

11.2 Forcing and resonance

If the world of oscillators was as we have described them so far, especially in Section 10.3, there wouldn't be much to talk about. The undamped oscillators (of which there are none) would be oscillating away and the damped oscillators (all the real ones) would be damped out to no motion. The reason vibrations exist is because they are somehow excited. This excitement is also called *forcing* whether or not it is due to a literal mechanical force.

The most important idea of this section is the following

If you shake something at about the same frequency at which it naturally oscillates you will eventually get large motions.

The rest of the section is largely a fleshing out of this idea.

The simplest example of a 'forced' harmonic oscillator is the mass-spring-dashpot system with an additional mechanical force applied to the mass. See fig. 11.14. Most of this section will be a study of this system. The governing equation for a forced damped oscillator can be derived from the free-body diagram as follows, where vector notation helps keep the signs right:

$$\begin{aligned}
 \sum \vec{F}_i &= m\vec{a} \\
 -F_s \hat{i} - F_d \hat{i} + F(t) \hat{i} &= m a \hat{i} \\
 \{ (-kx - c\dot{x} + F(t)) \hat{i} &= m \ddot{x} \hat{i} \} \\
 \{ \cdot \hat{i} \Rightarrow &-kx - c\dot{x} + F(t) = m\ddot{x}
 \end{aligned}$$

which is often re-arranged as

$$m\ddot{x} + c\dot{x} + kx = F(t). \quad (11.25)$$

When $F(t) = 0$, there is no forcing and the governing equation reduces to that of the un-forced damped harmonic oscillator, eqn. (11.2).

Equivalent ways to force an oscillator

There are many ways to "force" a system that all lead to the same forced-oscillator equation.

1. With a literal force as in fig. 11.14, shown again in fig. 11.15a.
2. By shaking the support, as in fig. 11.15b.
3. By displacing one end of the spring, but not the dashpot as in fig. 11.15c.

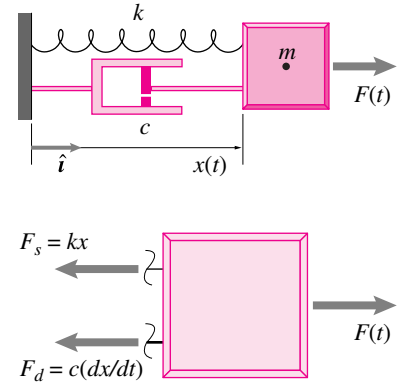


Figure 11.14: A forced mass-spring-dashpot is just a mass held in place by a spring and dashpot but pushed by a force $F(t)$ from some external source.

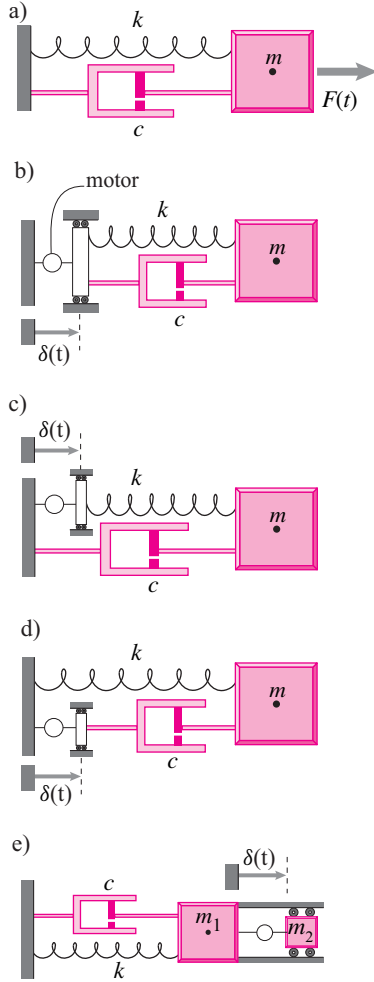


Figure 11.15: In all cases shown above the same forced oscillator eqn. (11.25) applies. In (a) a literal force is applied. In all the other cases the “forcing” is by a motor that moves something back and forth a distance δ . In (b) the support moves. In (c) and (d) just the spring or just the dashpot end is displaced. In (e) an extra mass is moved relative to the main mass.

① Note again that we use λ (lambda) here instead of the more commonly used ω (omega) because we want to avoid confusion with the magnitude of angular velocity $\omega = |\dot{\varphi}|$ which itself could have oscillatory motion. It would be confusing (and bad math) to write $\omega = A \sin \omega t$ with ω having two different meanings in the same equation.

4. By displacing one end of the dashpot, but not the spring as in fig. 11.15d.
5. By displacing a second mass attached to the first with a motor that controls relative position, as in fig. 11.15e.

That these five systems all lead to the same governing equation follows from drawing free-body diagrams, applying momentum balance, and collecting terms to match the form eqn. (11.25). Note that the meaning of some of the terms in the forced-oscillation equation is different for each system.

Types of forcing

In general this or that machine or structure could be forced in any number of complicated ways. But there are two special forcings of most common engineering interest:

- $F(t) = F_0$ (Constant force), and
- $F(t) = F \cos \lambda t$ (sinusoidal forcing).^①

Constant force idealizes situations where the force doesn’t vary much as due say, to gravity, a steady wind, or sliding dry friction. Sinusoidally varying forces are used to approximate oscillating forces as caused, say, by a vibrating support or earthquakes. Forces that are not sinusoidal can be thought of as sums of sine waves thus, in some sense, by knowing how a structure responds to sinusoidal forcing, at various frequencies, you know how it responds to all possible forcings^②. Let’s look at each of these two cases in detail.

Forcing with a constant force

The case of constant forcing is both common and easy to analyze, so easy that it is often ignored (see fig. 10.27 on page 524). If $F(t) = F_0 = \text{constant}$, then the general solution of eqn. (11.25) for $x(t)$ is the same as the unforced case but with a constant added. The constant is F_0/k . The usual way of accommodating this case is to describe a new equilibrium point at $x = F_0/k$ and to pick a new deflection variable that is zero at that point. If we pick a new variable z defined as $z = x - F_0/k$, then substituting into eqn. (11.25) we get

$$m\ddot{z} + c\dot{z} + kz = 0, \quad (11.26)$$

which is the unforced oscillator equation. That is, constant forcing reduces to the case of no forcing if one merely changes what one calls zero to be the place where the mass is in equilibrium, taking account of the spring stretch (or compression) caused by the constant applied force. Thus the solution of the forced equation for x is equivalent to the unforced solution for z :

$$z(t) = x(t) - F_0/k = e^{(-\frac{c}{2m})t} (A \cos(\lambda_d t) + B \sin(\lambda_d t)) \quad (11.27)$$

where $\lambda_d = \sqrt{\left(\frac{c}{2m}\right)^2 - k/m}$, as explained in box 11.3 on page 563.

An alternative approach is to use *superposition*. Here we say $x(t) = x_h(t) + x_p(t)$ where $x_h(t)$ satisfies $m\ddot{x} + c\dot{x} + kx = 0$ and $x_p(t)$ is any solution x_p of $m\ddot{x} + c\dot{x} + kx = F_0$. Any solution you like is called a “particular” solution. One easy solution is $x_p = F_0/k$. So the net solution is $x_p = F_0/k$ plus a solution x_h to the ‘homogeneous’ equation 11.26.

$$x(t) = \underbrace{e^{\left(-\frac{c}{2m}\right)t} (A \cos(\lambda_d t) + B \sin(\lambda_d t))}_{x_h} + \underbrace{F_0/k}_{x_{p1}} \quad (11.28)$$

Example: Hanging mass.

The mass hanging from the support shown in fig. 11.16 obeys the equation

$$m\ddot{x} + c\dot{x} + kx = \underbrace{k\ell_0 + mg}_{F_0}$$

One particular solution x_p , the easiest one, has the mass hanging still. In this solution, the mass position is the un-stretched length ℓ_0 of the spring plus the stretch of the spring due to gravity, $\Delta x = mg/k$. Because the mass is still in this solution, the dashpot constant c doesn’t appear. So

$$x_p = \ell_0 + mg/k.$$

The homogeneous solution x_h is given by (11.27) and the general motion is the sum

$$\begin{aligned} x(t) &= x_p + x_h \\ &= (\ell_0 + mg/k) + e^{-\frac{ct}{2m}} (C \cos(\lambda_d t) + D \sin(\lambda_d t)) \end{aligned}$$

where C and D are constants determined by the initial conditions. For any initial condition and corresponding values of C and D , the motion eventually decays to the stationary particular solution with the mass hanging still (because the exponentials go to zero as $t \rightarrow \infty$).

② The best approximation of a function as a sum of sine waves is a Fourier series, a topic you learn in math, advanced physics or linear systems courses.

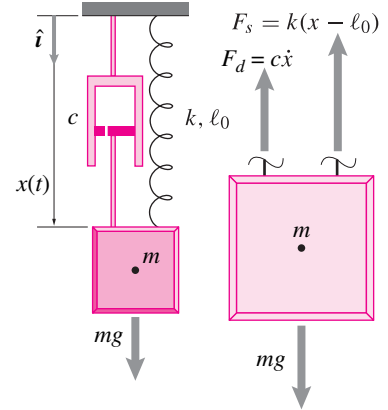


Figure 11.16: A mass hangs from a spring and dashpot. Its distance x from the support is due to the rest length ℓ_0 of the spring, and the stretch of the spring.

Forcing with a sinusoidally varying force

The motion resulting from sinusoidal forcing is of central interest in vibration analysis. In this case we imagine that $F(t) = F \cos pt$ where F is the amplitude of forcing and p is the angular frequency of the forcing. Note, we could just as well use $F(t) = F \sin pt$ for the forcing, sin and cos are both sinusoidal forcings.

The general solution of equation 11.25 is given by the sum of two parts. One is the general solution of equation 11.2, $x_h(t)$, and the other is *any* solution of equation 11.25, $x_p(t)$. The solution $x_h(t)$ of the damped oscillator equation 11.2 is called the ‘homogeneous’ or ‘complementary’ solution. Any solution $x_p(t)$ of the forced oscillator equation 11.25 is called a ‘particular’ solution.

We already know the solution $x_h(t)$ of the damped governing differential equation 11.2. This solution is equation 11.3, 11.5, or 11.6, depending

on the values of the mass, spring and damping constants. So the new problem is to find any solution to the forced equation 11.25. The easiest way to solve this (or any other) differential equation is to make a fortuitous guess (you may learn other methods in your math classes). In this case with

$$F(t) = F \cos(pt)$$

we make the guess that

$$x_p(t) = A \cos(pt) + B \sin(pt). \quad (11.29)$$

Basically this guess says “If you shake something with a sine wave it will probably move as a sine wave. But who knows the amplitude or phase?” Plugging this guess into the forced oscillator equation (11.25) we find values for A and B in box 11.2 on page 10.

Alternatively, a sum of sine waves can be written as a cosine wave (or sine wave) that has been shifted in phase as (see box 10.2 on page 521)

$$x_p(t) = A_0 \cos(pt - \phi),$$

The value of forced amplitude is simply $A_0 = \sqrt{A^2 + B^2}$. This is given in terms of m, c, k, p and F in box 11.2. The forced amplitude A_0 is the central subject of this section. It answers the question ‘How big are the oscillations when you shake something?’ Because the formula for A_0 is admittedly a mess, the answer is often given in a plot^③. The general solution, therefore, is

$$x(t) = x_h(t) + x_p(t). \quad (11.30)$$

The homogeneous solution $x_h(t)$, the motion of the unforced system, is just decaying oscillations and is usually not of primary interest in vibrating systems. The particular solution $x_p(t)$ is steady oscillations. These oscillations are of central interest. In particular most often in engineering one wants these oscillations to be big or small.

Example: MEMs devices.

One general type of “**M**icro **E**lectronic **M**achine” consists of, basically, a vibrating beam. A beam with an effective mass $50\mu\text{ gm}$ and effective stiffness of $k = 500\text{ N/m} = 5\mu\text{ N}/\mu\text{ m}$ has

$$\lambda_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{5\text{ N/m}}{50 \cdot 10^{-9}\text{ kg}}} = \sqrt{\frac{\frac{500\text{ N}}{\text{m}} \left(\frac{1\text{ kg m/s}^2}{1\text{ N}} \right)}{50 \cdot 10^{-6}\text{ kg}}} = \sqrt{10^{10}\text{ s}^{-1}} = 10^5/\text{s}$$

which corresponds to a frequency of $\lambda_n/2\pi \approx 15.7\text{ kHz}$. That is, such a MEMs device would be a good receiver (or ‘resonator’) for 15.7 kHz ultra-sonic vibrations. In this case resonance is useful to make the sensor sensitive.

The size of the oscillations scales with the size of the forcing F (this proportionality is known as ‘linearity’) and also depends on all the parameters m, c, k and p .

③ Another more important reason that a plot is used is that often in a physical system one can measure the vibrations while never knowing a detailed accurate set of differential equations which would describe the system accurately.

Frequency response and resonance

One way to show a structure's sensitivity to oscillatory loads is by a *frequency response* curve fig. 11.17. One curve shows the amplitude of vibrations vs the forcing frequency. The main idea of this section, resonance, shows as a peak in the frequency-response curve near the natural frequency $\lambda_n = \sqrt{k/m}$.

Recall that the natural frequency λ_n is the unforced frequency of undamped oscillation. The damped natural frequency λ_d , the frequency of decaying oscillations with damping present, is slightly slower (see, e.g., eqn. (11.3) on page 564.). The resonant frequency λ_{res} , the frequency of forcing for which the amplitude of motion is maximum (eqn. (11.2) on page 10), is slightly lower still. But, especially when the damping is low, there is only a small difference between the natural frequency, the damped frequency and the resonant frequency. So, in common language and engineering practice they are usually treated as one and the same.

In summary, the frequency response curve has a peak with forcing near to, but not exactly at, the natural frequency of unforced and undamped motion. But most engineers can reasonably assume, even though it's not exactly true, that resonance occurs when the forcing frequency is the natural vibration frequency.

Resonance is good and bad

Sometimes an engineer studies vibrations with the hope of minimizing them, sometimes with the hope of maximizing them. Resonance is sometimes the problem and sometimes the solution.

Resonant vibrations are usually undesirable in machinery or cars. The vibrations can lead to large stresses, undesirable motions, or unpleasant sounds. A building resonating to earthquake vibrations may be more likely to fall down.

On the other hand, nuclear Magnetic Resonance imaging is used for medical diagnosis. In the old days, the resonant excitation of a clock pendulum was used to keep time. The resonance of quartz crystals is used to time most watches now-a-days. Self excited resonance is what makes musical instruments have such clear pitches. And resonant vibrations are used to give a larger signal in micro-mechanical sensors. In the electrical domain, radio tuners depend on resonance to pick out just one radio band.

Other systems

Most machines and structures are not exactly a point mass moving in one direction and constrained by a single spring and single dashpot. On the other hand, almost all machines have mass, elastic give, and some dissipation when they move. So most machines have natural oscillations after they are banged or disturbed somehow. And so most structures and ma-

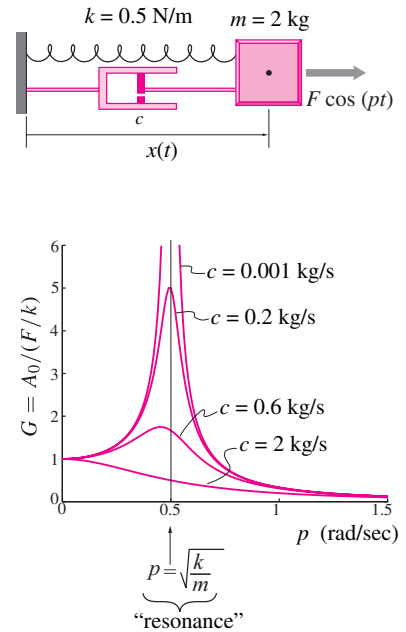


Figure 11.17: Amplitude of oscillation vs forcing frequency for various dampings. Each curve shows the gain G vs the forcing frequency for a fixed damping. Note that when the damping is small ($c \ll 1$) and the forcing is close to the natural frequency of vibration $p \approx \lambda_n$ there is a 'resonant' peak in the amplitude of the response. The smaller the damping the higher and narrower is this peak. For very high damping the peak is at a slightly lower frequency. The mass-spring-dashpot system shown was used to generate the plots using the formulas from box 11.2 on page 10.

④ **Golden Gate bridge cables.** On a walk a few years ago one of us noted that the vertical cables on the Golden Gate bridge could be induced to oscillate quite visibly if pushed by a person at the right frequency (about 0.5 hz). One cable, maybe 100 tons of steel, was nicely going back and forth about a half a meter. A police car pulled up and stopped. Through the megaphone the officer authoritatively and sternly threatened “If you break it you have to pay for it.”

chines can be shaken to large motions if the appropriate (or inappropriate, depending on your aims) frequency of force is applied^④.

So the concepts introduced here for a single mass-spring-dashpot system apply to much more complex machines and structures. In particular, they have natural vibration frequencies and they shake a lot (resonate) if forced at near those frequencies.

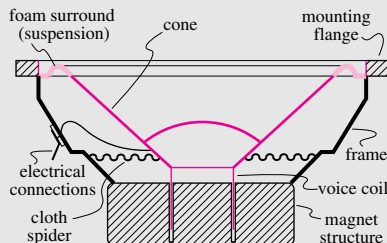
Experimental measurement

Because no real thing of interest is exactly a single mass-spring-dashpot the ideas of vibrations analysis are often not expressed in terms of (m , c and k). Rather, the more broad ideas of natural frequency, frequency response, and resonance are considered on their own. Using either a large-scale computer model (say a ‘finite-element’ model) or measurement of the physical system itself, one can draw a frequency-response curve like fig. 11.17 on page 5.

Here’s how. First you apply a sinusoidal force to the structure at the point of interest, say $F = F \cos(pt)$. Then you measure the motion of a part of the structure of interest. You might instead measure a strain or rotation, but for definiteness let’s assume you measure the displacement of some point on the structure δ .

If the structure is linear and has some damping, the eventual motion of the structure will eventually be a sinusoidal oscillation. In particular,

11.1 A Loudspeaker cone is a forced oscillator.



Cross-sectional view of a speaker.

A speaker, similar to the ones used in many home and auto speaker systems, is one of many devices which may be conveniently modeled as a one-degree-of-freedom mass-spring-dashpot system. A typical speaker has a paper or plastic cone, supported at the edges by a roll of plastic foam (the surround), and guided at the center by a cloth bellows (the spider). It has a large magnet structure, and (not visible from outside) a coil of wire attached to the point of the cone, which can slide up and down inside the magnet. (The device described above is, strictly speaking, the speaker driver. A complete speaker system includes an enclosure, one or more drivers, and various electronic components.) When you turn on your stereo, the amplifier forces a current through the coil in time with the music, causing the coil to alternately attract and

repel the magnet. This rapid oscillation of attraction and repulsion results in the vibration of the cone which you hear as sound.

In the speaker, the primary mass is comprised of the coil and cone, though the air near the cone also contributes as ‘added mass.’ The ‘spring’ and ‘dashpot’ effects in the system are due to the foam and cloth supporting the cone, and perhaps to various magnetic effects. Speaker system design is greatly complicated by the fact that the air surrounding the speaker must also be taken into account. Changing the shape of the speaker enclosure can change the effective values of all three mass-spring-dashpot parameters. (You may be able to observe this dependence by cupping your hands over a speaker (gently, without touching the moving parts), and observing amplitude or tone changes.) Nevertheless, knowledge of the basic characteristics of a speaker (e.g., resonance frequency), is invaluable in speaker system design.

Our approximate equation of motion for the speaker is identical to that of the ideal mass-spring-dashpot above, even though the forcing is from an electromagnetic force, rather than a direct mechanical force:

$$m\ddot{x} + c\dot{x} + kx = F(t) \text{ with } F(t) = \alpha i(t) \quad (11.31)$$

where $i(t)$ is the electrical current flow through the coil in amps, and α is the electro-mechanical coupling coefficient, in force per unit current.

you will measure that

$$\delta = A_0 \cdot \cos(pt - \phi). \quad (11.32)$$

If you had applied half as big a force, you would have measured half the displacement, still assuming the structure is linear, so the ratio of the displacement to the force A_0/F is independent of the size of the force F . Let's define:

$$G = \frac{A_0}{F}$$

That is, the amplification gain G is the ratio of the amplitude of the displacement sine wave to the amplitude of the forcing sine wave. Plotting p on the x axis and G on the y axis, this experiment gives one point on the frequency response curve. Repeating for a range of forcing frequencies one can plot up the frequency response $G = G(p)$.

Example: Shake table for earthquake response.

One way to get a frequency response curve for a building is to put a scale model on a “shake table”. The base is then moved sinusoidally through a range of frequencies and the motion of the model is observed. This way one can find peaks in the frequency-response curve. These are frequencies that, to the extent they are prevalent in a feared earthquake, are likely to cause damage.

Transient response

As discussed, the full solution of eqn. (11.25) with forcing $F(t) = F_0 + F \cos pt$ is the sum of three terms

$$x(t) = x_h + x_{p1} + x_{p2}$$

The first of these has decaying oscillations, the second is a constant, and the third has steady oscillations. When added up the motion can look quite complicated, as seen in fig. 11.19d. The main point is that after some initial complicated transient the motion eventually decays to steady oscillations ($x_{p2}(t) = A_0(\cos pt - \phi)$) plus an offset ($x_{p1} = F/k$).

The vocabulary of forced oscillations

Forced oscillations are so important and common that there is a specialized vocabulary for many of the terms and collections of commonly appearing terms. Here is a list, starting with the terms you know well.

m = the **mass** of the particle that is oscillating. For more complicated systems the mass m may represent an “effective” or “equivalent” mass.

c = the **damping coefficient**. c is used to describe the viscous drag, the resistance to motion $F_d = -c\dot{x}$.

k = the **spring constant**. k describes the elastic restoring “spring” force $F_s = -kx$.

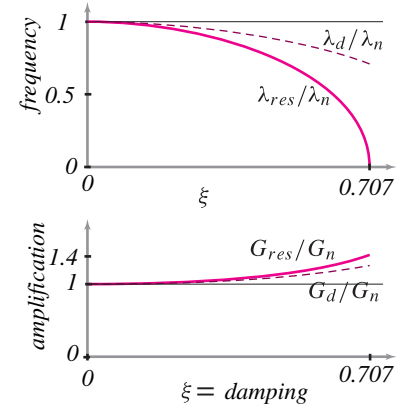


Figure 11.18: Natural, damped and resonant frequencies. The upper dashed plot shows the ratio of damped to natural frequency vs forcing frequency ratio ξ . The solid curve shows how the resonant frequency varies from the natural frequency as a function of frequency ratio. The lower plot shows how the amplification $G = A_0/(F/k)$ is nearly the same at the natural, damped and resonant frequencies. That the peak in the $c = 0.6\text{kg/s}$ curve of fig. 11.17 is slightly to the left of $p = \lambda_n = \sqrt{k/m}$ shows as λ_{res}/λ_n being slightly below 1 for $\xi \approx 0.3$. For low damping an engineer can treat the natural, damped and resonant frequencies as equal. Similarly the amplification when forced at the natural frequency is very close to the amplification when forcing is at the damped or at the resonant frequency.

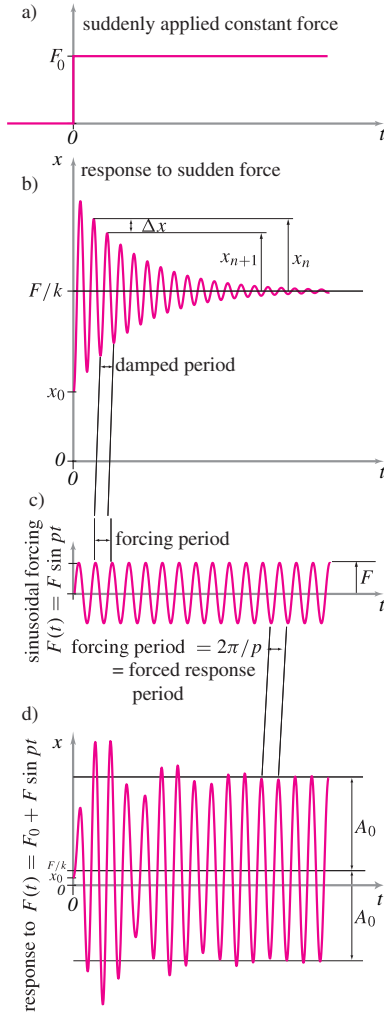


Figure 11.19: Transient response. (a) shows a suddenly applied force F_0 . The response (b) to this force is a motion that starts at the initial position x_0 ($x_0 > 0$ in this illustration) and then oscillates about the new equilibrium F_0/k . The motion is identical to unforced motion, but offset. Thus it can be used to evaluate the rate of decay of oscillation and the damped period of oscillation. (c) A sinusoidal forcing causes. (d) the response if the mass is released at $x = x_0$ and suddenly both a constant force F and a sinusoidal force $F \sin pt$ are applied. The motion eventually settles into a sinusoidal oscillation at the forcing frequency (which is a little longer period than the damped oscillation in this illustration) with amplitude A_0 .

F = the **forcing amplitude** for a sinusoidally varying applied force $F(t) = F \sin pt$ or $F(t) = F \cos pt$ or $F(t) = A \sin pt + B \cos pt$ with $F \equiv \sqrt{A^2 + B^2}$.

p = the **forcing frequency**. Some books will use the symbol ω for the forcing frequency.

The rest of the quantities below are completely determined by the quantities above (m, c, k, F and p).

$\lambda_n \equiv \sqrt{k/m}$ is the **natural frequency**. This is the frequency of oscillation if there is neither forcing nor damping. In that case $x(t) = A \cos \lambda_n t + B \sin \lambda_n t$. Many books use ω_n for the natural frequency.

$c_{\text{crit}} = 2 * \sqrt{km}$ is the **critical damping coefficient**. The relation of the actual damping c to the critical damping c_{crit} tells you whether a system is over-damped ($c > c_{\text{crit}} \Rightarrow$ decay to equilibrium, when unforced, that is exponential) or under-damped ($c < c_{\text{crit}} \Rightarrow$ decay to equilibrium, when unforced, that is oscillatory). See Fig. 11.2 on page 562. Sometimes c_{crit} is more simply written as c_c or c_{cr} .

$\xi \equiv c/c_{\text{crit}}$ is the **damping ratio**. The single number ξ ('ksee') tells you if a system is over damped ($\xi > 1$) or underdamped ($\xi < 1$).

$r \equiv p/\lambda_n = p/\sqrt{k/m}$ is the **frequency ratio**. If $r > 1$ then the forcing is faster than the frequency of natural unforced vibrations. If $r < 1$ then the forcing is slower than the natural vibrations.

A_0 = the **response amplitude**. When a steady oscillatory force is applied the motion is eventually oscillatory. The amplitude of the motions is A_0 , as in $x = A_0 \cos(pt - \phi)$ with

$$A_0 = (F/k) / \sqrt{(2\xi r)^2 + (1 - r^2)^2}.$$

$G \equiv A_0/(F/k)$ is the **gain or amplification**. G is the ratio of the eventual amplitude of the oscillator to the response that would occur if the same force was applied at zero frequency. It is the response amplitude scaled by the displacement that would occur if the same force was applied to a spring.

$\lambda_{\text{res}} = \lambda_n \sqrt{1 - 2\xi^2}$ is the **resonant frequency**. λ_{res} (also called λ_r or ω_r) is the frequency such that if $p = \lambda_{\text{res}}$ the amplification gain G is maximum. The resonant frequency is the frequency at which you force a system to get the biggest motions. The resonant frequency λ_{res} is rather close to the natural frequency λ_n in systems with small damping ratios. And these are also the systems that are prone to resonant vibrations.

λ_d is the **damped natural frequency**. If an underdamped system is released from rest it oscillates as the motions decay. The frequency of these oscillations is

$$\lambda_d = \lambda_n \sqrt{1 - \xi^2}.$$

The frequency λ_d of damped oscillations is a shade slower than the frequency λ_n of oscillation of the same system with no damping. When damping is small the natural frequency λ_n , the damped frequency λ_d and the resonant frequency λ_r are all close to each other (See fig. 11.17a).

G_n, G_{res} & G_d are the amplification **gains** (see G above) when forcing is at the natural, the resonant and the damped natural frequency respectively ($p = \lambda_n, \lambda_{\text{res}}$ & λ_d , see above). G_{res} is the biggest of these by definition. But it is not actually much bigger than G_n or G_d . These gains can be calculated using the formulas for G and A_0 above. They are plotted on fig. 11.17b.

D is the **logarithmic decrement**. D measures the rate of decay of unforced ($F = 0$) oscillations. The experimental definition, derivable from a graph of the motion, is

$$D = \ln\left(\frac{x_n}{x_{n+1}}\right).$$

In terms of m, c and k the logarithmic decrement is $D = \frac{cT}{2m} = 2\pi\xi$, as derived on page 563. If there is little damping, c is small ($\xi \ll 1$) and $D \approx (x_n - x_{n+1})/x_n$ is the fractional decrease in amplitude per oscillation. If $D = .1$ then each oscillation is about 10% smaller in amplitude than the previous one.

Q is the **quality factor**. For the mass-spring-dashpot system it is another way of describing the rate of decay of unforced oscillations.

$$\begin{aligned} Q &\equiv 2\pi(\text{energy of oscillator})/(\text{energy lost per cycle}) \\ &= 2\pi x_n^2 / (x_n^2 - x_{n+1}^2) \\ &\approx \pi/D = 1/(2\xi) \quad (\text{for small damping}) \end{aligned}$$

The π in the definition of Q makes it so there is no π in the formula for the quality factor Q in terms of the damping ratio ξ . Note that, so long as damping is small, ξ, D and Q can each be found approximately from the other. A system with low damping ($\xi \ll 1$) has high quality ($Q \gg 1$) and slowly decaying oscillations and hence a small logarithmic decrement ($D \ll 1$).

11.2 Solution of the forced oscillator equation

The main equation for understanding forced oscillations is:

$$m\ddot{x} + c\dot{x} + kx = F_0 + F \cos pt.$$

Because the equation is linear we look for a solution which is the sum of three terms

$$x(t) = x_h + x_{p1} + x_{p2}$$

where x_h is the homogeneous solution from Eqns 11.3 - 11.6 on page 564, depending on whether the system is underdamped (oscillatory decay), critically damped or over damped (non-oscillatory exponential decay). x_{p1} is a particular solution for the constant forcing F_0 . $x_{p1}(t)$ was found in eqn. (11.28) on page 3 to be, simply, $x_{p1} = F_0/k$.

The last part of the solution, finding an x_{p2} for the forcing term $F \cos pt$ is found by guessing

$$x_{p2} = A \cos pt + B \sin pt.$$

When this guess is plugged into the equation

$$m\ddot{x} + c\dot{x} + kx = F \cos pt$$

every term is either a multiple of $\sin pt$ or $\cos pt$. Thus we get

$$\{\text{A collection of constants}\} \cos pt + \{\text{Another collection}\} \sin pt = 0$$

The only way a sum of a sine wave and cosine wave can be zero for all time is for both coefficients to be zero. Setting the two collections of constants above both to zero gives two simultaneous equations for the unknowns A and B in terms of m, c, k and p . These can be solved to give

$$A = \frac{(F/k) \left(1 - \frac{p^2}{(k/m)}\right)}{\left(\frac{c^2}{km}\right) \left(\frac{p^2}{k/m}\right) + \left(1 - \frac{p^2}{(k/m)}\right)^2},$$

$$\text{and } B = \frac{(F/k)(cp/k)}{\left(\frac{c^2}{km}\right) \left(\frac{p^2}{k/m}\right) + \left(1 - \frac{p^2}{(k/m)}\right)^2}.$$

So we have found the particular solution for forcing with $F(t) = F \cos pt$, using A and B above, as

$$x_{p2} = A \cos pt + B \sin pt. \quad (11.33)$$

An alternative form for the solution is

$$x_{p2}(t) = A_0 \cos(pt - \phi), \quad (11.34)$$

for which we can find the constants A_0 and ϕ using the trig identity $\cos(\theta - \phi) = \cos \theta \cos \phi + \sin \theta \sin \phi$ described in box 10.2 on page 521. Applying this identity to the solution above we find the object of central interest, the forced amplitude

$$A_0 = \sqrt{A^2 + B^2} = \frac{F/k}{\sqrt{\left(\frac{c^2}{km}\right) \left(\frac{p^2}{k/m}\right) + \left(1 - \frac{p^2}{(k/m)}\right)^2}} \quad (11.35)$$

and also the phase angle

$$\phi = \tan^{-1} \left(\frac{B}{A} \right) = \tan^{-1} \left(\frac{cp/k}{\left(1 - \frac{p^2}{(k/m)}\right)} \right). \quad (11.36)$$

All of the expressions above can be somewhat simplified if we write them in terms of the *frequency ratio* $r = p/\lambda_n = p/\sqrt{k/m}$ and damping ratio $\xi = c/c_{\text{crit}} = c/2\sqrt{km}$ (The frequency ratio, damping ratio and some more specialized vibration words are defined on page 7.). Using these dimensionless quantities, the values of the constants in the solution $x_{p2}(t)$, namely eqn. (11.33) or eqn. (11.34), are:

$$A = \frac{(F/k)(1 - r^2)}{4\xi^2 r^2 + (1 - r^2)^2},$$

$$B = \frac{(F/k)(2\xi r)}{4\xi^2 r^2 + (1 - r^2)^2},$$

$$A_0 = \sqrt{A^2 + B^2} = \frac{F/k}{\sqrt{(2\xi r)^2 + (1 - r^2)^2}},$$

and

$$\phi = \tan^{-1} \left(\frac{B}{A} \right) = \tan^{-1} \left(\frac{2\xi r}{1 - r^2} \right).$$

These constants are for the particular forced solution $x_{p2}(t)$ of eqn. (11.33) or eqn. (11.34). Again, most important in all of this is the amplitude A_0 of the forced response. As you can see, the bottom of the fraction for A_0 gets quite small for small damping ($\xi \ll 1$) if the frequency ratio r is close to 1. That is,

the amplitude is big if the forcing is close to the natural frequency.

Resonant frequency

In detail, the frequency at which the vibration amplitude A_0 is maximum is not exactly the unforced undamped natural frequency $\lambda_n = \sqrt{k/m}$. The resonant frequency λ_{res} is found by maximizing A_0 with respect to $r = \lambda/\lambda_n$. Setting $dA_0/dr = 0$ and solving for r we find

$$r_{\text{res}} = \sqrt{1 - 2\xi^2} \Rightarrow \lambda_{\text{res}} = \lambda_n \sqrt{1 - 2\xi^2} \quad (11.37)$$

The ratio of $\lambda_{\text{res}}/\lambda_n$ is plotted on fig. 11.17 on page 5. Also plotted is the ratio of A_0 at resonance to A_0 if forcing is at the natural frequency. The morals are that a) for small damping the natural frequency and resonant frequency are very close, and b) for all dampings, there is little error in calculating the amplitude of the maximum vibration response by approximating resonance as being at the natural frequency. Even when resonance is barely a viable concept, for systems that are critically damped, the error is only 40%.

Similarly one might think the damped natural frequency

$$\lambda_d = \lambda_n \sqrt{1 - \xi^2}$$

would be a better approximation to the resonant frequency. Actually, it's about half way between the natural and resonant frequencies, as can be seen also on fig. 11.17.

SAMPLE 11.6 The mass-spring-dashpot system shown in the figure consists of a mass $m = 2 \text{ kg}$, a spring with stiffness $k = 3200 \text{ N/m}$ and a dashpot with damping coefficient $c = 10 \text{ kg/s}$.

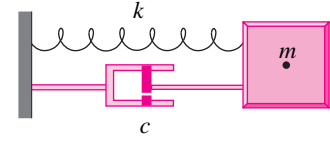


Figure 11.20

1. Is the system underdamped, critically damped or overdamped?
2. Find the damped natural frequency of the system.
3. What is the resonant frequency of the system.

Solution

1. The question about underdamped, critically damped, or overdamped can be answered conveniently by computing the damping ratio ξ . For an underdamped system, $\xi < 1$, for a critically damped system, $\xi = 1$, and for an overdamped system $\xi > 1$. So, let us compute ξ . We know that

$$\xi = \frac{c}{c_c} = \frac{c}{2\sqrt{km}}.$$

Thus, for the given system,

$$\xi = \frac{10 \text{ kg/s}}{2\sqrt{3200 \text{ N/m} \cdot 2 \text{ kg}}} = \frac{10 \text{ kg/s}}{160 \text{ kg/s}} = 0.062.$$

Since $\xi < 1$, the system is underdamped.

Underdamped ($\xi = 0.062$)

2. The damped natural frequency, λ_d , is given by

$$\lambda_d = \lambda_n \sqrt{1 - \xi^2}$$

where $\lambda_n = \sqrt{k/m}$ is the natural frequency of the system. Substituting the known values, we get

$$\lambda_d = \sqrt{\frac{3200 \text{ N/m}}{2 \text{ kg}}} \sqrt{1 - (0.062)^2} = 39.92 \text{ rad/s}$$

which is almost the same as the natural frequency $\lambda_n = 40 \text{ rad/s}$.

$\lambda_d = 39.92 \text{ rad/s}$

3. The resonant frequency of the system, λ_r , is given by

$$\lambda_r = \lambda_n \sqrt{1 - 2\xi^2}.$$

Substituting the known values of λ_n and ξ , we get

$$\lambda_r = 39.85 \text{ rad/s}$$

which is the smallest among the three characteristic frequencies of the system — natural frequency, damped natural frequency, and the resonant frequency. For small values of ξ , however, the three frequencies are practically indistinguishable as is the case here.

$\lambda_r = 39.85 \text{ rad/s}$

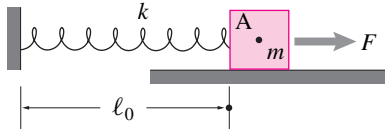


Figure 11.21

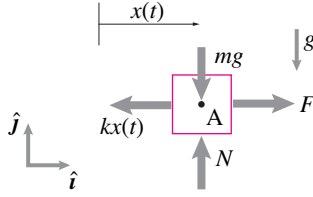


Figure 11.22: Free-body diagram of the mass.

SAMPLE 11.7 Response to a constant force: A constant force $F = 50 \text{ N}$ acts on a mass-spring system as shown in the figure. Let $m = 5 \text{ kg}$ and $k = 10 \text{ kN/m}$.

1. Write the equation of motion of the system.
2. If the system starts from the initial displacement $x_0 = 0.01 \text{ m}$ with zero velocity, find the displacement of the mass as a function of time.
3. Plot the response (displacement) of the system against time and describe how it is different from the unforced response of the system.

Solution

1. The free-body diagram of the mass is shown in fig. 11.22 at a displacement x (assumed positive to the right). Applying linear momentum balance in the x -direction, i.e., $(\sum \vec{F} = m\vec{a}) \cdot \hat{i}$, we get

$$\begin{aligned} F - kx &= m\ddot{x} \\ \Rightarrow m\ddot{x} + kx &= F \end{aligned} \quad (11.38)$$

which is the equation of motion of the system.

2. The equation of motion has a non-zero right hand side. Thus, it is a nonhomogeneous differential equation. A general solution of this equation is made up of two parts — the homogeneous solution x_h which is the solution of the unforced system (eqn. (11.38) with $F = 0$), and a particular solution x_p that satisfies the nonhomogeneous equation. Thus,

$$x(t) = x_h(t) + x_p(t). \quad (11.39)$$

Now, let us find $x_h(t)$ and $x_p(t)$.

Homogeneous solution: $x_h(t)$ has to satisfy the homogeneous equation

$$m\ddot{x} + kx = 0.$$

Let $\lambda = \sqrt{k/m}$. Then, from the solution of unforced harmonic oscillator, we know that

$$x_h(t) = A \sin(\lambda t) + B \cos(\lambda t)$$

where A and B are constants to be determined later from initial conditions.

Particular solution: x_p must satisfy eqn. (11.38). Since the nonhomogeneous part of the equation is a constant (F), we guess that x_p must be a constant too (of the same form as F). Let $x_p = C$. Now we substitute $x_p = C$ in eqn. (11.38) and solve the resulting equation to determine C :

$$m \underbrace{\ddot{C}}_0 + kC = F \Rightarrow C = F/k \quad \text{or} \quad x_p = F/k.$$

Substituting x_h and x_p in eqn. (11.39), we get

$$x(t) = A \sin(\lambda t) + B \cos(\lambda t) + F/k. \quad (11.40)$$

Now we use the given initial conditions to determine A and B .

$$\begin{aligned} x(t=0) &= B + F/k = x_0 \text{ (given)} \Rightarrow B = x_0 - F/k \\ \dot{x}(t) &= A\lambda \cos(\lambda t) - B\lambda \sin(\lambda t) \\ \Rightarrow \dot{x}(t=0) &= A = 0 \text{ (given)} \Rightarrow A = 0. \end{aligned}$$

Thus,

$$x(t) = (x_0 - F/k) \cos(\lambda t) + F/k, \quad (11.41)$$

$$\text{and} \quad \dot{x}(t) = -\lambda(x_0 - F/k) \sin(\lambda t). \quad (11.42)$$

3. Let us plug the given numerical values, $k = 10 \text{ kN/m}$, $m = 5 \text{ kg}$, (which gives $\lambda = \sqrt{k/m} = 44.72 \text{ rad/s}$), $F = 50 \text{ N}$ and $x_0 = 0.01 \text{ m}$ in eqn. (11.41) and (11.42). The displacement and the velocity are now given as

$$\begin{aligned} x(t) &= (0.005 \text{ m}) \cos(44.72 \text{ rad/s} \cdot t) + 0.005 \text{ m}, \\ \text{and} \quad \dot{x}(t) &= -(0.22 \text{ m/s}) \sin(44.72 \text{ rad/s} \cdot t). \end{aligned}$$

This response is plotted in fig. 11.23 against time. Note that the oscillations of the mass are about a non-zero mean value, $x_{\text{eq}} = 0.005 \text{ m}$. A little thought should reveal that this is what we should expect. When a mass hangs from a spring under gravity, the spring elongates a little, by mg/k to be precise, to balance the mass. Thus, the new static equilibrium position is not at the relaxed length ℓ_0 of the spring but at $\ell_0 + mg/k$. Any oscillations of the mass will be about this new equilibrium. The velocity, however, has a zero mean value which is what we expect from eqn. (11.42).

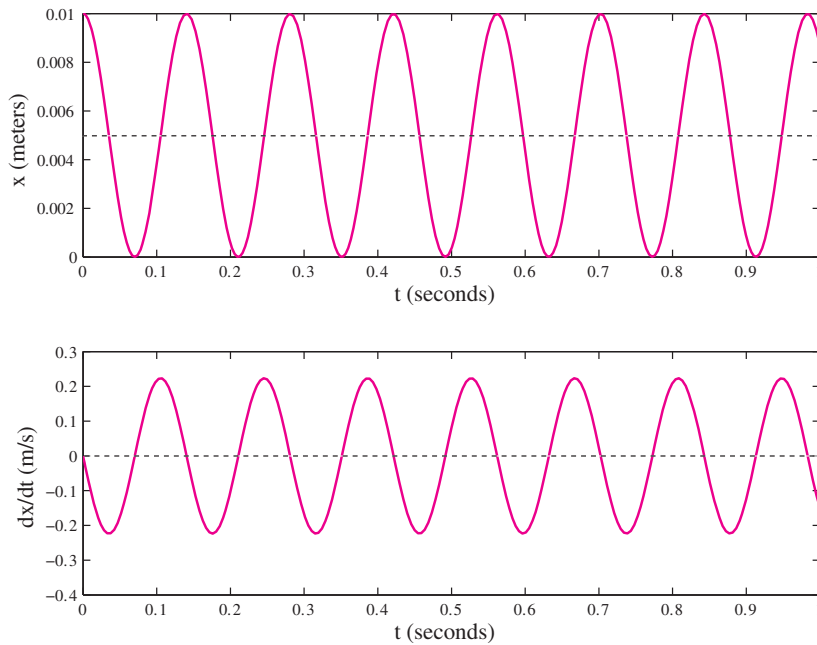


Figure 11.23: Displacement of the mass as a function of time. Note that the mass oscillates about a non-zero value of x .

This problem is exactly like a mass hanging from a spring under gravity, a constant force, but just rotated by 90° . The new static equilibrium is at $x_{\text{eq}} = F/k$ and any oscillations of the mass have to be around this new equilibrium.

We can rewrite the response of the system by measuring the displacement of the mass from the new equilibrium. Let $\tilde{x} = x - F/k$. Then, eqn. (11.41) becomes

$$\tilde{x} = \tilde{x}_0 \cos(\lambda t)$$

where $\tilde{x}_0 = x_0 - F/k$ is the initial displacement. Clearly, this is the response of an unforced harmonic oscillator. Thus the effect of a constant force on a spring-mass system is just a shift in its static equilibrium position.

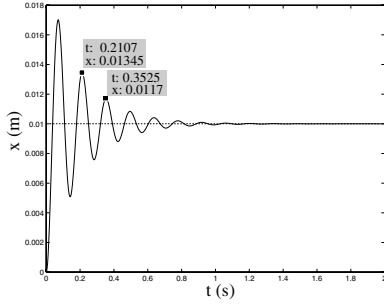


Figure 11.24

SAMPLE 11.8 A single degree of freedom damped oscillator has unknown mass, spring stiffness and damping coefficient. In order to find these quantities, the oscillator is subjected to a constant force $F_0 = 100$ N and its transient response is recorded. The response is shown in fig. 11.24. The two peaks marked in the response plot correspond to $(t, x) = (0.2107$ s, 0.01345 m) and $(0.3525$ s, 0.0117 m) respectively. Find the system parameters m , k , and c .

Solution Let the mass, stiffness, and damping coefficient of the system be m , k , and c , respectively. Then the equation of motion of the system, subjected to a constant force F_0 is,

$$m\ddot{x} + c\dot{x} + kx = F_0$$

where $x(t)$ is the displacement at some instant t . From the solution of this equation, we know that the steady state solution (after the transient oscillations die) is merely a shift in the static equilibrium position, given by F_0/k . From the given response, we see that

$$\frac{F_0}{k} = 0.01 \text{ m} \Rightarrow k = \frac{F_0}{0.01 \text{ m}} = \frac{100 \text{ N}}{0.01 \text{ m}} = 10 \text{ kN/m}.$$

Thus we have found one of the parameters, k . Now we need to find m and c .

Since two successive peaks are given in the transient response, we can use the logarithmic decrement to determine the damping ratio ξ from the relationship

$$\xi = \frac{1}{2\pi} \ln \left(\frac{x_n}{x_{n+1}} \right).$$

From the given data, $x_n = 0.01345$ m and $x_{n+1} = 0.0117$ m. Therefore, ①

$$\xi = \frac{1}{2\pi} \ln \left(\frac{0.01345 \text{ m}}{0.0117 \text{ m}} \right) = 0.022.$$

Since $\xi = c/c_c = c/(2\sqrt{km})$, we have

$$c = 2\xi\sqrt{km} = 0.044\sqrt{km} \quad (11.43)$$

This is just one equation in two unknowns, m and c (we already know k). So, we need another equation. From the peak to peak distance (in time), we can find the damped time period. That is $T_d = T_2 - T_1 = 0.3525 \text{ s} - 0.2107 \text{ s} = 0.1418 \text{ s}$. But, $T_d = 2\pi/\lambda_d$, and $\lambda_d = \lambda_n\sqrt{1 - \xi^2}$. Therefore,

$$\begin{aligned} \lambda_n^2 \equiv \frac{k}{m} &= \frac{\lambda_d^2}{1 - \xi^2} = \frac{4\pi^2}{T_d^2(1 - \xi^2)} \\ \Rightarrow m &= \frac{kT_d^2(1 - \xi^2)}{4\pi^2} \\ &= \frac{10000 \text{ N/m} \cdot (0.1418 \text{ s})^2(1 - 0.022^2)}{4\pi^2} \\ &= 5.09 \text{ kg}. \end{aligned}$$

Now substituting the value of m and k in eqn. (11.43), we get

$$c = 0.044\sqrt{10000 \text{ N/m} \cdot 5.09 \text{ kg}} = 9.92 \text{ kg/s}.$$

$$m = 5.09 \text{ kg}, \quad k = 10 \text{ kN/m}, \quad \text{and } c = 9.92 \text{ kg/s}.$$

① The calculation here is illustrative. In practice, you should not use just two data points for computing damping from the logarithmic decrement method. In general, we use several data points and determine the damping from taking an average of the decrements. See Sample 11.5 on page 574.

SAMPLE 11.9 Damping and forced response: When a single-degree-of-freedom damped oscillator (mass-spring-dashpot system) is subjected to a periodic forcing $F(t) = F_0 \sin(pt)$, then the response of the system is given by

$$x(t) = X \cos(pt - \phi)$$

where $X = \frac{F_0/k}{\sqrt{(2\xi r)^2 + (1-r^2)^2}}$, $\phi = \tan^{-1} \frac{2\xi r}{1-r^2}$, $r = \frac{p}{\lambda}$, $\lambda = \sqrt{k/m}$ and ξ is the damping ratio.

1. For $r \ll 1$, i.e., the forcing frequency p much smaller than the natural frequency λ , how does the damping ratio ξ affect the response amplitude X and the phase ϕ ?
2. For $r \gg 1$, i.e., the forcing frequency p much larger than the natural frequency λ , how does the damping ratio ξ affect the response amplitude X and the phase ϕ ?

Solution

1. If the frequency ratio $r \ll 1$, then r^2 will be even smaller; so we can ignore r^2 terms with respect to 1 in the expressions for X and ϕ . Thus, for $r \ll 1$,

$$\begin{aligned} X &= \frac{F_0/k}{\sqrt{(2\xi r)^2 + (1-r^2)^2}} \approx \frac{F_0/k}{1} = \frac{F_0}{k} \\ \phi &= \tan^{-1}(2\xi r) \approx \tan^{-1} 0 = 0 \end{aligned}$$

that is, the response amplitude does not vary with the damping ratio ξ , and the phase also remains constant at zero. As an example, we use the full expressions for X and ϕ for plotting them against ξ for $r = 0.01$ in fig. 11.25

For $r \ll 1$, $X \approx F_0/k$, and $\phi \approx 0$

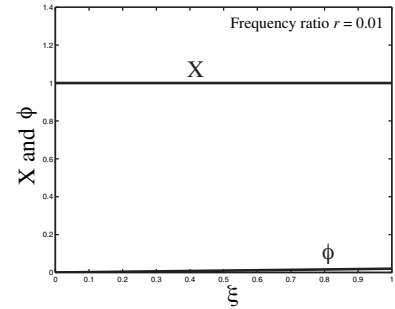


Figure 11.25

2. If $r \gg 1$, then the denominator in the expression for X , $4\xi^2 r^2 + (1-r^2)^2 \approx r^4$ (because we can ignore all other terms with respect to r^4). Similarly, we can ignore 1 with respect to r^2 in the expression for ϕ . Thus, for $r \gg 1$,

$$\begin{aligned} X &= \frac{F_0/k}{\sqrt{(2\xi r)^2 + (1-r^2)^2}} \approx \frac{F_0/k}{r^2} = 0 \\ \phi &= \tan^{-1} \frac{2\xi r}{-r^2} \approx \tan^{-1} \frac{2\xi}{-r} \approx \tan^{-1}(-0) = \pi. \end{aligned}$$

Once again, we see that the response amplitude and phase do not vary with ξ . This is also evident from fig. 11.26 where we plot X and ϕ using their full expressions for $r = 10$. The slight variation in ϕ around π goes away as we take higher values of r .

For $r \gg 1$, $X \approx 0$, and $\phi \approx \pi$

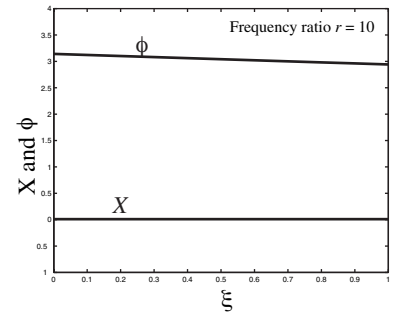


Figure 11.26

Thus, we see that the damping in a system does not affect the response of the system much if the forcing frequency is far away from the natural frequency.

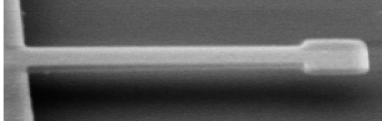


Figure 11.27: A MEMS cantilever resonator: Such resonators are typically silicon cantilever beams fabricated using micromachining process. These beams have geometric dimensions in micrometers and can be as small as a couple of micrometers long and a few nanometers in width and thickness.

SAMPLE 11.10 A MEMS (microelectromechanical system) cantilever resonator (shown in the figure) is modeled as a single degree of freedom oscillator (SDOF) oscillator. Using load deflection measurements, the stiffness of the beam (equivalent to the spring stiffness) is found to be 90 N/m . The beam is excited using electrical actuation and its resonant frequency is determined under two different conditions: (i) the beam vibrating in vacuum where the viscous damping is negligible, and (ii) the beam vibrating in ambient conditions where the airflow around it causes viscous damping. If the two frequencies are found to be 30 kHz and 28.4 kHz , respectively, find the equivalent mass m and the damping ratio for the SDOF model. If the beam is subjected to a periodic actuation at the free end by a force $F(t) = F \sin(2\pi f t)$ where $F = 50 \mu\text{N}$ and $f = 25 \text{ kHz}$, find the steady state displacement amplitude and the phase of the free end of the resonator.

Solution First we need to find m and c for the equivalent mass-spring-dashpot model. In the first case, where the resonant frequency is found in vacuum, we neglect damping, i.e., $c = 0$. Therefore, the given frequency is the natural frequency. However, it is f_n , not the circular natural frequency λ_n . Now, $\lambda_n = 2\pi f_n$, hence

$$\sqrt{\frac{k}{m}} = 2\pi f_n \Rightarrow m = \frac{k}{4\pi^2 f_n^2} = \frac{90 \text{ N/m}}{4\pi^2 (30000 \text{ s}^{-1})^2} = 2.533 \times 10^{-9} \text{ kg}.$$

We now use the damped natural frequency to find the damping ratio ξ . Since, we are given $f_d = 28.4 \text{ kHz}$, and we know that $\lambda_d = \lambda_n \sqrt{1 - \xi^2}$, we have

$$\begin{aligned} 2\pi f_d &= 2\pi f_n \sqrt{1 - \xi^2} \\ \xi &= \sqrt{1 - \left(\frac{f_d}{f_n}\right)^2} = \sqrt{1 - \left(\frac{28.4 \text{ kHz}}{30 \text{ kHz}}\right)^2} \\ &= 0.32. \end{aligned}$$

Now, we know the values of all system parameters for our SDOF model of the MEMS resonator — m , k and ξ (can find c if required from ξ , k and m). For the given sinusoidal forcing, the equation of motion of the SDOF oscillator is:

$$m\ddot{x} + c\dot{x} + kx = F \sin(2\pi f t).$$

We can write the steady state solution as the particular solution $x(t) = A_0 \sin(pt - \phi)$ where $p = 2\pi f$, and the displacement amplitude A_0 and the phase ϕ are given by the following expressions:

$$A_0 = \frac{F/k}{\sqrt{(2\xi r)^2 + (1 - r^2)^2}}, \quad \text{and} \quad \phi = \tan^{-1} \left(\frac{2\xi r}{1 - r^2} \right).$$

Since, $r = \frac{p}{\lambda_n} = \frac{2\pi f}{2\pi f_n} = \frac{25}{30} = 0.833$, we have,

$$\begin{aligned} A_0 &= \frac{(50 \times 10^{-6} \text{ N})/(90 \text{ N/m})}{\sqrt{(2 \cdot 0.32 \cdot 0.833)^2 + (1 - 0.833^2)^2}} \\ &= 9.04 \times 10^{-7} \text{ m} = 0.904 \mu\text{m}. \end{aligned}$$

Similarly, we find the phase as,

$$\begin{aligned} \phi &= \tan^{-1} \frac{2 \cdot 0.32 \cdot 0.833}{1 - 0.833^2} \\ &= 1.05 \text{ rad}. \end{aligned}$$

$$m = 2.533 \times 10^{-9} \text{ kg}, \xi = 0.32, A_0 = 0.904 \mu\text{m}, \text{ and } \phi = 1.05 \text{ rad}$$

SAMPLE 11.11 Energetics of resonance: Consider the response of a damped harmonic oscillator to a periodic forcing. Find the work done on the system by the periodic force during a single cycle of the force and show how this work varies with the forcing frequency and the damping ratio.

Solution Let us consider the damped harmonic oscillator shown in fig. 11.28 with $F(t) = F \sin(pt)$. The equation of motion of the system is $m\ddot{x} + c\dot{x} + kx = F \sin(pt)$ and the response of the system may be expressed as $X \sin(pt - \phi)$ where $X = (F/k)/\sqrt{(2\xi r)^2 + (1 - r^2)^2}$ and $\phi = \tan^{-1}(2\xi r/(1 - r^2))$, with $r = p/\lambda_n$, $\lambda_n = \sqrt{k/m}$ and $\xi = c/(2\sqrt{km})$.

We can compute the work done by the applied force on the system in one cycle by evaluating the integral

$$W = \int_{\text{one cycle}} F(t) dx$$

But, $x = X \sin(pt - \phi) \Rightarrow dx = Xp \cos(pt - \phi) dt$. Therefore,

$$\begin{aligned} W &= \int_0^{2\pi/p} F \sin(pt) \cdot Xp \cos(pt - \phi) dt \\ &= FXp \int_0^{2\pi/p} \sin(pt) \cos(pt - \phi) dt \\ &= FXp \int_0^{2\pi/p} \sin(pt) (\cos(pt) \cos \phi + \sin(pt) \sin \phi) dt \\ &= FXp \left[\cos \phi \cdot \frac{1}{2} \int_0^{2\pi/p} \sin(2pt) dt + \sin \phi \cdot \frac{1}{2} \int_0^{2\pi/p} (1 - \cos(2pt)) dt \right] \\ &= \frac{FXp}{2} \left[\cos \phi \left(-\frac{\cos(2pt)}{2p} \right) \Big|_0^{2\pi/p} + \sin \phi \left(t - \frac{\sin(2pt)}{2p} \right) \Big|_0^{2\pi/p} \right] \\ &= \frac{FXp}{2} \left[\frac{\cos \phi}{2p} (-1 + 1) + \frac{2\pi}{p} \sin \phi + 0 \right] \\ &= \frac{FXp}{2} \cdot \frac{2\pi}{p} \sin \phi \\ &= F\pi X \sin \phi \end{aligned}$$

Although the expression obtained above for W looks simple, we must substitute for X and ϕ to see the dependence of W on the damping ratio ξ and the frequency ratio r .

$$W = \frac{(F\pi) \cdot (F/k)}{\sqrt{(2\xi r)^2 + (1 - r^2)^2}} \cdot \frac{2\xi r}{\sqrt{(2\xi r)^2 + (1 - r^2)^2}} = \frac{(2\pi F^2 \xi r)/k}{(2\xi r)^2 + (1 - r^2)^2} \quad (11.44)$$

Unfortunately, this expression is too complicated to see the dependence of W on ξ and r . However, we know that for small $r (\ll 1)$, $\phi \approx 0$ and for large $r (\gg 1)$, $\phi \approx \pi$, implying that W is almost zero in both these cases. On the other hand, for r close to one, that is, close to resonance, $\phi \approx \pi/2 \Rightarrow \sin \phi \approx 1$, but the response amplitude X is large (for small ξ), which makes W to be big near the resonance. Figure 11.29 shows a plot of W against r , using eqn. (11.44), for different values of ξ . It is clear from the plot that the work done on the system in a single cycle is much larger close to the resonance for lightly damped systems. This explains why the response amplitude keeps on growing near resonance.

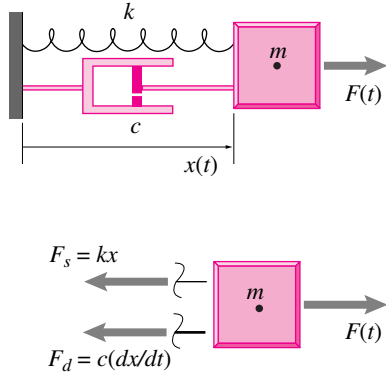


Figure 11.28

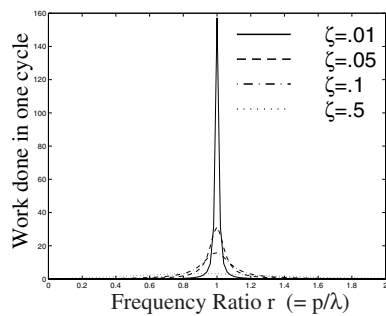


Figure 11.29

$$W = \pi F X \sin \phi$$

11.2 Forced Vibration and resonance

Preparatory Problems

11.2.1 Given that $\ddot{\theta} + k^2\theta = \beta \sin \omega t$, $\theta(0) = 0$, and $\dot{\theta}(0) = \dot{\theta}_0$, find $\theta(t)$. *

11.2.2 Three SDOF systems, each with the same mass but different stiffnesses, $k_1 = k$, $k_2 = 2k$, and $k_3 = 4k$, and different damping, $c_1 = c$, $c_2 = 2c$, and $c_3 = c/2$, are subjected to the same periodic forcing, $F = F_0 \sin pt$ where p is less than the resonant frequency of each of the systems. Sketch approximately the response of each of the three oscillators assuming all of them to be underdamped. Clearly mark the transient and steady state part of the response, and indicate the relative values of the response amplitudes.

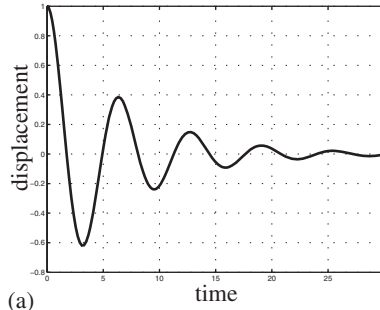
11.2.3 A 3 kg mass is suspended by a spring ($k = 10 \text{ N/m}$) and forced by a 5 N sinusoidally oscillating force with a period of 1 s. What is the amplitude of the steady-state oscillations (ignore the “homogeneous” solution) *

More-Involved Problems

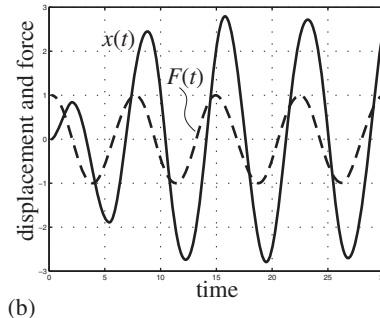
11.2.4 A machine that can be modeled as a SDOF system is put under vibration test for estimating the system parameters m , k , and c . First, a transient test is conducted by disturbing the machine from its equilibrium and letting it settle down to equilibrium again. The transient response is recorded as a displacement versus time plot and is shown in fig. 11.2.4(a). Next, a sinusoidal forcing of amplitude F_0 and angular frequency p is applied on the machine and its steady state response is recorded along with the forcing function. This response is shown in fig. 11.2.4(b).

- Mark the relevant points on the transient response plot and explain, with equations, which system parameters can be determined using what information from this plot.
- On the steady state plot, mark the phase difference between the response and the forcing function. From the given phase, can you find out whether $p > \lambda_n$ or $p < \lambda_n$?

- From the phase difference of the steady state response and the information obtained from the transient response, can you determine the frequency ratio r ? Explain with appropriate equations.
- From the amplitude of the steady state response, and the rest of the information obtained above, find the rest of the system parameters.



(a)



(b)

Problem 11.2.4

11.2.5 A machine produces a steady-state vibration due to a forcing function described by $Q(t) = Q_0 \sin \omega t$, where $Q_0 = 5000 \text{ N}$. The machine rests on a circular concrete foundation. The foundation rests on an isotropic, elastic half-space. The equivalent spring constant of the half-space is $k = 2,000,000 \text{ N/m}$ and has a damping ratio $d = c/c_c = 0.125$. The machine operates at a frequency of $\omega = 4 \text{ Hz}$.

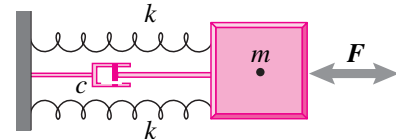
- What is the natural frequency of the system?
- If the system were undamped, what would the steady-state displacement be?
- What is the steady-state displacement given that $d = 0.125$?
- How much additional thickness of concrete should be added to the footing to reduce the damped steady-state amplitude by 50%? (The diameter must be held constant.)

11.2.6 The transient response of an oscillatory system shows exponential decay of the peak displacements at each cycle. The second peak is found to be twice as big as the fifth peak. Find the damping ratio ξ for the system. How many cycles does it take for the peak displacement to drop below 5% of the first peak displacement?

11.2.7 A 50 kg engine is mounted on springs with an equivalent single spring stiffness of 1200 N/m. Using various means, enough damping needs to be provided so that any unwanted vibration dies quickly. Assume that this objective is met by dissipating 80% of the available energy in a single cycle of vibration. Find the damping coefficient of the system.

11.2.8 Consider the system shown in the figure. You are given that $m = 10 \text{ kg}$, $k = 50 \text{ N/m}$, and $c = 5 \text{ kg/s}$. A periodic force $F = F_0 \cos pt$ acts on the system as shown where $F_0 = 25 \text{ N}$ and $p = 2.5 \text{ rad/s}$.

- Find the resonant frequency of the system.
- Find the steady state response of the system, specifying the amplitude and phase of the motion.
- What is the displacement amplification ($G = A/(F_0/k)$)?
- Find the work done by the force on the system in one cycle.
- Find the energy lost to the damper in one cycle.
- Find the quality factor, Q , of the system using the energy calculations.

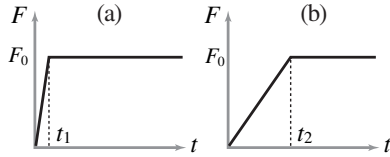


Problem 11.2.8

11.2.9 A MEMS cantilever beam resonator is used for mass measurement of biological molecules by comparing the shift in the resonant frequency of the beam after the test molecule is attached to the free end of the beam. In a SDOF model of the resonator, it is equivalent to finding the difference in the resonant

frequency of the system with mass m and $m + \Delta m$. If the 'effective mass' of the beam (mass to be used in the SDOF model) is 2.05×10^{-15} kg, the stiffness is 0.625 N/m, and the Q of the resonator is 900, find the shift in the resonant peak in Hz when a biological molecule of mass 1.36×10^{-21} kg is attached to the end of the beam (equivalently to the mass m).

11.2.10 A damped mass-spring system is subjected to a constant load $F_0 = 50$ N by ramping the load to the constant level in (a) $t_1 = 2$ s and (b) $t_2 = 10$ s. If the mass of the system $m = 1$ kg, the natural frequency $\lambda_n = 62$ rad/s, and the damping ratio $\xi = 0.2$, find the difference in the settling time of the system to the steady state between the two given cases.



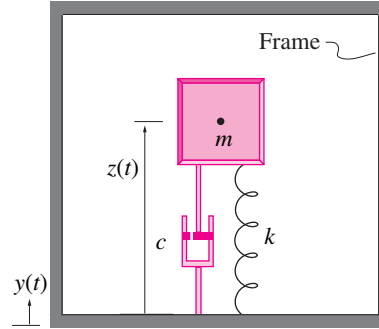
Problem 11.2.10

11.2.11 An accelerometer is a sensor that is used to measure acceleration of a body. It can be modeled as a single degree of freedom spring-mass-dashpot system that is attached to a body frame as shown in the figure. Assume that the body undergoes vertical motion denoted by $y(t)$ and, as a result, the mass of the accelerometer undergoes vertical motion $z(t)$ relative to the frame. From a measurement of $z(t)$ we want to know if we can determine the acceleration $\ddot{y}$ of the frame. Neglect gravity.

- What is the absolute or inertial acceleration of the mass in terms of $z(t)$ and $y(t)$ and their derivatives?
- Write the equation of motion of the mass in terms of z and y .
- Assume that $y(t) = y_0 \sin pt$. What is the magnitude of acceleration of the frame (that is, the peak acceleration of this sine wave)?
- Find the steady state response $z(t)$ of the accelerometer when $y(t) = y_0 \sin pt$ (a big mess). Plot the amplitude of the response vs the amplitude of the frame acceleration as the frequency p is varied.
- One would like a signal (z) that is proportional to acceleration $\ddot{y}$

independent of the frequency of shaking. Show that this requires that the frequency of frame motion p must be much smaller than the natural frequency λ_n .

- What is the amplitude of the response (the magnitude of z) of the accelerometer when $p \gg \lambda_n$?



Problem 11.2.11: A single degree of freedom spring-mass model of an accelerometer.

11.2.12 Consider the accelerometer described in Problem 11.2.11. Assume that the frame undergoes a sinusoidal motion given by $y(t) = y_0 \sin pt$.

- Find the response $z(t)$ of the accelerometer.
- Given that $m = 0.5$ kg, $k = 5$ kN/m, and $c = 10$ kg/s, find the maximum acceleration that the accelerometer can sense, assuming the accelerometer to work in the frequency range much below its natural frequency (i.e., $p/\lambda_n \ll 1$). Express your answer in terms of the gravitational acceleration g (it is customary to talk about acceleration of various things in terms of 'so many g 's').