SAMPLE 11.6 The mass-spring-dashpot system shown in the figure consists of a mass $m = 2\,\text{kg}$, a spring with stiffness $k = 3200\,\text{N/m}$ and a dashpot with damping coefficient $c = 10\,\text{kg/s}$.

1. Is the system underdamped, critically damped or overdamped?
2. Find the damped natural frequency of the system.
3. What is the resonant frequency of the system.

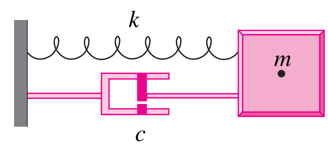

Figure 11.20

Solution

1. The question about underdamped, critically damped, or overdamped can be answered conveniently by computing the damping ratio ξ. For an underdamped system, $\xi < 1$, for a critically damped system, $\xi = 1$, and for an overdamped system $\xi > 1$. So, let us compute ξ. We know that

$$\xi = \frac{c}{c_c} = \frac{c}{2\sqrt{km}}.$$

Thus, for the given system,

$$\xi = \frac{10\,\text{kg/s}}{2\sqrt{3200\,\text{N/m}\cdot 2\,\text{kg}}} = \frac{10\,\text{kg/s}}{160\,\text{kg/s}} = 0.062.$$

Since $\xi < 1$, the system is underdamped.

$$\boxed{\text{Underdamped } (\xi = 0.062)}$$

2. The damped natural frequency, λ_d, is given by

$$\lambda_d = \lambda_n\sqrt{1-\xi^2}$$

where $\lambda_n = \sqrt{k/m}$ is the natural frequency of the system. Substituting the known values, we get

$$\lambda_d = \sqrt{\frac{3200\,\text{N/m}}{2\,\text{kg}}}\sqrt{1-(0.062)^2} = 39.92\,\text{rad/s}$$

which is almost the same as the natural frequency $\lambda_n = 40\,\text{rad/s}$.

$$\boxed{\lambda_d = 39.92\,\text{rad/s}}$$

3. The resonant frequency of the system, λ_r, is given by

$$\lambda_r = \lambda_n\sqrt{1-2\xi^2}.$$

Substituting the known values of λ_n and ξ, we get

$$\lambda_r = 39.85\,\text{rad/s}$$

which is the smallest among the three characteristic frequencies of the system — natural frequency, damped natural frequency, and the resonant frequency. For small values of ξ, however, the three frequencies are practically indistinguishable as is the case here.

$$\boxed{\lambda_r = 39.85\,\text{rad/s}}$$

SAMPLE 11.7 Response to a constant force: A constant force $F = 50\,\mathrm{N}$ acts on a mass-spring system as shown in the figure. Let $m = 5\,\mathrm{kg}$ and $k = 10\,\mathrm{kN/m}$.

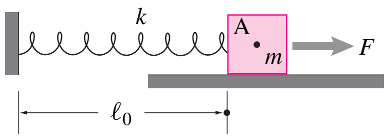

Figure 11.21

1. Write the equation of motion of the system.
2. If the system starts from the initial displacement $x_0 = 0.01\,\mathrm{m}$ with zero velocity, find the displacement of the mass as a function of time.
3. Plot the response (displacement) of the system against time and describe how it is different from the unforced response of the system.

Solution

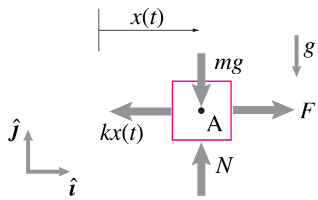

Figure 11.22: Free-body diagram of the mass.

1. The free-body diagram of the mass is shown in fig. 11.22 at a displacement x (assumed positive to the right). Applying linear momentum balance in the x-direction, *i.e.*, $(\sum \vec{F} = m\vec{a}) \cdot \hat{\imath}$, we get

$$\begin{aligned} F - kx &= m\ddot{x} \\ \Rightarrow \quad m\ddot{x} + kx &= F \end{aligned} \tag{11.38}$$

which is the equation of motion of the system.

2. The equation of motion has a non-zero right hand side. Thus, it is a nonhomogeneous differential equation. A general solution of this equation is made up of two parts — the homogeneous solution x_h which is the solution of the unforced system (eqn. (11.38) with $F = 0$), and a particular solution x_p that satisfies the nonhomogeneous equation. Thus,

$$x(t) = x_h(t) + x_p(t). \tag{11.39}$$

Now, let us find $x_h(t)$ and $x_p(t)$.

Homogeneous solution: $x_h(t)$ has to satisfy the homogeneous equation

$$m\ddot{x} + kx = 0.$$

Let $\lambda = \sqrt{k/m}$. Then, from the solution of unforced harmonic oscillator, we know that

$$x_h(t) = A\sin(\lambda t) + B\cos(\lambda t)$$

where A and B are constants to be determined later from initial conditions.

Particular solution: x_p must satisfy eqn. (11.38). Since the nonhomogeneous part of the equation is a constant (F), we guess that x_p must be a constant too (of the same form as F). Let $x_p = C$. Now we substitute $x_p = C$ in eqn. (11.38) and solve the resulting equation to determine C:

$$m\underbrace{\ddot{C}}_{0} + kC = F \quad \Rightarrow \quad C = F/k \quad \text{or} \quad x_p = F/k.$$

Substituting x_h and x_p in eqn. (11.39), we get

$$x(t) = A\sin(\lambda t) + B\cos(\lambda t) + F/k. \tag{11.40}$$

Now we use the given initial conditions to determine A and B.

$$\begin{aligned} x(t=0) &= B + F/k = x_0 \text{ (given)} \quad \Rightarrow \quad B = x_0 - F/k \\ \dot{x}(t) &= A\lambda\cos(\lambda t) - B\lambda\sin(\lambda t) \\ \Rightarrow \quad \dot{x}(t=0) &= A = 0 \text{ (given)} \quad \Rightarrow \quad A = 0. \end{aligned}$$

Thus,

$$x(t) = (x_0 - F/k)\cos(\lambda t) + F/k, \tag{11.41}$$

$$\text{and} \quad \dot{x}(t) = -\lambda(x_0 - F/k)\sin(\lambda t). \tag{11.42}$$

3. Let us plug the given numerical values, $k = 10\,\mathrm{kN/m}, m = 5\,\mathrm{kg}$, (which gives $\lambda = \sqrt{k/m} = 44.72\,\mathrm{rad/s}$), $F = 50\,\mathrm{N}$ and $x_0 = 0.01\,\mathrm{m}$ in eqn. (11.41) and (11.42). The displacement and the velocity are now given as

$$\begin{aligned} x(t) &= (0.005\,\mathrm{m})\cos(44.72\,\mathrm{rad/s}\cdot t) + 0.005\,\mathrm{m}, \\ \text{and} \quad \dot{x}(t) &= -(0.22\,\mathrm{m/s})\sin(44.72\,\mathrm{rad/s}\cdot t). \end{aligned}$$

This response is plotted in fig. 11.23 against time. Note that the oscillations of the mass are about a non-zero mean value, $x_{\mathrm{eq}} = 0.005\,\mathrm{m}$. A little thought should reveal that this is what we should expect. When a mass hangs from a spring under gravity, the spring elongates a little, by mg/k to be precise, to balance the mass. Thus, the new static equilibrium position is not at the relaxed length ℓ_0 of the spring but at $\ell_0 + mg/k$. Any oscillations of the mass will be about this new equilibrium. The velocity, however, has a zero mean value which is what we expect from eqn. (11.42).

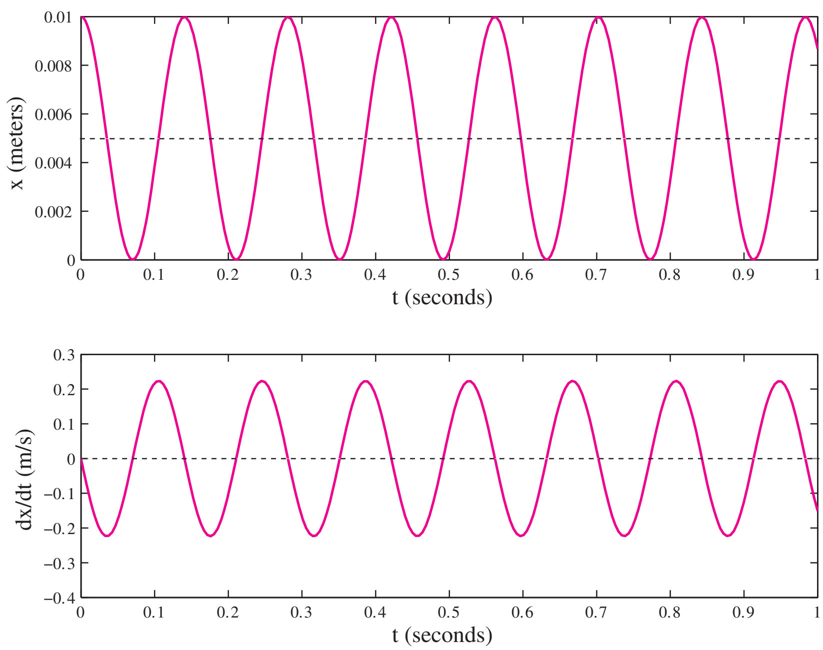

Figure 11.23: Displacement of the mass as a function of time. Note that the mass oscillates about a non-zero value of x.

This problem is exactly like a mass hanging from a spring under gravity, a constant force, but just rotated by 90°. The new static equilibrium is at $x_{\mathrm{eq}} = F/k$ and any oscillations of the mass have to be around this new equilibrium.

We can rewrite the response of the system by measuring the displacement of the mass from the new equilibrium. Let $\tilde{x} = x - F/k$. Then, eqn. (11.41) becomes

$$\tilde{x} = \tilde{x}_0 \cos(\lambda t)$$

where $\tilde{x}_0 = x_0 - F/k$ is the initial displacement. Clearly, this is the response of an unforced harmonic oscillator. Thus the effect of a constant force on a spring-mass system is just a shift in its static equilibrium position.

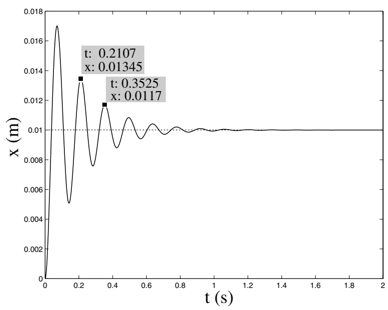

Figure 11.24

SAMPLE 11.8 A single degree of freedom damped oscillator has unknown mass, spring stiffness and damping coefficient. In order to find these quantities, the oscillator is subjected to a constant force $F_0 = 100$ N and its transient response is recorded. The response is shown in fig. 11.24. The two peaks marked in the response plot correspond to $(t, x) = (0.2107$ s, 0.01345 m$)$ and $(0.3525$ s, 0.0117 m$)$ respectively. Find the system parameters m, k, and c.

Solution Let the mass, stiffness, and damping coefficient of the system be m, k, and c, respectively. Then the equation of motion of the system, subjected to a constant force F_0 is,

$$m\ddot{x} + c\dot{x} + kx = F_0$$

where $x(t)$ is the displacement at some instant t. From the solution of this equation, we know that the steady state solution (after the transient oscillations die) is merely a shift in the static equilibrium position, given by F_0/k. From the given response, we see that

$$\frac{F_0}{k} = 0.01\,\mathrm{m} \quad \Rightarrow \quad k = \frac{F_0}{0.01\,\mathrm{m}} = \frac{100\,\mathrm{N}}{0.01\,\mathrm{m}} = 10\,\mathrm{kN/m}.$$

Thus we have found one of the parameters, k. Now we need to find m and c.

Since two successive peaks are given in the transient response, we can use the logarithmic decrement to determine the damping ratio ξ from the relationship

$$\xi = \frac{1}{2\pi} \ln\left(\frac{x_n}{x_{n+1}}\right).$$

From the given data, $x_n = 0.01345$ m and $x_{n+1} = 0.0117$ m. Therefore, (1)

(1) The calculation here is illustrative. In practice, you should not use just two data points for computing damping from the logarithmic decrement method. In general, we use several data points and determine the damping from taking an average of the decrements. See Sample 11.5 on page 574.

$$\xi = \frac{1}{2\pi} \ln\left(\frac{0.01345\,\mathrm{m}}{0.0117\,\mathrm{m}}\right) = 0.022.$$

Since $\xi = c/c_c = c/(2\sqrt{km})$, we have

$$c = 2\xi\sqrt{km} = 0.044\sqrt{km} \tag{11.43}$$

This is just one equation in two unknowns, m and c (we already know k). So, we need another equation. From the peak to peak distance (in time), we can find the damped time period. That is $T_d = T_2 - T_1 = 0.3525\,\mathrm{s} - 0.2107\,\mathrm{s} = 0.1418\,\mathrm{s}$. But, $T_d = 2\pi/\lambda_d$, and $\lambda_d = \lambda_n\sqrt{1-\xi^2}$. Therefore,

$$\begin{aligned}
\lambda_n^2 \equiv \frac{k}{m} &= \frac{\lambda_d^2}{1-\xi^2} = \frac{4\pi^2}{T_d^2(1-\xi^2)} \\
\Rightarrow \quad m &= \frac{kT_d^2(1-\xi^2)}{4\pi^2} \\
&= \frac{10000\,\mathrm{N/m}\cdot(0.1418\,\mathrm{s})^2(1-0.022^2)}{4\pi^2} \\
&= 5.09\,\mathrm{kg}.
\end{aligned}$$

Now substituting the value of m and k in eqn. (11.43), we get

$$c = 0.044\sqrt{10000\,\mathrm{N/m}\cdot 5.09\,\mathrm{kg}} = 9.92\,\mathrm{kg/s}.$$

$m = 5.09\,\mathrm{kg}, \quad k = 10\,\mathrm{kN/m}, \text{ and } c = 9.92\,\mathrm{kg/s}.$

SAMPLE 11.9 **Damping and forced response:** When a single-degree-of-freedom damped oscillator (mass-spring-dashpot system) is subjected to a periodic forcing $F(t) = F_0 \sin(pt)$, then the response of the system is given by

$$x(t) = X \cos(pt - \phi)$$

where $X = \frac{F_0/k}{\sqrt{(2\xi r)^2 + (1-r^2)^2}}$, $\phi = \tan^{-1} \frac{2\xi r}{1-r^2}$, $r = \frac{p}{\lambda}$, $\lambda = \sqrt{k/m}$ and ξ is the damping ratio.

1. For $r \ll 1$, *i.e.,* the forcing frequency p much smaller than the natural frequency λ, how does the damping ratio ξ affect the response amplitude X and the phase ϕ?
2. For $r \gg 1$, *i.e.,* the forcing frequency p much larger than the natural frequency λ, how does the damping ratio ξ affect the response amplitude X and the phase ϕ?

Solution

1. If the frequency ratio $r \ll 1$, then r^2 will be even smaller; so we can ignore r^2 terms with respect to 1 in the expressions for X and ϕ. Thus, for $r \ll 1$,

$$\begin{aligned} X &= \frac{F_0/k}{\sqrt{(2\xi r)^2 + (1-r^2)^2}} \approx \frac{F_0/k}{1} = \frac{F_0}{k} \\ \phi &= \tan^{-1}(2\xi r) \approx \tan^{-1} 0 = 0 \end{aligned}$$

that is, the response amplitude does not vary with the damping ratio ξ, and the phase also remains constant at zero. As an example, we use the full expressions for X and ϕ for plotting them against ξ for $r = 0.01$ in fig. 11.25

For $r \ll 1, X \approx F_0/k,$ and $\phi \approx 0$

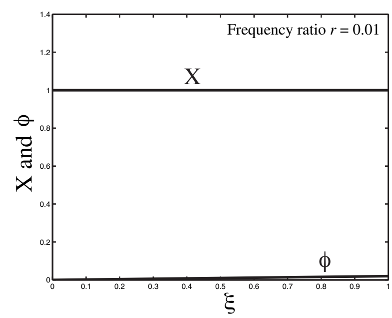

Figure 11.25

2. If $r \gg 1$, then the denominator in the expression for X, $4\xi^2 r^2 + (1-r^2)^2 \approx r^4$ (because we can ignore all other terms with respect to r^4. Similarly, we can ignore 1 with respect to r^2 in the expression for ϕ. Thus, for $r \gg 1$,

$$\begin{aligned} X &= \frac{F_0/k}{\sqrt{(2\xi r)^2 + (1-r^2)^2}} \approx \frac{F_0/k}{r^2} = 0 \\ \phi &= \tan^{-1} \frac{2\xi r}{-r^2} \approx \tan^{-1} \frac{2\xi}{-r} \approx \tan^{-1}(-0) = \pi. \end{aligned}$$

Once again, we see that the response amplitude and phase do not vary with ξ. This is also evident from fig. 11.26 where we plot X and ϕ using their full expressions for $r = 10$. The slight variation in ϕ around π goes away as we take higher values of r.

For $r \gg 1, X \approx 0,$ and $\phi \approx \pi$

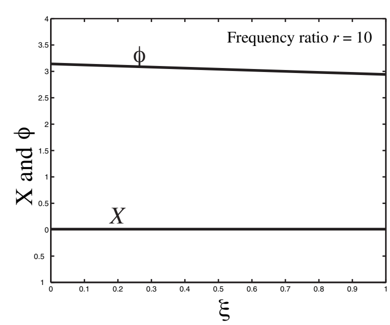

Figure 11.26

Thus, we see that the damping in a system does not affect the response of the system much if the forcing frequency is far away from the natural frequency.

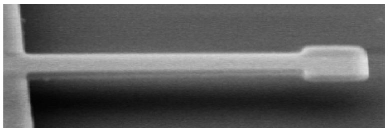

Figure 11.27: A MEMS cantilever resonator: Such resonators are typically silicon cantilever beams fabricated using micromachining process. These beams have geometric dimensions in micrometers and can be as small as a couple of micrometers long and a few nanometers in width and thickness.

SAMPLE 11.10 A MEMS (microelectromechanical system) cantilever resonator (shown in the figure) is modeled as a single degree of freedom oscillator (SDOF) oscillator. Using load deflection measurements, the stiffness of the beam (equivalent to the spring stiffness) is found to be 90 N/m. The beam is excited using electrical actuation and its resonant frequency is determined under two different conditions: (i) the beam vibrating in vacuum where the viscous damping is negligible, and (ii) the beam vibrating in ambient conditions where the airflow around it causes viscous damping. If the two frequencies are found to be 30 kHz and 28.4 kHz, respectively, find the equivalent mass m and the damping ratio for the SDOF model. If the beam is subjected to a periodic actuation at the free end by a force $F(t) = F\sin(2\pi f t)$ where $F = 50\mu$ N and $f = 25$ kHz, find the steady state displacement amplitude and the phase of the free end of the resonator.

Solution First we need to find m and c for the equivalent mass-spring-dashpot model. In the first case, where the resonant frequency is found in vacuum, we neglect damping, *i.e.,* $c = 0$. Therefore, the given frequency is the natural frequency. However, it is f_n, not the circular natural frequency λ_n. Now, $\lambda_n = 2\pi f_n$, hence

$$\sqrt{\frac{k}{m}} = 2\pi f_n \quad \Rightarrow \quad m = \frac{k}{4\pi^2 f_n^2} = \frac{90\,\mathrm{N/m}}{4\pi^2(30000\,\mathrm{s^{-1}})^2} = 2.533\times10^{-9}\,\mathrm{kg}.$$

We now use the damped natural frequency to find the damping ratio ξ. Since, we are given $f_d = 28.4$ kHz, and we know that $\lambda_d = \lambda_n\sqrt{(1-\xi^2)}$, we have

$$\begin{aligned}
2\pi f_d &= 2\pi f_n\sqrt{1-\xi^2}\\
\xi &= \sqrt{1-\left(\frac{f_d}{f_n}\right)^2} = \sqrt{1-\left(\frac{28.4\,\mathrm{kHz}}{30\,\mathrm{kHz}}\right)^2}\\
&= 0.32.
\end{aligned}$$

Now, we know the values of all system parameters for our SDOF model of the MEMS resonator — m, k and ξ (can find c if required from ξ, k and m). For the given sinusoidal forcing, the equation of motion of the SDOF oscillator is:

$$m\ddot{x} + c\dot{x} + kx = F\sin(2\pi f t).$$

We can write the steady state solution as the particular solution $x(t) = A_0\sin(pt-\phi)$ where $p = 2\pi f$, and the displacement amplitude A_0 and the phase ϕ are given by the following expressions:

$$A_0 = \frac{F/k}{\sqrt{(2\xi r)^2 + (1-r^2)^2}}, \quad \text{and} \quad \phi = \tan^{-1}\left(\frac{2\xi r}{1-r^2}\right).$$

Since, $r = \frac{p}{\lambda_n} = \frac{2\pi f}{2\pi f_n} = \frac{25}{30} = 0.833$, we have,

$$\begin{aligned}
A_0 &= \frac{(50^{-6}\,\mathrm{N})/(90\,\mathrm{N/m})}{\sqrt{(2\cdot0.32\cdot0.833)^2 + (1-0.833^2)^2}}\\
&= 9.04\times10^{-7}\,\mathrm{m} = 0.904\mu\mathrm{m}.
\end{aligned}$$

Similarly, we find the phase as,

$$\begin{aligned}
\phi &= \tan^{-1}\frac{2\cdot0.32\cdot0.833}{1-0.833^2}\\
&= 1.05\,\mathrm{rad}.
\end{aligned}$$

$m = 2.533 \times 10^{-9}$ kg, $\xi = 0.32$, $A_0 = 0.904 \mu\text{m}$, and $\phi = 1.05$ rad

SAMPLE 11.11 **Energetics of resonance:** Consider the response of a damped harmonic oscillator to a periodic forcing. Find the work done on the system by the periodic force during a single cycle of the force and show how this work varies with the forcing frequency and the damping ratio.

Solution Let us consider the damped harmonic oscillator shown in fig. 11.28 with $F(t) = F\sin(pt)$. The equation of motion of the system is $m\ddot{x} + c\dot{x} + kx = F\sin(pt)$ and the response of the system may be expressed as $X\sin(pt-\phi)$ where $X = (F/k)/\sqrt{(2\xi r)^2 + (1-r^2)^2}$ and $\phi = \tan^{-1}(2\xi r/(1-r^2))$, with $r = p/\lambda_n$, $\lambda_n = \sqrt{k/m}$ and $\xi = c/(2\sqrt{km})$.

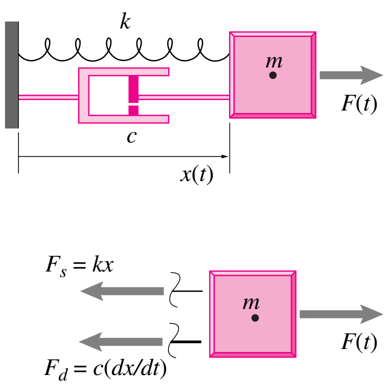

Figure 11.28

We can compute the work done by the applied force on the system in one cycle by evaluating the integral

$$W = \int_{\text{onecycle}} F(t)\,dx$$

But, $x = X\sin(pt-\phi) \quad \Rightarrow \quad dx = Xp\cos(pt-\phi)dt$. Therefore,

$$\begin{aligned}
W &= \int_0^{2\pi/p} F\sin(pt)\cdot Xp\cos(pt-\phi)\,dt \\
&= FXp\int_0^{2\pi/p} \sin(pt)\cos(pt-\phi)\,dt \\
&= FXp\int_0^{2\pi/p} \sin(pt)\left(\cos(pt)\cos\phi + \sin(pt)\sin\phi\right)\,dt \\
&= FXp\left[\cos\phi\cdot\frac{1}{2}\int_0^{2\pi/p}\sin(2pt)\,dt + \sin\phi\cdot\frac{1}{2}\int_0^{2\pi/p}(1-\cos(2pt))\,dt\right] \\
&= \frac{FXp}{2}\left[\cos\phi\left(-\frac{\cos(2pt)}{2p}\right)\Bigg|_0^{2\pi/p} + \sin\phi\left(t - \frac{\sin(2pt)}{2p}\right)\Bigg|_0^{2\pi/p}\right] \\
&= \frac{FXp}{2}\left[\frac{\cos\phi}{2p}(-1+1) + \frac{2\pi}{p}\sin\phi + 0\right] \\
&= \frac{FXp}{2}\cdot\frac{2\pi}{p}\sin\phi \\
&= F\pi X\sin\phi
\end{aligned}$$

Although the expression obtained above for W looks simple, we must substitute for X and ϕ to see the dependence of W on the damping ratio ξ and the frequency ratio r.

$$W = \frac{(F\pi)\cdot(F/k)}{\sqrt{(2\xi r)^2 + (1-r^2)^2}}\cdot\frac{2\xi r}{\sqrt{(2\xi r)^2 + (1-r^2)^2}} = \frac{(2\pi F^2\xi r)/k}{(2\xi r)^2 + (1-r^2)^2} \tag{11.44}$$

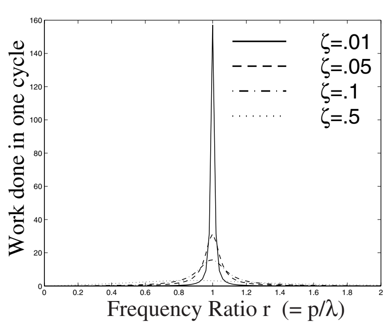

Figure 11.29

Unfortunately, this expression is too complicated to see the dependence of W on ξ and r. However, we know that for small $r(\ll 1)$, $\phi \approx 0$ and for large $r(\gg 1)$, $\phi \approx \pi$, implying that W is almost zero in both these cases. On the other hand, for r close to one, that is, close to resonance, $\phi \approx \pi/2 \Rightarrow \sin\phi \approx 1$, but the response amplitude X is large (for small ξ), which makes W to be big near the resonance. Figure 11.29 shows a plot of W against r, using eqn. (11.44), for different values of ξ. It is clear from the plot that the work done on the system in a single cycle is much larger close to the resonance for lightly damped systems. This explains why the response amplitude keeps on growing near resonance.

$$\boxed{W = \pi FX\sin\phi}$$