV_x\sqrt{V_x^2+V_y^2}\,\hat{i} - cV_y\sqrt{V_x^2+V_y^2}\,\hat{j}$$

$$m\ddot{x} = -c\dot{x}\sqrt{\dot{x}^2+\dot{y}^2}$$

$$m\ddot{y} = -mg - c\dot{y}\sqrt{\dot{x}^2+\dot{y}^2}$$

$$u = \dot{x} \;,\; v = \dot{y} \longrightarrow \text{(i),(ii)}$$

$$\dot{u} = -\frac{c}{m}u\sqrt{u^2+v^2} \longrightarrow \text{(iii)}$$

$$\dot{v} = -g - \frac{c}{m}v\sqrt{u^2+v^2} \longrightarrow \text{(iv)}$$

Matlab :- height at t = 1 sec

from matlab solution height at t = 1 .

is h = 46.5047 m

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

28

b) Assume 'g' is negligible

If 'g' is assumed to be negligible drag force is always in opposite direction of velocity. Hence no force component $\perp$ to the current trajectory acts on it.

ie it is a straight line motion, with $\theta = 30^\circ$ along the line

$$ma = -CV^2$$

$$m\frac{dv}{dt} = -CV^2$$

$$\int \frac{dv}{v^2} = -\int \frac{c}{m}\,dt$$

$$-\frac{1}{V} = -\frac{ct}{m} + A$$

$$V\Big|_{t=0} = u_o \Rightarrow A = -\frac{1}{u_o}$$

$$\frac{1}{V} = \frac{ct}{m} + \frac{1}{u_o}$$

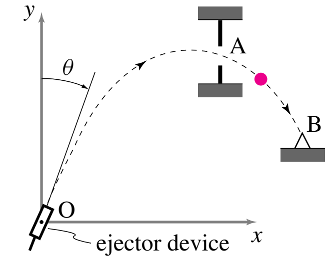
$$V = \frac{mu_o}{Cu_o t + 1} = \frac{ds}{dt}$$

$$S = \int \frac{mu_o}{Cu_o t + 1}\,dt = \frac{mu_o}{Cu_o}\log(Cu_o t + 1) = \frac{m}{c}\log(Cu_o t + 1)$$

for $t = 1$ $\quad S = 100\log(2-72)$

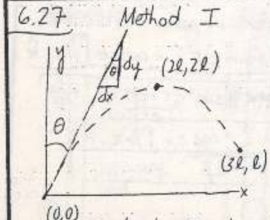
$$h = 100\sin 30^\circ = 50\,m$$

12.1.32 In the arcade game shown, the object of the game is to propel the small ball from the ejector device at O in such a way that it passes through the small aperture at A and strikes the contact point at B. The player controls the angle θ at which the ball is ejected and the initial velocity v_o. The trajectory is confined to the frictionless xy-plane, which may or may not be vertical. Find the value of θ that gives success. The coordinates of A and B are $(2\ell, 2\ell)$ and $(3\ell, \ell)$, respectively, where ℓ is your favorite length unit.



Problem 12.32

6.27 Method I



x-y plane not necessarily vertical

find the angle θ which causes the projectile to pass through $(2l, 2l)$ and end up at $(3l, l)$

as long as the x-y plane is not horizontal, the particle's trajectory is parabolic

gen'l eqn of parabola → $y = Ax^2 + Bx + C$

we know 3 pts. on the curve, so we can eval. the consts. A, B, C: $(0,0)$: $0 = A(0) + B(0) + C \Rightarrow C = 0$

$(2l, 2l)$: $2l = A(4l^2) + 2Bl$

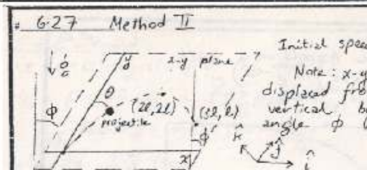
$(3l, l)$: $l = A(9l^2) + 3Bl$

⇒ (after some algebra), $A = -\frac{2}{3l}$, $B = 7/3$

Thus, $y = \frac{-2}{3l}x^2 + 7/3\,x$; (path of particle)

$\theta = \cot^{-1}\left(\frac{dy}{dx} \text{ at } x=0\right) = \cot^{-1}(7/3) = \boxed{23.2^\circ}$

6.27 Method II p. 11



Initial speed v_0

Note: x-y plane displaced from the vertical by an angle ϕ

$\hat{k}$ = unit normal to x-y plane

FBD of projectile

$\sum \underset{\sim}{F} = N\hat{k} + mg(-\cos\phi\,\hat{j} - \sin\phi\,\hat{k})$

LMB of particle

$N\hat{k} + mg(-\cos\phi\,\hat{j} - \sin\phi\,\hat{k}) = m(\ddot{x}\,\hat{i} + \ddot{y}\,\hat{j} + \ddot{z}\,\hat{k})$

$\ddot{z} = 0$ particle moves in x-y plane

$\hat{i}\cdot(\text{LMB}) = m\ddot{x} = 0$ ①

$\hat{j}\cdot(\text{LMB}) = m\ddot{y} = -mg\cos\phi$ ②

Integrate eqns of motion ①, ② twice to get $x(t)$, $y(t)$ (ϕ does not change w/ time!)

$\ddot{x} = 0$ $\quad\quad$ $\ddot{y} = -g\cos\phi = \text{const.}$

$\Rightarrow \dot{x} = A$ $\quad\quad$ $\Rightarrow \dot{y} = -g\cos\phi\, t + C$

$\Rightarrow x = At + B$ $\quad\quad$ $\Rightarrow y = -g\cos\phi \frac{t^2}{2} + Ct + D$

Initial conditions $\quad\quad$ Initial conditions

$x(t=0) = 0$ $\quad\quad$ $y(t=0) = 0$

$\dot{x}(t=0) = V_0\sin\theta$ $\quad\quad$ $\dot{y}(t=0) = v_0\cos\theta$

$\Rightarrow B = 0,\ A = V_0\sin\theta$ $\quad\quad$ $\Rightarrow D = 0\ \ C = V_0\cos\theta$

so $x = (v_0\sin\theta)t$ ③, $y = (-g\cos\phi)\frac{t^2}{2} + (v_0\cos\theta)t$ ④

put ③→④ to eliminate t:

$y = \frac{-g\cos\phi\, x^2}{2v_0^2\sin^2\theta} + \frac{x\cos\theta}{\sin\theta}$ $\quad$ note that $\frac{g\cos\phi}{2v_0^2}$ is a constant

let $K = -\frac{g\cos\phi}{2v_0^2}$ (to clean things up. also, we want to avoid having θ be a fn. of these quantities in our answer.)

$\therefore\ y = \frac{Kx^2}{\sin^2\theta} + \frac{x\cos\theta}{\sin\theta}$ ⑤

use points $(2l, 2l)$ & $(3l, l)$ in ⑤:

$2l = \frac{K(4l^2)}{\sin^2\theta} + 2l\frac{\cos\theta}{\sin\theta} \rightarrow \sin\theta = \frac{K(2l)}{\sin\theta} + \cos\theta$ ⑥

contd

6.27 contd p. 12

now for $(3l, l)$

$\Rightarrow l = \frac{K(9l^2)}{\sin^2\theta} + \frac{3l\cos\theta}{\sin\theta}$

$\Rightarrow \sin\theta = \frac{9Kl}{\sin\theta} + 3\cos\theta$ ⑦

using eqns ⑥ + ⑦, eliminate Kl:

from ⑥ $\quad Kl = \frac{(\sin\theta - \cos\theta)\sin\theta}{2}$

put into ⑦

$\sin\theta = \frac{9}{2}(\sin\theta - \cos\theta) + 3\cos\theta$

$\Rightarrow \frac{7}{2}\sin\theta = \frac{3}{2}\cos\theta$

$\Rightarrow \theta = \tan^{-1}(3/7) = \boxed{23.2^\circ = \theta}$

The same result as method I but more work!

(Note that there is one unique initial velocity v_0 (that depends on the angle ϕ that the x-y plane makes with the vertical) for which the particle will follow this path. Using eqns ⑥ & ⑦, we could solve for this velocity in terms of g & ϕ).

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*