10.1 Force and motion in 1D

Now we focus on special problems in which one particle moves on a straight line. With motion in only one direction, the kinematics is simple. It's essentially a rehash of freshman calculus. Even in 1D, vectors can be useful because of their help with signs. But vectors are not really needed and we will not be zealous in their use (in this one chapter).

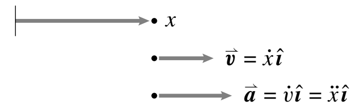


Figure 10.1: One-dimensional position, velocity, and acceleration. Positive values for $x, \dot{x}$ and $\ddot{x}$ describe positions, velocities and accelerations in the positive x direction.

Position, velocity, and acceleration in one dimension

We can call the direction of motion the $\hat{\imath}$ direction and the position of the particle, the distance of a particle in the $+\hat{\imath}$ direction from a reference point, x (see fig. 10.1). The particle might have some spacial extent, so to be precise we can define x as the x coordinate of the particle's center-of-mass. We can write the position $\vec{r}$, velocity $\vec{v}$ and acceleration $\vec{a}$ as

$$\begin{array}{llllll} & \text{position} & \vec{r} = x\hat{\imath} & & & = x\hat{\imath}, \\ & \text{velocity} & \vec{v} = v\hat{\imath} & = \dfrac{dx}{dt}\hat{\imath} & & = \dot{x}\hat{\imath}, \\ \text{and} & \text{acceleration} & \vec{a} = a\hat{\imath} & = \dfrac{dv}{dt}\hat{\imath} & = \dfrac{d^2x}{dt^2}\hat{\imath} & = \ddot{x}\hat{\imath}. \end{array} \tag{10.1}$$

Figure 10.2 shows example graphs of $x(t)$ and $v(t)$ versus time.

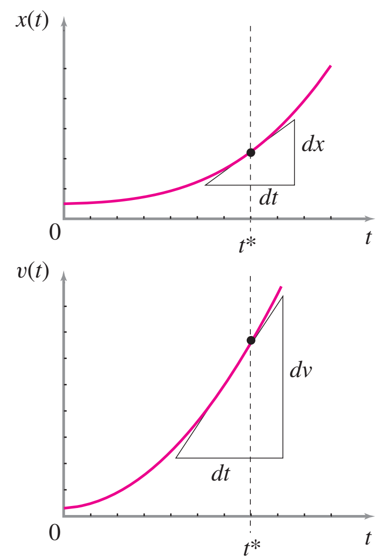


Figure 10.2: Graphs of $x(t)$ and $v(t) = \frac{dx}{dt}$ versus time. The slope of the position curve dx/dt at t^* is $v(t^*)$. And the slope of the velocity curve dv/dt at t^* is $a(t^*)$.

Signs. Vectors help with signs. When not using vectors we will take v and a to be positive if they have the same direction as increasing x (or y or whatever coordinate describes position). Even though we pedantically declare that 'velocity is a vector' and 'acceleration is a vector', we will loosely use the words 'velocity' and 'acceleration' to stand for scalars using these sign conventions.

Example: Position, velocity, and acceleration in one dimension
If position is given as

$$\begin{array}{ll} & x(t) = 3e^{4t/\mathrm{s}}\,\mathrm{m} \\ \text{then} & v(t) = dx/dt = 12e^{4t/\mathrm{s}}\,\mathrm{m/s} \\ \text{and} & a(t) = dv/dt = 48e^{4t/\mathrm{s}}\,\mathrm{m/s^2}. \end{array}$$

So at, say, time $t = 2\,\mathrm{s}$ the acceleration is

$$a\big|_{t=2\,\mathrm{s}} = 48e^{4\cdot 2\,\mathrm{s}/\mathrm{s}}(\mathrm{m/s^2}) = 48 \cdot e^8\,\mathrm{m/s^2} \approx 1.43 \cdot 10^5\,\mathrm{m/s^2}.$$

Fussing with units. In this example above we used 1/s as part of the argument of the exponential function. Thus when the exponential $e^{4t/\mathrm{s}}$ is differentiated with respect to time t the 1/s is grouped with the 4 as the coefficient of t in the exponential. So the factor 4/s comes out front

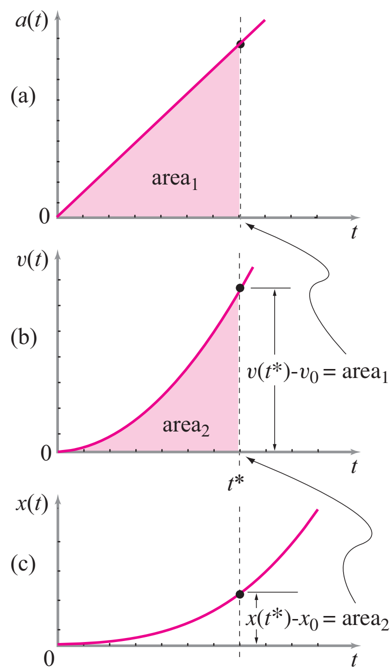


Figure 10.3: One-dimensional kinematics of a particle: (a) is a graph of the acceleration of a particle $a(t)$; (b) is a graph of the particle velocity $v(t)$ and the integral of $a(t)$ from $t_0 = 0$ to t^*, the shaded area under the acceleration curve; (c) is the position of the particle $x(t)$ and the integral of $v(t)$ from $t_0 = 0$ to t^*, the shaded area under the velocity curve.

① To cover the range of calculus problems you need to be a very good rider, however, and be able to ride frontwards, backwards, at zero speed and infinitely fast.

according to the chain rule of differentiation. Treating units as quantities, manipulated like all others makes the answer come out with the right units also. Note in the last line that, when the dimensional quantity (2 s) is substituted in for the variable t, the units cancel. For more on units see appendix 4.

1D kinematics ⇔ calculus. One-dimensional kinematics problems can include almost all of the skills in elementary calculus. In kinematics you are often given position, velocity or acceleration as function of time and you have to differentiate or integrate to find one of the other quantities. For example, if you are given the velocity $v(t)$ as a function of time and are asked to find the acceleration $a(t)$, you have to differentiate. If instead you were asked to find the position $x(t)$, you would calculate an integral (see fig. 10.3). Using the fundamental theorem of calculus, we get the integral versions of the relations between position, velocity, and acceleration (see fig. 10.3).

$$\begin{aligned} x(t) &= x_0 + \int_{t_o}^{t} v(\tau)\,d\tau \qquad \text{with} \qquad x_0 = x(t_0), \text{ and} \\ v(t) &= v_0 + \int_{t_o}^{t} a(\tau)\,d\tau \qquad \text{with} \qquad v_0 = v(t_0). \end{aligned}$$

With indefinite notation, these equations can also be written as:

$$\begin{aligned} x &= \textstyle\int v\,dt \\ v &= \textstyle\int a\,dt. \end{aligned}$$

If acceleration is given as a function of time, then position is found by integrating twice.

1D kinematics, bicycles and calculus. To put it another way, almost every calculus question could be phrased as a question about a bicycle speedometer①. With a bicycle speedometer (which includes a distance-measuring odometer) you can read your speed and distance travelled as functions of time. And given one of those two functions you could find the other using calculus. Acceleration is also of interest, but few bicycle speedometers also measure acceleration.

Differential equations

A *differential equation* is an equation that involves derivatives. Thus the equation relating position to velocity is

$$\frac{dx}{dt} = v \qquad \text{or, more explicitly} \qquad \frac{dx(t)}{dt} = v(t),$$

is a differential equation. An *ordinary differential equation* (ODE) is an equation that contains some terms that are ordinary derivatives (as opposed to partial derivatives and partial differential equations which we don't use in this book).

Example: Calculating a derivative solves an ODE
Given that the height of an elevator as a function of time on its 5 seconds long 3 meter trip from the first to second floor is

$$y(t) = (3\,\mathrm{m})\frac{\left(1-\cos\left(\frac{\pi t}{5\,\mathrm{s}}\right)\right)}{2}$$

we can solve the differential equation $v = \frac{dy}{dt}$ by differentiating to get

$$v = \frac{dy}{dt} = \frac{d}{dt}\left[(3\,\mathrm{m})\frac{\left(1-\cos\left(\frac{\pi t}{5\,\mathrm{s}}\right)\right)}{2}\right] = \frac{3\pi}{10}\sin\left(\frac{\pi t}{5\,\mathrm{s}}\right)\ \mathrm{m/s}$$

Note: this would be a harsh elevator because of the jump in the acceleration (not calculated above) at the start and stop.

A little less trivial is the case when you want to find a function when you are given the derivative.

Example: Integration solves a simple ODE
Assume that you start at home ($x = 0$) and, over about 30 seconds, you accelerate towards a steady-state speed of $4\,\mathrm{m/s}$ according to (see fig. 10.4)

$$v(t) = 4(1 - e^{-t/(30\,\mathrm{s})})\,\mathrm{m/s}.$$

Your ride lasts 1000 seconds. We can find how far you go by solving

$$\dot{x} = v(t) \quad \text{with the initial condition} \quad x(0) = 0.$$

This is simply solved by integration. Say, after 1000 seconds

$$\begin{aligned}
x(t = 1000\,\mathrm{s}) &= \int_0^{1000\,\mathrm{s}} v(t)\,dt = \int_0^{1000\,\mathrm{s}} 4(1 - e^{-t/(30\,\mathrm{s})})(\,\mathrm{m/s})\,dt \\
&= \left(4t + (120\,\mathrm{s})e^{-t/(30\,\mathrm{s})}\right)\Big|_0^{1000\,\mathrm{s}}\ \mathrm{m/s} \\
&= \left((4\cdot 1000\,\mathrm{s} + (120\,\mathrm{s})e^{-100/3}) - (0 + (120\,\mathrm{s})e^{0})\right)\ \mathrm{m/s} \\
&= \left(4000 - 120 + 120e^{-100/3})\right)\ \mathrm{m} \\
&\approx 3880\,\mathrm{m} \qquad \text{(to within an angstrom or so)}
\end{aligned}$$

This is only $120\,\mathrm{m}$ less than if the whole trip was travelled at a steady $4\,\mathrm{m/s}$ (then $x = 4\,\mathrm{m/s} \times 1000\,\mathrm{s} = 4000\,\mathrm{m}$).

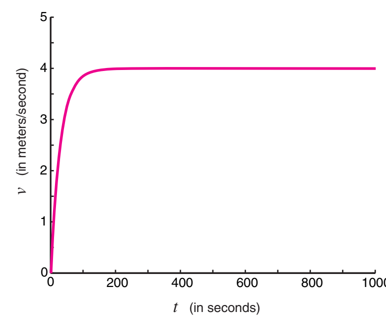


Figure 10.4: Velocity (speed) vs time for a 1000 s bike trip. The rider increases speed to exponentially approach 4 m/s.

Unlike the integral above, many integrals cannot be evaluated by hand (analytically).

Example: An ODE that leads to an intractable integral
Assume now that

$$v(t) = \frac{4t}{t + e^{-t/(30\,\mathrm{s})}\,\mathrm{s}}\ \mathrm{m}.$$

Again we have a bike trip where you start at zero speed and approach a steady speed of $4\,\mathrm{m/s}$. So your position as a function of time should be similar. Following the last example, we have

$$\dot{x} = v(t) \quad \text{with the initial condition} \quad x(0) = 0$$

with the given $v(t)$. The integral for position is then

$$x(t = 1000\,\mathrm{s}) \;=\; \int_0^{1000\,\mathrm{s}} v(t)\,dt = \int_0^{1000\,\mathrm{s}} \frac{4t}{t+e^{-t/(30\mathrm{s})}\,\mathrm{s}}\,\mathrm{m}\,dt \qquad (10.2)$$

$$= \;\ldots$$

which is the kind of thing you have nightmares about seeing on an exam. You couldn't solve this integral if your life depended on it. No one could. There is no formula for $x(t)$ that solves the differential equation, unless you regard eqn. (10.2) as a formula. In days of old they would say 'the problem has been reduced to quadrature' meaning that the remaining work was evaluating an integral②, even if they didn't know how to evaluate it exactly.

②Literally *quadrature* means finding a square (a 'quad') with area equal to the area under a given curve. The phrase 'reduced to quadrature' is used more generally to mean integration, even if the integration is of several variables.

Just because a differential equation can't be solved analytically with pencil and paper doesn't mean it can't be solved numerically. Most often the setup for numerical solution is not that difficult. Note that for numerical solution you either need dimensionless calculations, or at least need all variables to be in consistent units.

Example: Numerical solution of ODE
One of many ways to evaluate the integral of the above example numerically is by the following pseudo code.

```
ODE = { xdot = 4 t / (t+e^(-t/30)) }
IC  = { x(0) = 0  }
solve ODE with IC and evaluate at t=1000
```

The result is $x \approx 3988\,\mathrm{m}$ which is also, as expected because of the similarity with the previous example, only slightly shy of the steady-speed approximation of 4000 m. ③

③**Numerical error vs real difference.** When you notice such small differences (12m out of 4000m) based on computer calculation you need to question whether the difference is something real in the problem or, rather, is due to numerical errors. Two ways to check are with so-called convergence tests and by using your canned package's error estimate. We checked that the numerical integration is accurate to about 10^{-3}m, less than the 1m resolution that we printed (no need to type lots of digits with little information). Thus the $\approx$ 12m difference between the constant v solution and the solution where v approaches a constant, is a real difference. The startup is genuinely quicker in the second example (12 m behind constant v versus 120 m behind constant v), and not a numerical artifact.

More differential equations.

As mentioned, because dynamics equations contain derivatives they are all differential equations. A catalogue of the simplest differential equations and their solutions is given in box 3.1 on page 203.

The equations of dynamics

We want to understand kinetics (mechanics, dynamics), not just kinematics. The subject of mechanics is held up by the three pillars of material properties, geometry, and 'Newton's laws' (see page 28). Here we begin to flesh out the 'Newton's laws' pillar beyond statics (the first 8 chapters of this book), using kinematics (we just started with that above) to the third pillar, dynamics.

Linear momentum balance

For a particle moving in the x direction the velocity and acceleration are $\vec{\boldsymbol{v}} = v\hat{\boldsymbol{\imath}}$ and $\vec{\boldsymbol{a}} = a\hat{\boldsymbol{\imath}}$. Thus the linear momentum and its rate of change are

$$\begin{aligned}\vec{\boldsymbol{L}} &\equiv \sum m_i\vec{\boldsymbol{v}}_i &= m\vec{\boldsymbol{v}} = mv\hat{\boldsymbol{\imath}}, \text{ and}\\ \dot{\vec{\boldsymbol{L}}} &\equiv \sum m_i\vec{\boldsymbol{a}}_i &= m\vec{\boldsymbol{a}} = ma\hat{\boldsymbol{\imath}}.\end{aligned}$$

Using any of the free-body diagrams in fig. 10.5, where $\vec{\boldsymbol{F}} = F\hat{\boldsymbol{\imath}}$, the equation of linear momentum balance, ④ eqn. I from the front inside cover, or equation 12.1 reduces to:

$$F\hat{\boldsymbol{\imath}} = ma\hat{\boldsymbol{\imath}} \tag{10.3}$$

which in scalar form is the central subject of this section.

$$F = ma$$

In scalar form, F is the net force to the right and a is the acceleration to the right. For the equation $F = ma$ to have content each of the terms must have some meaning in other contexts. And, at least intuitively, each does (see box 10.1 on page 9).

Force. The force F could come from a spring, or a fluid or from your hand pushing the thing to the right or left, or any combination of these things. The most general case we want to consider here is that the force is determined by the position and velocity of the particle as well as the present time. Thus

$$F = f(x, v, t). \tag{10.4}$$

What do we mean 'determined by'? We mean that we have an independent way of knowing the force from its position, velocity and time, even without thinking yet about the linear momentum balance equation $F = ma$. Special cases would be, say,

$$\begin{array}{lclll}
F & = & f(t) & = F_0 \sin(\beta t) & \text{an oscillating load,} \\
F & = & mg & & \text{the force of earth's gravity} \\
F & = & f(v) & = -cv & \text{a linear viscous drag,} \\
F & = & f(x) & = -kx & \text{a linear spring, and} \\
F & = & f(x,v,t) & = -kx - cv + F_0 \sin(\beta t) & \text{a combination of forces.}
\end{array}$$

All elementary 1D particle mechanics problems can be reduced to the solution of this pair of coupled first order differential equations,

$$\begin{aligned}
\frac{dv}{dt} &= \underbrace{f(x,v,t)/m}_{a(t)} && (a) \\
\frac{dx}{dt} &= v(t) && (b)
\end{aligned} \tag{10.5}$$

where the function $f(x, v, t)$ is given and $x(t)$ and $v(t)$ are to be found.

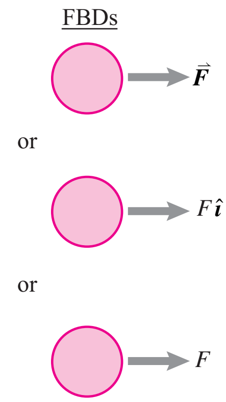


Figure 10.5: Three possible free-body diagrams for 1D particle motion. It is assumed that $\vec{\boldsymbol{F}}$ represents the sum of all forces acting on the particle. Note, even though the inertial term ma is of special interest, it does not show on the FBD. See the free-body diagram Chapter 3 for the prescription about this. See box 15.2 on page 730 for a further warning about d'Alembert 'forces'.

④**Angular momentum?** We skip thinking about angular momentum balance in this section. Why? If we pick an origin on the line of travel, all terms on both sides of all angular momentum balance equations are zero, and the equation $0 = 0$ tells us nothing new.

Know a solution when you see one. How can you tell if candidate functions solve a differential equation? First you can tell that the initial conditions are satisfied by evaluating the expressions at $t = 0$. To check that the differential equations are satisfied, you plug the candidate solutions into the equation and see that an identity results. Differential equations are satisfied when the unknown functions therein are replaced with specific functions that make the equations correct.

Viscous drag. If the only applied force is a viscous drag, $F = -cv$ (see fig. 10.6), then linear momentum balance ($F = ma$) would be $-cv = ma$ and Eqns. 10.5 are

$$\frac{dv}{dt} = -cv/m$$

$$\frac{dx}{dt} = v$$

where c and m are constants and $x(t)$ and $v(t)$ are yet to be determined functions of time. Because the force only slows the particle there will be no motion, unless the particle was given some initial velocity. In general, you need to specify the initial position and velocity to find a solution. So we complete the problem statement by specifying initial conditions

Example: Slowing with viscous drag
Find $x(t)$ given that

$$x(0) = x_0 \quad \text{and} \quad v(0) = v_0$$

where x_0 and v_0 are given constants. Before worrying about how to solve such equations, you should know how to recognize a solution. The following two functions, assume for now that they fell from the sky, solve the differential equations.

$$v(t) = v_0 e^{-ct/m}, \qquad \text{and}$$

$$x(t) = x_0 + mv_0(1 - e^{-ct/m})/c$$

Plugging this presumed solution for $v(t)$ into $\dot{v} = -cv/m$ gives, and this is what we want, $0 = 0$. And similarly, when the presumed solution for $x(t)$ is plugged in to $\dot{x} = v$ you also get the 'satisfying' result that $0 = 0$.

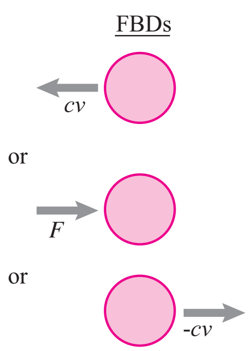


Figure 10.6: Three acceptable free-body diagrams showing the viscous drag on a particle moving to the right at speed v. The second FBD depends on being told on the side, say here, that $F = -cv$.

Replacing the unknown functions $v(t)$ and $x(t)$ with the given formulas gives an identity. Thus the given formulas satisfy (or solve) the differential equations. Just like the case of integration (or equivalently the solution for x of the ODE $\dot{x} = v(t)$), one often cannot find formulas for the solutions of differential equations.

Example: A dynamics problem with no pencil and paper solution
Consider the following case which models a particle in a sinusoidal force field with a second applied force that oscillates in time. Using the dimensional constants c, d, F_0, β, and m,

$$\frac{dv}{dt} = \left(c\sin(x/d) + F_0\sin(\beta t)\right)/m$$

$$\frac{dx}{dt} = v$$

with initial conditions $x(0) = 0$ and $v(0) = 0$.

There is no known formula for $x(t)$ that solves this ODE.

Just writing the ordinary differential equations and initial conditions is analogous to setting up an integral in freshman calculus. The solution is reduced to quadrature. Because numerical solution of sets of ordinary differential equations is a standard part of all modern computation packages you are in some sense done when you get this far. A computer can finish up for you.

Some special cases in 1D mechanics

There are various special cases of eqn. (10.5) which have simple solutions.

Example: The simplest dynamics problem.
1D, particle, no force. Formally working out the details,

$$\begin{array}{lcrcl}
F = ma \text{ with } F = 0 & \Rightarrow & 0 & = & ma \\
\text{definition of } a & \Rightarrow & \dot{v} & = & 0 \\
\text{integrating} & \Rightarrow & v & = & v_0 \quad (= \text{any constant}) \\
\text{integrating again} & \Rightarrow & x & = & x_0 + v_0 t
\end{array}$$

In a sense we have thus derived Newton's first law, 'an object in motion tends to stay in motion unless acted upon by a force'.

Constant force. Another simple case is constant force F which leads to constant acceleration $a = F/m$. Using calculus you should know well by now, you get the following formulas:

$$\begin{aligned}
a = const \Rightarrow \quad x \ &= x_0 + v_0 t + at^2/2 \\
a = const \Rightarrow \quad v \ &= v_0 + at \\
a = const \Rightarrow \quad v \ &= \pm\sqrt{v_0^2 + 2a(x - x_0)}.
\end{aligned}$$

These are much seen in high school physics because, by permuting what is given and what is unknown, one can make up 100 homework problems that can be solved with these formulas and without calculus.

Force given as a function of time. Say F is given as $F = F(t)$. This general case shows up when some kind of motor force is controlled by a human or computer to vary in time is some predetermined manner.

$$\begin{array}{lcrcl}
F = ma \text{ with } F = F(t) & \Rightarrow & F(t) & = & ma \\
\text{definition of } a & \Rightarrow & \dot{v} & = & F(t)/m \\
\text{integrating} & \Rightarrow & v & = & v_0 + \frac{1}{m}\int_0^t F(\tau)\, d\tau
\end{array}$$

And we have to integrate once again to get position.

Example: Ramping up the acceleration at the start
If you get a car going by gradually depressing the 'accelerator' so that its acceleration increases linearly with time, we have

$$\begin{array}{rrclclc} & a & = & ct & & \text{(take } t=0 \text{ at the start)} & \\ \Rightarrow & v(t) & = & \int_0^t a d\tau + v_0 & = & \int_0^t c\tau d\tau & = & ct^2/2 \\ & & & & & \text{(since } v_0 = 0) & \\ \Rightarrow & x(t) & = & \int_0^t v d\tau + x_0 & = & \int_0^t (c\tau^2/2) d\tau & = & ct^3/6 \\ & & & & & \text{(since } x_0 = 0). & \end{array}$$

The distance the car travels is proportional to the cube of the time that has passed from dead stop.

The overall subject of 'vibrations' is in some sense about what happens when something is shaken. We can think of 'shaking' as applying a force which varies sinusoidally in time.

Example: Force varies sinusoidally in time.
Assume a 1 kg mass starts from rest and has a force of $F = 2\cos(2\pi t/\,\mathrm{s})\,\mathrm{N}$ applied. That's a force that oscillates once per second with an amplitude of 2 N. What is the position at $t = 10\,\mathrm{s}$?

$$\begin{array}{llrcl} F = ma \text{ with } F = F(t) & \Rightarrow & 2\cos(2\pi t/\,\mathrm{s})\,\mathrm{N} & = & m\dot{v} \\ \text{integrating} & \Rightarrow & v & = & v_0 + \frac{1}{m}\int_0^t 2\cos(2\pi t/\,\mathrm{s})\,\mathrm{N}\,d\tau \\ \text{using freshman calculus} & \Rightarrow & v & = & v_0 + \frac{1}{\pi m}\sin(2\pi t/\,\mathrm{s})\,\mathrm{N\,s} \\ \text{IC: } v(0) = 0 \Rightarrow v_0 = 0 & \Rightarrow & v & = & \frac{1}{\pi m}\sin(2\pi t/\,\mathrm{s})\,\mathrm{N\,s} \\ \text{integrate again, using } \dot{x} = v & \Rightarrow & x & = & x_0 - \frac{1}{2\pi^2 m}\cos(2\pi t/\,\mathrm{s})\,\mathrm{N\,s^2} \\ \text{IC: } x(0) = 0 \Rightarrow x_0 = \frac{1}{2\pi^2 m} & \Rightarrow & x & = & \frac{1}{2\pi^2 m}\left(1 - \cos(2\pi t/\,\mathrm{s})\right)\,\mathrm{N\,s^2} \end{array}$$

Now we can substitute in $m = 1\,\mathrm{kg}$ and $t = 10\,\mathrm{s}$ to get $x = 0.0\,\mathrm{m}$. The algebraic cancellation of units came about naturally from substituting in the definition of a Newton $1\,\mathrm{N} = 1\,\mathrm{kg\,m/\,s^2}$. We carried the units through even though the final answer was 0.

Force depends on velocity. This case is encountered when, say, an object moves through a fluid and other forces, say gravity, are negligible. Here we have

$$F = ma \quad \Rightarrow \quad F(v) = m\dot{v}.$$

This is solved by multiplying both sides by dt and dividing both sides by $F(v)$ and integrating to get

$$\int dt = m \int \frac{dv}{F(v)} \quad \Rightarrow \quad t = m \int_{v_0}^{v} \frac{dv'}{F(v')}$$

If we want to know position vs time we have to integrate once again.

Example: The slowing of a bullet.
The main force on a bullet after it leaves the gun and before it hits its mark is from air drag. This drag is roughly proportional to the speed squared, thus

$$F = ma \quad \Rightarrow \quad -cv^2 = m\dot{v} \quad \Rightarrow \quad -c\int dt = m\int_{v_0}^{v}\frac{dv'}{v'^2}.$$

Carrying out the integrals ($\int dt = t$ and $\int v^{-2}\,dv = -v^{-1}$) we get

$$ct = m\left(\frac{1}{v} - \frac{1}{v_0}\right) \quad \Rightarrow \quad v = \frac{v_0}{cv_0 t/m + 1}$$

To get position we would integrate again to get:

$$x = \int_0^t v(t')\,dt' = \int_0^t \frac{v_0}{cv_0 t'/m + 1}\,dt' = (m/c)\ln(1 + cv_0 t/m)$$

Interestingly, according to this equation (which becomes less and less accurate as the bullet slows and gravity and eventually viscous forces become important) the bullet goes an infinite distance before stopping.

Force varies with position. This case, where $F = F(x)$ will be treated in some detail in the next section on energy.

The simplest ODEs

The simplest and most common ODEs in dynamics, and in the rest of science and engineering, are

Linear: e.g., no functions squared.

First or second order: Have only first or second derivatives, respectively, and

Constant coefficient: All multiples of the derivatives are constants, not functions of time.

The most essential of these are

$$\dot{x} = 0, \dot{x} = C, \ddot{x} = 0, \ddot{x} = C, \dot{x} = Cx, \dot{x} = -Cx, \ddot{x} = Cx \;\&\; \ddot{x} = -Cx.$$

These are discussed in appendix 3.1 on page 203.

What do the terms in $F = ma$ mean?

Some say '$F = ma$ is the definition of force'. This is a legitimate point of view. But, if we care about force for other reasons, which we do, it is not useful. For the equation $F = ma$ to be useful we need to be able to think about F, m and a independently from each other.

Mass, m. Now that we know about atoms (these centuries) and what they are made of (these decades) we can approximately (with about one percent accuracy) define the mass of a system by (in principle) by counting up the total number of protons and neutrons and multiplying by $1.67 \cdot 10^{-27}$ kg. That is, mass is a measure of the extent of matter. Given that we think of mass as the amount of matter, we could more accurately and more easily use a reference volume of a pure chemical substance as a reference. This way, with a good balance and some trouble, we could get an accuracy of parts per million. Officially,

mass is measured in comparison to a fancy piece of metal

locked in a box in some basement in some government building. That calibrated kilogram is accurate to about a part in 20,000,000 (See Appendix 4 starting on page 225). We can find the mass of a more complicated thing using that reference mass and a (very good) balance.

Acceleration, a. Because this is a course in mechanics and not in philosophy of science, we will just accept the concepts of space (x) and time (t) as given and measurable (using rulers and timers). So acceleration is operationally well defined as

$$a = d^2x/dt^2$$

with no use of $F = ma$. We have presupposed that we measure position relative to a Newtonian reference system. This definition is sensible to at least to about one part per billion for most engineering purposes (see page 33).

Force, F. Here is where people argue. It's easiest to define force using the deformation of solids. When one thing pushes on another, think of your little finger as caught in between. How much your finger is squeezed, as measured by how loud you yell, is a measure of force. More technically, we could look at the small amounts of deformation occurring where the bodies contact, and use the deformation as a measure of force. Or, more practically, we could interpose a calibrated material and measure its deformation. Such a chunk of material with deformation-measuring electronics is called a *load cell*. Load cells are sold by the millions (say, in bathroom scales). A load cell uses nothing about $F = ma$ to operate accurately.

One reason it is nice to think of force as having a life away from $F = ma$ is that the whole coherent and useful subject of statics, useful for designing bridges and airplane landing gear, has little or no use for $F = ma$. Alternatively, still without thinking about $F = ma$, one could define force in terms of the net effect of earth's gravitational pull on a calibrated mass at some officially-ordained location like Potsdam.

However you like to define force, the great result is that with any one of several possible independent definitions, things mostly work out. Miraculously, the same concept of force F works no matter which of these three you take as defining:

$$F = mg, \tag{10.6}$$

$$F = kx, \text{or} \tag{10.7}$$

$$F = ma. \tag{10.8}$$

Pick your favorite as fundamental and use the others with confidence.

Units of force. Most beginners prefer that forces be measured in Newtons. One Newton is $1\ \mathrm{kg\,m/s^2}$. But people not living in the SI world tend instead to use the kgf, also called a kilogram force or kp or kilopond. One kgf is the weight of a kilogram. It's about 9.8 N. In the English system the force most analogous to the Newton is called the Poundal, it's $1\ \mathrm{lbm\,ft/s^2}$ and is little used. More commonly used is the pound force, lbf. It's the weight of a pound mass and is about 32.2 Poundals. Read more about units of force in box 4.3 on page 235.

SAMPLE 10.1 Time derivatives: The position of a particle varies with time as $\vec{\boldsymbol{r}}(t) = (C_1 t + C_2 t^2)\hat{\boldsymbol{\imath}}$, where $C_1 = 4\,\mathrm{m/s}$ and $C_2 = 2\,\mathrm{m/s^2}$.

1. Find the velocity and acceleration of the particle as functions of time.
2. Sketch the position, velocity, and acceleration of the particle against time from $t = 0$ to $t = 5\,\mathrm{s}$.
3. Find the position, velocity, and acceleration of the particle at $t = 2\,\mathrm{s}$.

Solution

1. We are given the position of the particle as a function of time. We need to find the velocity (time derivative of position) and the acceleration (time derivative of velocity).

$$\vec{\boldsymbol{r}} = (C_1 t + C_2 t^2)\hat{\boldsymbol{\imath}} = (4\,\mathrm{m/s}\,t + 2\,\mathrm{m/s^2}\,t^2)\hat{\boldsymbol{\imath}} \tag{10.9}$$

$$\vec{\boldsymbol{v}} \equiv \frac{d\vec{\boldsymbol{r}}}{dt} = \frac{d}{dt}(C_1 t + C_2 t^2)\hat{\boldsymbol{\imath}}$$

$$= (C_1 + 2C_2 t)\hat{\boldsymbol{\imath}} = (4\,\mathrm{m/s} + 4\,\mathrm{m/s^2}\;t)\hat{\boldsymbol{\imath}} \tag{10.10}$$

$$\vec{\boldsymbol{a}} \equiv \frac{d\vec{\boldsymbol{v}}}{dt} = \frac{d}{dt}(C_1 + 2C_2 t)\hat{\boldsymbol{\imath}}$$

$$= 2C_2\hat{\boldsymbol{\imath}} = (4\,\mathrm{m/s^2})\hat{\boldsymbol{\imath}} \tag{10.11}$$

$$\boxed{\vec{\boldsymbol{v}} = (4\,\mathrm{m/s} + 4\,\mathrm{m/s^2}\;t)\hat{\boldsymbol{\imath}},\ \vec{\boldsymbol{a}} = (4\,\mathrm{m/s^2})\hat{\boldsymbol{\imath}}.}$$

Thus, we find that the velocity is a linear function of time and the acceleration is time-independent (a constant).

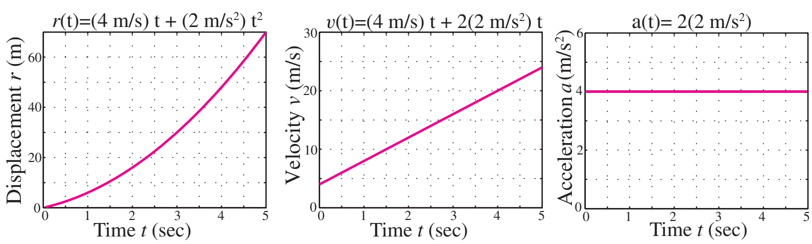


Figure 10.7:

2. We plot eqns. (10.9, 10.10, and 10.11) against time by taking 100 points between $t = 0$ and $t = 5\,\mathrm{s}$, and evaluating $\vec{\boldsymbol{r}}$, $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{a}}$ at those points. The plots are shown in fig. 10.7.
3. We can find the position, velocity, and acceleration at $t = 2\,\mathrm{s}$ by evaluating their expressions at the given time instant:

$$\vec{\boldsymbol{r}}|_{t=2\,\mathrm{s}} = [(4\,\mathrm{m/s})\cdot(2\,\mathrm{s}) + (2\,\mathrm{m/s^2})\cdot(2\,\mathrm{s})^2]\hat{\boldsymbol{\imath}}$$

$$= (16\,\mathrm{m})\hat{\boldsymbol{\imath}}$$

$$\vec{\boldsymbol{v}}|_{t=2\,\mathrm{s}} = [(4\,\mathrm{m/s}) + (4\,\mathrm{m/s^2})\cdot(2\,\mathrm{s})]\hat{\boldsymbol{\imath}}$$

$$= (12\,\mathrm{m/s})\hat{\boldsymbol{\imath}}$$

$$\vec{\boldsymbol{a}}|_{t=2\,\mathrm{s}} = (4\,\mathrm{m/s^2})\hat{\boldsymbol{\imath}} = \vec{\boldsymbol{a}}(\text{for all t})$$

$$\boxed{\text{At } t = 2\,\mathrm{s},\ \vec{\boldsymbol{r}} = (16\,\mathrm{m})\hat{\boldsymbol{\imath}},\ \vec{\boldsymbol{v}} = (12\,\mathrm{m/s})\hat{\boldsymbol{\imath}},\ \vec{\boldsymbol{a}} = (4\,\mathrm{m/s^2})\hat{\boldsymbol{\imath}}.}$$

SAMPLE 10.2 Math review: Solving simple differential equations. For the following differential equations, find the solution for the given initial conditions.

1. $\frac{dv}{dt} = a,\ v(t=0) = v_0$, where a is a constant.
2. $\frac{d^2x}{dt^2} = a,\ x(t=0) = x_0,\ \dot{x}(t=0) = \dot{x}_0$, where a is a constant.

Solution

1.

$$\begin{aligned} \frac{dv}{dt} &= a \quad \Rightarrow \quad dv = a\,dt \\ \text{or} \qquad \int dv &= \int a\,dt = a\int dt \\ \text{or} \qquad v &= at + C, \quad \text{where C is a constant of integration} \end{aligned}$$

Now, substituting the initial condition into the solution,

$$v(t=0) \equiv v_0 = a\cdot 0 + C \quad \Rightarrow \quad C = v_0.$$

Therefore,

$$v = at + v_0.$$

$$\boxed{v = v_0 + at}$$

Alternatively, we can use definite integrals:

$$\int_{v_0}^{v} dv = \int_0^t a\,dt \quad \Rightarrow \quad v - v_0 = at \quad \Rightarrow \quad v = v_0 + at.$$

2. This is a second order differential equation in x. We can solve this equation by first writing it as a first order differential equation in $v \equiv dx/dt$, solving for v by integration, and then solving again for x in the same manner.

$$\begin{aligned} \frac{d^2x}{dt^2} &= a \quad \text{or} \quad \frac{dv}{dt} = a \\ \text{or} \qquad \int dv &= \int a\,dt \\ \Rightarrow \quad v \equiv \dot{x} &= at + C_1 \qquad (10.12) \\ \text{but, } v \equiv \frac{dx}{dt}, \quad \Rightarrow \quad \int dx &= \int at\,dt + \int C_1\,dt \\ \text{or} \qquad x &= \frac{1}{2}at^2 + C_1 t + C_2, \qquad (10.13) \end{aligned}$$

where C_1 and C_2 are constants of integration. Substituting the initial condition for $\dot{x}$ in Eqn. (10.12), we get

$$\dot{x}(t=0) = \dot{x}_0 = a\cdot 0 + C_1 \quad \Rightarrow \quad C_1 = \dot{x}_0.$$

Similarly, substituting the initial condition for x in Eqn. (10.13), we get

$$x(t=0) = x_0 = \frac{1}{2}a\cdot 0 + \dot{x}_0\cdot 0 + C_2 \quad \Rightarrow \quad C_2 = x_0.$$

Therefore,

$$x(t) = x_0 + \dot{x}_0 t + \frac{1}{2}at^2.$$

$$\boxed{x(t) = x_0 + \dot{x}_0 t + \tfrac{1}{2}at^2}$$

SAMPLE 10.3 Constant speed motion: A ship cruises at a constant speed of 15 knots (15 nautical miles per hour) due Northeast. It passes a lighthouse at 8:30 am. The next lighthouse is approximately 35 nautical miles straight ahead. At what time does the ship pass the next lighthouse?

Solution We are given the distance s and the speed of travel v. We need to find how long it takes to travel the given distance.

$$\begin{aligned} s &= vt \\ \Rightarrow \quad t &= \frac{s}{v} = \frac{35\text{nautical miles}}{15\text{(nautical miles)/hour}} = 2.33\text{ hrs.} \end{aligned}$$

Now, the time at $t = 0$ is 8:30 am. Therefore, the time after 2.33 hrs (2 hours 20 minutes) will be 10:50 am.

$$\boxed{10 : 50\text{ am}}$$

SAMPLE 10.4 Constant velocity motion: A particle travels with constant velocity $\vec{\boldsymbol{v}} = 5\,\text{m/s}\hat{\boldsymbol{\imath}}$. The initial position of the particle is $\vec{\boldsymbol{r}}_0 = 2\,\text{m}\hat{\boldsymbol{\imath}} + 3\,\text{m}\hat{\boldsymbol{\jmath}}$. Find the position of the particle at $t = 3\,\text{s}$.

Solution Here, we are given the velocity, *i.e.*, the time derivative of position:

$$\vec{\boldsymbol{v}} \equiv \frac{d\vec{\boldsymbol{r}}}{dt} = v_0\hat{\boldsymbol{\imath}}, \qquad \text{where } v_0 = 5\,\text{m/s}.$$

We need to find $\vec{\boldsymbol{r}}$ at $t = 3\,\text{s}$, given that $\vec{\boldsymbol{r}}$ at $t = 0$ is $\vec{\boldsymbol{r}}_0$.

$$\begin{aligned} d\vec{\boldsymbol{r}} &= v_0\hat{\boldsymbol{\imath}}dt \\ \Rightarrow \quad \int_{\vec{\boldsymbol{r}}_0}^{\vec{\boldsymbol{r}}(t)} d\vec{\boldsymbol{r}} &= \int_0^t v_0\hat{\boldsymbol{\imath}}dt = v_0\hat{\boldsymbol{\imath}}\int_0^t dt \\ \vec{\boldsymbol{r}}(t) - \vec{\boldsymbol{r}}_0 &= v_0\hat{\boldsymbol{\imath}}t \\ \vec{\boldsymbol{r}}(t) &= \vec{\boldsymbol{r}}_0 + v_0t\hat{\boldsymbol{\imath}} \\ \vec{\boldsymbol{r}}(3\,\text{s}) &= (2\,\text{m}\hat{\boldsymbol{\imath}} + 3\,\text{m}\hat{\boldsymbol{\jmath}}) + (5\,\text{m/s}) \cdot (3\,\text{s})\hat{\boldsymbol{\imath}} \\ &= 17\,\text{m}\hat{\boldsymbol{\imath}} + 3\,\text{m}\hat{\boldsymbol{\jmath}}. \end{aligned}$$

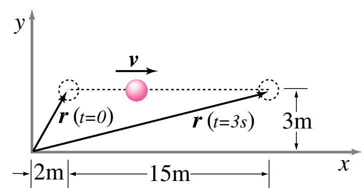


Figure 10.8

$$\boxed{\vec{\boldsymbol{r}} = 17\,\text{m}\hat{\boldsymbol{\imath}} + 3\,\text{m}\hat{\boldsymbol{\jmath}}}$$

Comments: We could solve this problem more compactly by working with scalars or components. It is given that the velocity is constant and is only in the x-direction. Therefore, the y-component of particle position will remain the same, *i.e.*, $r_y = r_{0_y} = 3\,\text{m}$, and $r_x = r_{0_x} + v_xt = 2\,\text{m} + (5\,\text{m/s}) \cdot (3\,\text{s}) = 17\,\text{m}$. Thus, $\vec{\boldsymbol{r}}(3\,\text{s}) = r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}} = 17\,\text{m}\hat{\boldsymbol{\imath}} + 3\,\text{m}\hat{\boldsymbol{\jmath}}$.

SAMPLE 10.5 **Constant acceleration:** A 0.5 kg mass starts from rest and attains a speed of $20\,\mathrm{m/s}\hat{\boldsymbol{\imath}}$ in 4 s. Assuming that the mass accelerates at a constant rate, find the force acting on the mass.

Solution Here, we are given the initial velocity $\vec{\boldsymbol{v}}(0) = \vec{\boldsymbol{0}}$ and the final velocity $\vec{\boldsymbol{v}}$ after $t = 4\,\mathrm{s}$. We have to find the force acting on the mass. The net force on a particle is given by $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$. Thus, we need to find the acceleration $\vec{\boldsymbol{a}}$ of the mass to calculate the force acting on it. Now, the velocity of a particle under constant acceleration is given by

$$\vec{\boldsymbol{v}}(t) = \vec{\boldsymbol{v}}_0 + \vec{\boldsymbol{a}}t.$$

Therefore, we can find the acceleration $\vec{\boldsymbol{a}}$ as

$$\begin{aligned} \vec{\boldsymbol{a}} &= \frac{\vec{\boldsymbol{v}}(t) - \vec{\boldsymbol{v}}(0)}{t} \\ &= \frac{20\,\mathrm{m/s}\hat{\boldsymbol{\imath}} - \vec{\boldsymbol{0}}}{4\,\mathrm{s}} \\ &= 5\,\mathrm{m/s^2}\hat{\boldsymbol{\imath}}. \end{aligned}$$

The force on the particle is

$$\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}} = (0.5\,\mathrm{kg}) \cdot (5\,\mathrm{m/s^2}\hat{\boldsymbol{\imath}}) = 2.5\,\mathrm{N}\hat{\boldsymbol{\imath}}.$$

$$\boxed{\vec{\boldsymbol{F}} = 2.5\,\mathrm{N}\hat{\boldsymbol{\imath}}}$$

SAMPLE 10.6 **Time of travel for a given distance:** A ball of mass 200 gm falls freely under gravity from a height of 50 m. Find the time taken to fall through a distance of 30 m, given that the acceleration due to gravity $g = 10\,\mathrm{m/s^2}$.

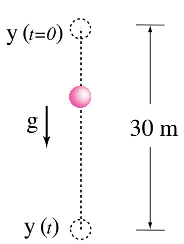


Figure 10.9

Solution The entire motion is in one dimension — the vertical direction. We can, therefore, use scalar equations for distance, velocity, and acceleration. Let y denote the distance travelled by the ball. Let us measure y vertically downwards, starting from the height at which the ball starts falling (see fig. 10.9). Under constant acceleration g, we can write the distance travelled as

$$y(t) = y_0 + v_0 t + \frac{1}{2}gt^2.$$

Note that at $t = 0$, $y_0 = 0$ and $v_0 = 0$. We are given that at some instant t (that we need to find) $y = 30\,\mathrm{m}$. Thus,

$$\begin{aligned} y &= \frac{1}{2}gt^2 \\ t &= \sqrt{\frac{2y}{g}} = \sqrt{\frac{2 \times 30\,\mathrm{m}}{10\,\mathrm{m/s^2}}} = 2.45\,\mathrm{s}. \end{aligned}$$

$$\boxed{t = 2.45\,\mathrm{s}}$$

SAMPLE 10.7 **Time varying acceleration:** A force $F(t) = F_0 \sin\lambda t$ acts on an initially still cart of mass m in a particular direction. Find the speed and the distance travelled by the cart as functions of time. Plot the acceleration, the speed and the displacement of the cart against time for $0 \leq t \leq \pi$ s, assuming $\lambda = 1/$ s. What are the speed and the displacement of the cart at $t = \pi$ s if $F_0 = 1$ N and $m = 1$ kg?

Solution We are given the applied force and the mass of the cart. Therefore, we know the acceleration ($a = F/m$). Thus,

$$\begin{aligned} a \equiv \frac{dv}{dt} &= \frac{F_0}{m}\sin\lambda t \\ \Rightarrow \quad dv &= a_0 \sin\lambda t dt \end{aligned}$$

where $a_0 = F_0/m$. Hence,

$$\begin{aligned} \int_0^{v(t)} dv &= \int_0^t a_0 \sin\lambda\tau d\tau \\ \Rightarrow \quad v(t) &= -\frac{a_0}{\lambda}(\cos\lambda t - 1) \\ &= \frac{a_0}{\lambda}(1 - \cos\lambda t). \end{aligned}$$

Since the speed $v = \frac{dx}{dt}$, we have,

$$\begin{aligned} dx &= \frac{a_0}{\lambda}(1 - \cos\lambda t)dt \\ \int_0^{x(t)} dx &= \int_0^t \frac{a_0}{\lambda}(1 - \cos\lambda\tau)d\tau \\ \Rightarrow \quad x(t) &= \frac{a_0}{\lambda}\left(t - \frac{1}{\lambda}\sin\lambda t\right). \end{aligned}$$

$$\boxed{v(t) = \frac{F_0/m}{\lambda}(1 - \cos\lambda t), \quad x(t) = \frac{F_0/m}{\lambda}\left(t - \frac{1}{\lambda}\sin\lambda t\right)}$$

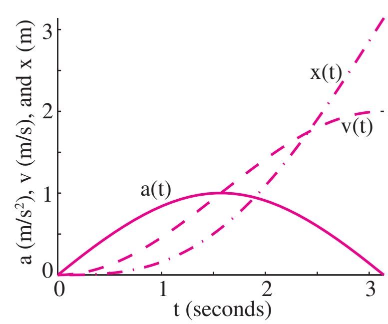


Figure 10.10: Plots of acceleration $a(t)$, speed $v(t)$ and displacement $x(t)$ as functions of time: $a(t) = a_0 \sin\lambda t$, $v(t) = \frac{a_0}{\lambda}(1 - \cos\lambda t)$, and $x(t) = \frac{a_0}{\lambda}(t - \frac{1}{\lambda}\sin\lambda t)$ where $a_0 = 1\,\mathrm{m/s^2}$ and $\lambda = 1/$ s.

Substituting $a_0 = F_0/m = 1\,\mathrm{N}/1\,\mathrm{kg} = 1\,\mathrm{m/s^2}$, $\lambda = 1/$ s and $t = \pi$ s in the expressions for v and x above, we find the speed and the displacement (distance travelled by the cart) at $t = \pi$ seconds as follows.

$$\begin{aligned} v(t = \pi\,\mathrm{s}) &= \frac{1\,\mathrm{m/s^2}}{1/\,\mathrm{s}}(1 - \cos\pi) \\ &= 2\,\mathrm{m/s}, \\ x(t = \pi\,\mathrm{s}) &= 1\,\mathrm{m/s}\left(\pi\,\mathrm{s} - \frac{1}{1/\,\mathrm{s}}\sin(\frac{1}{\mathrm{s}} \cdot \pi\,\mathrm{s})\right) \\ &= \pi\,\mathrm{m}. \end{aligned}$$

$$\boxed{\text{At } t = \pi\,\mathrm{s}, \quad v = 2\,\mathrm{m/s}, \quad x = \pi\,\mathrm{m}}$$

The graph of $a(t)$, $v(t)$, and $x(t)$ are shown in fig. 10.10 for $0 \leq t \leq \pi$ s assuming the given values of m, λ, and F_0. Note the behavior of $v(t)$ and $x(t)$ close to $t = 0$. Since the cart starts from rest, the speed builds up slowly, and the displacement builds up even more slowly because the speed is very low in the beginning.

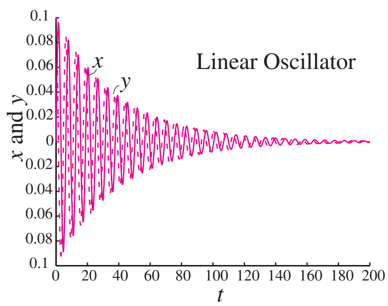


Figure 10.11: Numerical solution of the linear ODE $\ddot{x} + c\dot{x} + kx = 0$ with initial conditions $x(0) = 0$ and $\dot{x}(0) = 0.1$.

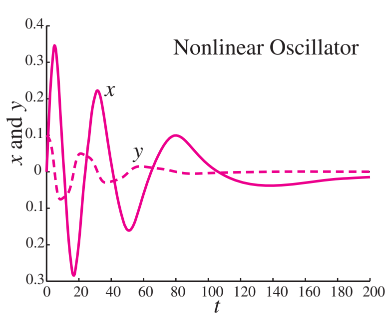


Figure 10.12: Numerical solution of the nonlinear ODE $\ddot{x} + c\dot{x}^2 + kx^3 = 0$ with initial conditions $x(0) = 0$ and $\dot{x}(0) = 0.1$.

SAMPLE 10.8 Numerical integration of ODE's:

1. Write the second order linear nonhomogeneous differential equation, $\ddot{x} + c\dot{x} + kx = a_0 \sin \omega t$, as a set of first order equations that can be used for numerical integration.
2. Write the second order nonlinear homogeneous differential equation, $\ddot{x} + c\dot{x}^2 + kx^3 = 0$, as a set of first order equations that can be used for numerical integration.
3. Solve the nonlinear equation given in (b) by numerical integration taking $c = 0.05$, $k = 1$, $x(0) = 0$, and $\dot{x}(0) = 0.1$. Compare this solution with that of the linear equation in (a) by setting $a_0 = 0$ and taking other values to be the same as for (b).

Solution

1.

$$\begin{array}{lrcl} \text{If we let} & \dot{x} & = & y, \\ \text{then} & \dot{y} & = & \ddot{x} = -c\dot{x} - kx + a_0 \sin \omega t \\ & & = & -cy - kx + a_0 \sin \omega t \end{array}$$

$$\text{or} \quad \left\{ \begin{array}{c} \dot{x} \\ \dot{y} \end{array} \right\} = \left[\begin{array}{cc} 0 & 1 \\ -k & -c \end{array} \right] \left\{ \begin{array}{c} x \\ y \end{array} \right\} + \left\{ \begin{array}{c} 0 \\ a_0 \sin \omega t \end{array} \right\}. \tag{10.14}$$

Equation (10.14) is written in matrix form to show that it is a set of *linear* first-order ODE's. In this case linearity means that the dependent variables only appear linearly, not as powers *etc.*

2.

$$\begin{array}{lrcl} \text{If} & \dot{x} & = & y \\ \text{then} & \dot{y} & = & \ddot{x} = c\dot{x}^2 - kx^3 = -cy^2 - kx^3 \end{array}$$

$$\text{or} \quad \left\{ \begin{array}{c} \dot{x} \\ \dot{y} \end{array} \right\} = \left\{ \begin{array}{c} y \\ -cy^2 - kx^3 \end{array} \right\}. \tag{10.15}$$

Equation (10.15) is a set of *nonlinear* first order ODE's. It cannot be arranged as Eqn. 10.14 because of the nonlinearity in x and y. It is, however, in an appropriate form for numerical integration.

3. Now we solve the set of first order equations obtained in (b) using a numerical ODE solver with the following pseudocode.

```
ODEs = {xdot = y,  ydot = -c y^2 - k x^3}
IC   = {x(0) = 0, y(0) = 0.1}
Set  k=1, c=0.05
Solve ODEs with IC for t=0 to t=200
Plot x(t) and y(t)
```

The plot obtained from numerical integration using a Runge-Kutta based integrator is shown in fig. 10.12. A similar program used for the equation in (a) with $a_0 = 0$ gives the plot shown in fig. 10.11. The two plots show how a simple nonlinearity changes the response drastically.

10.1 Force and motion in 1D

Preparatory Problems

10.1.1 Give three examples of real life objects where you might use the idealization, for dynamic calculations, that the object is a particle in unconstrained 1D motion. *

10.1.2 A car is going downhill on a constant slope straight road. For finding out the car's speed at the end of the road you model it as a particle. For specifying initial velocity, which point on the car would you consider? *

10.1.3 The acceleration of a particle is given as a function of time, $a(t)$. Is this information sufficient to find the speed of the particle at the end of, say, T seconds? *

10.1.4 If a particle has constant acceleration, its linear momentum (a) remains constant, (b) changes linearly with time, or (c) changes quadratically with time. Which one is true? *

10.1.5 In a motorcycle race on a straight track, the speed v^* of a motorcyclist at the d=200 m mark is recorded. Given that the rider started from rest, find the initial acceleration of the motorcycle. Assume the acceleration (a) is constant or (b) (challenge) decreases linearly with time to zero at the end. *

10.1.6 The force acting on a particle with mass m is a given function of time. If you plot the force vs time and find the area under the graph, from that area can you determine (a) the net displacement of the particle?, (b) the average velocity of the particle?, or (c) the change in linear momentum of the particle? *

10.1.7 If the linear momentum of a body remains constant in time, it must have (pick one): (a) a constant force acting on it, (b) no net force acting on it, or (c) a sinusoidal force acting on it. *

10.1.8 The distance between two points in a bicycle race is 10 km. How many minutes does a bicyclist take to cover this distance if he/she maintains a constant speed of 15 mph. *

10.1.9 A 5 kN constant force acts on a 1 kg object that was initially at rest for 5 seconds and then stops. Find the speed of the object at the end of (a) 5 seconds, and (b) 10 seconds.

10.1.10 Given that $\dot{x} = k_1 + k_2 t$, $k_1 = 1\,\mathrm{ft/s}$, $k_2 = 1\,\mathrm{ft/s^2}$, and $x(0) = 1\,\mathrm{ft}$, what is the displacement at the end of 10 seconds? *

10.1.11 Find $x(3\,\mathrm{s})$ given that

$$\dot{x} = x/(1\,\mathrm{s}) \quad \text{and} \quad x(0\,\mathrm{s}) = 1\,\mathrm{m}$$

or, expressed slightly differently,

$$\dot{x} = cx \quad \text{and} \quad x(0\,\mathrm{s}) = x_0,$$

where $c = 1\,\mathrm{s}^{-1}$ and $x_0 = 1\,\mathrm{m}$. Make a sketch of x versus t. *

10.1.12 A ball of mass m is dropped from rest at a height h above the ground. Find the position and velocity as a function of time (as well as m and g, if needed). Neglect air friction. When does the ball hit the ground? What is the velocity of the ball just before it hits? *

10.1.13 The speed of a particle varies sinusoidally as $v = A\sin[ct]$, where $A = 0.5\,\mathrm{m/s}$ and $c = 3\,\mathrm{rad/s}$. Let the initial position of the particle be $x(0) = 0$. Find the position of the particle at $t = \pi/2$ s. *

10.1.14 The speed of a particle is directly proportional to its position and is given as $\dot{x} = x/\mathrm{s}$. If the initial position, $x(0) = 1\,\mathrm{m}$, how far would the particle be from the origin in 5 seconds? *

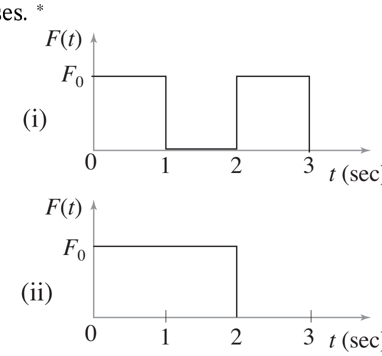
More-Involved Problems

10.1.15 Consider a force $F(t)$ acting on a cart over a 3 second span. In case (a), the force acts in two impulses of one second duration each as shown in fig. 10.1.15. In case (b), the force acts continuously for two seconds and then is zero for the last second. Given that the mass of the cart is 10 kg, $v(0) = 0$, and $F_0 = 10\,\mathrm{N}$, for each force profile,

a) Find the speed of the cart at the end of 3 seconds, and

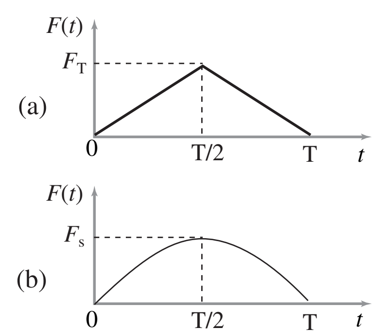
b) Find the distance travelled by the cart in 3 seconds.

Comment on your answers for the two cases. *



Problem 10.1.15

10.1.16 A car of mass m is accelerated by applying a triangular force profile shown in fig. 10.1.16(a). Find the speed of the car at $t = T$ seconds. If the same speed is to be achieved at $t = T$ seconds with a sinusoidal force profile, $F(t) = F_s \sin\frac{\pi t}{T}$, find the required force magnitude F_s. Is the peak higher or lower? Why? *
*



Problem 10.1.16

10.1.17 A particle of mass $m = 1\,\mathrm{kg}$ is acted upon by a short duration force given by

$$F(t) = \begin{cases} F_0 t/\mathrm{s} & 0 \le t \le 1\,\mathrm{s} \\ F_0(2 - t/\mathrm{s}) & 1\,\mathrm{s} < t \le 2\,\mathrm{s} \end{cases}$$

where $F_0 = 5\,\mathrm{N}$. If the particle starts from rest, find the speed of the particle as a function of time. Sketch the given force profile as a function of time and draw the corresponding speed $v(t)$ as a function of time. What is the speed of the particle at $t = 2\,\mathrm{s}$? *

10.1.18 A ball of mass m is dropped vertically from rest at a height h above the ground. Air resistance causes a drag force on the ball directly proportional to the speed v of the ball, $F_d = bv$. Find the velocity and position of the ball as a function of time. Find the velocity as a function of position. Gravity is non-negligible, of course.

10.1.19 A sinusoidal force acts on a 1 kg mass as shown in the figure and graph below. The mass is initially still; i. e.,

$$x(0) = v(0) = 0$$

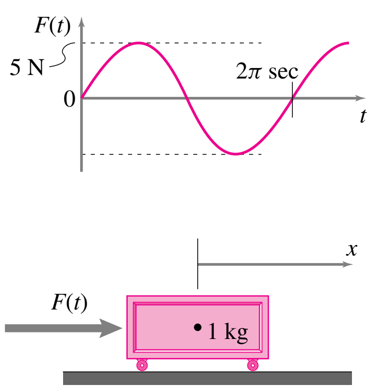
.

a) What is the velocity of the mass after 2π seconds?

b) What is the position of the mass after 2π seconds?

c) Plot position x versus time t for the motion.

*



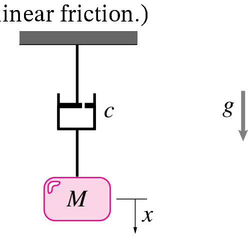
Problem 10.1.19

10.1.20 A motorcycle accelerates from 0 mph to 60 mph in 5 seconds. Find the average acceleration in $\mathrm{m/s^2}$. How does this acceleration compare with g, the acceleration of an object falling near the earth's surface?

10.1.21 A car moves on a straight road with an initial velocity $v_0 = 30\,\mathrm{m/s}$. Let its position at $t = 0$ be $x = 0$. For the first 5 s it has no acceleration, and thereafter it brakes with a retarding force that gives it a constant acceleration $a_x = -10m/s^2$. Calculate the velocity and the x-coordinate of the car when $t = 8\,\mathrm{s}$ and when $t = 12\,\mathrm{s}$, and find the distance travelled by the car from start until it comes to a final stop.

10.1.22 A grain of sugar falling through honey has a negative acceleration proportional to the difference between its velocity and its 'terminal' velocity, which is a known constant v_t. Write this sentence as a differential equation, defining any constants you need. Solve the equation assuming some given initial velocity v_0.

10.1.23 The mass-dashpot system shown below is released from rest at $x = 0$. Determine an equation of motion for the particle of mass m that involves only $\dot{x}$ and x (a first-order ordinary differential equation). The dashpot opposes the motion of mass m with a force $F = c\dot{x}$ where c is the damping coefficient of the dashpot. (You can think of the dashpot as a model for linear friction.)



Problem 10.1.23

10.1.24 Due to gravity, a particle falls in air with a drag force proportional to the speed squared.

1. Write $\sum \vec{F} = m\vec{a}$ in terms of variables you clearly define,
2. find a constant speed motion that satisfies your differential equation,
3. pick numerical values for your constants and for the initial height. Assume the initial speed is zero
   a) set up the equation for numerical solution,
   b) solve the equation on the computer,
   c) make a plot with your computer solution and show how that plot supports your answer to (2).

10.1.25 In quadratic drag problems, the deceleration is proportional to the square of velocity, *i.e.*, $a = \frac{dv}{dt} = -kv^2$. Assume that a particle with initial velocity $v(0) = v_0$ experiences quadratic drag.

a) How long does it take for the particle to reduce its speed to half of its initial speed (*i.e.*, find t such that $v(t) = \frac{1}{2}v_0$)?

b) Find the position of the particle as a function of velocity. How far does the particle move from its initial position when its velocity drops to half its initial value?

10.1.26 A bullet penetrating flesh slows approximately as it would if penetrating water. The drag on the bullet is about $F_D = c\rho_w v^2 A/2$ where ρ_w is the density of water, v is the instantaneous speed of the bullet, A is the cross sectional area of the bullet, and c is a drag coefficient which is about $c \approx 1$. Assume that the bullet has mass $m = \rho_l AL$ where ρ_l is the density of lead, A is the cross sectional area of the bullet and L is the length of the bullet (approximated as cylindrical). Assume $m = 2$ grams, entering velocity $v_0 = 400\,\mathrm{m/s}$, $\rho_l/\rho_w = 11.3$, and bullet diameter $d = 5.7\,\mathrm{mm}$.

a) Plot the bullet position vs time.

b) Assume the bullet has effectively stopped when its speed has dropped to $5\,\mathrm{m/s}$, what is its total penetration distance?

c) According to the equations implied above, what is the penetration distance in the limit $t \to \infty$?

d) How would you change the model to make it more reasonable in its predictions for long time?

10.1.27 A force pulls a particle of mass m towards the origin according to the law (assume same equation works for $x > 0, x < 0$)

$$F = Ax + Bx^2 + C\dot{x}$$

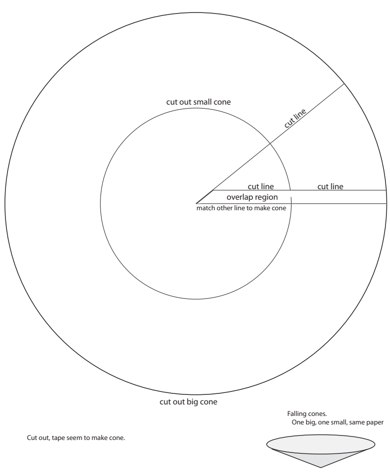
Assume $\dot{x}(0) = 0$.
Using numerical solution, find values of A, B, C, m, and x_0 so that

1. the mass never crosses the origin,
2. the mass crosses the origin once,
3. the mass crosses the origin many times.

[Hint: Vary one parameter at a time and choose a different set of parameter values for each case.]

10.1.28 Do the falling cone experiment: Make two cones, drop them simultaneously and see which falls faster. Explain the result in a way that would convince you if it was written by another student who you did not already trust (and you had not already seen the experiment done).

More details. Copy the drawing below onto two pieces of paper. Make one cone from each page. One has twice the radius as the other. They both have the same conical angle. Which falls faster? Use $F = ma$ to predict the result. Hint 1: the cutout lets you think about how to calculate the area of the paper used to make the cone. Hint 2: The mass of the cone is proportional to its area. The result holds for linear or quadratic drag, or a combination of both. It holds for the steady state or for the transient response. *



Problem 10.1.28

10.1.1 Give three examples of real life objects where you might use the idealization, for dynamic calculations, that the object is a particle in unconstrained 1D motion.

Answer:
Some possibilities: a) a car with given thrust and drag, b) a person falling vertically during bungy jumping, c) a speaker cone oscillating due to magnetic forces on its coils and resistance from air pressure.

Answer: 9.1

- Body in pure translatory Motion (e.g. sliding object)
- Motion of car can be also described by assuming as a particle if our interest is to find displacement, velocity & Accel^n.
- Motion of lift (Kinetics).

10.1.2 A car is going downhill on a constant slope straight road. For finding out the car's speed at the end of the road you model it as a particle. For specifying initial velocity, which point on the car would you consider?

Answer:
All points have equal velocity so all have the same velocity as the center of mass. Any point on the car can be used to measure the car's position.

9.2 Centre of mass.

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

10.1.3 The acceleration of a particle is given as a function of time, $a(t)$. Is this information sufficient to find the speed of the particle at the end of, say, T seconds?

Answer:
No. You need also to know $v(0)$. Then $v(T) = v(0) + \int_0^T a(t)\,dt$. Knowing $a(t)$ over a given time interval determines the change of v over that interval, but not the value $v(T)$.

9.3 $\frac{dv}{dt} = a(t)$

$\int dv = \int a\,dt \Rightarrow v = \int a\,dt + C$

No, we need initial velocity to find the const C.

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10.1.4 If a particle has constant acceleration, its linear momentum (a) remains constant, (b) changes linearly with time, or (c) changes quadratically with time. Which one is true?

Answer:
(b) changes linearly in time

9.4 $\frac{m\,dv}{dt} = F = ma$

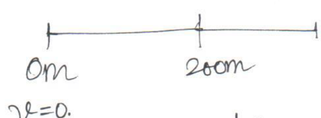
rate of change of linear momentum is constant.

Hence, linear momentum changes linearly with time (b)

10.1.5 In a motorcycle race on a straight track, the speed v^* of a motorcyclist at the d=200 m mark is recorded. Given that the rider started from rest, find the initial acceleration of the motorcycle. Assume the acceleration (a) is constant or (b) (challenge) decreases linearly with time to zero at the end.

Answer:
a) $a = {v^*}^2/(2d)$; b) Assume $a = a_0 - ct$. If you knew a_0 and c then you could find $T = a_0/c$, $d = a_0^3/(3c^2)$, and $v^* = a_0^2/(2c)$. But you are given v^* and d and can solve to get $a_0 = 4{v^*}^2/3d$.

9.5



$$a = \frac{dv}{dt} \Rightarrow v_f = v_i + \int a\,dt$$

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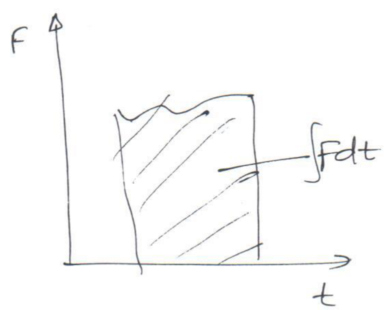
10.1.6 The force acting on a particle with mass m is a given function of time. If you plot the force vs time and find the area under the graph, from that area can you determine (a) the net displacement of the particle?, (b) the average velocity of the particle?, or (c) the change in linear momentum of the particle?

Answer:
Only (c), the change in linear momentum. You could find the displacement only if the initial velocity is also given.

9.6



$\int F dt = \int d(mv)$

c

10.1.7 If the linear momentum of a body remains constant in time, it must have (pick one): (a) a constant force acting on it, (b) no net force acting on it, or (c) a sinusoidal force acting on it.

Answer:
(b)

9.7 b No net force acting on it.

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10.1.8 The distance between two points in a bicycle race is 10 km. How many minutes does a bicyclist take to cover this distance if he/she maintains a constant speed of 15 mph.

Answer:
$t = 24.8\,\text{min}$

9.1.8

$d = 10\,\text{Km}$; $V = 15\,\text{mph}$

$$t = \frac{d}{V} = \frac{10 \times 10^3}{15 \times 1609.34} \times 60 = 24.85 \text{ mins}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

10.1.9 A 5 kN constant force acts on a 1 kg object that was initially at rest for 5 seconds and then stops. Find the speed of the object at the end of (a) 5 seconds, and (b) 10 seconds.

From the given information:

$$F = \begin{cases} 5\ \text{kN} & 0 \le t \le 5\,\text{s} \\ 0 & t > 5\,\text{s} \end{cases}$$

$$a = F/m \equiv \frac{dv}{dt}$$

$$\Rightarrow \quad v = \int_0^t a\,dt + v_0 = v_0 + \frac{F}{m} t$$

Here $v_0 = 0$, $F = 5\,\text{kN}$ for $t \le 5\,\text{s}$, $m = 1\,\text{kg}$

$$\Rightarrow \quad v(5\,\text{s}) = 0 + \frac{5\,\text{kN}}{1\,\text{kg}} \cdot 5\,\text{s} = 25000\ \text{m/s}$$

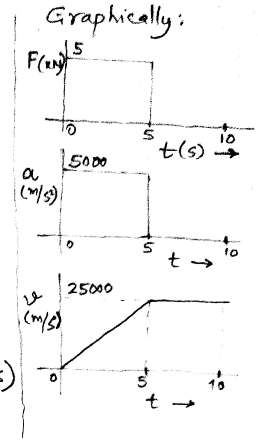
$$v(10\,\text{s}) = v(5\,\text{s}) + \int_5^{10} \frac{F}{m} dt$$

$$= 25000\ \text{m/s} + \int_5^{10} 0\,dt \quad (\because F = 0 \text{ for } t > 5\text{s})$$

$$= 25000\ \text{m/s}.$$

$$\boxed{\text{Ans: } v(5\text{s}) = 25000\ \text{m/s} = v(10\text{s})}$$

Graphically:



10.1.10 Given that $\dot{x} = k_1 + k_2 t$, $k_1 = 1\,\mathrm{ft/s}$, $k_2 = 1\,\mathrm{ft/s^2}$, and $x(0) = 1\,\mathrm{ft}$, what is the displacement at the end of 10 seconds?

Answer:
$x(10\,\mathrm{s}) = 61\,\mathrm{ft}$

(2)

9.10

Given

$\dot{x} = K_1 + K_2 t$

$K_1 = 1\,\mathrm{ft\,s^{-1}}, K_2 = 1\,\mathrm{ft\,s^{-2}}$

$x(0) = 1\,\mathrm{ft}$

$$\frac{dx}{dt} = K_1 + K_2 t \Rightarrow \int_1^x dx = \int_0^t (K_1 + K_2 t)\,dt$$

$$\Rightarrow (x-1) = K_1 t + K_2 t^2/2$$

$$\therefore x\Big|_{t=10\,\mathrm{sec}} = 1 + 1\times 10 + 1\times\frac{10^2}{2}$$

$$= 1 + 10 + 50$$

$$= 61\,\mathrm{ft}. \quad \underline{\underline{\text{Ans}}}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

10.1.11 Find $x(3\,\mathrm{s})$ given that

$$\dot{x} = x/(1\,\mathrm{s}) \quad \text{and} \quad x(0\,\mathrm{s}) = 1\,\mathrm{m}$$

or, expressed slightly differently,

$$\dot{x} = cx \quad \text{and} \quad x(0\,\mathrm{s}) = x_0,$$

where $c = 1\,\mathrm{s}^{-1}$ and $x_0 = 1\,\mathrm{m}$. Make a sketch of x versus t.

Answer:
$x(3\,\mathrm{s}) = 20.08\,\mathrm{m}$

9.11

given,

$$\dot{x} = \frac{dx}{dt} = cx$$

$$\Rightarrow \frac{dx}{x} = c\,dt$$

Integrating both sides, we get

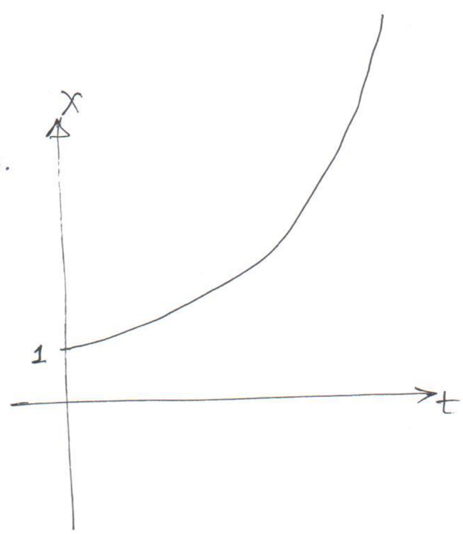
$$\ln x = ct + C_1$$

at $t=0$, $x(0)=1$, $c=1$

$$\Rightarrow 0 = 0 + C_1 \Rightarrow C_1 = 0.$$

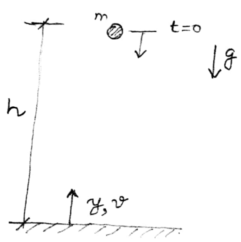
$$\therefore \ln x = t$$

$$\Rightarrow \boxed{x = e^t}$$



10.1.12 A ball of mass m is dropped from rest at a height h above the ground. Find the position and velocity as a function of time (as well as m and g, if needed). Neglect air friction. When does the ball hit the ground? What is the velocity of the ball just before it hits?

Answer:
$v = -gt$, $y = h - \frac{1}{2}gt^2$, $y = 0$ when $t = \sqrt{2g/h}$, $v = -\sqrt{2gh}$



$$a = \frac{dv}{dt} = -g$$

$$\Rightarrow v = \cancel{v_0}^{\,0} - gt$$

$$= -gt$$

Now $v = \frac{dy}{dt}$

$$\Rightarrow \frac{dy}{dt} = -gt \Rightarrow \int dy = \int -gt\,dt$$

$$\Rightarrow y = y_0 - \tfrac{1}{2}gt^2$$

at $t=0$, $y=h \Rightarrow y_0 = h$

$$\Rightarrow \boxed{y = h - \tfrac{1}{2}gt^2}$$

when the ball hits the ground, $y=0$

thus $0 = h - \frac{1}{2}gt^2 \Rightarrow t = \sqrt{\frac{2h}{g}}$

Substituting in $v = -gt$, we get the velocity at impact:

$$v = -g\sqrt{\frac{2h}{g}} = -\sqrt{2gh}$$

Thus $y=0$ (i.e., ball hits the ground) at $\boxed{t = \sqrt{\frac{2h}{g}} \text{ and } v = -\sqrt{2gh}}$

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10.1.13 The speed of a particle varies sinusoidally as $v = A\sin[ct]$, where $A = 0.5\,\mathrm{m/s}$ and $c = 3\,\mathrm{rad/s}$. Let the initial position of the particle be $x(0) = 0$. Find the position of the particle at $t = \pi/2$ s.

Answer:
$x(\pi/2) = 0.17\,\mathrm{m}$

1.129 Given that $\dot{x} = A\sin(\omega t)$ with $\omega = 3\,\mathrm{rad/s}$, $A = .5\,\mathrm{m/s}$ and initial condition $x(0) = 0\,\mathrm{m}$.

Find $x(\pi/2\ \mathrm{s.})$.

$$\frac{dx}{dt} = A\sin\omega t \Rightarrow \int dx = \int A\sin\omega t\,dt$$

$$\Rightarrow x = -\frac{A}{\omega}\cos\omega t + C$$

use initial condition $x(t=0\,\mathrm{s.}) = 0$ to evaluate C:

$$0 = -\frac{A}{\omega}\cos(0) + C \Rightarrow C = A/\omega$$

Thus: $x(t) = \frac{A}{\omega}(1 - \cos\omega t)$

at $t = \pi/2$ sec., $x = \frac{.5}{3}\left(1 - \underbrace{\cos 3\pi/2}_{0}\right)$ m. $\Rightarrow$

$$\boxed{x(t = \pi/2\ \mathrm{s.}) = \frac{1}{6}\ \mathrm{m.}}$$

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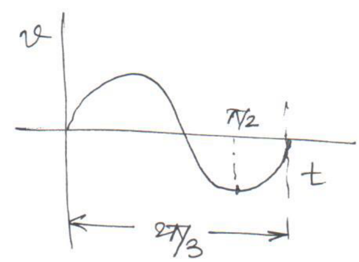
3

9.13

$v = A\sin[ct]$

$A = 0.5\,ms^{-1}.$

$c = 3\,rad\,s^{-1}.$



$$\frac{dx}{dt} = A\sin(ct)$$

$$\Rightarrow \int_0^x dx = \int_0^{\pi/2} A\sin(ct)\,dt$$

$$\Rightarrow x = \left[\frac{-A\cos[ct]}{c}\right]_0^{\pi/2}$$

$$= A/c = \frac{0.5}{3}$$

$$\Rightarrow x = 0.1667\,m. \text{ Ans.}$$

$$x = -A/c\,[\cos(ct) - 1]$$

$$\boxed{x = A/c\,[1-\cos(ct)]}$$

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10.1.14 The speed of a particle is directly proportional to its position and is given as $\dot{x} = x/\,\mathrm{s}$. If the initial position, $x(0) = 1\,\mathrm{m}$, how far would the particle be from the origin in 5 seconds?

Answer:
$x(5\,\mathrm{s}) = 148.41\,\mathrm{m}$

9.14 Given,

$\dot{x} = x/s$

$x(0s) = 1m.$

$x(t = 5sec) = ?$

$\frac{dx}{dt} = x \Rightarrow \int dx/x = \int dt$

$\cancel{\Rightarrow x = e^t}$ $\Rightarrow \ln x = t + C$

$\Rightarrow \boxed{C = 0}$ $\because t = 0, x = 1.$

$\therefore x = e^t$

$\therefore x\Big|_{t=5sec} = e^5\, m$

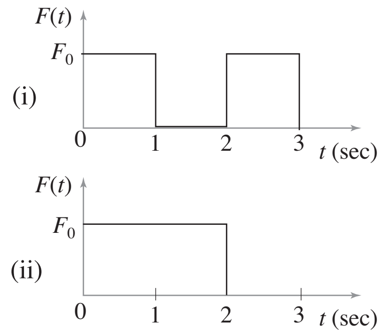
$\Rightarrow x(5s) = 148.413m.$ Ans

10.1.15 Consider a force $F(t)$ acting on a cart over a 3 second span. In case (a), the force acts in two impulses of one second duration each as shown in fig. 10.1.15. In case (b), the force acts continuously for two seconds and then is zero for the last second. Given that the mass of the cart is 10 kg, $v(0) = 0$, and $F_0 = 10\,\text{N}$, for each force profile,

a) Find the speed of the cart at the end of 3 seconds, and

b) Find the distance travelled by the cart in 3 seconds.

Comment on your answers for the two cases.

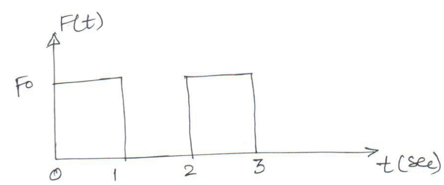


Problem 10.15

Answer:
(a) $v(3\,\text{s}) = 2\,\text{m/s}$ in each case. (b) $x(3\,\text{s}) = 3\,\text{m}$ for case (a), $x(3\,\text{s}) = 4\,\text{m}$ for case (b). During the one second with no force, in case (b) it has already got up to a higher speed.

9.15

(a)



$m\,dv = F_0\,dt$ — (1)

$\Rightarrow m(v_1 - v_0) = F_0(t_1 - t_0)$ [$\because F_0$ is constt].

$\Rightarrow v_1 = \dfrac{10 \times (1-0)}{10} = 1\,\text{ms}^{-1}$.

Since external force acting b/w point (1) & (2) is 0,

$v_1 = v_2$.

Again,

$v_3 = v_2 + \dfrac{F_0}{m}[t_3 - t_2]$ (from (1))

$\Rightarrow v_3 = 1 + \dfrac{10}{10}(1)$
$= 2\,\text{ms}^{-1}$.

(b) $v_2 = v(0) + \dfrac{F_0}{m}[t_2 - t_0]$
$= 2\,\text{ms}^{-1}$.

$\therefore v_3 = v_2 = 2\,\text{ms}^{-1}$. [Since no force acting b/w point (2) & (3)].

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

9.15

$m = 10\ kg$ $\quad$ $F = ma \;\therefore\; a_0 = F_0/m_0 = 10N/10kg$

$v(0) = 0$ $\quad$ $= 1\ m/s^2$

$F_0 = 10\ N$

Force profile (a):

@ $t = 1s$: $a = 1\ m/s^2$, $v = v_0 + at = 0 + 1 m/s^2 (1s) = 1\ m/s$

$x = x_0 + v_0 t + \frac{1}{2}at^2 = 0 + 0 + \frac{1}{2}(1 m/s^2)(1s)^2 = 0.5m$

@ $t = 2s$: $a = 0$, $v = v_0 + at = 1\ m/s + 0 = 1\ m/s$

$x = x_0 + v_0 t + \frac{1}{2}at^2 = 0.5 + 1(1) + 0 = 1.5\ m$

@ $t = 3s$: $a = 1\ m/s^2$, $v = v_0 + at = 1 + 1(1) = \underline{2.0\ m/s}$

$x = x_0 + v_0 t + \frac{1}{2}at^2 = 1.5 + 1(1) + \frac{1}{2}(1)(1)^2 = \underline{3.0\ m}$

Force profile (b):

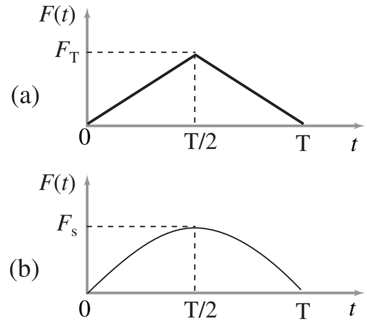
@ $t = 2s$: $a = 1\ m/s^2$, $v = v_0 + at = 0 + 1(2) = 2\ m/s$

$x = x_0 + v_0 t + \frac{1}{2}at^2 = 0 + 0 + \frac{1}{2}(1)(2)^2 = 2\ m$

@ $t = 3s$: $a = 0$, $v = v_0 + at = 2 + 0 = \underline{2.0\ m/s}$

$x = x_0 + v_0 t + \frac{1}{2}at^2 = 2 + 2(1) + 0 = \underline{4.0\ m}$

10.1.16 A car of mass m is accelerated by applying a triangular force profile shown in fig. 10.1.16(a). Find the speed of the car at $t = T$ seconds. If the same speed is to be achieved at $t = T$ seconds with a sinusoidal force profile, $F(t) = F_s \sin \frac{\pi t}{T}$, find the required force magnitude F_s. Is the peak higher or lower? Why?



Problem 10.16

Answer:
b) $F_s = \frac{\pi}{4} F_T$. This is lower than for (a) because for a given peak force the sinusoidal force is bigger at every instant in time. So, to have the same effect (same impulse) the peak must be lower.
$F_s = \frac{\pi}{4} F_T$

9.16

4

$$F = m\frac{dv}{dt}$$

$$\Rightarrow m\,dv = F\,dt$$

$$m(v_f - v_i) = \int F\,dt$$

To get same final velocity, v_f. $\int_0^T F\,dt$ must be same for both cases.

$$\therefore \frac{1}{2} \times F_T \times T = \int_0^T F_s \sin\left(\frac{\pi t}{T}\right) dt$$

$$\Rightarrow \frac{F_T \cdot T}{2} = \left[F_s \cdot -\cos\left(\frac{\pi t}{T}\right) \cdot \frac{T}{\pi}\right]_0^T$$

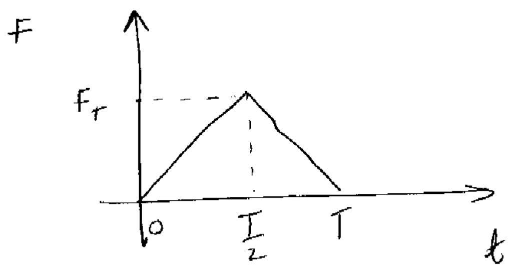
$$= \frac{T F_s}{\pi}\left[1 - \cos\frac{\pi \cancel{T}}{\cancel{T}}\right]$$

$$\Rightarrow \frac{F_T \cdot \cancel{T}}{2} = \frac{2\cancel{T} F_s}{\pi}$$

$$\Rightarrow \boxed{F_s = \frac{\pi}{4} F_T.}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

9.16



a) $V = \int a\,dt$

$= \frac{1}{m}\underbrace{\int F\,dt}_{\text{area under } F-t \text{ curve}}$

$= \frac{1}{m}\left[\frac{1}{2}\times T \times F_T\right]$ $\left\{\text{Area of triangle} = \frac{1}{2}\times \text{base} \times \text{height}\right\}$

$\therefore \boxed{V = \frac{T F_T}{2m}}$

b) $F(t) = F_s \sin\left(\frac{\pi t}{T}\right)$

$V = \frac{1}{m}\int_0^T F\,dt = \frac{F_s}{m}\int_0^T \sin\left(\frac{\pi t}{T}\right)dt$

$= \frac{F_s}{m}\,\frac{\left[-\cos\frac{\pi t}{T}\right]_0^T}{\frac{\pi}{T}}$

$= \frac{T F_s}{m\pi}\left[-\cos\pi + \cos 0\right] = \frac{2T F_s}{m\pi}$

But $\frac{T F_T}{2m} = \frac{2T F_s}{\pi m} \Rightarrow \boxed{F_s = \frac{\pi}{4} F_T}$

10.1.17 A particle of mass $m = 1\,\text{kg}$ is acted upon by a short duration force given by

$$F(t) = \begin{cases} F_0 t/\text{s} & 0 \le t \le 1\,\text{s} \\ F_0(2 - t/\text{s}) & 1\,\text{s} < t \le 2\,\text{s} \end{cases}$$

where $F_0 = 5\,\text{N}$. If the particle starts from rest, find the speed of the particle as a function of time. Sketch the given force profile as a function of time and draw the corresponding speed $v(t)$ as a function of time. What is the speed of the particle at $t = 2\,\text{s}$?

Answer:
$v(2\,\text{s}) = 5\,\text{m/s}$

9.17

Given,

$m = 1\,\text{Kg}$.

$$F(t) = \begin{cases} F_0 t & 0 \le t \le 1s \\ F_0(2-t) & 1s \le t \le 2s \end{cases}$$

$F_0 = 5\,\text{N}$.

$v(t=0) = 0$.

For $0 \le t \le 1S$

$$m\frac{dv}{dt} = F_0 t$$

$$\Rightarrow dv = \frac{F_0}{m}\, t\,dt$$

Integrating both sides, we get

$$\int_0^v dv = \frac{F_0}{m}\int_0^t t\,dt$$

$$\Rightarrow \boxed{v = \frac{F_0}{2m} t^2} \Rightarrow \boxed{v = 2.5 t^2}$$

$$\therefore v\Big|_{t=1\,\text{sec}} = 2.5\,\text{m/s}.$$

$1s \le t \le 2s$.

$F(t) = F_0(2-t)$

$$\Rightarrow m\int_{v_1}^{v_2} dv = F_0 \int_1^2 (2-t)\,dt$$

$$\Rightarrow v_2 = v_1 + \frac{F_0}{m}\cdot\left[2t - \frac{t^2}{2}\right]_1^2$$

$$\Rightarrow v_2 = 2.5 + \frac{5}{1}\left[(4-2) - (2 - \tfrac{1}{2})\right]$$

$$\Rightarrow v_2 = 5\,\text{ms}^{-1}. \quad \text{Ans}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

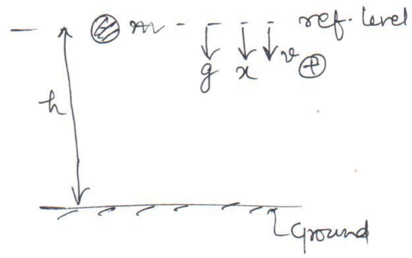
10.1.18 A ball of mass m is dropped vertically from rest at a height h above the ground. Air resistance causes a drag force on the ball directly proportional to the speed v of the ball, $F_d = bv$. Find the velocity and position of the ball as a function of time. Find the velocity as a function of position. Gravity is non-negligible, of course.

5

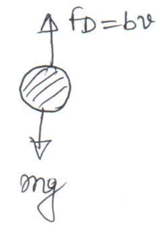
9.18

$v(t) = ?$

$x(t) = ?$



Free body diagram:—



Net force on body of mass 'm' in x-dirn.

$$mg - bv = m\dot{x}$$

$$\Rightarrow \quad m\frac{dv}{dt} = mg - b\dot{x} \qquad [v = \dot{x}]$$

$$\Rightarrow \quad \frac{\cancel{m}\,dv}{(\cancel{m}g - \frac{bv}{m})} = dt$$

$$\Rightarrow \quad \ln\left(\cancel{m}g - \frac{bv}{m}\right) = t + C$$

at $t=0,\ v=0$

$$\Rightarrow \quad C = \ln(g).$$

$$\therefore \quad \ln\left(1 - \frac{bv}{mg}\right) = t$$

$$\Rightarrow \quad 1 - \frac{bv}{mg} = e^t$$

$$\Rightarrow \quad \frac{bv}{mg} = 1 - e^t$$

$$\Rightarrow \quad \boxed{v = \frac{mg}{b}\left[1 - e^t\right]}$$

Ans

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

$$v = \frac{mg}{b}\left[1 - e^{t}\right]$$

$$\Rightarrow \frac{dx}{dt} = \frac{mg}{b}\left[1 - e^{t}\right]$$

$$\Rightarrow \int_0^x dx = \frac{mg}{b}\int_0^t (1 - e^{t})\,dt$$

$$\Rightarrow x = \frac{mg}{b}\left[t - e^{t}\right]_0^t$$

$$= \frac{mg}{b}\left[t - e^{t} + 1\right]$$

$$\Rightarrow \boxed{x = \frac{mg}{b}\left[1 + t - e^{t}\right]} \quad \underline{\text{Ans}}$$

10.1.19 A sinusoidal force acts on a 1 kg mass as shown in the figure and graph below. The mass is initially still; i. e.,

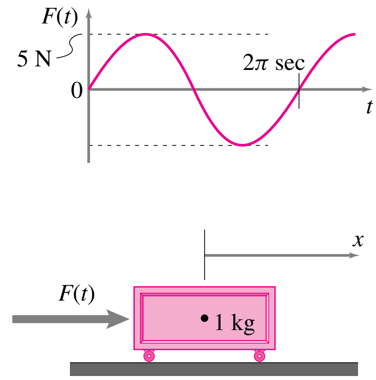
$$x(0) = v(0) = 0$$

.

a) What is the velocity of the mass after 2π seconds?

b) What is the position of the mass after 2π seconds?

c) Plot position x versus time t for the motion.



Problem 10.19

Answer:
(a) $v = 0$, (b) $x = 10\pi$ m

6

9.19

$x(0) = v(0) = 0$, $m = 1\,kg$.

$F(t) = 5\sin t$

$m\frac{dv}{dt} = 5\sin t$

$\Rightarrow \int_0^v dv = \frac{5}{m}\int_0^t \sin t\, dt$

$\Rightarrow v = 5\left[-\cos t\right]_0^t$

$\Rightarrow \boxed{v = 5[1-\cos t]}$ Ans

(a) $v\big|_{t=2\pi} = 0$.

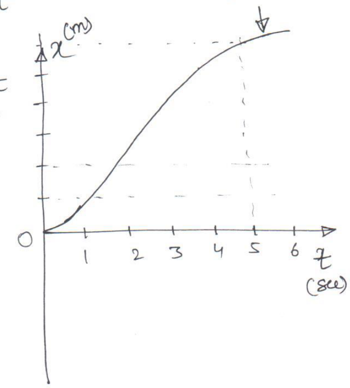
(b) $v = 5(1-\cos t)$

$\frac{dx}{dt} = 5(1-\cos t)$

$\Rightarrow \int_0^x dx = \int_0^t 5(1-\cos t)\,dt$

$\Rightarrow x = 5\left[t - \sin t\right]_0^t$

$\Rightarrow \boxed{x = 5[t-\sin t]}$.



Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

10.1.20 A motorcycle accelerates from 0 mph to 60 mph in 5 seconds. Find the average acceleration in $\mathrm{m/s^2}$. How does this acceleration compare with g, the acceleration of an object falling near the earth's surface?

9.20

$v_i = 0\,\mathrm{mph}$

1 mile $\simeq$ 1.6 Km.

$v_f = v(t = 5\,\mathrm{sec}) = 60\,\mathrm{mph} \simeq \dfrac{\cancel{60} \times 1.6 \times 10^3}{\cancel{60} \times 60}$

$= 26.67\,\mathrm{m s^{-1}}$.

Average acceleration: $a_{avg} = \dfrac{v_f - v_i}{t_f - t_i}$

$= \dfrac{26.67}{5}\,\mathrm{m s^{-2}}$

$\Rightarrow \boxed{a_{avg} = 5.334\,\mathrm{m s^{-2}}}$ Ans

$\Rightarrow \boxed{a_{avg} = 0.5437\,g.}$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

10.1.21 A car moves on a straight road with an initial velocity $v_0 = 30\,\text{m/s}$. Let its position at $t = 0$ be $x = 0$. For the first 5 s it has no acceleration, and thereafter it brakes with a retarding force that gives it a constant acceleration $a_x = -10 m/s^2$. Calculate the velocity and the x-coordinate of the car when $t = 8$ s and when $t = 12$ s, and find the distance travelled by the car from start until it comes to a final stop.

9.21

$0 \leq t \leq 5\,\text{sec}$

$v_0 = 30\,\text{ms}^{-1}$.

$\therefore\ x(5\,\text{sec}) = v_0 t$

$= 150\,\text{m}$.

for $t \geq 5\,\text{sec}$, $a_x = -10\,\text{ms}^{-2}$.

$$\Rightarrow \frac{dv}{dt} = -10$$

$$\Rightarrow \int_{30}^{v} dv = \int_{5}^{t} -10\,dt$$

$$\Rightarrow v - 30 = -10(t-5)$$

$$\Rightarrow \boxed{v = 80 - 10t}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

9.21 Continue ... 7

$v = 80 - 10t$ — (1)

$\frac{dx}{dt} = 80 - 10t$

$\Rightarrow \int_{150}^{x} dx = \int_{5}^{t} (80 - 10t)\,dt$

$\Rightarrow x = 150 + \left(80t - 10t^2/2\right)_5^t$

$= 150 + 80t - 5t^2 - 400 + 125$

$\Rightarrow \boxed{x = -5t^2 + 80t - 125}$

$\therefore v\Big|_{t=8\,sec} = 0$ Ans

$v\Big|_{t=12\,sec} = 80 - 10 \times 12$

$= -40\,ms^{-1}$. Ans

body started moving in negative x-dirn. with 40 ms⁻¹ at t = 12 sec.

$x\Big|_{t=8\,sec} = -5 \times 8^2 + 80 \times 8 - 125$

$= 195\,m$. Ans

$x\Big|_{t=12\,sec} = 115\,m$. Ans

10.1.22 A grain of sugar falling through honey has a negative acceleration proportional to the difference between its velocity and its 'terminal' velocity, which is a known constant v_t. Write this sentence as a differential equation, defining any constants you need. Solve the equation assuming some given initial velocity v_0.

9.22

$$a \propto -(v - v_t)$$

$$a = -b(v - v_t) \quad ; \; b = \text{proportionality Constant.}$$

$$\Rightarrow \frac{m\,dv}{dt} = -b\,(v - v_t)$$

$$\Rightarrow \frac{dv}{dt} = -\frac{b}{m}(v - v_t)$$

$$\Rightarrow \frac{dv}{(v - v_t)} = -\frac{b}{m}\,dt$$

Integrating, we get

$$\int_{v_0}^{v} \frac{dv}{(v - v_t)} = -\frac{b}{m}\int_{t=0}^{t} dt$$

$$\Rightarrow \left[\ln(v - v_t)\right]_{v_0}^{v} = -\frac{b}{m}t$$

$$\Rightarrow \left(\frac{v - v_t}{v_0 - v_t}\right) = e^{-b/m\,t}$$

$$\Rightarrow v = (v_0 - v_t)\,e^{-b/m\,t} + v_t$$

$$\Rightarrow \boxed{v = v_t + (v_0 - v_t)\,e^{-b/m\,t}}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

8

9.22 Continue –

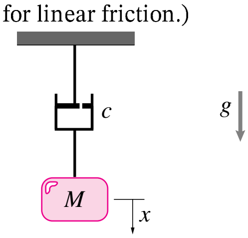
$$\frac{dx}{dt} = v_t + (v_0 - v_t)\, e^{-b/m\, t}$$

$$\int_{x_0}^{x} dx = \int_0^t \left[v_t + (v_0 - v_t)\, e^{-b/m\, t} \right] dt$$

$$\Rightarrow \quad x = x_0 + v_t \cdot t + (v_0 - v_t)\frac{m}{b}\left[1 - e^{-b/m\, t}\right]$$

Ans

10.1.23 The mass-dashpot system shown below is released from rest at $x = 0$. Determine an equation of motion for the particle of mass m that involves only $\dot{x}$ and x (a first-order ordinary differential equation). The dashpot opposes the motion of mass m with a force $F = c\dot{x}$ where c is the damping coefficient of the dashpot. (You can think of the dashpot as a model for linear friction.)

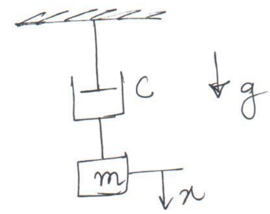


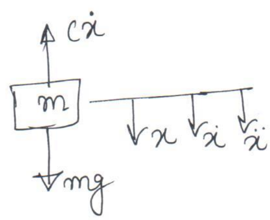
Problem 10.23

9.23

9

Free body diagram of mass "m"





$\cancel{m\ddot{x}+mg=c\dot{x}}$ $\quad F_{net} = ma$

$\Rightarrow \quad mg - c\dot{x} = m\ddot{x}$

$\Rightarrow \quad \boxed{m\ddot{x} + c\dot{x} - mg = 0.}$

$\dot{x} = y$

$\boxed{m\dot{y} + cy = mg}$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

10.1.24 Due to gravity, a particle falls in air with a drag force proportional to the speed squared.

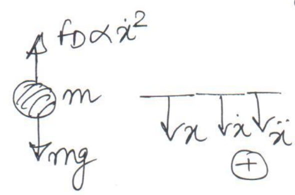
1. Write $\sum \vec{F} = m\vec{a}$ in terms of variables you clearly define,
2. find a constant speed motion that satisfies your differential equation,
3. pick numerical values for your constants and for the initial height. Assume the initial speed is zero
   a) set up the equation for numerical solution,
   b) solve the equation on the computer,
   c) make a plot with your computer solution and show how that plot supports your answer to (2).

9.24 1. $F_D \propto \dot{x}^2$ (10)

$\Rightarrow F_D = +b\dot{x}^2$

$m\ddot{x} = mg - b\dot{x}^2$

$$\Rightarrow \boxed{m\ddot{x} + b\dot{x}^2 - mg = 0}$$



(2) Constant speed motion, $\ddot{x} = 0$

$$\Rightarrow b\dot{x}^2 = mg.$$

$$\dot{x}^2 = \frac{mg}{b}.$$

$$\Rightarrow \dot{x} = \sqrt{\frac{mg}{b}}.$$

(3) $\ddot{x} = -\frac{b}{m}\dot{x}^2 + mg.$

Matlab: For Numerical solution

```
~~function [xdot~~
function [xdot] = myeqn(t,x)

xdot = [x(2) ;(-b*x(2)^2)/m + m*g.]

(t,x) = ode45(@myeqn ,[tspan], [zo]);
```

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

10.1.26 A bullet penetrating flesh slows approximately as it would if penetrating water. The drag on the bullet is about $F_D = c\rho_w v^2 A/2$ where ρ_w is the density of water, v is the instantaneous speed of the bullet, A is the cross sectional area of the bullet, and c is a drag coefficient which is about $c \approx 1$. Assume that the bullet has mass $m = \rho_l AL$ where ρ_l is the density of lead, A is the cross sectional area of the bullet and L is the length of the bullet (approximated as cylindrical). Assume $m = 2$ grams, entering velocity $v_0 = 400\,\mathrm{m/s}$, $\rho_l/\rho_w = 11.3$, and bullet diameter $d = 5.7\,\mathrm{mm}$.

a) Plot the bullet position vs time.

b) Assume the bullet has effectively stopped when its speed has dropped to 5 m/s, what is its total penetration distance?

c) According to the equations implied above, what is the penetration distance in the limit $t \to \infty$?

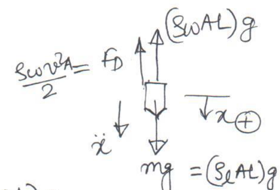
d) How would you change the model to make it more reasonable in its predictions for long time?

9.26

Free body diagram of bullet –

$\rho_L = 11.3\rho_w$

$\Rightarrow \rho_w = 0.0885\rho_l.$



$$\Rightarrow \ddot{x}(\rho_l AL) = (\rho_l AL)g - (\rho_w AL)g - \frac{\rho_w v^2 A}{2}$$

$$\Rightarrow \cancel{\rho_l A}L\frac{dv}{dt} = 0.9115(\cancel{\rho_l A}L)g - \frac{0.0885(\cancel{\rho_l A})v^2}{2}$$

$$\Rightarrow \frac{L\,dv}{dt} = 0.9115Lg - \frac{0.0885v^2}{2}$$

$$\Rightarrow \frac{l\,dv}{(8.942l - 0.04425v^2)} = dt$$

$$\Rightarrow \frac{dv}{\left(8.942 - \frac{0.04425v^2}{L}\right)} = dt$$

Integrate to find velocity & position as a function of time.

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

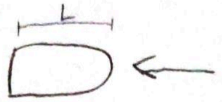
MARIGOT FACKENTHAL
MKF47
2/16/21

10.1.26

BULLET WITH DRAG

GIVEN:

(force sense only correct with positive velocity)



$F_D = c\rho_w V^2 \frac{A}{2}$

where ρ_w = density of water = 1000 kg/m³

V = instantaneous speed

A = cross-sectional area = $\pi r^2 = \pi\left(\frac{0.0057m}{2}\right)^2$

c = drag coefficient = 1

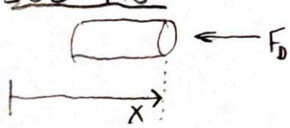
also ρ_l/ρ_w = density of lead / density of H_2O = 11.3

$V_0 = 400$ m/s

$m = 0.002$ kg $= \rho_l A L$

FIND: a) position vs. time, b) position at $V = 5$ m/s c) position at $t \to \infty$, d) model change for large-t predictions

SOLUTION:



FORCE BALANCE: $\sum \vec{F} = m\vec{a}$

1D problem, so I dot with $\hat{\imath}$, no more vector notation from here on out

$-c\rho_w V^2 \frac{A}{2} = ma$

$\underbrace{-c\rho_w \frac{A}{2}\left(\frac{1}{m}\right)}_{\text{let's call this constant "B"}} V^2 = \frac{dv}{dt}$

$(B > 0)$

NOTE: because of the square, this equation only works for $V > 0$. (imagine if V were negative - the square would get rid of the sign & cause the force to have the wrong sense.)

$\therefore$ $\boxed{-Bv^2 = \frac{dV}{dt}}$

and $\boxed{V = \frac{dx}{dt}}$

two 1st-order ODES

PG 1

okay, we have our 2 ODEs:

$$-Bv^2 = \frac{dv}{dt} \qquad V = \frac{dx}{dt}$$

we can start addressing the specific questions now :)

a) position vs. time

2 strategies:

NUMERICAL

plan: use MATLAB to estimate the position based on last position & rate of change of position @ last position

"Euler method"

code structure:

1. parameters
2. define time array
3. initial conditions
4. use numerical integrator (takes in EoM, params, time array, IC's) (returns integrated state vector)
5. plot

FUNCTIONS:

① numerical integrator

↓

ANALYTICAL

plan 1: solve by hand :(

let's use the 1st order $-Bv^2 = \frac{dv}{dt}$ ODE and try separation of variables:

$$-B\,dt = \frac{1}{V^2}\,dV$$

integrate both sides

$$-Bt + C_1 = -\frac{1}{V}$$

$$V = -\frac{1}{-Bt+C_1} = \frac{1}{Bt-C_1}$$

use initial condition V_0 to solve for integration constant:

$$V_0 = \frac{1}{B(0)-C_1} = -\frac{1}{C_1}$$

$$C_1 = -\frac{1}{V_0}$$

now we have:

$$\boxed{V = \frac{1}{Bt+\frac{1}{V_0}}}$$

which we can rewrite as:

↓

PG 2

② EoM function

see $\boxed{\text{p10\_1\_26.m}}$ (pg.____) for full code.

see $\boxed{\text{figure (1)}}$ (pg. ____) for position vs. time plot.

$$\frac{dx}{dt} = \frac{1}{Bt + \frac{1}{V_0}}$$ another 1st order ODE!

separate variables again:

$$dx = \frac{1}{Bt + \frac{1}{V_0}}\,dt$$

integrate both sides

$$x = \frac{\ln(BV_0t+1)}{B} + C_2$$

use initial condition X_0 to solve for integration constant:

$$X_0 = \frac{\ln(BV_0(0)+1)}{B} + C_2$$

$$X_0 = \cancel{\frac{\ln(1)}{B}}^{0} + C_2$$

$$C_2 = X_0$$

so finally:

$$\boxed{x = \frac{\ln(BV_0t+1)}{B} + X_0}$$

plan 2: solve in MATLAB symbolic toolbox... cool, but not necessary for this class. see $\boxed{\text{p10\_1\_26.m}}$ (pg.____) if interested.

phew! let's go to part b

pg ③

b) position @ $V = 5\,m/s$

strategies:

① NUMERICAL: open zarray in workspace.

zarray =

X_0	X_1	X_2	X_3	....
V_0	V_1	V_2	V_3	...

find where $V = 5\,m/s$, select corresponding X.

② NUMERICAL: plot V vs. t. click around on plot until you find $V = 5\,m/s$. note the time.
now plot x vs. t. click around on plot until you find the time you just noted. now note the position.

③ ANALYTICAL: use the analytical solution $V = \frac{1}{Bt + \frac{1}{V_0}}$ to solve for t when $V = 5\,m/s$.
then use the solution $x = \frac{\ln(BV_0 t + 1)}{B} + x_0$ to solve for x at the t you found.

④ ANALYTICAL: use your two analytical solutions

$$V = \frac{1}{Bt + \frac{1}{V_0}} \quad \& \quad x = \frac{\ln(BV_0 t + 1)}{B} + x_0$$

to produce ONE equation that eliminates t. (i.e. X as a function of V. solve.)

there are many methods. Result should be around $\boxed{x = 0.68\,m.}$

PG ④

c) What happens to position as $t \to \infty$?? $x \to \infty$... but why?

ANALYTICAL: look at solution

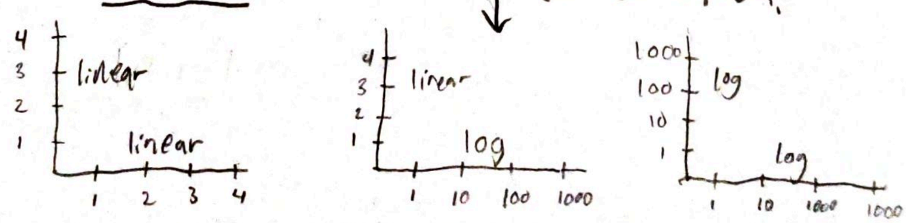
$$x = \frac{\ln(BV_0 t + 1)}{B} + x_0$$

take the limit! (or recognize that the behavior of $\ln(t)$ does not reach a horizontal asymptote. $\lim_{t\to\infty}(\ln(t)) = \infty$)

NUMERICAL: fancy MATLAB! the linear scaling on your plot makes it hard to see if $x(t)$ is approaching a horizontal asymptote or just increasing very slowly. But what if we change the t-axis to log scale? see $\boxed{\text{figure (1)}}$ & $\boxed{\text{figure (2)}}$ (pg. ___)

review:

this one works well for our system.



command: plot( ) command: semilogx( ) command: loglog( )

not super important for this class – if you are interested, spend some time thinking about why log scales better display long-term behavior :)

RESULT: as $t \to \infty$, $\boxed{x \to \infty}$

what?? this doesn't happen in real life... see part (d)!

pg 5

```
% p10_1_26.m

% DYNAMICS HW 2
% PROBLEM 5
% RUINA-PRATAP 10.1.26

% MARIGOT FACKENTHAL
% MKF47@CORNELL.EDU


clear; clf; close all; clc;

% SETUP
 *******************************************************************

% parameters [CHANGE THESE]
m = 0.002;
c = 1;
rhoW = 1000;
rhoL = 11.3*rhoW;
d = 0.0057;
A = pi*(d/2)^2;
L = m/(rhoL*A);

% for this problem, I'll group some of the parameters into a variable
% called "B" (it's a constant that will get carried around)
B = c*rhoW*A / (2*m);
p.B = B;  % pack B into a parameter struct for ode45

% timing
t0 = 0;  tf = 1; n = 5000;
tarray = linspace(t0,tf,n);

% initial conditions
rx0 = 0;
vx0 = 400;
z0 = [rx0; vx0];  % use semicolons to make it a 4x1
% also, make sure the entries are in the standard order (ex. x1, x2,
 v1, v2)



% NUMERICAL SOLUTION
 ******************************************************

[tarray, zarray] = myEulerSolver(@eqn, tarray, z0, p);

% outputs
rx = zarray(:,1);
vx = zarray(:,2);

% lets also find F(t)
```

```
Farray = -B.*vx.^2 .* m;


% ANALYTICAL SOLUTION
 **************************************************

syms x(t) v(t) B_

disp('Analytical solutions w/o initial conditions or parameters:')
ode1 = diff(x) == v;
ode2 = diff(v) == -B_*v^2;
odes = [ode1; ode2];
S = dsolve(odes);
xSol(t) = S.x
vSol(t) = S.v

disp('Analytical solutions with initial conditions and parameters:')
ode1 = diff(x) == v;
ode2 = diff(v) == -B*v^2;
odes = [ode1; ode2];
cond1 = x(0) == 0;
cond2 = v(0) == 400;
conds = [cond1; cond2];
S = dsolve(odes, conds);
xSol(t) = real(S.x)
vSol(t) = S.v

xAnalytical = xSol(tarray)';
vAnalytical = vSol(tarray)';


% lets also find F(t) [analytical]
FarrayAnalytical = (-B.*vAnalytical.^2 .* m)';


disp('Find x when v = 5m/s:')
xSol(0.031)



% PLOTTING
 ************************************************************

% ---------------
% POSITION V. TIME
figure(1)
hold on
grid on

% plot numerical solution
plot(tarray, rx, '-r', 'LineWidth', 3)

% plot analytical solution
```

```
plot(tarray, xAnalytical, '--b', 'LineWidth', 3)

% figure settings
xlabel('t [s]')
ylabel('x [m]')
title('(1) bullet with drag (position)')
legend('numerical', 'analytical')
hold off


% ---------------
% POSITION V. LOG(TIME)
figure(2)

% plot numerical solution
semilogx(tarray, rx, '-r', 'LineWidth', 3)
hold on
grid on

% plot analytical solution
semilogx(tarray, xAnalytical, '--b', 'LineWidth', 3)

% figure settings
xlabel('t [s]')
ylabel('x [m]')
title('(2) bullet with drag (logx scale, position)')
legend('numerical', 'analytical')
hold off


% ---------------
% NUMERICAL TO ANALYTICAL COMPARISON FOR POSITION V. TIME
figure(3)
hold on
grid on

% plot difference between analytical and numerical solutions
plot(tarray, xAnalytical-rx)

% figure settings
xlabel('t [s]')
ylabel('analytical - numerical [m]')
title('(3) numerical to analytical comparison (position)')
hold off


% ---------------
% FORCE V. VELOCITY
figure(4)

% plot numerical solution
loglog(vx, Farray, '-r', 'LineWidth', 3)
hold on
grid on
```

```
% plot analytical solution
loglog(vAnalytical, FarrayAnalytical, '--b', 'LineWidth', 3)

% figure settings
xlabel('v [m/s]')
ylabel('F [N]')
title('(4) bullet with drag (F vs v)')
legend('numerical', 'analytical')
hold off



% FUNCTIONS
 ****************************************************************

% equations of motion
function dz = eqn(t,z,p) % time, state, struct of matricies

    % unpack parameters **********************************
    B = p.B;

    % unpack z vector ************************************
    % positions
    rx = z(1);
    % velocities
    vx = z(2);

    % fill dz vector *************************************
    % first entries become velocities
    dz(1) = vx;
    % next entries become accelerations
    % CHANGE THESE 2 LINES ACCORDING TO EOM
    dz(2) = -B*vx^2;

    % transpose dz from row to column vector **************
    dz = dz';

end


% Solver using Euler's method (Stolen from Teaya Yang)
function  [tarray, zarray] = myEulerSolver(eqn, tarray, z0, p)
% Generating zarray matrix
ntimes = length(tarray);
zarray = zeros(ntimes, length(z0));
zarray(1,:) = z0';
h = tarray(2) - tarray(1); % step size


    % Applying Euler's method at each step
    for i  = 1:ntimes-1
```

```
        znow    = zarray(i,:)';   tnow = tarray(i);  % set present
 values
        zdot = eqn(tnow,znow,p);   % evaluate RSH (right hand side) at
 present time

        % Euler's method
        znew = znow + zdot * h;

        % Append to zarray
        zarray(i+1,:) = znew'; % Add a row to the array of solutions.
                           % Recall, z is a column, zarray is in rows.
    end % loop

end  % function
```

Analytical solutions w/o initial conditions or parameters:

xSol(t) =

```
                     C3
 C2 + log(B_*t - C1)/B_
```

vSol(t) =

```
              0
 -1/(C1 - B_*t)
```

Analytical solutions with initial conditions and parameters:

xSol(t) =

*0.1568*log(abs(6.3794*t + 0.0025)) + 0.9392*

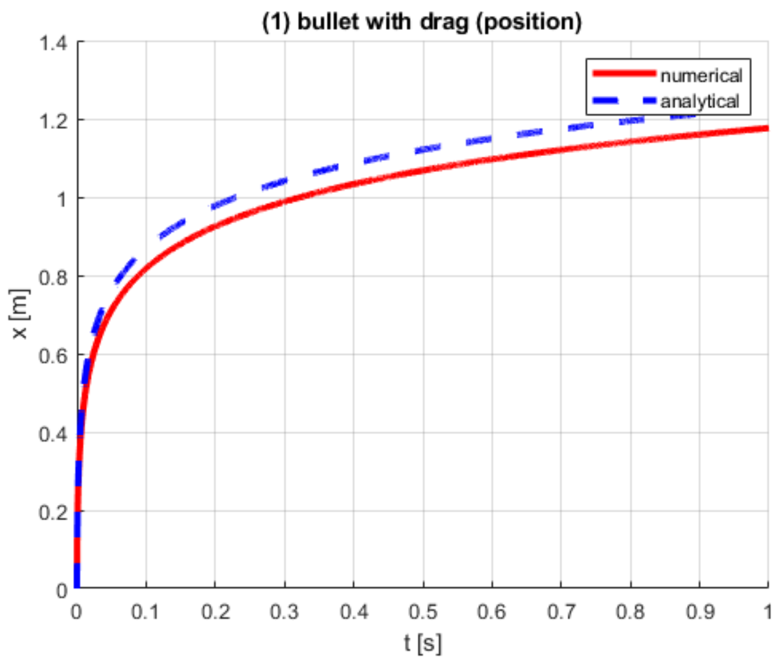
vSol(t) =

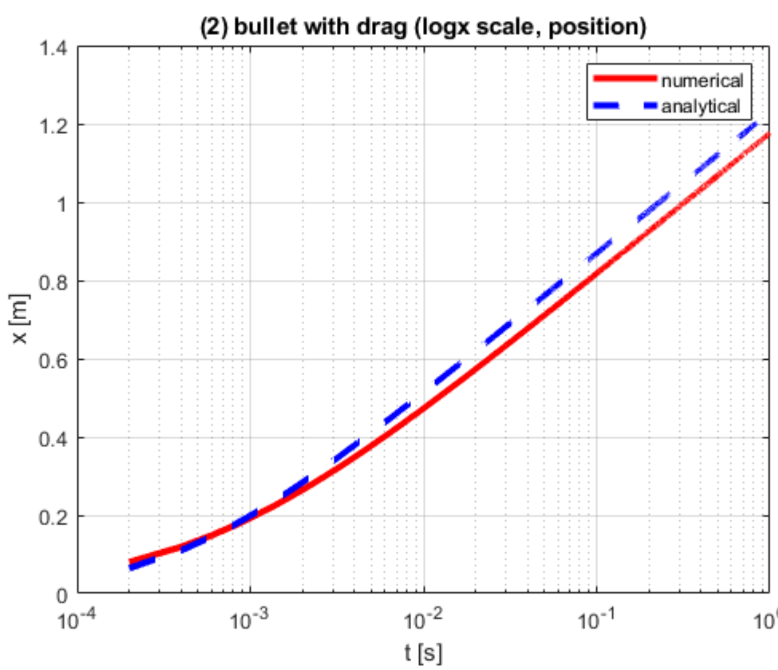
*1/(6.3794*t + 0.0025)*

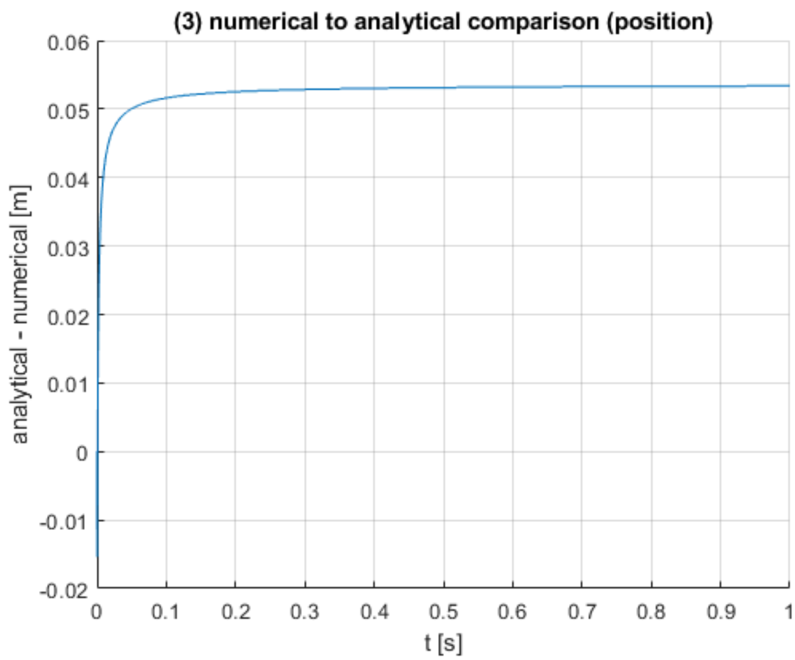
Find x when v = 5m/s:

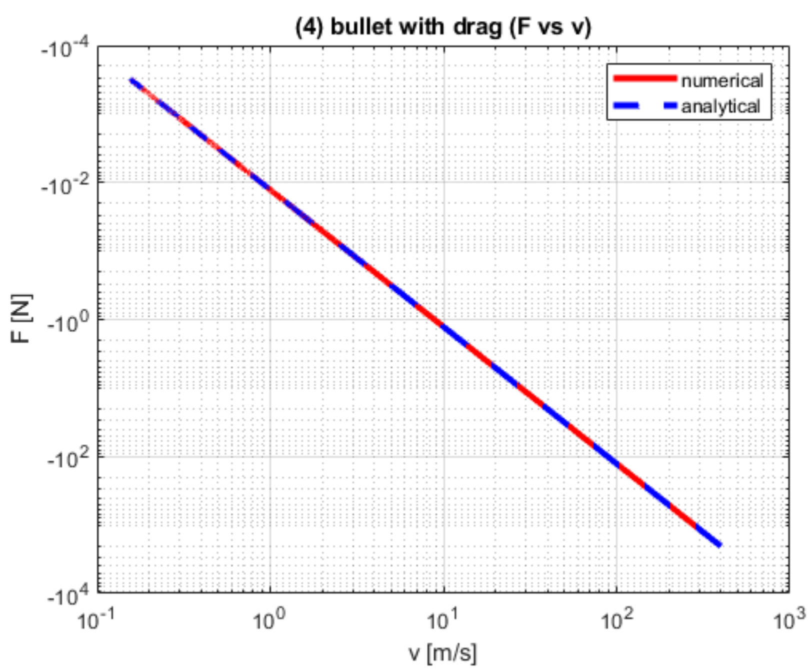
ans =

0.6871


(1) bullet with drag (position)
numerical
analytical
x [m]
t [s]
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

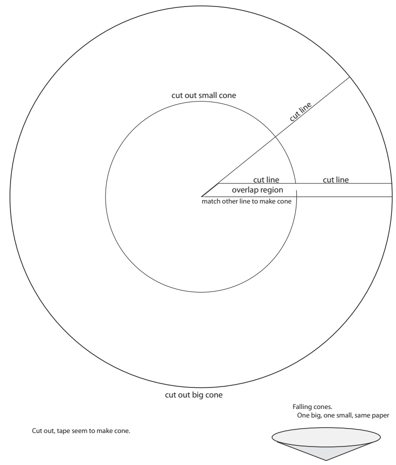

(2) bullet with drag (logx scale, position)
numerical
analytical
x [m]
t [s]
1.4
1.2
1
0.8
0.6
0.4
0.2
0
10^-4
10^-3
10^-2
10^-1
10^0


(3) numerical to analytical comparison (position)
analytical - numerical [m]
t [s]
0.06
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


(4) bullet with drag (F vs v)
numerical
analytical
F [N]
v [m/s]
-10^{-4}
-10^{-2}
-10^{0}
-10^{2}
-10^{4}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}

10.1.28 Do the falling cone experiment: Make two cones, drop them simultaneously and see which falls faster. Explain the result in a way that would convince you if it was written by another student who you did not already trust (and you had not already seen the experiment done).

More details. Copy the drawing below onto two pieces of paper. Make one cone from each page. One has twice the radius as the other. They both have the same conical angle. Which falls faster? Use $F = ma$ to predict the result. Hint 1: the cutout lets you think about how to calculate the area of the paper used to make the cone. Hint 2: The mass of the cone is proportional to its area. The result holds for linear or quadratic drag, or a combination of both. It holds for the steady state or for the transient response.



Problem 10.28

Answer:
The cones fall the same. All of the different forces (gravity, linear drag, quadratic drag) are proportional to either the mass or the area. And, for the cones, mass is proportional to area.