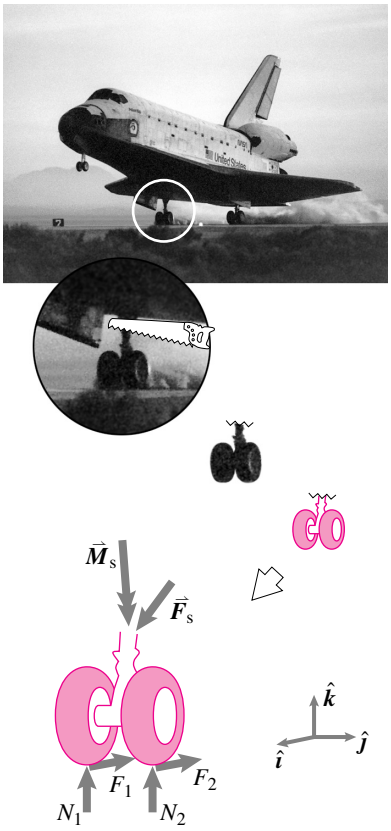
*Only for Cornell class use.*

Introduction to

# STATICS and DYNAMICS




**Rudra Pratap** and **Andy Ruina**

Oxford University Press (Preprint)
Most recent modifications on January 23, 2007.

**Reference Tables:** Now and again look at the front and back pages. These tables concisely cover much of mechanics.

# Summary of Mechanics

**0) The laws of mechanics apply to any collection of material or 'body.'** This body could be the overall system of study or any part of it. In the equations below, the forces and moments are those that show on a free body diagram. Interacting bodies cause equal and opposite forces and moments on each other.

**I) Linear Momentum Balance (LMB)/Force Balance**

| | | |
|---|---|---|
| Equation of Motion | $\sum \vec{F}_i = \dot{\vec{L}}$ | The total force on a body is equal to its rate of change of linear momentum. (I) |
| Impulse-momentum (integrating in time) | $\int_{t_1}^{t_2} \sum \vec{F}_i \cdot dt = \Delta \vec{L}$ | Net impulse is equal to the change in momentum. (Ia) |
| Conservation of momentum (if $\sum \vec{F}_i = \vec{0}$ ) | $\dot{\vec{L}} = \vec{0} \Rightarrow$ $\Delta \vec{L} = \vec{L}_2 - \vec{L}_1 = \vec{0}$ | When there is no net force the linear momentum does not change. (Ib) |
| Statics (if $\dot{\vec{L}}$ is negligible) | $\sum \vec{F}_i = \vec{0}$ | If the inertial terms are zero the net force on system is zero. (Ic) |

**II) Angular Momentum Balance (AMB)/Moment Balance**

| | | |
|---|---|---|
| Equation of motion | $\sum \vec{M}_{\mathrm{C}} = \dot{\vec{H}}_{\mathrm{C}}$ | The sum of moments is equal to the rate of change of angular momentum. (II) |
| Impulse-momentum (angular) (integrating in time) | $\int_{t_1}^{t_2} \sum \vec{M}_{\mathrm{C}} dt = \Delta \vec{H}_{\mathrm{C}}$ | The net angular impulse is equal to the change in angular momentum. (IIa) |
| Conservation of angular momentum (if $\sum \vec{M}_{\mathrm{C}} = \vec{0}$) | $\dot{\vec{H}}_{\mathrm{C}} = \vec{0} \Rightarrow$ $\Delta \vec{H}_{\mathrm{C}} = \vec{H}_{\mathrm{C}2} - \vec{H}_{\mathrm{C}1} = \vec{0}$ | If there is no net moment about point C then the angular momentum about point C does not change. (IIb) |
| Statics (if $\dot{\vec{H}}_{\mathrm{C}}$ is negligible) | $\sum \vec{M}_{\mathrm{C}} = \vec{0}$ | If the inertial terms are zero then the total moment on the system is zero. (IIc) |

**III) Power Balance (1st law of thermodynamics)**

| | | |
|---|---|---|
| Equation of motion | $\dot{Q} + P = \underbrace{\dot{E}_{\mathrm{K}} + \dot{E}_{\mathrm{P}} + \dot{E}_{\mathrm{int}}}_{\dot{E}}$ | Heat flow plus mechanical power into a system is equal to its change in energy (kinetic + potential + internal). (III) |
| for finite time | $\int_{t_1}^{t_2} \dot{Q} dt + \int_{t_1}^{t_2} P dt = \Delta E$ | The net energy flow going in is equal to the net change in energy. (IIIa) |
| Conservation of Energy (if $\dot{Q} = P = 0$) | $\dot{E} = 0 \Rightarrow$ $\Delta E = E_2 - E_1 = 0$ | If no energy flows into a system, then its energy does not change. (IIIb) |
| Statics (if $\dot{E}_{\mathrm{K}}$ is negligible) | $\dot{Q} + P = \dot{E}_{\mathrm{P}} + \dot{E}_{\mathrm{int}}$ | If there is no change of kinetic energy then the change of potential and internal energy is due to mechanical work and heat flow. (IIIc) |
| Pure Mechanics (if heat flow and dissipation are negligible) | $P = \dot{E}_{\mathrm{K}} + \dot{E}_{\mathrm{P}}$ | In a system well modeled as purely mechanical the change of kinetic and potential energy is due to mechanical work on the system. (IIId) |

# Some Definitions

(See also the index and glossary in the back.)

| | | |
|---|---|---|
| $\vec{r}$ or $\vec{x}$ | Position | *e.g.*, $\vec{r}_i \equiv \vec{r}_{i/O}$ is the position of a point $i$ relative to the origin, O) |
| $\vec{v} \equiv \dfrac{d\vec{r}}{dt}$ | Velocity | *e.g.*, $\vec{v}_i \equiv \vec{v}_{i/O}$ is the velocity of a point $i$ relative to O, measured in a non-rotating reference frame) |
| $\vec{a} \equiv \dfrac{d\vec{v}}{dt} = \dfrac{d^2\vec{r}}{dt^2}$ | Acceleration | *e.g.*, $\vec{a}_i \equiv \vec{a}_{i/O}$ is the acceleration of a point $i$ relative to O, measured in a Newtonian frame) |
| $\vec{\omega}$ | Angular velocity | A measure of rotational velocity of a rigid body. |
| $\vec{\alpha} \equiv \dot{\vec{\omega}}$ | Angular acceleration | A measure of rotational acceleration of a rigid body. |
| $\vec{L} \equiv \begin{cases} \sum m_i \vec{v}_i & \text{discrete} \\ \int \vec{v} dm & \text{continuous} \end{cases}$ $= m_{tot}\vec{v}_{cm}$ | Linear momentum | A measure of a system's net translational rate (weighted by mass). |
| $\dot{\vec{L}} \equiv \begin{cases} \sum m_i \vec{a}_i & \text{discrete} \\ \int \vec{a} dm & \text{continuous} \end{cases}$ $= m_{tot}\vec{a}_{cm}$ | Rate of change of linear momentum | The aspect of motion that balances the net force on a system. |
| $\vec{H}_C \equiv \begin{cases} \sum \vec{r}_{i/C} \times m_i \vec{v}_i & \text{discrete} \\ \int \vec{r}_{/C} \times \vec{v} dm & \text{continuous} \end{cases}$ | Angular momentum about point C | A measure of the rotational rate of a system about a point C (weighted by mass and distance from C). |
| $\dot{\vec{H}}_C \equiv \begin{cases} \sum \vec{r}_{i/C} \times m_i \vec{a}_i & \text{discrete} \\ \int \vec{r}_{/C} \times \vec{a} dm & \text{continuous} \end{cases}$ | Rate of change of angular momentum about point C | The aspect of motion that balances the net torque on a system about a point C. |
| $E_K \equiv \begin{cases} \frac{1}{2}\sum m_i v_i^2 & \text{discrete} \\ \frac{1}{2}\int v^2 dm & \text{continuous} \end{cases}$ | Kinetic energy | A *scalar* measure of net system motion. |
| $E_{int} =$ (heat-like terms) | Internal energy | The non-kinetic non-potential part of a system's total energy. |
| $P \equiv \sum \vec{F}_i \cdot \vec{v}_i + \sum \vec{M}_i \cdot \vec{\omega}_i$ | Power of forces and torques | The mechanical energy flow into a system. Also, $P \equiv \dot{W}$, rate of work. |
| $[\boldsymbol{I}^{cm}] \equiv \begin{bmatrix} I_{xx}^{cm} & I_{xy}^{cm} & I_{xz}^{cm} \\ I_{xy}^{cm} & I_{yy}^{cm} & I_{yz}^{cm} \\ I_{xz}^{cm} & I_{yz}^{cm} & I_{zz}^{cm} \end{bmatrix}$ | Moment of inertia matrix about cm | A measure of how mass is distributed in a rigid body. |

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How did we do it? Software we have used to prepare this book includes *TeXshop* (a mac freeware implementation of *LaTeX*), *Adobe Illustrator*, *GraphicsConverter* (a good simple photo editor) and *MATLAB*.

Modified January 23, 2007.

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*amplifying force are usually based on principals idealized by pulleys, levers, wedges and toggles. Force-analysis of transmissions and mechanisms is done by drawing free body diagrams of the parts, writing equilibrium equations for these, and solving the equations for desired unknowns.*

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# Preface

*General issues about content, level, organization and style, motivation, how to study and how the role of computers.*

**Contents**


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This is an engineering statics and dynamics text aimed primarily at middle-level engineering students. The book emphasizes use of vectors, free body diagrams, momentum and energy balance, and computation. These are taught alongside intuitive approaches. The book is intended as an introduction for various kinds of students, and as a reference.

**Prerequisite and co-requisite skills.** We assume some students start with some skills.

- Freshman calcululus. Readers are assumed to have facility with the basic geometry, algebra, trigonometry, differentiation and integration used in elementary calculus. Some of these topics are briefly reviewed in this book, but not as *ab initio* tutorials.

This books teaches students to set-up algebraic and differential equations for computer solution. The book uses a pseudo-language easily translated by a student into any other language.

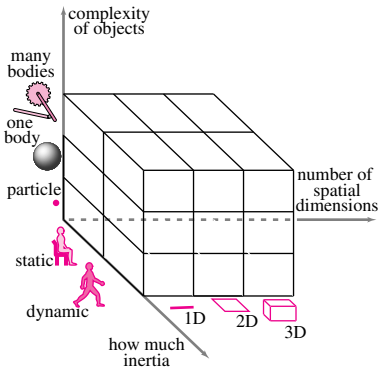
- We assume the student knows or is learning a computer language or package in which they can solved sets of linear algebraic equations, make plots and numerically solve simple ordinary differential equations.

Many students will have had exposure to other useful subjects detailed foreknowledge of which this book does not assume.

- Completion of freshman physics may help but is not needed.
- Vector topics, especially dot and cross products, are introduced here from scratch in the context of mechanics.
- A background in linear algebra wouldn't hurt, but the reduction of linear equations to matrix form is taught here. A key fact from linear algebra, also presented here, is that linear algebraic equations are generally amenable to simple computer solution.
- A course in differential equations would also add context. But the basic concepts of differential equations are presented here as needed.

## Organization

Mechanics can be subdivided into statics *vs* dynamics, particle *vs* rigid object *vs* many bodies ('multi-object'), and 1 *vs* 2 *vs* 3 spatial dimensions (1D, 2D & 3D). Thus a mechanics table of contents might have one chunk of text for each of the $2 \times 3 \times 3 = 18$ combinations:




I. Statics
A. particle
1) 1D
2) 2D
3) 3D
B. rigid object
4) 1D
5) 2D
6) 3D
C. many bodies
7) 1D
8) 2D
9) 3D

II. Dynamics
A. particle
10) 1D
11) 2D
12) 3D
B. rigid object
13) 1D
14) 2D
15) 3D
C. many bodies
16) 1D
17) 2D
18) 3D.

However, these 18 chunks vary greatly in difficulty; 1D statics is low-level high school material and 3D multi-object dynamics is difficult graduate material. Further, the chunks use various overlapping concepts and skills. So it is not sensible to organize a book into 18 corresponding chapters. Nonetheless, some vestiges of the scheme above are used in all books, and the general flow of this book is from the bottom back left corner of the box in the figure on the side, towards the diagonal opposite. The details of the organization, as visible in the annotated table of contents on the previous pages, has evolved through trial and error, review and revision, and many semesters of student testing.

The first eight chapters cover the basics of statics and the rest of the book covers the basics of engineering dynamics. Relatively harder topics, which might be skipped in quicker or less-advanced courses, are identifiable by chapter, section or subsection titles like “three-dimensional” or “advanced”.

**Coverage for courses.** The sections have been divided so that the homework problems selected from one section might be about half of a typical weekly homework assignment. The theory and examples from one section might be adequately covered in about one lecture, your mileage may vary.

A leisurely one semester statics course, or a more fast-paced half-semester prelude to strength of materials should use chapters 1-8, excluding topics of less interest. A typical one semester dynamics course will cover most of of chapters 9-16, reviewing chapters 1-3 at the start. A lower-level one-semester statics and dynamics course can cover the less advanced parts of chapters 1-6 and 9-14. An advanced full-year statics and dynamics course could cover most of the book. That is, the statics portion of the book fits easily in a semester and the whole of the dynamics portion in a bit more than a semester. Chapters 15-18 can also be used as a start for a second advanced dynamics course. A student who has learned the statics part of this book is well-prepared for using statics in engineering practice, for learing Strength of Materials and for going on to Dynamics. A student who has learned the dynamics portion is well prepared to go on to learn Vibrations, Systems Dy-

namics or more advanced Multi-object Dynamics.

## Organization and formatting

Each subject is covered in various ways.

- Every section starts with **descriptive text** and short *examples* motivating and describing the theory;
- More detailed explanations of the **theory** are in boxes interspersed in the text. For example, one box explains the common derivation of angular momentum balance from $\vec{F} = m\vec{a}$ (page **??**), one explains the genius of the wheel (page 187), and another connects $\vec{\omega}$ based kinematics to $\hat{e}_r$ and $\hat{e}_\theta$ based kinematics (page **??**);
- **Sample problems** (marked with a gray border) at the end of each section show how to do homework-like calculations. These set an example to the student by their consistent use of free-body diagrams, systematic application of basic principles, vector notation, units, and checks against both intuition and special cases;
- **Homework problems** at the end of each chapter give students a chance to practice mechanics calculations. The first problems for each section build a student's confidence with the basic ideas. The problems are ranked in approximate order of difficulty, with theoretical problems coming later. Problems marked with an * have an answer at the back of the book;
- **Reference tables** on the inside covers and end pages concisely summarize much of the content in the book. These tables can save students the time of hunting for formulas and definitions.

## Notation

Clear vector notation helps students do problems. One common class of student errors comes from copying a textbook's printed bold vector $\boldsymbol{F}$ the same way as a plain-text scalar $F$. We reduce this error by use a redundant vector notation, a bold and harpooned $\vec{\boldsymbol{F}}$.

As for all authors and teachers concerned with motion in two and three dimensions we have struggled with the tradeoffs between a precise notation and a simple notation. Perfectly precise notations are complex and intimidating. Simple notations are ambiguous and hide key information. Our attempt at clarity without too-much clutter is summarized in the box on page 27.

# Relation to other mechanics books

The bulk of the content of this book can be found in other places including freshman physics texts, other engineering texts, and hundreds of classics. Nonetheless this book is in some ways original in organization and approach. It also uses some important but not well-enough known concepts*. Mastery of freshman physics (*e.g.*, from Halliday, Resnick & Walker, Tipler, or Serway) would encompass some of this book's contents. However after

* For example, we use angular momentum balance (appropriately expressed) with respect to any possibly-accelerating point, not just points selected from an arcane list.

freshman physics students often have only a vague notion of what mechanics is, and how it can be used. For example many students leave freshman physics with the sense that a free body diagram (or 'force diagram') is a vague conceptual picture with arrows for various forces and motions drawn on it this way and that. Even the freshman-text illustrations sometimes do not make clear which force is acting on which object. Also, because freshman physics tends to avoid use of college math, many students leave freshman physics with little sense of how to use vectors or calculus to solve mechanics problems. This book aims to lead students who may start with these fuzzy freshman-physics notions into a world of precise, yet still intuitive, mechanics.

Various statics and dynamics textbooks cover much of the same material as this one. These textbooks have modern applications, ample samples, lots of pictures, and lots of homework problems. Many are excellent in some ways. Most of today's engineering professors learned from one of these books. Nonetheless we were dissatisfied and wrote this book hoping to do still better. Some of our goals include

- showing the unity of the subject,
- presenting a complete description of the subject,
- clear notation in figures and equations,
- integration of the applicability of computers,
- consistent use of units throughout,
- introduction of various insights into how things work,
- a friendly writing style.

Between about 1689 and 1960 hundreds of books were written with titles like *statics*, *engineering mechanics*, *dynamics*, *machines*, *mechanisms*, *kinematics*, or *elementary physics*. Many thoughtfully cover most aspects of the material here. Unfortunately none are good modern textbooks. They lack an appropriate pace, style of language, and organization. They are too reliant on geometry skills and not enough on vectors and numerical computation skills. And they lack sufficient modern applications, sample calculations, illustrations, and homework problems for a modern text book. But much good mechanics can be found only in these older books*. If you love the subject you will enjoy glancing at these books.

## What do you think?

We have tried to make it as easy as possible for you to learn basic mechanics from this book. We present truth as we know it and as we think it is effectively communicated. Nonetheless we have undoubtedly left some technical and strategic errors. Please let us know your thoughts so that we can improve future editions.

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Andy Ruina, ruina@cornell.edu

* Here are three good and universally respected classics:

J.P. Den Hartog's *Mechanics* originally published in 1948 but still available as an inexpensive reprint (well written and insightful);

J.L. Synge and B.A. Griffith, *Principles of Mechanics* through page 408. Originally published in 1942, reprinted in 1959 (good pedagogy but dry); and

E.J. Routh's, *Dynamics of a System of rigid bodies*, Vol 1 (the "elementary" part through chapter 7. Originally published in 1905, but reprinted in 1960). Routh also has 5 other idea- packed statics and dynamics books. Routh shared college graduation honors with the now-more-famous physicist James Clerk Maxwell.

# 0.1 To the student (please read)

Mother nature holds her family, the collection of all physical things, to strict rules. She is so strict that, to the extent we know her rules, we can make reliable predictions about how her family, all things, behave. In particular, most objects of concern to engineers obediently follow a subset of Nature's rules called the laws of Newtonian mechanics. So, if you learn the laws of mechanics, as this book should help you to do, you will be able to make quantitative predictions about how things stand, move, and fall. You will also gain intuition about the mechanics part of Nature's rule book.

## How to use this book

Here is some general guidance.

### Check your own understanding

Most likely you want a decent grade by successfully getting through the homework assignements and exams. You will naturally get help by looking at examples and samples in the text or lecture notes, by looking up formulas in the front and back covers of this book, and by asking questions of friends, teaching assistants and professors. What good are books, notes, classmates or teachers if they don't help you do the homework? All the examples and sample problems in this book, for example, are just for this purpose.

But watch out. Too-much use of help from books, notes and people can lead to self deception*. After you have got through a problem using such help you should, at least sometimes, check that you have actually learned to solve the problem.

> To see if you have learned to do a problem, do it again, justifying each step, without looking up *even one small oh-I-almost-knew-that thing*.

If you can't do this, you have a new opportunity to learn at two levels. First, you can learn the missing skill or idea. More deeply, by getting stuck after you have been able to get through a problem with guidance, you can learn things about your learning process. Often the real source of difficulty isn't a key formula or fact, but something more subtle. We have tried to bring out some of these more subtle ideas in the text. We know that the text is usually a last resort for a time-pressured student juggling 5 courses. But we think you may find it useful.

### Read the parts that are at your level

Some of you are science and math school-smart, mechanically inclined, or are especially motivated to learn mechanics. Others of you are reluctantly taking this class to fulfil a requirement. This book for both of you. The sections start with generally accessible introductory material and include simple

* **"Exams are harder than homework."** Some students say "I can do the homework problems. I just can't do the exams. Exams are harder and trickier." However, we write both the homework and exam problems. Many of our homework problems (e.g., in this book) are recycled exam problems, and our course is not getting harder over the years. So we have checked the students' claims. At least 9 out of 10 of these students, when free from all help, can not do simplified versions of the exact same homework problems that they said they could do. That is, 9 out of 10 students who think they can do the homework, even though they can't do the exams, are fooling themselves.

examples. The early sample problems in each section are also easy. But we also have discussions of the theory and other more advanced applications and asides to challenge more motivated students. If you are a nerd, please be patient with the slow introductions and the calculations that often go line by line without skipping steps. On the other hand, if you are just trying to get through this course you need not stop and admire every one of the scenic detours.

## Calculation strategies and skills

In this book we try to demonstrate a systematic approach to solving problems. But its impossible to reduce all mechanics problem solutions to one clear recipe (despite the attempt to do so on the inside back cover). If it were possible we would write a computer that did that, and not a textbook, thus freeing the human mind, your mind in particular, from mechanics problem solutions like a calculator frees you from the tedium of long division. There is an art to solving problems, whether homework problems or engineering design problems. Art and human insight, as opposed to precise algorithm or recipe, is what makes engineering require humans and not just computers*. Throughout the book, in discussion and examples, we will try to teach you some of this art. For starters here are a few general tips.

### Understand the question

You may be tempted to start writing equations and quoting principles when you first see a problem. However, it is usually worth a few minutes (and sometimes a few hours) to try to get an intuitive sense of a problem before jumping to equations. Before you draw any sketches or write equations, think: does the problem make sense? What information has been given? What are you trying to find? Is what you are trying to find determined by what is given? What physical laws make the problem solvable? What extra information do you think you need? What information have you been given that you don't need? You should first get a general sense of the problem to steer you through the technical details.

Some students find they can read every line of sample problems yet cannot do test problems, or, later on, cannot do applied design work effectively. This failing may come from following details without spending time, thinking, gaining an overall sense of the problems.

### Think through your solution strategy

For problem solutions, like those we present in this book or when we teach, there was a time when we had to think about the order of our work. You also have to think about the order of your work. You will find some tips in the text and samples. But it is your job to own the material, to learn how to think about it your own way, to become an expert in your own style, and to do the work in the way that makes things most clear to you and your readers.

* **Long division analogy.** To be honest, this book does teach many methods which computers *can* handle. Once a problem has been reduced to a precise mechanical model a computer code (a finite-element program or a rigid-body dynamics program) could take over. But there are issues of setup and interpretation that you will do better if you are able to do some calculations yourself, at least simple ones. Consider the long-division analogy. Division by a 3 (or more) digit number is usually done by calculators, not long-division, at least since about 1975. But competence at division, at least at division by one digit numbers, allows one to quickly catch calculator-entry errors. And detailed knowledge about division (that, for example, its the inverse of multiplication, or that division by zero is problematic) is useful. And such knowledge seems to come better by practice with numbers manipulated in one's head and on paper than just on a calculator. Similarly it is useful to know well mechanics-problem recipes, even if they are precise and thus in principle code-able.

## The order of calculation is often backwards from the order of thinking

When working out how to solve a problem you often start with general principles, then look at terms you need to know. If these are not given you think how to figure those from other terms and so on. On the other hand, when you go to calculate an answer you have to start with the information given and work your way backwards into the equation which has your answer*. To find the net worth of a corporation you add the value of the various divisions. To get the value of a division you add up the values of the factories. For each factory you add up the value of the pieces of machinery. But to get an actual corporate value you have to start by evaluating the pieces of machinery in each factory and working back up from the known towards the answer. Beware that

> When you read the minimal write-up of a calculation, especially an algorithmic recipe or computer program, you often are reading in the inverse order of the thinking that went in to generating the solution.

Of course real problem solving goes both ways. You think about what you need in order to calculate what you want. But you also think about what you can calculate easily from what is given plainly to you. You reach from the broad towards the details. And you work with known details towards answers of any kind, wanted or not. And you thus hunt out, building from details and reaching back from the goal, a route leading all the way from the details to the goal.

## Look for equations containing unkowns, not formulas for unknowns

In elementary science and math we often learn formulas like

$$V = LWH, \qquad d = \frac{1}{2}at^2, \quad \text{and} \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to find $V$, $d$, or $x$. And such formulas are nice if you have to calculate something often. Most often though, you will not know a formula where the thing you want is on the left and everything given is on the right. You will have, say,

$$\begin{aligned} V &= LWH && \text{when you want to find } W \text{ from } V, L, \text{ and } H, \\ d &= \frac{1}{2}at^2 && \text{when you want to find } t \text{ from } a \text{ and } d, \text{ and} \\ ax^2 + bx + c &= 0 && \text{when you want to find } x \text{ from } a, b, \text{ and } c. \end{aligned}$$

Once you have got this far the only problem is math*. Here are two tricks of the mind

* **A tree analogy.** Energy gets stored in the roots of a tree. It gets there from the trunk. The branches feed the trunk, the twigs feed the branches, and the leaves feed the twigs with energy from the sun. But the flow goes the opposite way, from the leaves on down to the roots. But if you try to invent a tree by starting at the leaves with no knowledge of the root you could easily get lost and connect leaves to electric wires to gas pipes — all nonsense. There's no point in connecting the leaves to anything until you have a sense of the whole tree.

* For this and other courses, you should be good at solving math problems with you pencil and with a computer. But you should distinguish between the task of setting up a math problem and solving it. The math often takes the bulk of the time and paper, but it's not where your thoughts should start. The new material that you are attempting to master is largely about setting up the math problem.

1) **The math genius.** Imagine you have a math and computer genius friend who doesn't know any mechanics. Your first task is writing things down so she could finish up for you. She doesn't want to help? Then realize that finishing up without her is a separate later job.
2) **Be an egotist.** Pretend you are omniscient and know everything. Then write down true statements about those things; equations that contain terms that omniscient-you know. Then relax your ego a bit. Count equations and unknowns to see if you, or at least your math genius friend, could solve for the things you previously faked knowing.

A common beginner's error is to hunt for a formula that generates the sought unknown in terms of given quantities. Rather, you should

> Find relations that contain variables of interest; don't worry about whether they are on the right or left side of an equation. Don't worry about whether the variables are alone or isolated.

## Vectors and free body diagrams

In the toolbox of someone who can solve lots of mechanics problems are two well-worn tools:

- A vector calculator that always keeps vectors and scalars distinct, and
- A reliable and clear free body diagram drawing tool.

Because many of the terms in mechanics equations are vectors, the ability to do vector calculations is essential. Because the concept of an isolated system is at the core of mechanics, every mechanics practitioner needs the ability to draw a good free body diagram. The second and third chapters will help you build your own set of these two most-important tools.

> **Guarantee:** If you learn to do clear correct vector algebra and to draw good free body diagrams you will do well at mechanics. Assuming, you don't totally stop studying then and there.

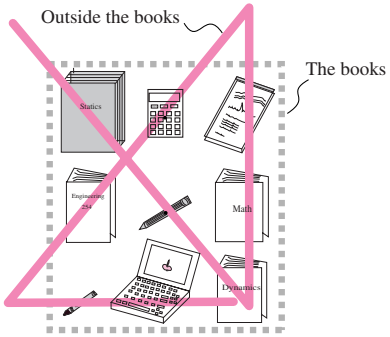



Figure 0.1: Thinking outside of the books. A famous puzzle asks: using 4 contiguous straightline segments connect all 9 dots that are in a square $3 \times 3$ array. The only solution has segments extending outside the "box" of 9 points. Hence the expression "thinking outside of the box".

Filename:tfigure-outsidethebooks

## Thinking outside the books

We do mechanics because we like mechanics. We hope you will too. It's fun to puzzle out how things work. Its satisfying to do calculations that make realistic predictions. Mechanics is interesting in its own right and it feels good to take pride in new skills. We wrote this book because we want to help you get through the subject. But we don't know a straightforward path through

your resources (say a path with 4 straight segments) that really covers what you need to know. You need to think outside of the confines of your usual study resources. Like when you are relaxed, away from the pressures of books, notes, pencils or paper, say when you are walking, showering or lying down*. These are the places where you naturally work out life problems, but they are good places to work out mechanics problems too.

*Having an animated mechanics discussion with friends is also good. You should enjoy your inner nerd in public. Are your friends turned off by tech-talk? There are billions of people out there, find one or two that like to have fun with you this way.

## 0.2 A note on computation

Mechanics is a physical subject. The concepts in mechanics do not depend on computers. But mechanics is also a quantitative subject; relevant amounts (of length, mass, force, moment, time, etc) are described with numbers, and relations are described using equations and formulas. Computers are very good with numbers and formulas. Thus the modern practice of engineering mechanics uses computers. The most-needed computer skills for mechanics are:

- solution of simultaneous linear algebraic equations,
- plotting, and
- numerical solution of ODEs (Ordinary Differential Equations).

More basically, an engineer also needs the ability to routinely evaluate standard functions ($x^3$, $\cos^{-1}\theta$, $etc.$), to enter and manipulate lists and arrays of numbers, and to write short programs.

### Classical languages, applied packages, and simulators

Programming in standard languages such as Fortran, Basic, Pascal, C++, or Java probably take too much time to use in solving simple mechanics problems. Thus an engineer needs to learn to use one or another widely available computational package ($e.g.$, MATLAB, O-MATRIX, SCI-LAB, OCTAVE, MAPLE, MATHEMATICA, MATHCAD, TKSOLVER, LABVIEW, etc). We assume that students have learned, or are learning such a package. Although none of the homework here depends on such, we also encourage you to play with packaged mechanics simulators ($e.g.$, INVENTOR, WORKING MODEL, ADAMS, DADS, ODE, etc) for testing and building your intuition.

### How we explain computation in this book.

Solving a mechanics problem involves

1. Reducing a physical problem to a well posed mathematical problem;
2. Solving the math problem using some combination of pencil and paper and numerical computation; and
3. Giving physical interpretation of the mathematical solution.

This book is primarily about setup (a) and interpretation (c), which are rather the same, no matter what method is used to solve the equations. If a problem requires computation, the exact computer commands vary from package to

package. And we don't know which one you are using. So in this book we express our computer calculations using an informal pseudo computer language. For reference, typical commands are summarized on page xviii.

### Required computer skills.

Here, in a little more detail, are the primary computer skills you need.

- **Linear algebraic equatinos.** Many mechanics problems are statics or 'instantaneous mechanics' problems. These problems involve trying to find some forces or accelerations at a given configuration of a system. These problems can generally be reduced to the **solution of linear algebraic equations** of this general type: solve

$$\begin{aligned} 3\ x\ +\ \ 4\ y\ &=\ 8 \\ -7\ x\ +\ \sqrt{2}\ y\ &=\ 3.5 \end{aligned}$$

for $x$ and $y$. In practice the number of variables and equations can be quite large. Some computer packages will let you enter equations almost as written above. In our pseudo language we would write:

```
set = { 3*x +       4*y = 8
        -7*x + sqrt(2)*y = 3.5 }
solve set for x and y
```

Other packages may require you to set up your equations in matrix form

$$\underbrace{\begin{bmatrix} 3 & 4 \\ -7 & \sqrt{2} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{z} = \underbrace{\begin{bmatrix} 8 \\ 3.5 \end{bmatrix}}_{b} \quad \text{or} \quad Az = b$$

which in computer-speak might look something like this:

```
A = [ 3       4
      -7  sqrt(2) ]
b = [ 8 3.5 ]'
solve  A*z=b for z
```

where `A` is a $2 \times 2$ matrix, `b` is a column of 2 numbers (the `'` indicates that the row of numbers `b` should be transposed into a column), and the two elements of `z` are $x$ and $y$. For systems of two equations, like above, a computer is hardly needed. But for systems of three equations pencil and paper work is sometimes error prone. Given the tedium, the propensity for error, and the availability of electronic alternatives, pencil and paper solution of four or more equations is an anachronism.

- **Plotting.** In order to see how a result depends on a parameter, or to see how a quantity varies with position or time, it is useful to see a **plot**. Any plot based on more than a few data points or a complex formula is far more easily drawn using a computer than by hand. Most often you can organize your data into a set of $(x, y)$ pairs stored in an `x` list and a corresponding `y` list. A simple computer command will then plot $x$ *vs* $y$. The pseudo-code below, for example, plots a circle using 100 points

```
npoints = [0 1 2 3 ... 100]
theta   = npoints * 2 * pi / 100
x       = cos(theta)
y       = sin(theta)
plot y vs x
```

where `npoints` is the list of numbers from 1 to 100, `theta` is a list of 100 numbers evenly spaced between 0 and $2\pi$ and `x` and `y` are lists of 100 corresponding $x$, $y$ coordinate points on a circle.

- **ODEs** The result of using the laws of dynamics is often a set of differential equations which need to be solved. A simple example would be:

  Find $x$ at $t = 5$ given that $\dfrac{dx}{dt} = x$ and that at $t = 0$, $x = 1$.

  The solution to this problem can be found easily enough by hand to be $x(5) = e^5$. But often the differential equations are just too hard for pencil and paper solution. Fortunately the **numerical solution of ordinary differential equations** (ODEs) is already programmed into scientific and engineering computer packages. The simple problem above is solved with computer code equivalent to these informal commands:

  ```
  ODES =  { xdot  = x }
  ICS  =  { xzero = 1 }
  solve ODES with ICS until t=5
  ```

  which will yield a list of values for paired values for $t$ and $x$ the last of which will be $t = 5$ and $x$ close to $e^5 \approx 148.4$.

## Examples of informal computer commands

In this book computer commands are given informally using commands that are not as strict as any real computer package. You will need to translate the informal commands below into commands your package understands. This reference table uses mathematical ideas which you may or may not know before you read this book, but these are introduced in the text when needed.

| | |
|---|---|
| `x=7` | Set the variable $x$ to 7. |
| `omega=13` | Set $\omega$ to 13. |
| `u=[1 0 -1 0]`<br>`v=[2 3 4 pi]` | Define $u$ and $v$ to be the lists shown. |
| `t= [.1 .2 .3 ... 5]` | Set $t$ to the list of 50 numbers implied by the expression. |
| `y=v(3)` | sets $y$ to the third value of $v$ (in this case 4). |
| `A=[1 2 3 6.9`<br>`5 0 1 12 ]` | Set $A$ to the array shown. |
| `z= A(2,3)` | Set z to the element of $A$ in the second row and third column. |
| `w=[3`<br>`4`<br>`2`<br>`5]` | Define $w$ to be a column vector. |
| `w = [3 4 2 5]'` | Same as above. `'` means transpose. |
| `u+v` | Vector addition. In this case the result is [3 3 3 $\pi$]. |
| `u*v` | Element by element multiplication, in this case [2 0 $-$4 0]. |
| `sum(w)` | Add the elements of $w$, in this case 14. |
| `cos(w)` | Make a new list, each element of which is the cosine of the corresponding element of $[w]$. |
| `mag(u)` | The square root of the sum of the squares of the elements in $[u]$, in this case 1.41421... |
| `u dot v` | The vector dot product of component lists $[u]$ and $[v]$, (we could also write `sum(A*B)`. |
| `C cross D` | The vector cross product of $\vec{C}$ and $\vec{D}$, assuming the three element component lists for $[C]$ and $[D]$ have been defined. |
| `A matmult w` | Use the rules of matrix multiplication to multiply $[A]$ and $[w]$. |
| `eqset = {3x + 2y = 6`<br>`6x + 7y = 8}` | Define 'eqset' to stand for the set of 2 equations in braces. |
| `solve eqset`<br>`for x and y` | Solve the equations in 'eqset' for $x$ and $y$. |
| `solve Ax=b for x` | Solve the matrix equation $[A][x] = [b]$ for the list of numbers $x$. This assumes $A$ and $b$ have already been defined. |
| `for i = 1 to N`<br>`such and such`<br>`end` | Execute the commands 'such and such' $N$ times, the first time with $i = 1$, the second with $i = 2$, etc |
| `plot y vs x` | Assuming $x$ and $y$ are two lists of numbers of the same length, plot the $y$ values *vs* the $x$ values. |
| `solve ODEs`<br>`with ICs`<br>`until t=5` | Assuming a set of ODEs and ICs have been defined, use numerical integration to solve them and evaluate the result at $t = 5$. |

With an informality consistent with what is written above, other commands are introduced as needed.

# Part I: Basics for Mechanics

CHAPTER 1

# What is mechanics?

*We use mechanics to predict forces and motions. We do this using the three pillars of the subject: I. models of physical behavior, II. geometry, and III. the basic mechanics balance laws. The laws of mechanics are informally summarized in this introductory chapter. The extreme accuracy of Newtonian mechanics is emphasized, despite relativity and quantum mechanics supposedly having 'overthrown' seventeenth century physics. Various uses of the word 'model' are described.*

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1.2 Mechanics is wrong, why study it? . . . . . **9**
1.3 The heirarchy of models . . . . . **11**

Mechanics is the study of force, deformation, motion, and the relations between them. We care about forces because we want to know how hard to push something to make it move or whether it will break when we push. We care about deformation and motion because we want things to move or not move in certain ways. Towards these ends our goals are to solve special versions of this general mechanics problem:

**The general mechanics problem:** Given some (possibly idealized) information about the properties, forces, deformations, and motions of a mechanical system, make useful predictions about other aspects of its properties, forces, deformations, and motions.

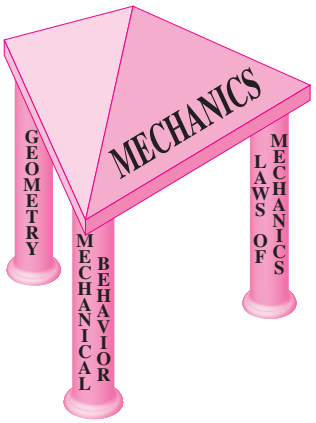
By *system*, we mean a tangible thing such as a wheel, a gear, a car, a bridge, a human finger, a butterfly, a skateboard and rider, a quartz-watch timing crystal, a building in an earthquake, a rocket, or the piston in an engine. Will a wheel slip? a gear tooth break? a car tip over? What is the biggest truck that can cross a given bridge? What muscles are used when you hit a key on your computer? How do people balance on skateboards? How does size effect the frequency of crystal vibration? Which buildings are more likely to fall in what kinds of earthquakes? What is the relation between gas-ejection rate and thrust in a rocket? What forces are on the connecting rod in an engine?

For each special case of the general mechanics problem we need to identify the system(s) of interest, idealize the system(s), use classical (high-school, Euclidean) geometry to describe the layout, deformation and motion, and finally use the laws of Newtonian mechanics. Although you may make more-or-less large approximations in your system modeling, both classical geometry and Newtonian mechanics have held up, with minor refinement mostly in notation, for over three hundred years. Those who want to know how machines, structures, plants, animals and planets hold together and move about need to know Newtonian mechanics. In another two or three hundred years people who want to design robots, buildings, airplanes, boats, prosthetic devices, and large or microscopic machines will probably still use the equations and principles we now call Newtonian mechanics*.

* The laws of classical mechanics, however expressed, are named for Isaac Newton because his theory of the world, the *Principia* published in 1689, contains much of the still-used theory. Newton used his theory to explain the motions of planets, the trajectory of a cannon ball, why there are tides, and many other things.

## 1.1 The three pillars

Any mechanics problem can be divided into 3 parts which we think of as the 3 pillars that hold up the subject:




1. *constitutive laws*: the mechanical behavior of objects and materials;
2. *kinematics*: the geometry of motion and distortion; and
3. *kinetics*: the laws of mechanics ($\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$, etc.).

Let's discuss each of these ideas a little more so you can get an overview before digging into the details in later chapters.

## Pillar 1: Mechanical behavior, constitutive laws

The first pillar of mechanics is mechanical behavior. The *Mechanical behavior* of something is the description of how loads cause deformation (or vice versa). When something carries a force it stretches, shortens, shears, bends, or breaks. Your finger tip squishes when you poke something. Too large a force on a gear in an engine causes it to break. The force of air on an insect wing makes it bend. Various geologic forces bend, compress and break rock. This relation between force and deformation can be viewed in a few ways.

**Definition of force.** First, the relation between force and deformation gives us a definition of force. Force can be defined by the amount of spring stretch it causes. Thus most modern force measurement devices measure force indirectly by measuring the deformation it causes in a calibrated spring of some kind. That force can be defined in terms of deformation is one justification for calling 'mechanical behavior' the first pillar. It gives us a notion of force even before we introduce the laws of mechanics.

**Steel vs chewing gum.** Second, a piece of steel distorts under a given load differently than a same-sized piece of chewing gum. This observation, that different objects deform differently with the same loads, implies that an object's properties affect its mechanics. The relations of an object's deformations to the forces that are applied are called the *mechanical properties* of the object. Mechanical properties are sometimes called *constitutive laws* because the mechanical properties describe how an object is constituted (mean-

ing 'what it is made from') at least from a mechanics point of view. The classic example of a constitutive law is that of a linear spring which you remember from your elementary physics classes: '$F = kx$' (spring tension is proportional to stretch). To do mechanics we have to make assumptions and idealizations about the constitutive laws applicable to the parts of a system. How stretchy (elastic) or gooey (viscous) or otherwise deformable is an object? The set of assumptions about the mechanical behavior of the system is sometimes called the *constitutive model*.

**Deformation is often hard to see.** Distortion in the presence of forces is easy to see on squeezed fingertips, in chewing gum between finger tips or when a piece of paper bends. But pieces of rock or metal have deformation that is essentially invisible and sometimes hard to imagine. With the exceptions of things like rubber, flesh, or objects that are very small in one or two dimensions (thin sheets and wires), solid objects that are not in the process of breaking typically change their dimensions much less than 1% when loaded. Most structural materials deform less than one part per thousand with working loads. These small deformations, even though essentially invisible, are important because they are enough to break bones and collapse bridges.

**Rigid-object mechanics.** Part of good engineering is to idealize away things that are not important, so as to make calculations as easy as possible. So, when deformations are not of consequence engineers usually wish them away. Mechanics, where deformation is neglected, is called *rigid-object* mechanics because a rigid (infinitely stiff) solid would not deform*. Rigidity, the assumption of infinite stiffness, is an extreme constitutive assumption. However, the assumption of rigidity greatly simplifies many calculations while still generating adequate predictions for many practical problems. The assumption of rigidity also simplifies the introduction of more general mechanics concepts. Thus for understanding the steering dynamics of a car we might treat the car as a rigid object, whereas for crash analysis where rigidity is clearly a poor approximation, we might treat a car as highly deformable.

**Contact behavior.** Most constitutive models describe the material inside an object. But to solve a mechanics problem involving friction or collisions one also has to have a constitutive model for the contact interactions. The standard friction model (or idealization) '$F \leq \mu N$' is an example of a contact constitutive model, as is the elementary 'restitution' model for collisions '$v^+ = ev^-$'.

**Summarizing,** we need a model of a system's mechanical behavior before we can make useful predictions. Useful models can sound absurdly extreme, as in the assumption that a piece of a human body is rigid.

* **Rigid body vs rigid object.** Traditionally 'rigid-objects' were called 'rigid-bodies', using the old-fashioned language that physical things were abstractly called 'bodies'. Now that mechanics is used widely to describe biological things, like people, the word 'body' can be confusing. 'Rigid-body' biomechanics might convey the study of people with rigid *rigamortis* muscles, so called 'stiffs'. Often in biomechanics we think of the parts of the body as rigid, say the fore-arm or the shank of the leg. It is confusing to say that the human body is modeled as a collection of rigid bodies, rather than as a collection of rigid objects. So we often, although not religiously, adopt the ordinary English that things are objects and things whose deformation we neglect are rigid objects.

## Pillar 2: The geometry of motion and deformation, kinematics

In mechanics we use classical Greek (Euclidean) geometry to describe the layout, deformation and large motions of objects. Deformation is defined by changes of lengths and angles between various pairs and triplets of points. Motion is defined by the changes of the position of points in time. Length, angle, similar triangles, the curves that particles follow and so on can be studied and understood without Newton's laws and thus make up the second independent pillar.

**Large motions.** Many machines and machine parts are designed to move something relatively far. Bicycles, planes, elevators, and hearses are designed to move people; a clockwork, to move clock hands; insect wings, to move insect bodies; and forks, to move potatoes. A connecting rod is designed to move a crankshaft; a crankshaft, to move a transmission; and a transmission, to move a wheel. And wheels are designed to move bicycles, cars, and skateboards.

The description of the motion of these things, of how the positions of the pieces change with time, of how the connections between pieces restrict the motions, of the curves traversed by the parts of a machine, and of the relations of these curves to each other is called *kinematics*. Kinematics is the study of the geometry of motion (or geometry in motion).

**Motion verses deformation.** Think of deformations (as in the mis-spelling deform-motions) as involving small changes of distance between points on one object, and of net motion (paragraph above) as involving large changes of distance between points on different objects. We often need to understand deformation of individual parts to predict when they will break. Sometimes motions associated with deformation are important in itself, say you would like the stretch between the two ends of a wing brace to be small. And sometimes the larger net transport motion is of interest; for example we would like all points on a plane to travel about the same large distance from New York to Bangalore. Really, deformation and motion are not distinct topics, both involve keeping track of the positions of points. The distinction we have made is for simplicity. Trying to simultaneously describe deformations and large motions is just too complicated for beginners to understand and too complicated for most engineering practice. So the ideas are kept (somewhat artificially) separate in elementary mechanics courses such as this one. As separate topics, the geometry needed to understand small deformations (called 'strains') and the geometry needed to understand large motions of rigid bodies ('particle and rigid-object kinematics') are both basic parts of mechanics. This book, however, has little about deformation and strain.

## Pillar 3: Relation of force to motion, the laws of mechanics, kinetics

The same intuitive 'force' that causes deformation also causes motion, or more precisely, acceleration of mass. The relation between force and acceleration of mass makes up the third pillar holding up mechanics. We loosely call this *Newton's laws*; synonyms include *the laws of mechanics*, *momentum and energy balance* and *kinetics*.* Force is related to deformation by material properties (elasticity, viscosity, etc.) and force is related to motion by the laws of mechanics summarized in the front cover. In words and informally, these are:*

0) The laws of mechanics apply to any system (rigid or not):
   a) Force and moment are *the* measure of mechanical interaction; and
   b) Action = minus reaction applies to all interactions, ( 'every action has an equal and opposite reaction');
I) The net force on a system causes a net linear acceleration (*linear momentum balance*),
II) The net turning effect of forces on system causes it to rotationally accelerate (*angular momentum balance*), and
III) The change of energy of a system is due to the energy flow into the system (*energy balance*).

**A non-minimal set of assumptions.** The principles of action and reaction, linear momentum balance, angular momentum balance, and energy balance, are actually redundant various ways. Linear momentum balance can be derived from angular momentum balance and sometimes vice-versa (see page **??**). Energy balance equations can often be derived from the momentum balance equations. The principle of action and reaction can be derived from the momentum balance equations. In engineering practice, however, we worry little about which idea could be derived from the others for the problem under consideration. The four assumptions in O-III above are not a mathematically minimal set, but they are all accepted truths by practitioners of mechanics.

A lot follows from the laws of Newtonian mechanics, including the contents of this book. When these ideas are supplemented with idealizations of the mechanical behavior of particular systems (e.g., of machines, buildings or human bodies), they lead to predictions about motions and forces. There is an endless stream of results about the mechanics of one or another special system. Some of these results are classified into entire fields of research such as 'fluid mechanics,' 'vibrations,' 'seismology,' 'granular flow,' 'biomechanics,' or 'celestial mechanics.'

* It is easy to confuse the similar looking and sounding words *kinematics* and *kinetics*. *Kinematics* concerns geometry with no mention of force and *kinetics* concerns the relation of force to motion. The following backwards pnemonic device might help you. Adding 'ma' to the middle of the word *kinetics* gives the word 'kinematics', whereas adding the concept $m\vec{a}$ (as in mass times acceleration) to the concept of kinematics gives the concept called kinetics.

* **Newton's laws vs the modern appraoch.** Isaac Newton's original three laws are:

1) an object in motion tends to stay in motion,
2) $\vec{F} = m\vec{a}$ for a particle, and
3) the principle of action and reaction.

These three Newton laws could be used as a starting point for the study of mechanics. The more modern approach here leads to the same ends. Why bother? One confusion in using Newton's original statements is trying to understand how the first law is not just a special case of the second law. One thought of modern historian's of Science is that Newton's first law is implicitly, by describing what happens when there is no force, defining force. In this view Newton's first law is somewhat equivalent to what we call law (0a). Another advantage to the more modern approach is that we can think of angular momentum and energy as fundamental quantities with general import, not just quantities relevant to the particular models or systems for which we can make derivations based on Newton's particle mechanics.

The four basic ideas also lead to mathematically advanced formulations of mechanics with names like 'Lagrange's equations,' 'Hamilton's equations,' 'virtual work', and 'variational principles.' If you go on in mechanics, you may learn some of these things in more advanced courses.

## Statics, dynamics, and strength of materials

Elementary mechanics is sometimes partitioned into three courses named 'statics', 'dynamics', and 'strength of materials'. These subjects vary in how much they emphasize material properties, geometry, and Newton's laws.

**Statics** is mechanics with the idealization that the acceleration of mass is negligible in Newton's laws. The first eight chapters of this book provide a thorough introduction to statics. Things need not be standing exactly still, nothing is, to be well idealized with statics. But, as the name implies, statics is generally about things that don't move much. The first pillar of mechanics, constitutive laws, is generally introduced without fanfare into statics problems by the (implicit) assumption of rigidity. Other constitutive assumptions used include inextensible ropes, linear springs, and frictional contact. The material properties used as examples in elementary statics are very simple. Also, because things don't move or deform much in statics, the geometry of deformation and motion are all but ignored. Despite the commonly applied vast simplifications, statics is useful for the analysis of natural and engineered structures, slow machines or the light parts of fast machines, and other things (say, the stability of boats).

**Dynamics** concerns the non-negligible acceleration of mass. Chapters 9-18 of this book introduce dynamics. As with statics, the first pillar of mechanics, constitutive laws, is given a relatively minor role in the elementary dynamics presented here. For the most part, the same library of elementary properties are used with little fanfare (rigidity, inextensibility, linear elasticity, and friction). Dynamics thus concerns kinematics and kinetics. Once one has mastered statics, the hard part of dynamics is the kinematics. Dynamics is useful for the analysis of, for example, fast machines, vibrations, and ballistics.

**Strength of materials** expands statics to include material properties and also pays more attention to distributed forces (*e.g.*, 'traction' and 'stress'). This book only occasionally touches lightly on strength-of-materials topics like stress (loosely, force per unit area), strain (a way to measure deformation), and linear elasticity (a commonly used constitutive idealization of solids that generalizes the concept of a spring). Strength of materials gives equal emphasis to all three pillars of mechanics. Strength of materials is useful for predicting the amount of deformation in a structure or machine, where it is most likely to break with a given load, and whether or not it is likely to break with that load.

## 1.2 Why study Newtonian mechanics when it has been overthrown by modern physics?

We are repeatedly reminded that Newtonian ideas have been replaced by relativity and quantum mechanics. So why should you read this book and learn ideas, remnants of the nineteenth century, which are known to be wrong?

First off, this criticism is maybe a bit off base: general relativity and quantum mechanics are inconsistent with each other, not yet united by a universally-accepted deeper theory of everything. So srict consistency with modern physics, as we know it, isn't possible. But how big are the errors we make when we do classical mechanics, neglecting various more modern physics discoveries?

**Special relativity.** The errors from neglecting the effects of special relativity are on the order of $v^2/c^2$ where $v$ is a typical speed in your problem and $c$ is the speed of light. The biggest errors are associated with the fastest objects. For, say, calculating space shuttle trajectories this leads to an error of about

$$\frac{v^2}{c^2} \approx \left(\frac{5\,\text{mi/ s}}{3\times 10^8\,\text{m/s}}\right)^2 \approx .000000001 \approx \text{one millionth of one percent}$$

**General relativity** errors having to do with the non-flatness of space are so small that the genius Einstein had trouble finding a place where the deviations from Newtonian mechanics could possibly be observed. Finally he predicted a small, barely measurable effect on the predicted motion of the planet Mercury. Newtonion mechanics predicts a fixed eliptical orbit. Einstein's equations correctly predicted that the eliptical path itself rotates (precesses) once every 3 million years (or 43 arcsec per century). So the Newtonian 'error' is about one part in $10^8$ (like a one cent error in a millionaire's bank balance). Global positioning sattelites (GPS) do actually take general relativity into account to prevent errors of about one part in a billion (a millimeter error over a thousand kilometers).

**Uncertainty principle.** In classical mechanics we assume we can know exactly where something is and how fast it is going. But according to quantum mechanics this is impossible. The product of the uncertainty $\delta x$ in position of an object and the the uncertainty $\delta p$ of its momentum must be greater than Planck's constant $\hbar$. Planck's constant is small; $\hbar \approx 1\times 10^{-34}$joule$\cdot$ s. The fractional error in position is biggest for small objects moving slowly. So if one measures the location of a computer chip with mass $m = 10^{-4}$ kg to within $\delta x = 10^{-6}$ m $\approx$ a twenty fifth of a thousands of an inch, the uncertainty in its velocity $\delta v = \delta p/m$ is only

$$\delta x\delta p = \hbar \Rightarrow \delta v = m\hbar/\delta x \approx 10^{-24}\,\text{m/ s} \approx 10^{-15}\text{inches per year.}$$

**Brownian motion.** In classical mechanics we usually (although not always) neglect fluctuations associated with the thermal vibrations of atoms. But any object in thermal equilibrium with its surroundings constantly undergoes changes in size, pressure, and energy, as it interacts with the environment. For example, the internal energy per particle of a sample at temperature $T$ fluctuates with amplitude cal

$$\frac{\Delta E}{N} = \frac{1}{\sqrt{N}}\sqrt{k_B T^2 c_V},$$

where $k_B$ is Boltzmann's constant, $T$ is the absolute temperature, $N$ is the number of particles in the sample, and $c_V$ is the specific heat. Water has a specific heat of 1 cal/K, or around 4 Joule/K. At room temperature of 300 Kelvin, for $10^{23}$ molecules of water, these values lead to an uncertainty of only $7.2 \times 10^{-21}$ Joule in the the internal energy of the water. Thermal fluctuations are big enough to visibly move pieces of dust in an optical microscope (Brownian motion), and to generate variations in electric currents that are easily measured, but for most engineering mechanics purposes they are negligible. But if thermal fluctuations are of interest, they can be modeled reasonably accurately using Newtonian mechanics at the atomic scale.

**Physics errors vs modeling errors.** Classical Newtonian physics is an accurate approximation of Nature for engineers, with errors typically on the order of parts per billion. On the other hand, the errors within mechanics, due to imperfect modeling or inaccurate measurement, are, except in extreme situations (like GPS), *far* greater than the errors due to the imperfection of Newtonian mechanics theory. For example, mechanical force measurements are typically off by a percent or so, distance measurements by a part in a thousand, and material properties are rarely known to one part in a hundred and often not even one part in 10. That is, even if you are good, your mechanics will typically be off by 100,000 times more than the laws of mechanics themselves.

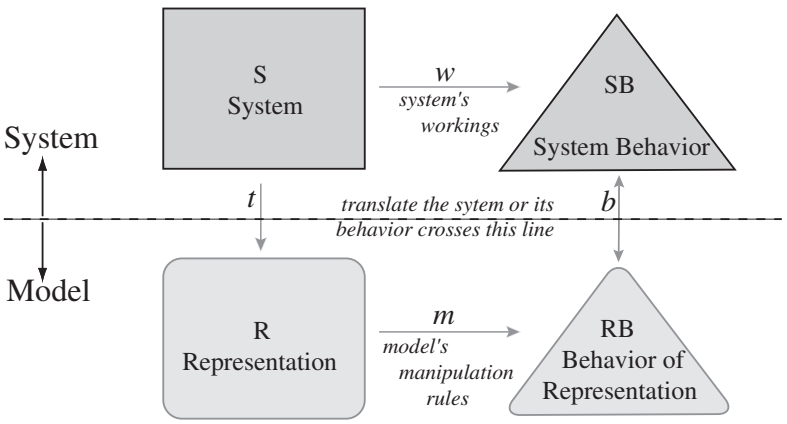
If your engineering mechanics calculations make inaccurate predictions it will surely be because of errors in modeling or measurement (lets assume no math mistakes on your part here), not inaccuracies in the laws of mechanics. Only in the rarest of circumstances are mechanics predictions off because of neglect of relativity, quantum mechanics, or statistical mechanics.

**You can trust Newtonian mechanics.** Newtonian mechanics is accurate enough, and also much simpler to use than the theories which have 'overthrown' it. You have trusted your life many times to engineers who treated classical mechanics as 'truth' and in turn, your engineering mechanics work will justly be based on the laws of classical mechanics. Although philosophically objectionable, it is reasonable engineering practice to think of the laws of mechanics as absolute truth.

## 1.3 Models, modeling, and the heirarchy of models

A plastic toy car guided by a child's hand crashes into another toy car. In common English the toys are models of cars. But the word *model* has a broader meaning in Engineering and Science. In this broader sense, for example, the toy crash is a model of a real car crash. The model of the crash *event* is that two plastic things are guided together by human hands. Its as if there are two parallel universes, the 'real' one and the 'model' one. And the whole real process of car collision is 'modeled by' the model process of thecrashing of toy cars. The word model then means that cars are replaced by plastic toys and the laws of mechanics replaced by the guiding of the childs hands. And the results of the collision are replaced by whatever damage occurs to the plastic toys.

What is a model? A commuting diagram.




**The system** S has behaviors SB that are intrinsic to the system by its own workings *w*.

**What is a model?** Broadly speaking the model includes the Representation R of the system , the manipulation rules *m* which yield the behavior of the representation RB, and the translation rules *t* and *b*. In science and engineering the rules of manipulation *m* are often math relations.

**Some model merits:** Broadness of systems to which it applies; broadness of features predicted; accuracy of predictions ( eg, the model *commutes* in that route S-> *t*-> R-> *m*-> RB agrees well with route S->*w*->SB->*b*->RB); simplicity and unambiguity in the rules *t*, *m*, and *b;* unambiguity in interpretation of predictions.

Figure 1.1: The abstract commuting diagram definition of a model. The laws of Newtonian mechanics make up a model for the motions of objects depending on many sub-models, such as the concept of a force and of a rigid object.

Filename:tfigure1-commute

**The commuting diagram.** The idea of a model, in this broader sense, is reflected in a so-called commuting diagram, as shown abstractly in Fig. 1.1. The top row is the system to be modeled, say the real cars. The real car collision is the workings of the system $w$ as dictated by natures laws in their full subtlety and complexity, taking account all known and as-yet unknown physics. And the way the cars move and deform and end up damaged is the system behavior SB. Parallel to this in the bottom row of the figure is the model universe. A plastic car R represents a real car by having about the same shape and coloring. The laws of nature $w$ are 'modeled by' the mapipulation rules in the model $m$, in this case the guidance of the child's hands. And the result of the real crash SB is 'modeled by' the result of the play crash. The model is compared to reality by making an association between bent and twist car metal with cracked and scratched toy plastic.

In this case we may or may not think that the model is a 'good model' depending on how well the damage to the plastic mimics the damage to real cars. This is expressed by the success at 'commuting', in the mathematical sense of the word commuting. Is the result of making a model and then carrying out the model process (down then right) the same as the result of the process then modelded (right then down)? In the language of the diagram the question is, does $S\xrightarrow{t} R\xrightarrow{m}RB$ give the same result as $S\xrightarrow{w}SB\xrightarrow{b}RB$? For example, we compare the prediction of damaged plastic to what the real car damage would translate to as cracks and scratches on the plastic? If they agree well then the model 'commutes'. That is, starting with the real system do you get the same answer by applying the real workings and then the translate to what you would expect to see in the model as you would by modeling the car in plastic and applying the model workings. Using the toy care example we can see aspects that commute and aspects that don't. That both the real cars and the toy cars have lots of damage at the front is a sign of the model "commuting". Thats a good feature of the model. That the toy car passengers have no scratches and that the people in the real cars were severely injured is a lack of commuting, and a defect in 'the model'.

**Mathematical vs physical models.** In the toy crash example above the 'model' included a physical object, the toy car. More commonly in science and engineering the model is a constellation of ideas with no physical object involved. For example, if a solid 'is modeled as' a rigid object that means the motion of the object will be calculated by assuming that the solid does not deform. No piece of plastic representing the object is needed.*

**Models in engineering.** In engineering we use models to make predictions about reality. So the 'commuting' ability is usually expressed by comparing the model predictions to reality, wrapping three quarters of the way counterclockwise around the diagram from the system (at the upper left) down to its model representation through the model manipulations to the model behavior and back up to the prediction for reality (at the upper right).

* An exception to the non-material nature of the word 'model' in science (which is the definition we use here) is the use of the word model in biological experiments. In biology an 'animal model', for example, would be a monkey whose response to a carcinogen is supposed to mimic the response in a human. In experimental biology the commuting definition of the word model is more-or-less ignored and the word now basically means 'experimental subject'. For example some biologists do experiments on 'human models' where they are thinking of a human as a model of him or herself.

**Models are pervasive.** This abstraction about modeling is confusing partly because we are surrounded by it all the time. Explaining it is like explaining water to a fish. For example, language and thought are themselves, in a sense, models of reality.

## Mechanics models

In a course like this we are concerned with a hierarchy of models.

Most basically we model space, time and matter as having all the common-sense features that we are used to. For example we assume that the location of any point in space can be described by its $x$, $y$, and $z$ coordinates relative to some origin.

Second, we model all of natures rules for motion with the basic laws of mechanics. As stated above with reference to modern physics concepts, this is a high-quality model whose errors (or lack of ability to commute) will likely be of no significance to you ever in your life. Third, we have models of objects and forces. In this course, as opposed to a course in structural mechanics, we generally 'model' solid things as particles and rigid objects with an error ranging from a few percent to a small fraction of a percent. Models of forces can be very accurate, for example you can know gravity forces, if you know where you are on the earth (see page A), to about one part in $10^6$. Some force models are reasonably accurate like the description of linear springs (typically 1% accurate or so). And some force models are basically poor, like for friction and collisions (with typical errors of 20-50%), we just don't know good models for these things so we use and try to understand the bad models we have cooked up so far.

Given this hierarchical collection of mechanics models we next get to the engineers task of 'modeling'. Given a real machine, how do we 'model' it as made up of various mechanics models from the paragraphs above? Which parts do we approximate as rigid objects, which as massless linear springs, etc? This modeling task is an important part of engineering practice.

However, before one can develop the art of engineering modeling one needs to know how to work with the range of common engineeering models. In terms of the diagram in Fig. 1.1 you need to know how to do the manipulations $m$ for a given candidate models before you can develop the art of determining what particular models should be used to to representat your system of interest. Much more specifically for elementary mechanics you need to know how particles and rigid bodies interact and move if governed by the common models for their interactions. Understanding how particles and rigid bodies interact and move is the core of this book.

**Models in homework problems.** Most often we will be implicitly telling you what model to use for each problem, although sometimes in a mildly disguised language (in order to start training your modeling skills). Judging whether or not a given model is good (*i.e.*, commutes, corresponds well with reality) is an important part of engineering practice. So we will point out defficiencies in various models here and there. Further, because some of these

models are pretty good, you can use your intuition (another model!) to guide your learning of mechanics models and you can use your new understanding of mechanics models to improve your intuition about reality.

**Utility of rigid-object-mechanics models.** The bottom line is this. If you understand well how particles and rigid bodies interact and move according to the 'rigid-object-mechanics' model, you will pretty-well understand how real things move and hold together.

CHAPTER 2

# Vectors for mechanics

*The key vectors for statics, namely relative position, force, and moment, are used to motivate needed vector skills. Notational clarity is emphasized because correct calculation is impossible without distinguishing vectors from scalars. Vector addition is motivated by the need to add forces and relative positions, dot products are motivated as the tool which reduces vector equations to scalar equations, and cross products are motivated as the formula which correctly calculates the heuristically motivated quantities of moment and moment about an axis.*

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This book is about the laws of mechanics which were informally introduced in Chapter 1. The most fundamental quantities in mechanics, used to define all the others, are the two scalars, mass $m$ and time $t$, and the two vectors, relative position $\vec{r}_{i/O}$ (of point $i$ relative to point O), and force $\vec{F}$. Scalars are typed with an ordinary font ($t$ and $m$) and vectors are typed in bold with a harpoon on top ($\vec{r}_{i/O}$, $\vec{F}$). All of the other quantities we use in mechanics are defined in terms of these four. A list of all the scalars and vectors used in mechanics are given in boxes 2.1 and 2.2 on pages 27 and 27. We assume that anyone reading this book is competent at scalar arithmetic and algebra (that means adding, subtracting, multiplying and dividing ordinary numbers and symbols representing numbers). For mechanics you also need facility with vector arithmetic and algebra. Lets start at the beginning.

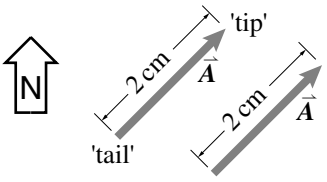



Figure 2.1: Vector $\vec{A}$ is 2 cm long and points Northeast. Two copies of $\vec{A}$ are drawn to show that where the drawing is placed doesn't effect the vector. Its the same vector drawn in two different places.

Filename:tfigure-northeast

## What is a vector?

> A vector is a (possibly dimensional) quantity that is fully described by its magnitude and direction.

whereas scalars are just (possibly dimensional) single numbers*. As a first vector example, consider a line segment with head and tail ends and a length (magnitude) of 2 cm and pointed Northeast. Lets call this vector $\vec{A}$ (see fig. 2.1).

$$\vec{A} \stackrel{def}{=} \text{2 cm long line segment pointed Northeast}$$

Every vector in mechanics is well visualized as an arrow. The direction of the arrow is the direction of the vector. The length of the arrow is proportional to the magnitude of the vector. The magnitude of $\vec{A}$ is a positive scalar indicated by $|\vec{A}|$. A vector does not lose its identity if it is picked up and moved around in space (so long as it is not rotated or stretched). Thus both vectors drawn in fig. 2.1 are the same vector $\vec{A}$.

## Vector arithmetic makes sense

We have oversimplified. We said that a vector is something with magnitude and direction. In fact, by common modern convention, that's not enough. A one way street sign, for example, is not considered a vector even though is has a magnitude (its mass is, say, half a kilogram) and a direction (the direction of

* By 'dimensional' we mean 'with units' like meters, Newtons, or kg. We don't mean having an abstract vector-space dimension, as in one, two or three dimensional.

most of the traffic). A thing is only called a vector if, additionally, elementary vector arithmetic, vector addition in particular, has a sensible meaning*.

The following sentence summarizes centuries of thought and also motivates this chapter:

> The vectors in mechanics have magnitude and direction *and* elementary vector arithmetic operations have sensible physical meanings.

This chapter is about vector arithmetic. In this chapter you will learn how to add and subtract vectors, how to stretch them, how to find their components, and how to multiply them with each other two different ways. Each of these operations has use in mechanics and, in particular, the concept of vector addition always has a physical interpretation.

## 2.1 Vector notation and vector addition

Facility with vectors has several aspects.

1. You must recognize which quantities are vectors (such as force) and which are scalars (such as length).
2. You have to use a notation that distinguishes between vectors and scalars using, for example, $\vec{\boldsymbol{a}}$, or $\underline{\boldsymbol{a}}$ for acceleration and $a$ for a scalar with the same magnitude $|a| = |\vec{\boldsymbol{a}}|$.
3. You need skills in vector arithmetic. Most students need to know a little more than they learned in their previous math and physics courses.

In this first section (2.1) we start with notation and go on to finding the relative position vectir from a picture, multiplication of a vector by a scalar, vector addition and vector subtraction.

### How to write vectors

A scalar is written as a single English or Greek letter. This book uses slanted type for scalars (*e.g.*, $m$ for mass) but ordinary printing is fine for hand work (*e.g.*, m for mass). A vector is also represented by a single letter of the alphabet, either English or Greek, but ornamented to indicate that it is a vector and not a scalar. The common ornamentations are described below. *Use one of these vector notations in all of your work.* *.

$\vec{\boldsymbol{F}}$ Putting a harpoon (or arrow) over the letter $\boldsymbol{F}$ is the suggestive notation used in in this book for vectors.

$\boldsymbol{F}$ In most texts a bold $\boldsymbol{F}$ represents the vector $\vec{\boldsymbol{F}}$. But bold face is inconvenient for hand written work. The lack of bold face pens and pencils tempts students to transcribe a bold $\boldsymbol{F}$ as $F$. But $F$ with no adornment represents a scalar and not a vector. Learning how to work with vectors and scalars is hard enough without the added confusion of not being able to tell at a glance which terms in your equations are vectors and which are scalars.

* In abstract mathematics they don't even bother with talking about magnitudes and directions. All they care about is vector arithmetic. So, to the mathematicians, anything which obeys simple vector arithmetic is a vector, arrow-like or not. In math talk lots of strange things are vectors, like arrays of numbers and functions. In this book vectors always have magnitude and direction.

* **Caution:** Be careful to distinguish vectors from scalars in all of your written work. Clear notation helps clear thinking and will help you solve problems. If you notice that you are not using clear vector notation, *stop*, determine which quantities are vectors and which scalars, and fix your notation. Rare is the student who consistently gets correct answers to exam questions without clear vector notation. And almost as rare is the student who has clear vector usage and can't do problems. For some students, accepting this vector language and syntax is a pain. Swallow it.

$\underline{\boldsymbol{F}}$ Underlining or undersquiggling ($\underline{\boldsymbol{F}}$) is an easy and unambiguous notation for hand writing vectors. Recent national polls found that 13 out of 17 mechanics professors use this notation. These professors would copy a $\vec{\boldsymbol{F}}$ from this book by writing $\underline{\boldsymbol{F}}$. The origin of the notation seems to be from old-fashioned typesetting; therein an author would indicate that a letter should be printed in bold by underlining it.

$\bar{\boldsymbol{F}}$ It is a stroke simpler to put a bar rather than a harpoon over a symbol. But the saved effort causes ambiguity since an over-bar is often used to indicate average. There could be confusion, say, between the velocity $\bar{v}$ and the average speed $\bar{v}$.

$\hat{\boldsymbol{\imath}}$ Over-hat. Putting a hat on top is like an over-arrow or over-bar. In this book we reserve the hat for unit vectors. For example, we use $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$, or $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, and $\hat{\boldsymbol{e}}_3$ for unit vectors parallel to the $x$, $y$, and $z$ axes, respectively. The same poll of 17 mechanics professors found that 11 of them used no special notation for unit vectors and just wrote them like, *e.g.*, $\underline{\imath}$.

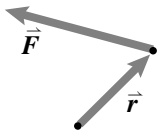



Figure 2.2: Position and force vectors are drawn with different scales.

Filename:tfigure-posandforce

## Drawing vectors

In Fig. 2.1 on page 17, the magnitude of $\vec{\boldsymbol{A}}$ was used as the drawing length. But drawing a vector using its magnitude as length would be awkward if, say, we were interested in vector $\vec{\boldsymbol{B}}$ that points Northwest and has a magnitude of 2 meters. To well contain $\vec{\boldsymbol{B}}$ in a drawing would require a piece of paper about 2 meters square (each edge the length of a basketball player). This situation moves from difficult to ridiculous if the magnitude of the vector of interest is 2 km and it would take half an hour to stroll from tail to tip dragging a purple crayon. Thus in pictures we merely make scale drawings of vectors with, say, one centimeter of graph paper representing 1 kilometer of vector magnitude.

The need for scale drawings to represent vectors is apparent for a vector whose magnitude is not length. Force is a vector since it has magnitude and direction. Say $\vec{\boldsymbol{F}}_{\text{gr}}$ is the 700 N force that the ground pushes up on your feet as you stand still. We can't draw a line segment with length 700 N for $\vec{\boldsymbol{F}}_{\text{gr}}$ because a Newton is a unit of force not length. A scale drawing is needed.

One often needs to draw vectors with different units on the same picture, as for showing the position $\vec{\boldsymbol{r}}$ at which a force $\vec{\boldsymbol{F}}$ is applied (see Fig. 2.2). In this case different scale factors are used for the drawing of the vectors that have different units.

Drawing and measuring are tedious and also not very accurate. And drawing in 3 dimensions is particularly hard (given the short supply of 3D graph paper nowadays). So the magnitudes and directions of vectors are usually defined with numbers and units rather than scale drawings. Nonetheless, the drawing rules and geometric descriptions define all the vector concepts.

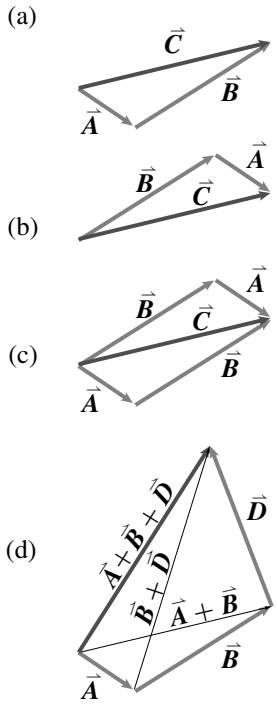



Figure 2.3: (a) tip to tail addition of $\vec{A}+\vec{B}$, (b) tip to tail addition of $\vec{B}+\vec{A}$, (c) the parallelogram interpretation of vector addition which shows the commutative law of vector addition: $\vec{A}+\vec{B}=\vec{B}+\vec{A}$, and (d) The associative law of vector addition: $(\vec{A}+\vec{B})+\vec{D}=\vec{A}+(\vec{B}+\vec{D})$.

This figure also makes sense in 3D. Drawings (a), (b) and (c) are all on a tilted planes. And the 6 vectors drawn in (d) lie on the edges of a tetrahedron.

Filename:tfigure-tiptotail

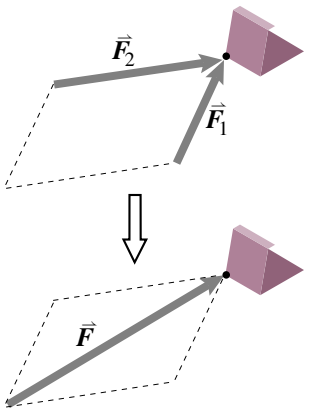



Figure 2.4: Two forces acting at a point may be replaced by their sum for all mechanics purposes.

Filename:tfigure-forcesadd

## Adding vectors

**Tip to tail rule.** The sum of two vectors $\vec{A}$ and $\vec{B}$ is defined by the *tip to tail* rule of vector addition shown in Fig. 2.3a for the sum $\vec{C}=\vec{A}+\vec{B}$. Vector $\vec{A}$ is drawn. Then vector $\vec{B}$ is drawn with its tail at the tip (or head) of $\vec{A}$. The sum $\vec{C}$ is the vector from the tail of $\vec{A}$ to the tip of $\vec{B}$.

**Parallelogram rule.** The same sum is achieved if $\vec{B}$ is drawn first, as shown in Fig. 2.3b. Putting both ways of adding $\vec{A}$ and $\vec{B}$ on the same picture draws a parallelogram as shown in Fig. 2.3c. Hence the tip to tail rule of vector addition is also called the *parallelogram* rule. The parallelogram construction shows the *commutative* property of vector addition, namely that $\vec{A}+\vec{B}=\vec{B}+\vec{A}$.

**3D.** Note that you can view Fig. 2.3a-c as 3D pictures. In 3D, the parallelogram will generally be on some tilted plane.

**Adding many vectors.** Three vectors are added by the same tip to tail rule. The construction shown in Fig. 2.3d shows that $(\vec{A}+\vec{B})+\vec{D}=\vec{A}+(\vec{B}+\vec{D})$ so that the expression $\vec{A}+\vec{B}+\vec{D}$ is unambiguous. This is the *associative* property of vector addition.

With these two laws we see that the sum $\vec{A}+\vec{B}+\vec{D}+\ldots$ can be permuted to $\vec{D}+\vec{A}+\vec{B}+\ldots$ or any which way without changing the result. So vector addition shares the associativity and commutivity of scalar addition that you are used to *e.g.*, that $3+(7+\pi)=(\pi+3)+7$.

**Concurrent forces.** We can reconsider the statement 'force is a vector' and see that it hides one of the basic assumptions in mechanics, namely:

> If forces $\vec{F}_1$ and $\vec{F}_2$ are applied to a point on a structure they can be replaced, for all mechanics considerations, with a single force $\vec{F}=\vec{F}_1+\vec{F}_2$ applied to that point

as illustrated in Fig. 2.4. The force $\vec{F}$ is said to be *equivalent* to the *concurrent* (acting at one point) force system consisting of $\vec{F}_1$ and $\vec{F}_2$ acting at the same point.

**Apples and oranges.** Note that two vectors with different dimensions cannot be added. Figure 2.2 on page 19 can no more sensibly be taken to represent meaningful vector addition than can the scalar sum of a length and a weight, "2 ft + 3 N", be taken as meaningful.

## Subtraction and the zero vector

Subtraction is most simply defined by inverse addition. Find $\vec{C} - \vec{A}$ means find the vector which when added to $\vec{A}$ gives $\vec{C}$. We can draw $\vec{C}$, draw $\vec{A}$ and then find the vector which, when added tip to tail to $\vec{A}$ give $\vec{C}$. Figure 2.3a shows that $\vec{B}$ answers the question. Another interpretation comes from defining the negative of a vector $-\vec{A}$ as $\vec{A}$ with the head and tail switched. Again you can see from Fig. 2.3b, by imagining that the head and tail on $\vec{A}$ were switched that $\vec{C} + (-\vec{A}) = \vec{B}$. The negative of a vector evidently has the expected property that $\vec{A} + (-\vec{A}) = \vec{0}$, where $\vec{0}$ is the vector with no magnitude so that $\vec{C} + \vec{0} = \vec{C}$ for all vectors $\vec{C}$.

## Relative position vectors

The concept of relative position permeates most mechanics equations. The position of point B relative to point A is represented by the vector $\vec{r}_{B/A}$ (pronounced 'r of B relative to A') drawn from A and to B (as shown in Fig. 2.5). An alternate notation for this vector is $\vec{r}_{AB}$ (pronounced 'r A B' or 'r A to B'). You can think of the position of B relative to A as being the position of B relative to you if you were standing on A. Similarly $\vec{r}_{C/B} = \vec{r}_{BC}$ is the position of C relative to B.

Figure 2.5a shows that relative positions add by the tip to tail rule. That is,

$$\vec{r}_{C/A} = \vec{r}_{B/A} + \vec{r}_{C/B} \qquad \text{or} \qquad \vec{r}_{AC} = \vec{r}_{AB} + \vec{r}_{BC}$$

so vector addition has a sensible meaning for relative position vectors.

Note that the position of B relative to A is the opposite (negative vector) of the position of A relative to B,

$$\vec{r}_{B/A} = -\vec{r}_{B/A}.$$

**Position relative to the origin.** Often when doing problems we pick a distinguished point in space, say a prominent point or corner of a machine or structure, and use it as the origin of a coordinate system O. The position of point A relative to O is $\vec{r}_{A/0}$ or $\vec{r}_{OA}$ but we often adopt the shorthand notation $\vec{r}_A$ (pronounced 'r A') leaving the reference point O as implied. Figure 2.5b shows that

$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$$

which rolls off the tongue easily and makes the concept of relative position easier to remember.*

## Multiplying by a scalar stretches a vector

Naturally enough $2\vec{F}$ means $\vec{F} + \vec{F}$ (see Fig. 2.6) and $127\vec{A}$ means $\vec{A}$ added to itself 127 times. Similarly $\vec{A}/7$ or $\frac{1}{7}\vec{A}$ means a vector in the direction of

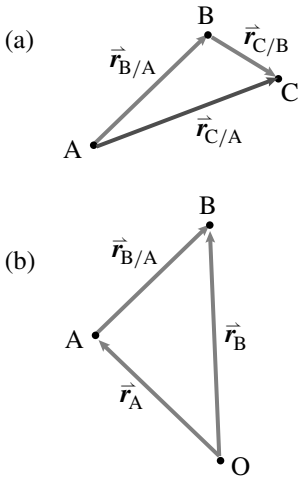



Figure 2.5: a) Relative position of points A, B, and C; b) Relative position of points O, A, and B.

Filename:tfigure-relpos

* For the first 7 chapters of this book you can just translate 'relative to' to mean 'minus' as in english. 'How much money does Rudra have *relative to* Andy?' means what is Rudra's wealth *minus* Andy's wealth? What is the position of B *relative to* A? It is the position of B *minus* the position of A.

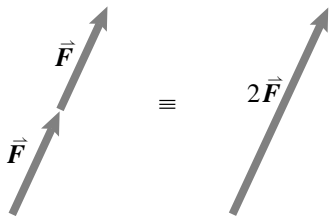



Figure 2.6: Multiplying a vector by a scalar stretches it.

Filename:tfigure-stretch

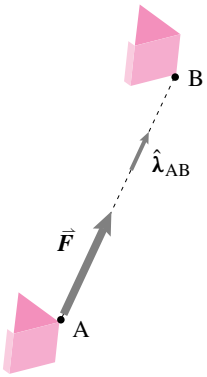



Figure 2.7: The force $\vec{F}$ that points from A to B can be represented as the product of a scalar $F$ and a unit vector $\hat{\boldsymbol{\lambda}}_{AB}$: $\vec{F} = F\hat{\boldsymbol{\lambda}}_{AB}$.

Filename:tfigure-FAtoB

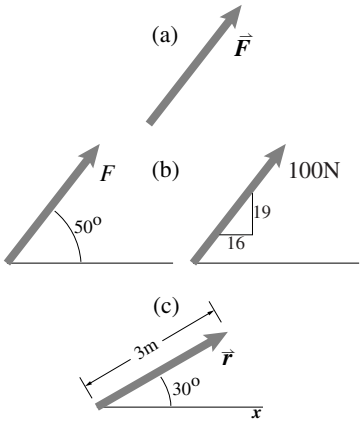



Figure 2.8: Two different ways of drawing a vector (a) Symbolic. The magnitude and direction of the vector is given by the symbol $\vec{F}$, the drawn arrow has no quantitative information; (b) "Scalar times arrow" shows an arrow with clearly indicated orientation next to the scalar $F$ or the scalar 100N. The vector indicated is the scalar multiplied by a unit vector in the direction drawn; (c) Combined. The symbol $vr$ is defined (set equal to) a vector with the magnitude and orientation shown.

Filename:tfigure1-d

$\vec{A}$ that when added to itself 7 times gives $\vec{A}$. By combining these two ideas we can define any rational multiple of $\vec{A}$. For example $\frac{29}{13}\vec{A}$ means add 29 copies of the vector that when added 13 times to itself gives $\vec{A}$. We skip the mathematical fine point of extending the definition to $c\vec{A}$ for $c$ that are irrational.

We can define $-17\vec{A}$ as $17(-\vec{A})$, combining our abilities to negate a vector and multiply it by a positive scalar. In general, for any positive scalar $c$ we define $c\vec{A}$ as the vector that is in the same direction as $\vec{A}$, or opposite if $c$ is negative, but whose magnitude is multiplied by $|c|$. Five times a 5 N force pointed Northeast is a 25 N force pointed Northeast. Minus 5 times a 5 N force pointed Northeast is a 25 N force pointed SouthWest.

**Distributive rule for scalar multiplication.** If you imagine stretching a whole vector addition diagram (*e.g.*, Fig. 2.3a on page 20) equally in all directions the distributive rule for scalar multiplication is apparent:

$$c(\vec{A} + \vec{B}) = c\vec{A} + c\vec{B}$$

## Unit vectors have magnitude 1

*Unit vectors* are vectors with a magnitude of one. Unit vectors are useful for indicating direction. Key examples are the unit vectors pointed in the positive $x$, $y$ and $z$ directions $\hat{\imath}$ (called 'i hat' or just 'i'), $\hat{\jmath}$, and $\hat{k}$. We distinguish unit vectors by hatting them but any undistinguished vector notation will do (*e.g.*, using $\underline{\imath}$).

An easy way to find a unit vector in the direction of a vector $\vec{A}$ is to divide $\vec{A}$ by its magnitude. Thus

$$\hat{\boldsymbol{\lambda}}_A \equiv \frac{\vec{A}}{|\vec{A}|}$$

is a unit vector in the $\vec{A}$ direction. You can check that this defines a unit vector by looking up at the rules for multiplication by a scalar. Multiplying $\vec{A}$ by the scalar $1/|\vec{A}|$ gives a new vector with magnitude $|\vec{A}|/|\vec{A}| = 1$.

**A vector as a scalar times a unit vector.** A common situation is to know that a force $\vec{F}$ is a yet unknown scalar $F$ multiplied by a unit vector pointing between known points A and B. (Fig. 2.7). We can then write $\vec{F}$ as

$$\vec{F} = F\hat{\boldsymbol{\lambda}}_{AB} = F\frac{\vec{r}_{AB}}{|\vec{r}_{AB}|} = F\frac{\vec{r}_B - \vec{r}_A}{|\vec{r}_B - \vec{r}_A|}$$

where we have used $\hat{\boldsymbol{\lambda}}_{AB}$ as the unit vector pointing from A to B. Note that in this usage, one we will use often, the scalar need not be positive. So the 'scalar part' might be plus or minus the magnitude of the vector.

## Vectors in pictures and diagrams.

Some options for drawing vectors are shown in sample 2.1 on page 28. The two notations below are the most common.

**Symbolic: labeling an arrow with a vector symbol.** Indicate a vector, say a force $\vec{F}$, by drawing an arrow and then labeling it with one of the symbolic notations above as in Fig. 2.8a. *In this notation, the arrow is only schematic*, the magnitude and direction are determined by the algebraic symbol $\vec{F}$. It is most clear if you draw the arrow roughly in the vector's direction and roughly to scale, but

> If the symbol and drawing disagree the symbol takes precedence (see sample sample 2.1j)

**Graphical: "scalar times arrow",** a scalar multiplies a unit vector in the direction of a drawn arrow (Fig. 2.8b). Indicate a vector's direction by drawing an arrow. The direction should be made clear with a marked angle or slope. The length drawn is irrelevant. Write a letter of the alphabet, say $F$, or a (possibly dimensional number, say 100N) near the vector. The vector indicated is a scalar $F$ (or the number) multiplying a unit vector in the direction of the arrow. Often you know that a force acts along a known line but you don't know which way. This is accomodated by allowing the scalar $F$ to be positive or negative (See examples in sample 2.1.)

**Combined: graphical representation used to define a symbolic vector.** The symbolic notation can be used with the graphical notation to define the vector symbol. In Fig. 2.8c $\vec{r}$ is being defined (being set equal) to the vector with magnitude 3m and direction $30^\circ$ CCW from the $+x$ axis.

## The cartesian components of a vector

A given vector, say $\vec{F}$, can be described as the sum of vectors each of which is parallel to a coordinate axis. Most often we use Cartesian axes, with the $x$, $y$, and $z$ axes all orthogonal to each other. Thus $\vec{F} = \vec{F}_x + \vec{F}_y$ in 2D and $\vec{F} = \vec{F}_x + \vec{F}_y + \vec{F}_z$ in 3D. Each of these vectors can in turn be written as the product of a scalar and a unit vector along the positive axes, *e.g.*, $\vec{F}_x = F_x\hat{\imath}$ (see Fig. 2.9). So

$$\vec{F} = \vec{F}_x + \vec{F}_y = F_x\hat{\imath} + F_y\hat{\jmath} \quad \text{(2D)}$$

or

$$\vec{F} = \vec{F}_x + \vec{F}_y + \vec{F}_z = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}. \quad \text{(3D)}$$

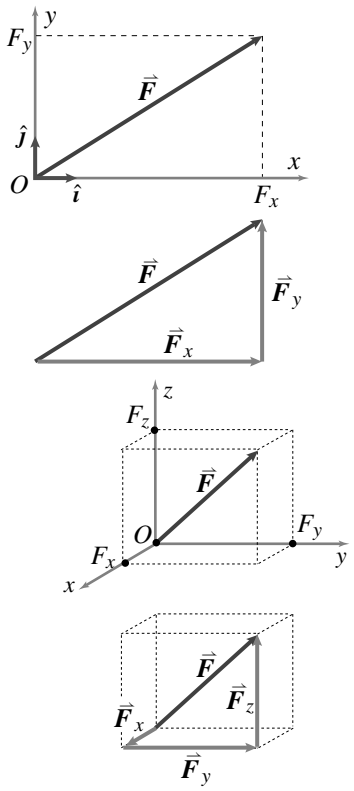



Figure 2.9: A vector can be broken into a sum of vectors, each parallel to the axis of a coordinate system. Each of these is a component multiplied by a unit vector along the coordinate axis, *e.g.*, $\vec{F}_x = F_x\hat{\imath}$.

Filename:tfigure-vectproject

The scalars $F_x$, $F_y$, and $F_z$ are called the components of the vector with respect to the axes $xyz$. The components may also be thought of as the orthogonal projections (the shadows) of the vector onto the coordinate axes.

Because the list of components is such a handy way to describe a vector we have a special notation for it. The bracketed expression $[\vec{F}]_{xyz}$ stands for the list of components of $\vec{F}$ presented as a horizontal or vertical array (depending on context), as shown below.

$$[\vec{F}]_{xyz} = [F_x, \quad F_y, \quad F_z] \quad \text{or} \quad [\vec{F}]_{xyz} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}.$$

If we had an $xy$ coordinate system with $x$ pointing East and $y$ pointing North we could write the components of a 5 N force pointed Northeast as $[\vec{F}]_{xy} = [(5/\sqrt{2})\,\text{N}, \quad (5/\sqrt{2})\,\text{N}]$.

Note that the components of a vector in some tilted coordinate system $x'y'z'$ are different from its components in the coordinate system $xyz$ because the projections are different. Even though $\vec{F} = \vec{F}$ it is *not true* that $[\vec{F}]_{xyz} = [\vec{F}]_{x'y'z'}$ (see Fig. 2.19 on page 37). In mechanics we often make use of multiple coordinate systems. So to define a vector by its components the coordinate system used must be specified.

Rather than using new letters to repeat the same concept we sometimes label the coordinate axes $x_1$, $x_2$ and $x_3$ and the unit vectors along them $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, and $\hat{\boldsymbol{e}}_3$ (thus freeing our minds from silently pronouncing the extra letters y,z,j, and k).

Note that non-Cartesian coordinates, most especially polar coordinates, are often useful in dynamics, as will be explained later.

## Manipulating vectors by manipulating components

Because a vector can be represented by its components (once given a coordinate system) we should be able to relate our geometric understanding of vectors to their components. In practice, when push comes to shove, most calculations with vectors are done with components.

### Adding and subtracting with components

Because a vector can be broken into a sum of orthogonal vectors, because addition is associative, and because each orthogonal vector can be written as a component times a unit vector we get the addition rule:

$$[\vec{A} + \vec{B}]_{xyz} = [(A_x + B_x), \quad (A_y + B_y), \quad (A_z + B_z)]$$

which can be described by the tricky words 'the components of the sum of two vectors are given by the sums of the corresponding components.' Similarly,

$$[\vec{A} - \vec{B}]_{xyz} = [(A_x - B_x), \quad (A_y - B_y), \quad (A_z - B_z)].$$

### Multiplying a vector by a scalar using components

The vector $\vec{A}$ can be decomposed into the sum of three orthogonal vectors. If $\vec{A}$ is multiplied by 7 then so must be each of the component vectors. Thus

$$[c\vec{A}]_{xyz} = [cA_x, \quad cA_y, \quad cA_z].$$

The cartesian components of a scaled vector are the corresponding scaled components. For example if $c = 3$ and $[\vec{A}]_{xyz} = [2, 4, -5]$ then $[c\vec{A}]_{xyz} = [6, 12, -15]$.

Often the components of vectors are written as columns rather than rows of numbers. Thus we would write

$$[\vec{A}]_{xyz} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = [2, 4, -5]' = \begin{bmatrix} 2 \\ 4 \\ -5 \end{bmatrix}.$$

The $'$ means 'matrix transpose', turning the rows into columns and *vice versa*. We can add the components of vectors using this notation, so if $d = -0.5$ and $[\vec{B}]_{xyz} = [100, 200, -300]'$ then

$$[c\vec{A} + d\vec{B}]_{xyz} = c[\vec{A}]_{xyz} + d[\vec{B}]_{xyz} = \begin{bmatrix} cA_x + dB_x \\ cA_y + dBy \\ cA_z + dBz \end{bmatrix} = \begin{bmatrix} -44 \\ -88 \\ 135 \end{bmatrix}$$

Finally we can use matrix notation and the definition of matrix multiplication to add multiples of vectors

$$\underbrace{\begin{bmatrix} A_x & B_x \\ A_y & By \\ A_z & Bz \end{bmatrix}}_{\text{A 3 by 2 matrix}} \begin{bmatrix} c \\ d \end{bmatrix} \underbrace{\equiv}_{\text{Is defined to mean}} c \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} + d \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} cA_x + dB_x \\ cA_y + dBy \\ cA_z + dBz \end{bmatrix}.$$

So, for example,

$$\begin{aligned} [c\vec{A} + d\vec{B}]_{xyz} &= \begin{bmatrix} 2 & 100 \\ 4 & 200 \\ -5 & -300 \end{bmatrix} \begin{bmatrix} 3 \\ -0.5 \end{bmatrix} \\ &= \begin{bmatrix} 3 \cdot 2 & + & -0.5 \cdot 100 \\ 3 \cdot 4 & + & -0.5 \cdot 200 \\ 3 \cdot (-5) & + & -0.5 \cdot (-300) \end{bmatrix} = \begin{bmatrix} -44 \\ -88 \\ 135 \end{bmatrix}. \end{aligned}$$

In the language of linear algebra (skip this sentence if you never took such a course), a matrix multiplied by a column vector is a linear combination of the matrix columns with weights (coefficients) given by the elements of the column vector.

## Adding vectors on a computer

Computers deal well with lists of numbers but not generally with units. So only the numerical part of a calculation shows in the computer work. For example, when we write on the computer

```
F = [ 3 5 -7]
```

we take that to be computereze for $[\vec{F}]_{xyz} = [3\,\text{N}, \quad 5\,\text{N}, -7\,\text{N}]$. To do computer work we have to be clear about what units and what coordinate system we are using. In particular, at this point in the course, we advise you to only use one coordinate system and one consistent set of units in any one problem that uses computer calculations. We can add multiples of vectors on a computer with commands something like this:

```
A = [   2    4    -5]'
B = [ 100  200  -300]'
c = 3
d = -0.5
C = c*A + d*B
```

or using the matrix notation, like this.

```
A = [   2    4    -5]'
B = [ 100  200  -300]'
M = [A  B]   %The matrix M is made from column A next
c =  3
d = -0.5
v =  [c d]'
C = M*v
```

Or, if you like to just put in the numbers and type as little as possible,

```
M = [  2    100
       4    200
      -5   -300]
C = M * [3  -0.5]'.
```

Although this last approach is compact, it makes deciphering your work later more difficult, so we generally advice against it.

## Magnitude of a vector using components

The Pythagorean Theorem for right triangles ('$A^2 + B^2 = C^2$') tells us that

$$|\vec{F}| = \sqrt{F_x^2 + F_y^2}, \tag{2D}$$

$$|\vec{F}| = \sqrt{F_x^2 + F_y^2 + F_z^2}. \tag{3D}$$

To get the result in 3D the 2D Pythagorean Theorem needs to be applied twice successively, first to get the magnitude of the sum $\vec{F}_x + \vec{F}_y$ and once more to add in $\vec{F}_z$. On a computer one might write something like this

```
   F    = [10 -20 30]
answer =  sqrt( F(1)^2 + F(2)^2 + F(3)^2 )
```

However this formula is so commonly needed that many computer languages will have a command like *norm* or *mag* so computer code something like *answer = norm(F)* or *answer = mag(F)* might replace the second line in the calculation above.

## 2.1 The scalars in mechanics

The scalar quantities used in this book, and their dimensional symbols in brackets [ ], are listed below ($M$ for mass, $L$ for length, $T$ for time, $F$ for force, and $E$ for energy).

- mass $m$, $[M]$;
- length or distance $\ell$, $w$, $x$, $r$, $\rho$, $d$, or $s$, $[L]$;
- time $t$, $[T]$;
- pressure $p$, $[F/L^2] = [M/(L \cdot T^2)]$;
- angles $\theta$ 'theta', $\phi$ 'phi', $\gamma$ 'gamma', and $\psi$ 'psi', [dimensionless];
- energy $E$, kinetic energy $E_{\mathrm{K}}$, potential energy $E_{\mathrm{P}}$, $[E]=[F \cdot L]=[M \cdot L^2/T^2]$;
- work $W$, $[E]=[F \cdot L]=[M \cdot L^2/T^2]$;
- tension $T$, $[M \cdot L/T^2] = [F]$;
- power $P$, $[E/T]=[M \cdot L^2/T^3]$;
- the magnitudes of all the vector quantities are also scalars, for example
  - speed $|\vec{\boldsymbol{v}}|$, $[L/T]$;
  - magnitude of acceleration $|\vec{\boldsymbol{a}}|$, $[L/T^2]$;
  - magnitude of angular momentum $|\vec{\boldsymbol{H}}|$, $[M \cdot L^2/T]$;
- the components of vectors, for example
  - $r_x$ (where $\vec{\boldsymbol{r}} = r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}}$), or
  - $L_{x'}$ (where $\vec{\boldsymbol{L}} = L_{x'}\hat{\boldsymbol{\imath}}' + L_{y'}\hat{\boldsymbol{\jmath}}'$);
- coefficient of friction $\mu$ 'mu', or friction angle $\phi$ 'phi';
- coefficient of restitution $e$;
- mass per unit length, area, or volume $\rho$;
- oscillation frequency $\beta$ or $\lambda$.

## 2.2 The Vectors in Mechanics

The vector quantities used in mechanics and the notations used in this book are shown below. The dimensional symbols of each are shown in brackets [ ]. Some of these quantities are also shown in figure **??**.

- position $\vec{\boldsymbol{r}}$ or $\vec{\boldsymbol{x}}$, $[L]$;
- velocity $\vec{\boldsymbol{v}}$ or $\dot{\vec{\boldsymbol{x}}}$ or $\dot{\vec{\boldsymbol{r}}}$, $[L/t]$;
- acceleration $\vec{\boldsymbol{a}}$ or $\dot{\vec{\boldsymbol{v}}}$ or $\ddot{\vec{\boldsymbol{r}}}$, $[L/t^2]$;
- angular velocity $\vec{\boldsymbol{\omega}}$ 'omega' (or, if aligned with the $\hat{\boldsymbol{k}}$ axis, $\dot{\theta}\hat{\boldsymbol{k}}$), $[1/t]$;
- rate of change of angular velocity $\vec{\boldsymbol{\alpha}}$ 'alpha' or $\dot{\vec{\boldsymbol{\omega}}}$ (or, if aligned with the $\hat{\boldsymbol{k}}$ axis, $\ddot{\theta}\hat{\boldsymbol{k}}$), $[1/t^2]$;
- force $\vec{\boldsymbol{F}}$ or $\vec{\boldsymbol{N}}$, $[m \cdot L/t^2]=[F]$;
- moment or torque $\vec{\boldsymbol{M}}$, $[m \cdot L^2/t^2]=[F \cdot L]$;
- linear momentum $\vec{\boldsymbol{L}}$, $[m \cdot L/t]$ and its rate of change $\dot{\vec{\boldsymbol{L}}}$, $[m \cdot L/t^2]$;
- angular momentum $\vec{\boldsymbol{H}}$, $[m \cdot L^2/t]$; and its rate of change $\dot{\vec{\boldsymbol{H}}}$, $[m \cdot L^2/t^2]$.
- unit vectors to help write other vectors [dimensionless]:
  - $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ for cartesian coordinates,
  - $\hat{\boldsymbol{\imath}}'$, $\hat{\boldsymbol{\jmath}}'$, and $\hat{\boldsymbol{k}}'$ for crooked cartesian coordinates,
  - $\hat{\boldsymbol{e}}_r$ and $\hat{\boldsymbol{e}}_\theta$ for polar coordinates,
  - $\hat{\boldsymbol{e}}_t$ and $\hat{\boldsymbol{e}}_n$ for path coordinates, and
  - $\hat{\boldsymbol{\lambda}}$ 'lambda' and $\hat{\boldsymbol{n}}$ as miscellaneous unit vectors.

**Ornamentation of vectors** Subscripts and superscripts are often added to indicate the point, points, object, or objects the vectors are describing. Upper case letters (O, A, B, C,...) are used to denote points. Upper case calligraphic (or script if you are writing by hand) letters ($\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$...$\mathcal{F}$...) are for labeling rigid objects or reference frames. $\mathcal{F}$ is the fixed, Newtonian, or 'absolute' reference frame (think of $\mathcal{F}$ as the ground if you are a first time reader). For example, $\vec{\boldsymbol{r}}_{\mathrm{AB}}$ or $\vec{\boldsymbol{r}}_{\mathrm{B/A}}$ is the position of the point $B$ relative to point $A$. $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$ is the absolute angular velocity of the object called $\mathcal{B}$ ($\vec{\boldsymbol{\omega}}_{\mathcal{B}}$ is short hand for $\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{F}}$). And $\vec{\boldsymbol{H}}_{\mathcal{A}/C}$ is the angular momentum of object $\mathcal{A}$ relative to point $C$.

The notation is further complicated when we want to take derivatives with respect to moving frames, a topic which comes up later in the book. For completeness here: ${}^{\mathcal{B}}\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}/\mathcal{E}}$ is the time derivative with respect to reference frame $\mathcal{B}$ of the angular velocity of object $\mathcal{D}$ with respect to object (or frame) $\mathcal{E}$. (If this paragraph doesn't read like gibberish to you, you have already studied dynamics. Its here for the experts who are deep into the book and are looking back.)

**SAMPLE 2.1 Various ways of representing a vector:** A vector $\vec{F} = 3\,\mathrm{N}\hat{\imath} + 3\,\mathrm{N}\hat{\jmath}$ is represented in various ways, some incorrect, in the following figures. The base vectors used are shown first. Comment on each representation, whether it is correct or incorrect, and why.

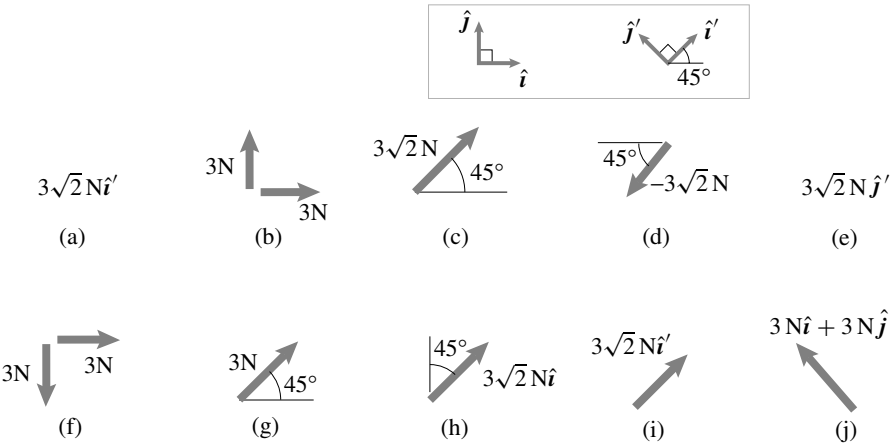



Figure 2.10:

Filename:sfig2-vectors-rep

**Solution** The given vector is a force with components of 3 N each in the positive $\hat{\imath}$ and $\hat{\jmath}$ directions using the unit vectors $\hat{\imath}$ and $\hat{\jmath}$ shown in the box above. The unit vectors $\hat{\imath}'$, and $\hat{\jmath}'$ are also shown. Note that the unit vectors $\hat{\imath}'$ and $\hat{\jmath}'$ can be expressed in terms of their components along $\hat{\imath}$ and $\hat{\jmath}$ as follows:

$$\begin{aligned}\hat{\imath}' &= |\hat{\imath}'|\cos 45^\circ\hat{\imath} + |\hat{\imath}'|\sin 45^\circ\hat{\jmath} \\ &= \frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath} \\ &= \frac{1}{\sqrt{2}}(\hat{\imath} + \hat{\jmath}). \qquad (2.1)\end{aligned}$$

Similarly,

$$\begin{aligned}\hat{\jmath}' &= |\hat{\jmath}'|\cos 135^\circ\hat{\imath} + |\hat{\jmath}'|\sin 135^\circ\hat{\jmath} \\ &= \frac{1}{\sqrt{2}}(-\hat{\imath} + \hat{\jmath}). \qquad (2.2)\end{aligned}$$

**a) Correct:** $3\sqrt{2}\,\mathrm{N}\hat{\imath}'$. From the picture defining $\hat{\imath}'$, you can see that $\hat{\imath}'$ is a unit vector with equal components in the $\hat{\imath}$ and $\hat{\jmath}$ directions; *i.e.*, it is parallel to $\vec{F}$. So $\vec{F}$ is given by its magnitude $\sqrt{(3\,\mathrm{N})^2 + (3\,\mathrm{N})^2}$ times a unit vector in its direction, in this case $\hat{\imath}'$. It is the same vector. Algebraically,

$$3\sqrt{2}\,\mathrm{N}\hat{\imath}' = 3\sqrt{2}\,\mathrm{N}\cdot\frac{1}{\sqrt{2}}(\hat{\imath} + \hat{\jmath}) = 3\,\mathrm{N}\hat{\imath} + 3\,\mathrm{N}\hat{\jmath} = \vec{F}.$$

**b) Correct:** Here two vectors are shown: one with magnitude 3 N in the direction of the horizontal arrow $\hat{\imath}$, and one with magnitude 3 N in the direction of the vertical arrow $\hat{\jmath}$. When two forces act on an object at a point, their effect is additive. So the net vector is the sum of the vectors shown. That is, $3\,\mathrm{N}\hat{\imath} + 3\,\mathrm{N}\hat{\jmath}$. It is the same vector.

**c) Correct:** Here we have a scalar $3\sqrt{2}\,\mathrm{N}$ next to an arrow. The vector described is the scalar multiplied by a unit vector in the direction of the arrow. Since the arrow's direction

is marked as the same direction as $\hat{\boldsymbol{\imath}}'$, which we already know is parallel to $\vec{\boldsymbol{F}}$, this vector represents the same vector $\vec{\boldsymbol{F}}$. Componentwise, we can write,

$$3\sqrt{2}\,\mathrm{N}(\cos 45^\circ \hat{\boldsymbol{\imath}} + \sin 45^\circ \hat{\boldsymbol{\jmath}}) = 3\,\mathrm{N}\hat{\boldsymbol{\imath}} + 3\,\mathrm{N}\hat{\boldsymbol{\jmath}} = \vec{\boldsymbol{F}}.$$

**d) Correct:** The scalar $-3\sqrt{2}\,\mathrm{N}$ is multiplied by a unit vector in the direction indicated, $-\hat{\boldsymbol{\imath}}'$. So we get $(-3\sqrt{2}\,\mathrm{N})(-\hat{\boldsymbol{\imath}}')$ which is $3\sqrt{2}\,\mathrm{N}\hat{\boldsymbol{\imath}}'$ as before. It is the same vector.

**e) Incorrect:** $3\sqrt{2}\,\mathrm{N}\hat{\boldsymbol{\jmath}}'$. The magnitude is right, but the direction is off by 90 degrees. It is a different vector. Algebraically,

$$3\sqrt{2}\,\mathrm{N}\hat{\boldsymbol{\jmath}}' = 3\sqrt{2}\,\mathrm{N}\cdot\frac{1}{\sqrt{2}}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}) = -3\,\mathrm{N}\hat{\boldsymbol{\imath}} + 3\,\mathrm{N}uj \neq \vec{\boldsymbol{F}}.$$

**f) Incorrect:** $3\,\mathrm{N}\hat{\boldsymbol{\imath}} - 3\,\mathrm{N}\hat{\boldsymbol{\jmath}}$. The $\hat{\boldsymbol{\imath}}$ component of the vector is correct but the $\hat{\boldsymbol{\jmath}}$ component is in the opposite direction. The vector is in the wrong direction by 90 degrees. It is a different vector.

**g) Incorrect:** Right direction but the magnitude is off by a factor of $\sqrt{2}$.

**h) Incorrect:** The magnitude is right. The direction indicated is right. But, the algebraic symbol $3\sqrt{2}\,\mathrm{N}\hat{\boldsymbol{\imath}}$ takes precedence and it is in the wrong direction ($\hat{\boldsymbol{\imath}}$ instead of $\hat{\boldsymbol{\imath}}'$). It is a different vector.

**i) Correct:** A labeled arrow. The arrow is only schematic. The algebraic symbol $3\sqrt{2}\,\mathrm{N}\hat{\boldsymbol{\imath}}'$ defines the vector. We draw the arrow to remind us that there is a vector to represent. The tip or tail of the arrow would be drawn at the point of the force application. In this case, the arrow is drawn in the direction of $\vec{\boldsymbol{F}}$, but strictly speaking, it need not.

**j) Correct:** Like (i) above, the directional and magnitude information are embedded in the algebraic symbol $3\,\mathrm{N}\hat{\boldsymbol{\imath}} + 3\,\mathrm{N}\hat{\boldsymbol{\jmath}}$. The arrow is there to indicate a vector. In this case, it points in the wrong direction so is not ideally communicative. In fact, it is confusing and therefore, not recommended. But it still correctly represents the given vector because the algebraic symbol takes precedence over the graphical symbol.

□

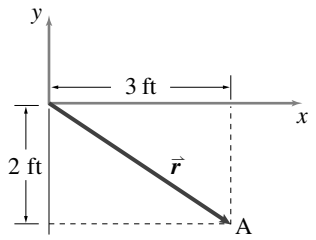



Figure 2.11: A vector $\vec{r} = 3\,\text{ft}\hat{\imath} - 2\,\text{ft}\hat{\jmath}$ is drawn by locating its end point which is 3 units away along the x-axis and 2 units away along the negative y-axis.

Filename:sfig1-2-4a

**SAMPLE 2.2 Drawing a vector from its components:** Draw the vector $\vec{r} = 3\,\text{ft}\hat{\imath} - 2\,\text{ft}\hat{\jmath}$ using its components.

**Solution** To draw $\vec{r}$ using its components, we first draw the axes and measure 3 units (any units that we choose on the ruler) along the $x$-axis and 2 units along the negative $y$-axis. We mark this point as $A$ (say) on the paper and draw a line from the origin to the point $A$. We write the dimensions '3 ft' and '2 ft' on the figure. Finally, we put an arrowhead on this line pointing towards $A$.

□

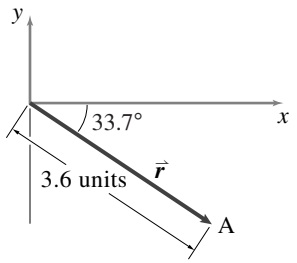



Figure 2.12: A vector $\vec{r}$ with a given length (3.6 ft ) and direction (slope angle $\theta = -33.7^\circ$ ) is drawn by measuring its length along a line drawn at angle $\theta$ from the positive x-axis.

Filename:sfig1-2-4b

**SAMPLE 2.3 Drawing a vector from its length and direction:** A vector $\vec{r}$ is 3.6 ft long and is directed $33.7^\circ$ from the $x$-axis towards the negative $y$-axis. Draw $\vec{r}$.

**Solution** We first draw the $x$ and $y$ axes and then draw $\vec{r}$ as a line from the origin at an angle $-33.7^\circ$ from the $x$-axis (minus sign means measuring clockwise), measure 3.6 units (magnitude of $\vec{r}$) along this line and finally put an arrowhead pointing away from the origin.

□

**Comments** Note that this is the same vector as in Sample 2.2. In fact, you can easily verify that

$$\begin{aligned} r_x &= r\cos\theta = 3.6\,\text{ft}\cdot\cos(-33.7^\circ) = 3\,\text{ft}, \\ r_y &= r\sin\theta = 3.6\,\text{ft}\cdot\sin(-33.7^\circ) = -2\,\text{ft}, \\ \Rightarrow\quad \vec{r} &= r_x\hat{\imath} + r_y\hat{\jmath} = (3\,\text{ft})\hat{\imath} - (2\,\text{ft})\hat{\jmath} \end{aligned}$$

as given in Sample 2.2.

**SAMPLE 2.4 Adding vectors:** Three forces, $\vec{F}_1 = 2\,\text{N}\hat{\imath} + 3\,\text{N}\hat{\jmath}$, $\vec{F}_2 = -10\,\text{N}\hat{\jmath}$, and $\vec{F}_3 = 3\,\text{N}\hat{\imath} + 1\,\text{N}\hat{\jmath} - 5\,\text{N}\hat{k}$, act on a particle. Find the net force on the particle.

**Solution** The net force on the particle is the vector sum of all the forces, *i.e.*,

$$\begin{aligned}
\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \\
&= (2\,\text{N}\hat{\imath} + 3\,\text{N}\hat{\jmath}) + (-10\,\text{N}\hat{\jmath}) + (3\,\text{N}\hat{\imath} + 1\,\text{N}\hat{\jmath} - 5\,\text{N}\hat{k}) \\
&= \begin{array}{rcrcr} 2\,\text{N}\hat{\imath} & + & 3\,\text{N}\hat{\jmath} & + & 0\hat{k} \\ 0\hat{\imath} & - & 10\,\text{N}\hat{\jmath} & + & 0\hat{k} \\ 3\,\text{N}\hat{\imath} & + & 1\,\text{N}\hat{\jmath} & - & 5\hat{k} \end{array} \\
&= (2\,\text{N} + 3\,\text{N})\hat{\imath} + (3\,\text{N} - 10\,\text{N} + 1\,\text{N})\hat{\jmath} + (-5\,\text{N})\hat{k} \\
&= 5\,\text{N}\hat{\imath} - 6\,\text{N}\hat{\jmath} - 5\,\text{N}\hat{k}.
\end{aligned}$$

$$\boxed{\vec{F}_{\text{net}} = 5\,\text{N}\hat{\imath} - 6\,\text{N}\hat{\jmath} - 5\,\text{N}\hat{k}}$$

**Comments:** In general, we do not need to write the summation so elaborately. Once you feel comfortable with the idea of summing only similar components in a vector sum, you can do the calculation in two lines.

---

**SAMPLE 2.5 Subtracting vectors:** Two forces $\vec{F}_1$ and $\vec{F}_2$ act on a body. The net force on the body is $\vec{F}_{\text{net}} = 2\,\text{N}\hat{\imath}$. If $\vec{F}_1 = 10\,\text{N}\hat{\imath} - 10\,\text{N}\hat{\jmath}$, find the other force $\vec{F}_2$.

**Solution**

$$\begin{aligned}
\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 \\
\Rightarrow \quad \vec{F}_2 &= \vec{F}_{\text{net}} - \vec{F}_1 \\
&= 2\,\text{N}\hat{\imath} - (10\,\text{N}\hat{\imath} - 10\,\text{N}\hat{\jmath}) \\
&= -8\,\text{N}\hat{\imath} + 10\,\text{N}\hat{\jmath}.
\end{aligned}$$

$$\boxed{\vec{F}_2 = -8\,\text{N}\hat{\imath} + 10\,\text{N}\hat{\jmath}}$$

---

**SAMPLE 2.6 Scalar times a vector:** Two forces acting on a particle are $\vec{F}_1 = 100\,\text{N}\hat{\imath} - 20\,\text{N}\hat{\jmath}$ and $\vec{F}_2 = 40\,\text{N}\hat{\jmath}$. If $\vec{F}_1$ is doubled, does the net force double?

**Solution**

$$\vec{F}_{\text{net}} = \vec{F}_1 + \vec{F}_2 = (100\,\text{N}\hat{\imath} - 20\,\text{N}\hat{\jmath}) + (40\,\text{N}\hat{\jmath}) = 100\,\text{N}\hat{\imath} + 20\,\text{N}\hat{\jmath}.$$

After $\vec{F}_1$ is doubled, the new net force $\vec{F}_{(\text{net})_2}$ is

$$\begin{aligned}
\vec{F}_{(\text{net})_2} &= 2\vec{F}_1 + \vec{F}_2 \\
&= 2(100\,\text{N}\hat{\imath} - 20\,\text{N}\hat{\jmath}) + (40\,\text{N}\hat{\jmath}) \\
&= 200\,\text{N}\hat{\imath} \neq 2\underbrace{(100\,\text{N}\hat{\imath} + 20\,\text{N}\hat{\jmath})}_{\vec{F}_{\text{net}}}
\end{aligned}$$

No, the net force does not double.

**SAMPLE 2.7 Magnitude and direction of a vector:** The velocity of a car is given by $\vec{v} = (30\hat{\imath} + 40\hat{\jmath})$ mph.

1. Find the speed (magnitude of $\vec{v}$) of the car.
2. Find a unit vector in the direction of $\vec{v}$.
3. Write the velocity vector as a product of its magnitude and the unit vector.

#### Solution

1. **Magnitude of $\vec{v}$:** The magnitude of a vector is the length of the vector. It is a scalar quantity, usually represented by the same letter as the vector but without the vector notation (i.e. no bold face, no underbar). It is also represented by the modulus of the vector (the vector written between two vertical lines). The length of a vector is the square root of the sum of squares of its components. Therefore, for

$$\begin{aligned} \vec{v} &= 30\,\text{mph}\hat{\imath} + 40\,\text{mph}\hat{\jmath}, \\ v = |\vec{v}| &= \sqrt{v_x^2 + v_y^2} \\ &= \sqrt{(30\,\text{mph})^2 + (40\,\text{mph})^2} \\ &= 50\,\text{mph} \end{aligned}$$

which is the speed of the car.

$$\boxed{\text{speed} = 50\ \text{mph}}$$

2. **Direction of $\vec{v}$ as a unit vector along $\vec{v}$:** The direction of a vector can be specified by specifying a unit vector along the given vector. In many applications you will encounter in dynamics, this concept is useful. The unit vector along a given vector is found by dividing the given vector with its magnitude. Let $\hat{\boldsymbol{\lambda}}_v$ be the unit vector along $\vec{v}$. Then,

$$\begin{aligned} \hat{\boldsymbol{\lambda}}_v &= \frac{\vec{v}}{|\vec{v}|} = \frac{30\,\text{mph}\hat{\imath} + 40\,\text{mph}\hat{\jmath}}{50\,\text{mph}} \\ &= 0.6\hat{\imath} + 0.8\hat{\jmath}. \qquad \text{(unit vectors have no units!)} \end{aligned}$$

$$\boxed{\hat{\boldsymbol{\lambda}}_v = 0.6\hat{\imath} + 0.8\hat{\jmath}}$$

3. **$\vec{v}$ as a product of its magnitude and the unit vector $\hat{\boldsymbol{\lambda}}_v$:** A vector can be written in terms of its components, as given in this problem, or as a product of its magnitude and direction (given by a unit vector). Thus we may write,

$$\vec{v} = |\vec{v}|\hat{\boldsymbol{\lambda}}_v = 50\,\text{mph}(0.6\hat{\imath} + 0.8\hat{\jmath})$$

which, of course, is the same vector as given in the problem.

$$\boxed{\vec{v} = 50(0.6\hat{\imath} + 0.8\hat{\jmath})\ \text{mph}}$$

**SAMPLE 2.8 Position vector from the origin:** In the $xyz$ coordinate system, a particle is located at the coordinate (3m, 2m, 1m). Find the position vector of the particle.

**Solution** The position vector of the particle at P is a vector drawn from the origin of the coordinate system to the position P of the particle. See Fig. 2.13. We can write this vector as

$$\begin{aligned} \vec{r}_{\rm P} &= (3\,{\rm m})\hat{\imath} + (2\,{\rm m})\hat{\jmath} + (1\,{\rm m})\hat{k} \\ \text{or} \qquad \vec{r}_{\rm P} &= (3\hat{\imath} + 2\hat{\jmath} + \hat{k})\,{\rm m}. \end{aligned}$$

$$\boxed{\vec{r}_{\rm P} = 3\,{\rm m}\hat{\imath} + 2\,{\rm m}\hat{\jmath} + 1\,{\rm m}\hat{k}}$$

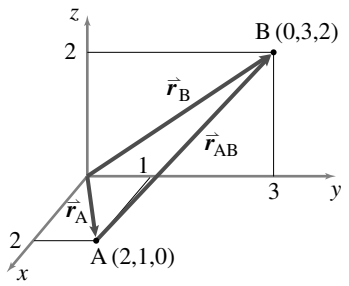



Figure 2.14: The position vector of B with respect to A is found from $\vec{r}_{\rm AB} = \vec{r}_{\rm B} - \vec{r}_{\rm A}$.

Filename:sfig2-vec1-7

**SAMPLE 2.9 Relative position vector:** Let A (2m, 1m, 0) and B (0, 3m, 2m) be two points in the $xyz$ coordinate system. Find the position vector of point B with respect to point A, *i.e.*, find $\vec{r}_{\rm AB}$ (or $\vec{r}_{\rm B/A}$).

**Solution** From the geometry of the position vectors shown in Fig. 2.14 and the rules of vector sums, we can write,

$$\begin{aligned} \vec{r}_{\rm B} &= \vec{r}_{\rm A} + \vec{r}_{\rm AB} \\ \Rightarrow \quad \vec{r}_{\rm AB} &= \vec{r}_{\rm B} - \vec{r}_{\rm A} \\ &= (0\hat{\imath} + 3\,{\rm m}\hat{\jmath} + 2\,{\rm m}\hat{k}) - (2\,{\rm m}\hat{\imath} + 1\,{\rm m}\hat{\jmath} + 0\hat{k}) \\ &= -2\,{\rm m}\hat{\imath} + 2\,{\rm m}\hat{\jmath} + 2\,{\rm m}\hat{k}. \end{aligned}$$

$$\boxed{\vec{r}_{\rm AB} \equiv \vec{r}_{\rm B/A} = -2\,{\rm m}\hat{\imath} + 2\,{\rm m}\hat{\jmath} + 2\,{\rm m}\hat{k}}$$

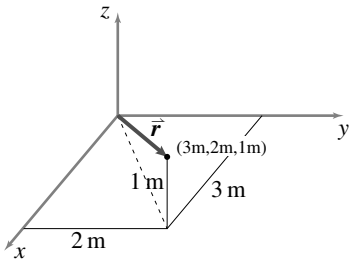



Figure 2.13: The position vector of the particle is a vector drawn from the origin of the coordinate system to the position of the particle.

Filename:sfig2-vec1-6

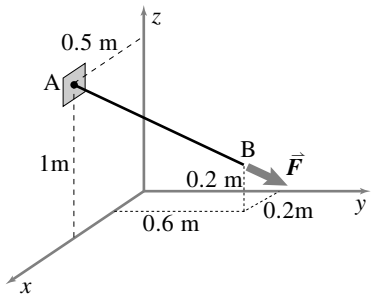



Figure 2.15:
Filename:sfig1-2-2

**SAMPLE 2.10 Finding a force vector given its magnitude and line of action:** A string is pulled with a force $F = 100\,\text{N}$ as shown in Fig. 2.15. Write $F$ as a vector.

**Solution** A vector can be written, as we just showed in the previous sample problem, as the product of its magnitude and a unit vector along the given vector. Here, the magnitude of the force is given and we know it acts along AB. Therefore, we may write

$$\vec{F} = F\hat{\boldsymbol{\lambda}}_{AB}$$

where $\hat{\boldsymbol{\lambda}}_{AB}$ is a unit vector along AB. So now we need to find $\hat{\boldsymbol{\lambda}}_{AB}$. We can easily find $\hat{\boldsymbol{\lambda}}_{AB}$ if we know vector AB. Let us denote vector AB by $\vec{r}_{AB}$ (sometimes we will also write it as $\vec{r}_{B/A}$ to represent the *position of B with respect to A* as a vector). Then,

$$\hat{\boldsymbol{\lambda}}_{AB} = \frac{\vec{r}_{AB}}{|\vec{r}_{AB}|}.$$

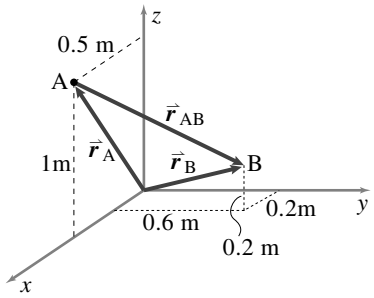



Figure 2.16: $\vec{r}_{AB} = \vec{r}_B - \vec{r}_A$.
Filename:sfig1-2-2b

To find $\vec{r}_{AB}$, we note that (see Fig. 2.16)

$$\vec{r}_A + \vec{r}_{AB} = \vec{r}_B$$

where $\vec{r}_A$ and $\vec{r}_B$ are the position vectors of point A and point B respectively. Hence,

$$\begin{aligned}
\vec{r}_{B/A} &= \vec{r}_{AB} = \vec{r}_B - \vec{r}_A \\
&= (0.2\,\text{m}\hat{\boldsymbol{\imath}} + 0.6\,\text{m}\hat{\boldsymbol{\jmath}} + 0.2\,\text{m}\hat{\boldsymbol{k}}) - (0.5\,\text{m}\hat{\boldsymbol{\imath}} + 1.0\,\text{m}\hat{\boldsymbol{k}}) \\
&= -0.3\,\text{m}\hat{\boldsymbol{\imath}} + 0.6\,\text{m}\hat{\boldsymbol{\jmath}} - 0.8\,\text{m}\hat{\boldsymbol{k}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\hat{\boldsymbol{\lambda}}_{AB} &= \frac{-0.3\,\text{m}\hat{\boldsymbol{\imath}} + 0.6\,\text{m}\hat{\boldsymbol{\jmath}} - 0.8\,\text{m}\hat{\boldsymbol{k}}}{\sqrt{(-0.3)^2 + (0.6)^2 + (-0.8)^2}\ \text{m}} \\
&= -0.29\hat{\boldsymbol{\imath}} + 0.57\hat{\boldsymbol{\jmath}} - 0.77\hat{\boldsymbol{k}},
\end{aligned}$$

and, finally,

$$\begin{aligned}
\vec{F} &= \overbrace{(100\,\text{N})}^{F}\,\hat{\boldsymbol{\lambda}}_{AB} \\
&= -29\,\text{N}\hat{\boldsymbol{\imath}} + 57\,\text{N}\hat{\boldsymbol{\jmath}} - 77\,\text{N}\hat{\boldsymbol{k}}.
\end{aligned}$$

$$\boxed{\vec{F} = -29\,\text{N}\hat{\boldsymbol{\imath}} + 57\,\text{N}\hat{\boldsymbol{\jmath}} - 77\,\text{N}\hat{\boldsymbol{k}}}$$

**SAMPLE 2.11 Adding vectors on computers:** The following six forces act at different points of a structure. $\vec{F}_1 = -3\,\text{N}\hat{\imath}$, $\vec{F}_2 = 20\,\text{N}\hat{\imath} - 10\,\text{N}\hat{\jmath}$, $\vec{F}_3 = 1\,\text{N}\hat{\imath} + 20\,\text{N}\hat{\jmath} - 5\,\text{N}\hat{k}$, $\vec{F}_4 = 10\,\text{N}\hat{\imath}$, $\vec{F}_5 = 5\,\text{N}(\hat{\imath} + \hat{\jmath} + \hat{k})$, $\vec{F}_6 = -10\,\text{N}\hat{\imath} - 10\,\text{N}\hat{\jmath} + 2\,\text{N}\hat{k}$.

1. Write all the force vectors in column form.
2. Find the net force by hand calculation.
3. Write a computer program to sum $n$ vectors, each with three components. Use your program to compute the net force.

**Solution**

1. The 3-D vector $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$ is represented as a column (or a row) as follows:

$$[\vec{F}] = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}_{xyz}$$

Following this convention, we write the given forces as

$$[\vec{F}_1] = \begin{pmatrix} 0 \\ -3\,\text{N} \\ 0 \end{pmatrix}_{xyz}, [\vec{F}_2] = \begin{pmatrix} 20\,\text{N} \\ -10\,\text{N} \\ 0 \end{pmatrix}_{xyz}, \cdots, [\vec{F}_6] = \begin{pmatrix} -10\,\text{N} \\ -10\,\text{N} \\ 2\,\text{N} \end{pmatrix}_{xyz}$$

2. The net force, $\vec{F}_{\text{net}} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 + \vec{F}_5 + \vec{F}_6$ or

$$\begin{aligned}[\vec{F}_{\text{net}}] &= \begin{pmatrix} 0 & 20 & 1 & 10 & 5 & -10 \\ -3 & -10 & 20 & 0 & 5 & -10 \\ 0 & 0 & -5 & 0 & 5 & 2 \end{pmatrix}_{xyz} \text{N} \\ &= \begin{pmatrix} 26 \\ 2 \\ 2 \end{pmatrix}_{xyz} \text{N}\end{aligned}$$

3. The steps to do this addition on computers are as follows.
   - Enter the vectors as rows or columns:

```
F1 = [0  -3  0]
F2 = [20  -10  0]
F3 = [1  20  -5]
F4 = [10  0  0]
F5 = [5  5  5]
F6 = [-10  -10  2]
```

   - Sum the vectors, using a summing operation that automatically does element by element addition of vectors:

```
Fnet = F1 + F2 + F3 + F4 + F5 + F6
```

   - The computer generated answer is:

```
Fnet = [26  2  2].
```

$$\boxed{\vec{F}_{\text{net}} = 26\,\text{N}\hat{\imath} + 2\,\text{N}\hat{\jmath} + 2\,\text{N}\hat{k}}$$

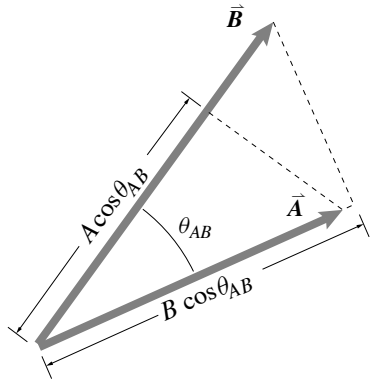



Figure 2.17: The dot product of $\vec{A}$ and $\vec{B}$ is a scalar and so is not easily drawn. It is given by $\vec{A}\cdot\vec{B} = AB\cos\theta_{AB}$ which is $A$ times the projection of $\vec{B}$ in the $A$ direction and also $B$ times the projection of $\vec{A}$ in the $B$ direction.
Filename:tfigure1-11

## 2.2 The dot product of two vectors

The dot product is used to project a vector in a given direction, to reduce a vector to components, to reduce vector equations to scalar equations, to define work and power, and to help solve geometry problems.

The *dot product* of two vectors $\vec{A}$ and $\vec{B}$ is written $\vec{A}\cdot\vec{B}$ (pronounced 'A dot B'). The dot product of $\vec{A}$ and $\vec{B}$ is the product of the magnitudes of the two vectors times a number that expresses the degree to which $\vec{A}$ and $\vec{B}$ are parallel: $\cos\theta_{AB}$, where $\theta_{AB}$ is the angle between $\vec{A}$ and $\vec{B}$. That is,

$$\vec{A}\cdot\vec{B} \stackrel{def}{=} |\vec{A}|\,|\vec{B}|\cos\theta_{AB}$$

which is sometimes written more concisely as $\vec{A}\cdot\vec{B} = AB\cos\theta$. One special case occurs when $\cos\theta_{AB} = 1$, $\vec{A}$ and $\vec{B}$ are parallel, and $\vec{A}\cdot\vec{B} = AB$. Another is when $\cos\theta_{AB} = 0$, $\vec{A}$ and $\vec{B}$ are perpendicular, and $\vec{A}\cdot\vec{B} = 0$.*

The dot product of two vectors is a scalar. So the dot product is sometimes called the *scalar product*. Using the geometric definition of dot product, and the rules for vector addition we have already discussed, you can convince yourself of (or believe) the following properties of dot products.

- $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$ — commutative law, $AB\cos\theta = BA\cos\theta$
- $(a\vec{A})\cdot\vec{B} = \vec{A}\cdot(a\vec{B}) = a(\vec{A}\cdot\vec{B})$ — a distributive law, $(aA)B\cos\theta = A(aB)\cos\theta$
- $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$ — another distributive law, the projection of $\vec{B}+\vec{C}$ onto $\vec{A}$ is the sum of the two separate projections
- $\vec{A}\cdot\vec{B} = 0$ if $\vec{A}\perp\vec{B}$ — perpendicular vectors have zero for a dot product, $AB\cos\pi/2 = 0$
- $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|$ if $\vec{A}\parallel\vec{B}$ — parallel vectors have the product of their magnitudes for a dot product, $AB\cos 0 = AB$. In particular, $\vec{A}\cdot\vec{A} = A^2$ or $|\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}}$
- $\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1$, $\hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{k} = \hat{k}\cdot\hat{\imath} = 0$ — The standard base vectors used with cartesian coordinates are unit vectors and they are perpendicular to each other. In math language they are 'orthonormal.'

* At this point you should know, almost without a thought, that $\cos 0 = 1, \cos\pi/2 = 0, \sin 0 = 0$, and $\sin\pi/2 = 1$. If you don't, stop now and draw as many triangles and unit circles as it takes to cement these 4 special cases into your head.

- $\hat{\imath}' \cdot \hat{\imath}' = \hat{\jmath}' \cdot \hat{\jmath}' = \hat{k}' \cdot \hat{k}' = 1$, $\hat{\imath}' \cdot \hat{\jmath}' = \hat{\jmath}' \cdot \hat{k}' = \hat{k}' \cdot \hat{\imath}' = 0$ — The standard tilted base vectors are orthonormal.

The identities above lead to the following equivalent ways of expressing the dot product of $\vec{A}$ and $\vec{B}$ (see box 2.2 on page 41 to see how the component formula follows from the geometric definition above):

$$\begin{aligned}
\vec{A} \cdot \vec{B} &= |\vec{A}||\vec{B}| \cos\theta_{AB} \\
&= A_x B_x + A_y B_y + A_z B_z \quad \text{(component formula for dot product)} \\
&= A_{x'} B_{x'} + A_{y'} B_{y'} + A_{z'} B_{z'} \\
&= |\vec{A}| \cdot [\text{projection of } \vec{B} \text{ in the } \vec{A} \text{ direction}] \\
&= |\vec{B}| \cdot [\text{projection of } \vec{A} \text{ in the } \vec{B} \text{ direction}]
\end{aligned}$$

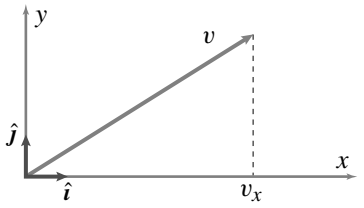



Figure 2.18: The dot product with unit vectors gives projection. For example, $v_x = \vec{v} \cdot \hat{\imath}$.
Filename:tfigure1-3-dotprod

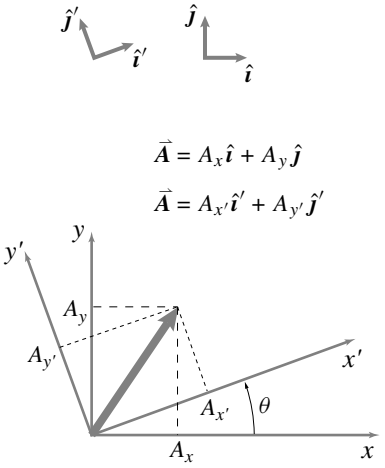



Figure 2.19: The dot product helps find components in terms of tilted unit vectors. For example, $A_{y'} = \vec{A} \cdot \hat{\jmath}' = A_x(\hat{\imath} \cdot \hat{\jmath}') + A_y(\hat{\imath} \cdot \hat{\jmath}') = A_x(-\sin\theta) + A_y(\cos\theta)$.
Filename:tfigure1-3-dotprod-a

## Using the dot product to find components

To find the $x$ component of a vector or vector expression one can use the dot product of the vector (or expression) with a unit vector in the $x$ direction as in figure 2.18. In particular,

$$v_x = \vec{v} \cdot \hat{\imath}.$$

This idea can be used for finding components in any direction. If one knows the orientation of the tilted unit vectors $\hat{\imath}'$, $\hat{\jmath}'$, $\hat{k}'$ relative to the standard bases $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ then all the angles between the base vectors are known. So one can evaluate the dot products between the standard base vectors and the tilted base vectors. In 2-D assume that the dot products between the standard base vectors and the vector $\hat{\jmath}'$ (*i.e.*, . $\hat{\imath} \cdot \hat{\jmath}'$, $\hat{\jmath} \cdot \hat{\jmath}'$) are known. One can then use the dot product to find the $x'y'$ components $(A_{x'}, A_{y'})$ from the $xy$ components $(A_x, A_y)$. For example, as shown in 2-D in figure 2.19, we can start with the obvious equation

$$\vec{A} = \vec{A}$$

and dot both sides with $\hat{\jmath}'$ to get:

$$\begin{aligned}
\vec{A} \cdot \hat{\jmath}' &= \vec{A} \cdot \hat{\jmath}' \\
\underbrace{(A_{x'}\hat{\imath}' + A_{y'}\hat{\jmath}')}_{\vec{A}} \cdot \hat{\jmath}' &= \underbrace{(A_x\hat{\imath} + A_y\hat{\jmath})}_{\vec{A}} \cdot \hat{\jmath}' \\
A_{x'} \underbrace{\hat{\imath}' \cdot \hat{\jmath}'}_{0} + A_{y'} \underbrace{\hat{\jmath}' \cdot \hat{\jmath}'}_{1} &= A_x\hat{\imath} \cdot \hat{\jmath}' + A_y\hat{\jmath} \cdot \hat{\jmath}' \\
A_{y'} &= A_x \underbrace{(\hat{\imath} \cdot \hat{\jmath}')}_{-\sin\theta} + A_y \underbrace{(\hat{\jmath} \cdot \hat{\jmath}')}_{\cos\theta}.
\end{aligned}$$

Similarly, one could find the component $A_{x'}$ using a dot product with $\hat{\imath}'$.

This technique of finding components is useful when one problem uses more than one base vector system.

## Using dot products with unit vectors other than $\hat{\imath}$, $\hat{\jmath}$, or $\hat{k}$

It is often useful to use dot products to get scalar equations using unit vectors other than $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.

Example: **Getting scalar equations without dotting with $\hat{\imath}$, $\hat{\jmath}$, or $\hat{k}$**
Given the vector equation

$$-mg\hat{\jmath} + N\hat{\boldsymbol{n}} = ma\hat{\boldsymbol{\lambda}}$$

where it is known that the unit vector $\hat{\boldsymbol{n}}$ is perpendicular to the unit vector $\hat{\boldsymbol{\lambda}}$, we can get a scalar equation by dotting both sides with $\hat{\boldsymbol{\lambda}}$ which we write as follows

$$\begin{aligned} \left\{ (-mg\hat{\jmath} + N\hat{\boldsymbol{n}}) \right. &= \left. (ma\hat{\boldsymbol{\lambda}}) \right\} \cdot \hat{\boldsymbol{\lambda}} \\ (-mg\hat{\jmath} + N\hat{\boldsymbol{n}}) \cdot \hat{\boldsymbol{\lambda}} &= (ma\hat{\boldsymbol{\lambda}}) \cdot \hat{\boldsymbol{\lambda}} \\ -mg\hat{\jmath} \cdot \hat{\boldsymbol{\lambda}} + N \underbrace{\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\lambda}}}_{0} &= ma \underbrace{\hat{\boldsymbol{\lambda}} \cdot \hat{\boldsymbol{\lambda}}}_{1} \\ -mg\hat{\jmath} \cdot \hat{\boldsymbol{\lambda}} &= ma. \end{aligned}$$

Then we find $\hat{\jmath} \cdot \hat{\boldsymbol{\lambda}}$ as the cosine of the angle between $\hat{\jmath}$ and $\hat{\boldsymbol{\lambda}}$. We have thus turned our vector equation into a scalar equation and eliminated the unknown $N$ at the same time.

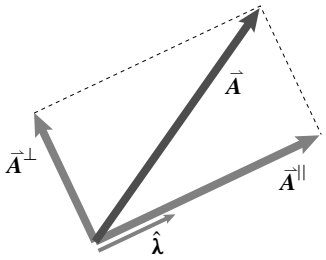



Figure 2.20: For any $\vec{A}$ and $\hat{\boldsymbol{\lambda}}$, $\vec{A}$ can be decomposed into a part parallel to $\hat{\boldsymbol{\lambda}}$ and a part perpendicular to $\hat{\boldsymbol{\lambda}}$.
Filename:tfigure-Graham1

## Using dot products to solve geometry problems

We have seen how a vector can be broken down into a sum of components each parallel to one of the orthogonal base vectors. Another useful decomposition is this: Given any vector $\vec{A}$ and a unit vector $\hat{\boldsymbol{\lambda}}$ vector $\vec{A}$ can be written as the sum of two parts,

$$\vec{A} = \vec{A}^{\parallel} + \vec{A}^{\perp}$$

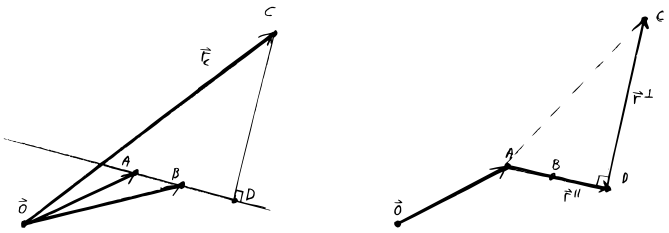
where $\vec{A}^{\parallel}$ is parallel to $\hat{\boldsymbol{\lambda}}$ and $\vec{A}^{\perp}$ is perpendicular to $\hat{\boldsymbol{\lambda}}$ (see fig. 2.20). The part parallel to $\hat{\boldsymbol{\lambda}}$ is a vector pointed in the $\hat{\boldsymbol{\lambda}}$ direction that has the magnitude of the projection of $\vec{A}$ in that direction,

$$\vec{A}^{\parallel} = (\vec{A} \cdot \hat{\boldsymbol{\lambda}})\hat{\boldsymbol{\lambda}}.$$

The perpendicular part of $\vec{A}$ is just what you get when you subtract out the parallel part, namely,

$$\vec{A}^{\perp} = \vec{A} - \vec{A}^{\parallel} = \vec{A} - (\vec{A} \cdot \hat{\boldsymbol{\lambda}})\hat{\boldsymbol{\lambda}}$$

The claimed properties of the decomposition can now be checked, namely that $\vec{A} = \vec{A}^{\parallel} + \vec{A}^{\perp}$ (just add the 2 equations above and see), that $\vec{A}^{\parallel}$ is in the direction of $\hat{\boldsymbol{\lambda}}$ (its a scalar multiple), and that $\vec{A}^{\perp}$ is perpendicular to $\hat{\boldsymbol{\lambda}}$ (evaluate $\vec{A}^{\perp} \cdot \hat{\boldsymbol{\lambda}}$ and find 0).




**Example.** Given the positions $\vec{r}_A$, $\vec{r}_B$, and $\vec{r}_C$ of three points what is the position of the point D on the line AB that is closest to C? The answer is,

$$\vec{r}_D = \vec{r}_A + \vec{r}_{C/A}{}^{\parallel}$$

where $\vec{r}_{C/A}{}^{\parallel}$ is the part of $\vec{r}_{C/A}$ that is parallel to the line segment AB. Thus,

$$\vec{r}_D = \vec{r}_A + (\vec{r}_C - \vec{r}_A) \cdot \frac{\vec{r}_B - \vec{r}_A}{|\vec{r}_B - \vec{r}_A|}.$$

Likewise we could find the parts of a vector $\vec{A}$ in and perpendicular to a given plane. If the plane is defined by two vectors that are not necessarily orthogonal we could follow these steps. First find two vectors in the plane that are orthogonal, using the method above. Next subtract from $\vec{A}$ the part of it that is parallel to each of the two orthogonal vectors in the plane. In math lingo the execution of this process goes by the intimidating name 'Graham Schmidt orthogonalization.' The next section will show how to solve this geometry problem with cross products.

## A Given vector can be written as various sums and products

A vector $\vec{A}$ has many representations. The equivalence of different representations of a vector is partially analogous to the case of a dimensional scalar which has the same value no matter what units are used (*e.g.*, the mass $m = 4.41$ lbm is equal to $m = 2$ kg). Here are some common representations of vectors.

**Scalar times a unit vector in the vector's direction.** $\vec{F} = F\hat{\lambda}$ means the scalar $F$ multiplied by the unit vector $\hat{\lambda}$.

**Sum of orthogonal component vectors.** $\vec{F} = \vec{F}_x + \vec{F}_y$ is a sum of two vectors parallel to the $x$ and $y$ axes, respectively. In three dimensions, $\vec{F} = \vec{F}_x + \vec{F}_y + \vec{F}_z$.

**Components times unit base vectors.** $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath}$ or $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$ in three dimensions. One way to think of this sum is to realize that $\vec{F}_x = F_x\hat{\imath}$, $\vec{F}_y = F_y\hat{\jmath}$ and $\vec{F}_z = F_z\hat{k}$.

**Components times rotated unit base vectors.** $\vec{F} = F'_x\mathbf{i}' + F'_y\mathbf{j}'$ or $\vec{F} = F'_x\mathbf{i}' + F'_y\mathbf{j}' + F'_z\mathbf{k}'$ in three dimensions. Here the base vectors marked with primes, $\mathbf{i}'$, $\mathbf{j}'$ and $\mathbf{k}'$, are unit vectors parallel to some mutually orthogonal $x'$, $y'$, and $z'$ axes. These $x'$, $y'$, and $z'$ axes may be tilted in relation to the $x$, $y$, and $z$ axes. That is, the $x'$ axis need not be parallel to the $x$ axis, the $y'$ not parallel to the $y$ axis, and the $z'$ axis not parallel to the $z$ axis.

**Components times other unit base vectors.** If you use polar or cylindrical coordinates the unit base vectors are $\hat{\boldsymbol{e}}_\theta$ and $\hat{\boldsymbol{e}}_R$, so in 2-D , $\vec{\boldsymbol{F}} = F_R\hat{\boldsymbol{e}}_R + F_\theta\hat{\boldsymbol{e}}_\theta$ and in 3-D, $\vec{\boldsymbol{F}} = F_R\hat{\boldsymbol{e}}_R + F_\theta\hat{\boldsymbol{e}}_\theta + F_z\hat{\boldsymbol{k}}$. If you use 'path' coordinates, you will use the path-defined unit vectors $\hat{\boldsymbol{e}}_t$, $\hat{\boldsymbol{e}}_n$, and $\hat{\boldsymbol{e}}_b$ so in 2-D $\vec{\boldsymbol{F}} = F_t\hat{\boldsymbol{e}}_t + F_n\hat{\boldsymbol{e}}_n$. In 3-D $\vec{\boldsymbol{F}} = F_t\hat{\boldsymbol{e}}_t + F_n\hat{\boldsymbol{e}}_n + F_b\hat{\boldsymbol{e}}_b$.

**A list of components.** $[\vec{\boldsymbol{F}}]_{xy} = [F_x, F_y]$ or $[\vec{\boldsymbol{F}}]_{xyz} = [F_x, F_y, F_z]$ in three dimensions. This form coincides best with the way computers handle vectors. The row vector $[F_x, F_y]$ coincides with $F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}}$ and the row vector $[F_x, F_y, F_z]$ coincides with $F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}} + F_z\hat{\boldsymbol{k}}$.

In summary:

$$
\begin{aligned}
\vec{\boldsymbol{A}} &= \vec{\boldsymbol{A}} \\
&= |\vec{\boldsymbol{A}}|\hat{\boldsymbol{\lambda}}_A = A\hat{\boldsymbol{\lambda}}_A, && \text{where } \hat{\boldsymbol{\lambda}}_A \parallel \vec{\boldsymbol{A}},\ A = |\vec{\boldsymbol{A}}| \text{ and } |\hat{\boldsymbol{\lambda}}_A| = 1 \\
&= \vec{\boldsymbol{A}}_x + \vec{\boldsymbol{A}}_y + \vec{\boldsymbol{A}}_z && \text{where } \vec{\boldsymbol{A}}_x, \vec{\boldsymbol{A}}_y, \vec{\boldsymbol{A}}_z \text{ are parallel to the } x, y, z \text{ axes} \\
&= A_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\jmath}} + A_z\hat{\boldsymbol{k}}, && \text{where } \hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}} \text{ are parallel to the } x, y, z \text{ axes} \\
&= A_{x'}\hat{\boldsymbol{\imath}}' + A_{y'}\hat{\boldsymbol{\jmath}}' + A_{z'}\hat{\boldsymbol{k}}', && \text{where } \hat{\boldsymbol{\imath}}', \hat{\boldsymbol{\jmath}}', \hat{\boldsymbol{k}}' \text{ are } \parallel \text{ to skewed } x', y', z' \text{ axes} \\
&= A_R\hat{\boldsymbol{e}}_R + A_\theta\hat{\boldsymbol{e}}_\theta + A_z\hat{\boldsymbol{k}}, && \text{using cylindrical coordinate basis vectors.} \\
[\vec{\boldsymbol{A}}]_{xyz} &= [A_x, A_y, A_z] && [\vec{\boldsymbol{A}}]_{xyz} \text{ stands for the component list in } xyz \\
[\vec{\boldsymbol{A}}]_{x'y'z'} &= [A_{x'}, A_{y'}, A_{z'}] && [\vec{\boldsymbol{A}}]_{x'y'z'} \text{ stands for the component list in } x'y'z'
\end{aligned}
$$

## Vector algebra

Vectors are algebraic quantities and manipulated algebraically in equations. The rules for vector algebra are similar to the rules for ordinary (scalar) algebra. For example, if vector $\vec{\boldsymbol{A}}$ is the same as vector $\vec{\boldsymbol{B}}$, $\vec{\boldsymbol{A}} = \vec{\boldsymbol{B}}$, for any scalar $a$ and any vector $\vec{\boldsymbol{C}}$, we then

$$
\begin{aligned}
\vec{\boldsymbol{A}} + \vec{\boldsymbol{C}} &= \vec{\boldsymbol{B}} + \vec{\boldsymbol{C}}, \\
a\vec{\boldsymbol{A}} &= a\vec{\boldsymbol{B}}, \text{ and} \\
\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{C}} &= \vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{C}}
\end{aligned}
$$

because performing the same operation on equal quantities maintains the equality. The vectors $\vec{\boldsymbol{A}}$, $\vec{\boldsymbol{B}}$, and $\vec{\boldsymbol{C}}$ might themselves be expressions involving other vectors.

The equations above show the allowable manipulations of vector equations: adding a common term to both sides, multiplying both sides by a common scalar, taking the dot product of both sides with a common vector. All the distributive, associative, and commutative laws of ordinary addition and multiplication hold but for when there is no sensible meaning to the expressions*.

* **Beware.** It does not make sense to add a vector and a scalar; $7 + \vec{\boldsymbol{A}}$ is a *nonsense* expression. And you cannot divide a vector by a vector or a scalar by a vector; $7/\hat{\boldsymbol{\imath}}$ and $\vec{\boldsymbol{A}}/\vec{\boldsymbol{C}}$ are *nonsense* expressions.

## Dot products on the computer

Computer use for vector addition was discussed on page 25. Most computer languages will allow vector addition by commands something like this: and

dot products like this:

```
A   = [  1   2   5 ]
B   = [ -2   4 19 ]
D   = A(1)*B(1) + A(2)*B(2) + A(3)*B(3).
```

In our pseudo code we could write `D = A dot B`. Many computer languages have a shorter way to write the dot product like `dot(A,B)`. In a language built for linear algebra `D = A*B'`* will work because the rules of matrix multiplication are then the same as the component formula for the dot product.

* Recall that `B'` means the transpose of `B`, in this case turning the row of numbers `B` into a column of numbers.

## 2.3 *THEORY*

### *Using the geometric definition of the dot product to find the dot product in terms of components*

Vectors are essentially a geometric concept and we have consequently defined the dot product geometrically as $\vec{A}\cdot\vec{B} = ABcos\theta$. Almost 400 years ago René Descartes discovered that you could do geometry by doing algebra on the coordinates of points.

So we should be able to figure out the dot product of two vectors by knowing their components. The central key to finding this component formula is the distributive law $(\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C})$. If we write $\vec{A} = A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k}$ and $\vec{B} = B_x\hat{\imath}+B_y\hat{\jmath}+B_z\hat{k}$ then we just repeatedly use the distributive law as follows.

$$\begin{aligned}
\vec{A}\cdot\vec{B} &= (A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k})\cdot(B_x\hat{\imath}+B_y\hat{\jmath}+B_z\hat{k}) \\
&= (A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k})\cdot B_x\hat{\imath} + \\
&\quad (A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k})\cdot B_y\hat{\jmath} + \\
&\quad (A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k})\cdot B_z\hat{k} \\
&= A_xB_x\hat{\imath}\cdot\hat{\imath} + A_yB_x\hat{\jmath}\cdot\hat{\imath} + A_zB_x\hat{k}\cdot\hat{\imath} + \\
&\quad A_xB_y\hat{\imath}\cdot\hat{\jmath} + A_yB_y\hat{\jmath}\cdot\hat{\jmath} + A_zB_y\hat{k}\cdot\hat{\jmath} + \\
&\quad A_xB_z\hat{\imath}\cdot\hat{k} + A_yB_z\hat{\jmath}\cdot\hat{k} + A_zB_z\hat{k}\cdot\hat{k} \\
&= A_xB_x(1) + A_yB_x(0) + A_zB_x(0) + \\
&\quad A_xB_y(0) + A_yB_y(1) + A_zB_y(0) + \\
&\quad A_xB_z(0) + A_yB_z(0) + A_zB_z(1)
\end{aligned}$$

$$\Rightarrow \quad \vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z \qquad \text{(3D).}$$

$$\Rightarrow \quad \vec{A}\cdot\vec{B} = A_xB_x + A_yB_y \qquad \text{(2D).}$$

The demonstration above could have been carried out using a different orthogonal coordinate system $x'y'z'$ that was tilted with respect to the $xyz$ system. By identical reasoning we would find that $\vec{A}\cdot\vec{B} = A_{x'}B_{x'} + A_{y'}B_{y'} + A_{z'}B_{z'}$. Even though all of the numbers in the list $[A_x, A_y, A_z]$ might be different from the numbers in the list $[A_{x'}, A_{y'}, A_{z'}]$ and similarly all the list $[\vec{B}]_{xyz}$ might be different than the list $[\vec{B}]_{x'y'z'}$, so (somewhat remarkably),

$$A_xB_x + A_yB_y + A_zB_z = A_{x'}B_{x'} + A_{y'}B_{y'} + A_{z'}B_{z'}.$$

If we call our coordinate $x_1, x_2$, and $x_3$; and our unit base vectors $\hat{e}_1$,$\hat{e}_2$, and $\hat{e}_3$ we would have $\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$ and $\vec{B} = B_1\hat{e}_1 + B_2\hat{e}_2 + B_3\hat{e}_3$ and the dot product has the tidy form: $\vec{A}\cdot\vec{B} = A_1B_1 + A_2B_2 + A_3B_3 = \sum_{i=1}^{3} A_iB_i$.

**SAMPLE 2.12 Calculating dot products:** Find the dot product of the two vectors $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} - 2\hat{k}$ and $\vec{r} = 5\,\text{m}\hat{\imath} - 2\,\text{m}\hat{\jmath}$.

**Solution** The dot product of the two vectors is

$$\begin{aligned}
\vec{a}\cdot\vec{r} &= (2\hat{\imath} + 3\hat{\jmath} - 2\hat{k})\cdot(5\,\text{m}\hat{\imath} - 2\,\text{m}\hat{\jmath}) \\
&= (2\cdot 5\,\text{m})\underbrace{\hat{\imath}\cdot\hat{\imath}}_{1} - (2\cdot 2\,\text{m})\underbrace{\hat{\imath}\cdot\hat{\jmath}}_{0} \\
&\quad + (3\cdot 5\,\text{m})\underbrace{\hat{\jmath}\cdot\hat{\imath}}_{0} - (3\cdot 2\,\text{m})\underbrace{\hat{\jmath}\cdot\hat{\jmath}}_{1} \\
&\quad - (2\cdot 5\,\text{m})\underbrace{\hat{k}\cdot\hat{\imath}}_{0} + (2\cdot 2\,\text{m})\underbrace{\hat{k}\cdot\hat{\jmath}}_{0} \\
&= 10\,\text{m} - 6\,\text{m} \\
&= 4\,\text{m}.
\end{aligned}$$

$$\boxed{\vec{a}\cdot\vec{r} = 4\,\text{m}}$$

**Comments:** Note that with just a little bit of foresight, we could totally ignore the $\hat{k}$ component of $\vec{a}$ since $\vec{r}$ has no $\hat{k}$ component, *i.e.*, $\hat{k}\cdot\vec{r} = 0$. Also, if we keep in mind that $\hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{\imath} = 0$, we could compute the above dot product in one line:

$$\vec{a}\cdot\vec{r} = (2\hat{\imath} + 3\hat{\jmath})\cdot(5\,\text{m}\hat{\imath} - 2\,\text{m}\hat{\jmath}) = (2\cdot 5\,\text{m})\underbrace{\hat{\imath}\cdot\hat{\imath}}_{1} - (3\cdot 2\,\text{m})\underbrace{\hat{\jmath}\cdot\hat{\jmath}}_{1} = 4\,\text{m}.$$

---

**SAMPLE 2.13** What is the $y$-component of $\vec{F} = 5\,\text{N}\hat{\imath} + 3\,\text{N}\hat{\jmath} + 2\,\text{N}\hat{k}$ ?

**Solution** Although it is perhaps obvious that the $y$-component of $\vec{F}$ is 3 N, the scalar multiplying the unit vector $\hat{\jmath}$, we calculate it below in a formal way using the dot product between two vectors. We will use this method later to find components of vectors in arbitrary directions.

$$\begin{aligned}
F_y &= \vec{F}\cdot(\text{a unit vector along } y\text{-axis}) \\
&= (5\,\text{N}\hat{\imath} + 3\,\text{N}\hat{\jmath} + 2\,\text{N}\hat{k})\cdot\hat{\jmath} \\
&= 5\,\text{N}\underbrace{\hat{\imath}\cdot\hat{\jmath}}_{0} + 3\,\text{N}\underbrace{\hat{\jmath}\cdot\hat{\jmath}}_{1} + 2\,\text{N}\underbrace{\hat{k}\cdot\hat{\jmath}}_{0} \\
&= 3\,\text{N}.
\end{aligned}$$

$$\boxed{F_y = \vec{F}\cdot\hat{\jmath} = 3\,\text{N}.}$$

**SAMPLE 2.14 Finding angle between two vectors using dot product:** Find the angle between the vectors $\vec{r}_1 = 2\hat{\imath} + 3\hat{\jmath}$ and $\vec{r}_2 = 2\hat{\imath} - \hat{\jmath}$.

**Solution** From the definition of dot product between two vectors

$$\begin{aligned} \vec{r}_1 \cdot \vec{r}_2 &= |\vec{r}_1||\vec{r}_2| \cos\theta \\ \text{or} \qquad \cos\theta &= \frac{\vec{r}_1 \cdot \vec{r}_2}{|\vec{r}_1||\vec{r}_2|} \\ &= \frac{(2\hat{\imath} + 3\hat{\jmath}) \cdot (2\hat{\imath} - \hat{\jmath})}{(\sqrt{2^2 + 3^2})(\sqrt{2^2 + 1^2})} \\ &= \frac{4 - 3}{\sqrt{13}\sqrt{5}} = 0.124 \\ \text{Therefore,} \qquad \theta &= \cos^{-1}(0.124) = 82.87^\circ. \end{aligned}$$

$$\boxed{\theta = 83^\circ}$$

---

**SAMPLE 2.15 Finding direction cosines from unit vectors:** Find the angles (from direction cosines) between $\vec{F} = 4\,\text{N}\hat{\imath} + 6\,\text{N}\hat{\jmath} + 7\,\text{N}\hat{k}$ and each of the three axes.

**Solution**

$$\begin{aligned} \vec{F} &= F\hat{\lambda} \\ \hat{\lambda} &= \frac{\vec{F}}{F} \\ &= \frac{4\,\text{N}\hat{\imath} + 6\,\text{N}\hat{\jmath} + 7\,\text{N}\hat{k}}{\sqrt{4^2 + 6^2 + 7^2}\,\text{N}} \\ &= 0.4\hat{\imath} + 0.6\hat{\jmath} + 0.7\hat{k}. \end{aligned}$$

Let the angles between $\hat{\lambda}$ and the $x$, $y$, and $z$ axes be $\theta$, $\phi$ and $\psi$ respectively. Then

$$\begin{aligned} \cos\theta &= \frac{\hat{\imath} \cdot \hat{\lambda}}{|\hat{\imath}||\hat{\lambda}|} = \frac{0.4}{|1||1|} = 0.4. \\ \Rightarrow \quad \theta &= \cos^{-1}(0.4) = 66.4^\circ. \end{aligned}$$

Similarly,

$$\begin{aligned} \cos\phi &= 0.6 \quad \text{or} \quad \phi = 53.1^\circ \\ \cos\psi &= 0.7 \quad \text{or} \quad \psi = 45.6^\circ. \end{aligned}$$

$$\boxed{\theta = 66.4^\circ,\ \phi = 53.1^\circ,\ \psi = 45.6^\circ}$$

**Comments:** The components of a unit vector give the direction cosines with the respective axes. That is, if the angle between the unit vector and the $x$, $y$, and $z$ axes are $\theta$, $\phi$ and $\psi$, respectively (as above), then

$$\hat{\lambda} = \underbrace{\cos\theta}_{\lambda_x}\hat{\imath} + \underbrace{\cos\phi}_{\lambda_y}\hat{\jmath} + \underbrace{\cos\psi}_{\lambda_z}\hat{k}.$$

**SAMPLE 2.16 Projection of a vector in the direction of another vector:** Find the component of $\vec{F} = 5\,\text{N}\hat{\imath} + 3\,\text{N}\hat{\jmath} + 2\,\text{N}\hat{k}$ along the vector $\vec{r} = 3\,\text{m}\hat{\imath} - 4\,\text{m}\hat{\jmath}$.

**Solution** The dot product of a vector $\vec{a}$ with a unit vector $\hat{\lambda}$ gives the projection of the vector $\vec{a}$ in the direction of the unit vector $\hat{\lambda}$. Therefore, to find the component of $\vec{F}$ along $\vec{r}$, we first find a unit vector $\hat{\lambda}_r$ along $\vec{r}$ and dot it with $\vec{F}$.

$$\begin{aligned}
\hat{\lambda}_r &= \frac{\vec{r}}{|\vec{r}|} = \frac{3\,\text{m}\hat{\imath} - 4\,\text{m}\hat{\jmath}}{\sqrt{3^2+4^2}\,\text{m}} = 0.6\hat{\imath} - 0.8\hat{\jmath} \\
F_r &= \vec{F}\cdot\hat{\lambda}_r \\
&= (5\,\text{N}\hat{\imath} + 3\,\text{N}\hat{\jmath} + 2\,\text{N}\hat{k})\cdot(0.6\hat{\imath} - 0.8\hat{\jmath}) \\
&= 3.0\,\text{N} + 2.4\,\text{N} = 5.4\,\text{N}.
\end{aligned}$$

$$\boxed{F_r = 5.4\,\text{N}}$$

---

**SAMPLE 2.17** Assume that after writing the equation $\sum\vec{F} = m\vec{a}$ in a particular problem, a student finds $\sum\vec{F} = (20\,\text{N} - P_1)\hat{\imath} + 7\,\text{N}\hat{\jmath} - P_2\hat{k}$ and $\vec{a} = 2.4\,\text{m/s}^2\hat{\imath} + a_3\hat{\jmath}$. Separate the scalar equations in the $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ directions.

**Solution** From a vector equation, separating the scalar equations is trivial as long as both sides of a vector equation are in the same basis — individual components on both sides must equal. That is

$$\begin{aligned}
\overbrace{(20\,\text{N} - P_1)\hat{\imath} + 7\,\text{N}\hat{\jmath} - P_2\hat{k}}^{\sum\vec{F}} &= m\overbrace{(2.4\,\text{m/s}^2\hat{\imath} + a_3\hat{\jmath})}^{\vec{a}} \\
\Rightarrow \quad 20\,\text{N} - P_1 &= m(2.4\,\text{m/s}^2) && (\hat{\imath}\text{ component}) \\
7\,\text{N} &= ma_3 && (\hat{\jmath}\text{ component}) \\
-P_2 &= 0. && (\hat{k}\text{ component})
\end{aligned}$$

$$\boxed{20\,\text{N} - P_1 = m(2.4\,\text{m/s}^2), \quad 7\,\text{N} = ma_3,\ P_2 = 0}$$

**Comments:** The results obtained by equating individual components on both sides of the vector equation are based on the general technique of taking the dot product of both sides of an equation with a vector. It gives a scalar equation valid in any direction that one desires. For the example at hand, the long but easily readable and illustrative calculation is as follows. Taking the dot product of both sides of $\sum\vec{F} = m\vec{a}$ equation with $\hat{\imath}$, we write

$$\begin{aligned}
\hat{\imath}\cdot\left[(20\,\text{N} - P_1)\hat{\imath} + 7\,\text{N}\hat{\jmath} - P_2\hat{k}\right] &= m(2.4\,\text{m/s}^2\hat{\imath} + a_3\hat{\jmath})\Big] \\
\Rightarrow \quad \underbrace{(20\,\text{N} - P_1)}_{F_x}\underbrace{\hat{\imath}\cdot\hat{\imath}}_{1} + 7\,\text{N}\underbrace{\hat{\jmath}\cdot\hat{\imath}}_{0} - P - 2\underbrace{\hat{k}\cdot\hat{\imath}}_{0} &= m(\underbrace{2.4\,\text{m/s}^2}_{a_x}\underbrace{\hat{\imath}\cdot\hat{\imath}}_{1} + a_3\underbrace{\hat{\jmath}\cdot\hat{\imath}}_{0}) \\
\Rightarrow \quad 20\,\text{N} - P_1 = m(2.4\,\text{m/s}^2) & \qquad (i.e.,\ F_x = ma_x)
\end{aligned}$$

Similarly,

$$\hat{\jmath}\cdot\left[\sum\vec{F} = m\vec{a}\right] \quad \Rightarrow \quad 7\,\text{N} = ma_3 \quad \text{and} \quad \hat{k}\cdot\left[\sum\vec{F} = m\vec{a}\right] \quad \Rightarrow \quad -P_2 = 0.$$

# 2.3 Cross product, moment, and moment about an axis

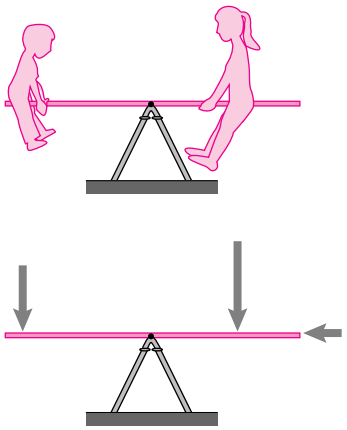



Figure 2.21: On a balanced teeter totter the bigger person gets the short end of the stick. A sideways force directed towards the hinge has no effect on the balance.

The poet Robert Burns suggested using a Teeter Totter to weigh hogs like this: 1) Get a perfetly symmetric plank and balance it accross a saw horse, 2) Put the hog on one end of the plank, 3) pile rocks on the other end of the plank until the plan is again perfectly balanced, 4) Carefully guess the weight of the rocks.

Filename:tfigure-teeter

When you try to move something you can push it and you can turn it. In mechanics, the measure of your pushing is the net force you apply. The measure of your turning is the net *moment*, also sometimes called the net *torque* or net *couple*. In this section we will define the moment of a force intuitively, geometrically, and finally using vector algebra. We will do this first in 2 dimensions and then in 3. The main mathematical tool here is the vector cross product, a second way of multiplying vectors together. The cross product is used to define (and calculate) moment and to calculate various quantities in dynamics. The cross product also sometimes helps solve three-dimensional geometry problems.

Although concepts involving moment (and rotation) are often harder for beginners than force (and translation), they were understood first. The ancient principle of the lever is the basic idea incorporated by moments. The principle of the lever can be viewed as the root of all mechanics.

Ultimately you can take on faith the vector definition of moment (given opposite the inside cover) and its role in eqs. II. But we can more or less deduce the definition by generalizing from common experience.

## Teeter totter mechanics

The two people weighing down on the teeter totter in Fig. 2.21 tend to rotate it about its hinge, the right one clockwise and the left one counterclockwise. We will now cook up a measure of the tendency of each force to cause rotation about the hinge and call it *the moment of the force about the hinge*.

As is verified a million times a year by young future engineering students, to balance a teeter-totter the smaller person needs to be further from the hinge. If two people are on one side then the teeter totter is balanced by two similar people an equal distance from the hinge on the other side. Two people can balance one similar person by scooting twice as close to the hinge. These proportionalities generalize to this: the tendency of a force to cause rotation is proportional to the size of the force and to its distance from the hinge (for forces perpendicular to the teeter totter).

If someone standing nearby adds a force that is directed towards the hinge it causes no tendency to rotate. Because any force can be decomposed into a sum of forces, one perpendicular to the teeter totter and the other towards the hinge, and because we assume that the affect of the sum of these forces is the sum of the affects of each separately, and because the force towards the hinge has no tendency to rotate, we have deduced:

> The moment of a force about a hinge is the product of its distance from the hinge and the component of the force perpendicular to the line from the hinge to the force.

Here then is the formula for 2D *moment about C* or *moment with respect to*

*The '/' in the subscript of $\vec{M}$ reads as 'relative to' or 'about'. For simplicity we often leave the / out and just write $\vec{M}_C$.

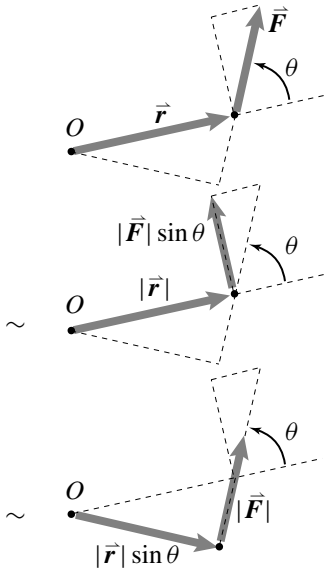



Figure 2.22: The moment of a force $|\vec{r}||\vec{F}|\sin\theta$ is either the product of its distance $|\vec{r}|$ with its perpendicular component $|\vec{F}|\sin\theta$ or of its lever arm (perpendicular distance) $|\vec{r}|\sin\theta$ and the full force $|\vec{F}|$. The $\sim$ indicates that the lower two forces and positions have the same moment as the force at top.

Filename:tfigure-slidevector

$C$.*

$$M_{/C} = |\vec{r}|\,(|\vec{F}|\sin\theta) = (|\vec{r}|\sin\theta)\,|\vec{F}|. \tag{2.3}$$

Here, $\theta$ is the angle between $\vec{r}$ (the position of the point of force application relative to the hinge) and $\vec{F}$ (see fig. 2.22). This formula for moment has all the teeter totter deduced properties. Moment is proportional to $r$, and to the part of $\vec{F}$ that is perpendicular to $\vec{r}$. The re-grouping as $(|\vec{r}|\sin\theta)$ shows that a force $\vec{F}$ has the same effect if it is applied at a new location that is displaced in the direction of $\vec{F}$. That is, the force $\vec{F}$ can slide along its length without changing its $M_{/C}$ and is *equivalent* in its effect on the teeter totter. The quantity $|\vec{r}|\sin\theta$ is sometimes called the *lever arm* of the force.

By common convention we define as positive a moment that causes a counterclockwise rotation. A moment that causes a clockwise rotation is negative. If we define $\theta$ appropriately then eqn. (2.3) obeys this sign convention. We define $\theta$ as the angle from the positive vector $\vec{r}$ to the positive vector $\vec{F}$ measured counterclockwise. Point the thumb of your right hand towards yourself. Point the fingers of your right hand along $\vec{r}$ and curl them towards the direction of $\vec{F}$ and see how far you have to rotate them. The force caused by the person on the left of the teeter totter has $\theta = 90^\circ$ so $\sin\theta = 1$ and the formula 2.3 gives a positive counterclockwise $M$. The force of the person on the right has $\theta = 270^\circ$ (3/4 of a revolution) so $\sin\theta = -1$ and the formula 2.3 gives a negative $M$.

In two dimensions moment is really a scalar concept, it is either positive or negative. In three dimensions moment is a vector. But even in 2D we find it easier to keep track of signs if we treat moment as a vector. In the $xy$ plane, the 2D moment is a vector in the $\hat{k}$ direction (straight out of the plane). So eqn. 2.3 becomes

$$\vec{M}_{/C} = |\vec{r}|\,|\vec{F}|\sin\theta\,\hat{k}. \tag{2.4}$$

If you curl the fingers of your *right hand* in the direction of rotation caused by a force your thumb points in the direction of the moment vector.

## The 2D cross product

The expression we have found for the right side of eqn. 2.4 is the 2D cross product of vectors $\vec{r}$ and $\vec{F}$. We can now apply the concept to any pair of vectors whether or not they represent force and position. The 2D cross product is defined as :

$$\underbrace{\vec{A}\times\vec{B}}_{\text{'A cross B'}} \stackrel{def}{=} |\vec{A}|\,|\vec{B}|\,\sin\theta\,\hat{k}. \tag{2.5}$$

where $\theta$ is the amount that $\vec{A}$ would need to be rotated counterclockwise to point in the same direction as $\vec{B}$. An equivalent alternative approach is to define the cross product as

$$\vec{A}\times\vec{B} \stackrel{def}{=} |\vec{A}|\,|\vec{B}|\,\sin\theta\,\hat{n}. \tag{2.6}$$

with $\theta$ defined to be less than $180^\circ$ and $\hat{\boldsymbol{n}}$ defined as the unit vector pointing in the direction of the thumb when the fingers are curled from the direction of $\vec{\boldsymbol{A}}$ towards the direction of $\vec{\boldsymbol{B}}$. For the $\vec{r}$ and $\vec{\boldsymbol{F}}$ on the right of the teeter totter this definition forces us to point our thumb into the plane (in the negative $\hat{\boldsymbol{k}}$ direction). With this definition $\sin\theta$ is always positive and the negative moments come from $\hat{\boldsymbol{n}}$ being in the $-\hat{\boldsymbol{k}}$ direction.

With a few sketches you could convince yourself that the definition of cross product in eqn.2.5 obeys these standard algebra rules (for any 3 2D vectors $\vec{\boldsymbol{A}}$, $\vec{\boldsymbol{B}}$, and $\vec{\boldsymbol{C}}$ and any scalar $d$):

$$\begin{aligned} d(\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}) &= (d\vec{\boldsymbol{A}}) \times \vec{\boldsymbol{B}} = \vec{\boldsymbol{A}} \times (d\vec{\boldsymbol{B}}) \\ \vec{\boldsymbol{A}} \times (\vec{\boldsymbol{B}} + \vec{\boldsymbol{C}}) &= \vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} + \vec{\boldsymbol{A}} \times \vec{\boldsymbol{C}}. \end{aligned}$$

A difference between the algebra rules for scalar multiplication and vector cross product multiplication is that for scalar multiplication $AB = BA$ whereas for the cross product $\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} \neq \vec{\boldsymbol{B}} \times \vec{\boldsymbol{A}}$ (because the definition of $\theta$ in eqn. 2.5 and $\hat{\boldsymbol{n}}$ in 2.6 depends on order). In particular $\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} = -\vec{\boldsymbol{B}} \times \vec{\boldsymbol{A}}$.

Because the magnitude of the cross product of $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ is the magnitude of $\vec{\boldsymbol{A}}$ times the magnitude of the projection of $\vec{\boldsymbol{B}}$ in the direction perpendicular to $\vec{\boldsymbol{A}}$ (as shown in the top two illustrations of fig. 2.22) you can think of the cross product as a measure of how much two vectors are perpendicular to each other. In particular

$$\begin{aligned} \text{if } \vec{\boldsymbol{A}} \perp \vec{\boldsymbol{B}} \quad &\Rightarrow \quad |\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}| = |\vec{\boldsymbol{A}}|\,|\vec{\boldsymbol{B}}|, \text{ and} \\ \text{if } \vec{\boldsymbol{A}} \parallel \vec{\boldsymbol{B}} \quad &\Rightarrow \quad |\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}| = \vec{\boldsymbol{0}}. \end{aligned}$$

For example, $\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}}$, $\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\imath}} = -\hat{\boldsymbol{k}}$, $\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}} = \vec{\boldsymbol{0}}$, and $\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}} = \vec{\boldsymbol{0}}$.

## Component form for the 2D cross product

Just like the dot product, the cross product can be expressed using components. As can be verified by writing $\vec{\boldsymbol{A}} = A_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\jmath}}$, and $\vec{\boldsymbol{B}} = B_x\hat{\boldsymbol{\imath}} + B_y\hat{\boldsymbol{\jmath}}$ and using the distributive rules:

$$\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} \quad = (A_xB_y - B_xA_y)\hat{\boldsymbol{k}}. \tag{2.7}$$

Some people remember this formula by putting the components of $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ into a matrix and calculating the determinant $\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$. If you number the components of $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ (*e.g.*, $[\vec{\boldsymbol{A}}]_{x_1x_2} = [A_1, A_2]$), the cross product is $\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} \quad = (A_1B_2 - B_2A_1)\hat{\boldsymbol{e}}_3$. This you might remember as "first times second minus second times first."

**Example**: Given that $\vec{\boldsymbol{A}} = 1\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}$ and $\vec{\boldsymbol{B}} = 10\hat{\boldsymbol{\imath}} + 20\hat{\boldsymbol{\jmath}}$ then $\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} = (1 \cdot 20 - 2 \cdot 10)\hat{\boldsymbol{k}} = 0\hat{\boldsymbol{k}} = \vec{\boldsymbol{0}}$.

For vectors with just a few components it is often most convenient to use the distributive rule directly.

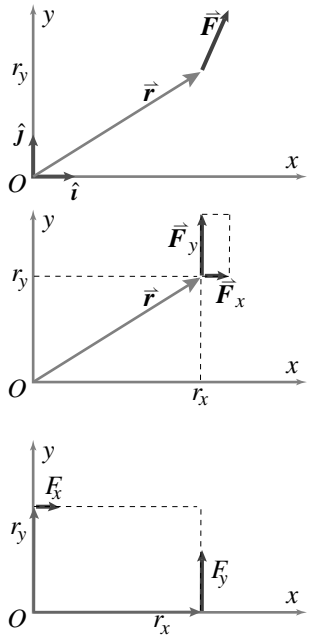



Figure 2.23: The component form of the 2D moment can be found by sequentially breaking the force into components, sliding each component along its line of action to the $x$ and $y$ axis, and adding the moments of the two components.

Filename:tfigure1-2Dcrosscomps

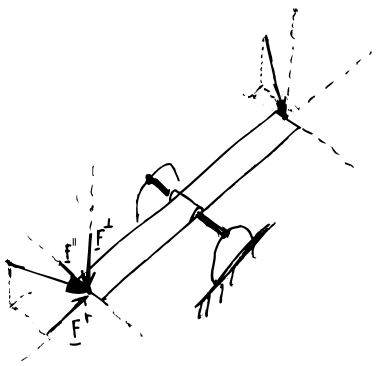


Figure 2.24: Teeter totter with applied forces broken into components parallel to the axis $\vec{F}^{\parallel}$, radial $\vec{F}^{r}$, and perpendicular to the plane containing the axis and the point of force application $\vec{F}^{\perp}$.

Filename:tfigure-3Dteeter

**Example**: Given that $\vec{A} = 7\hat{\imath}$ and $\vec{B} = 37.6\hat{\imath} + 10\hat{\jmath}$ then $\vec{A} \times \vec{B} = (7\hat{\imath}) \times (37.6\hat{\imath} + 10\hat{\jmath}) = (7\hat{\imath}) \times (37.6\hat{\imath}) + (7\hat{\imath}) \times (10\hat{\jmath}) = \vec{0} + 70\hat{k} = 70\hat{k}$.

### There are many ways of calculating a 2D cross product

You have several options for calculating the 2D cross product. Which you choose depends on taste and convenience. You can use the geometric definition directly, the first times the perpendicular part of the second (distance times perpendicular component of force), the second times the perpendicular part of the first (lever arm times the force), components, or break each of the vectors into a sum of vectors and use the distributive rule.

## 2D moment by components

We can use the component form of the 2D cross product to find a component form for the moment $\vec{M}_{/C}$ of eqn. 2.4. Given $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath}$ acting at P, where $\vec{r}_{P/C} = r_x\hat{\imath} + r_y\hat{\jmath}$, the moment of the force about C is

$$\vec{M}_{/C} = (r_x F_y - r_y F_x))\hat{k}$$

or the moment of $\vec{F}$ about the axis at C is

$$M_C = r_x F_y - r_y F_x. \tag{2.8}$$

We can derive this component formula with the sequence of vector manipulations shown graphically in fig. 2.23.

## 3D moment about an axis

The concept of moment about an axis is historically, theoretically, and practically important. Moment about an axis describes the principle of the lever, which far precedes Newton's laws. The net moment of a force system about enough different axes determines everything needed in mechanics about a force system. And one can sometimes quickly solve a statics or dynamics problem by considering moment about a judiciously chosen axis.

Lets start by thinking about a teeter totter again. Looking from the side we thought of a teeter totter as a 2D system. But the teeter totter really lives in the 3D world (see Fig. 2.24). We now re-interpret the 2D moment $M$ as the moment of the 2D forces about the $\hat{k}$ axis of rotation at the hinge. It is plain that a force $\vec{F}^{\parallel}$ pushing a teeter totter parallel to the axle causes no tendency to rotate. And we already agreed that a radial force $\vec{F}^{r}$ causes no rotation. So we see that the moment a force causes about an axis is the distance of the force from the axis times the part of the force that is neither parallel to the axis nor directed towards the axis.

Now look at this in the more 3-dimensional context of fig. 2.25. Here an imagined axis of rotation is defined as the line through C that is in the

$\hat{\boldsymbol{\lambda}}$ direction. A force $\vec{F}$ is applied at P. We can break $\vec{F}$ into a sum of three vectors

$$\vec{F} = \vec{F}^{\parallel} + \vec{F}^{r} + \vec{F}^{\perp}$$

where $\vec{F}^{\parallel}$ is parallel to the axis, $\vec{F}^{r}$ is directed along the shortest connection between the axis and P (and is thus perpendicular to the axis) and $\vec{F}^{\perp}$ is out of the plane defined by C, P and $\hat{\boldsymbol{\lambda}}$. By analogy with the teeter totter we see that $\vec{F}^{r}$ and $\vec{F}^{\parallel}$ cause no tendency to rotate about the axis. So only the $\vec{F}^{\perp}$ contributes.

**Example:** Try this. Stand facing a partially open door with the front of your body parallel to the plane of the door (a door with no springs is best). Hold the outer edge of the door with one hand. Press down and note that the door is not opened or closed. Push towards the hinge and note that the door is not opened or closed. Push and pull away and towards your body and note how easily you cause the door to rotate. Thus the only force component that tends to rotate the door is perpendicular to the plane of the door (which is the plane of the hinge and line from the hinge to your hand). Now move your hand to the middle of the door, half the distance from the hinge. Note that it takes more force to rotate the door with the same authority (push with your pinky if you have trouble feeling the difference).

Thus the only potent force for rotation is perpendicular to the plane of the hinge and point of force application, and its potence is increased with distance from the hinge.

We can also decompose $\vec{r} = \vec{r}_{\mathrm{P/C}}$ into two parts, one parallel to the hinge and one radial, as

$$\vec{r} = \vec{r}^{\parallel} + \vec{r}^{r}.$$

Clearly $\vec{r}^{\parallel}$ has no affect on how much rotation $\vec{F}$ causes about the axis. If for example the point of force application was moved parallel to the axis a few centimeters, the tendency to rotate would not be changed. Altogether, we have that the moment of the force $\vec{F}$ about the axis $\hat{\boldsymbol{\lambda}}$ through C is given by

$$M_{\lambda \mathrm{C}} = r^{r}\ F^{\perp}.$$

The perpendicular distance from the axis to the point of force application is $|\vec{r}^{\,r}|$ and $\vec{F}^{\perp}$ is the part of the force that causes right-handed rotation about the axis. A moment about an axis is defined as positive if curling the fingers of your right hand gives the sense of rotation when your outstretched thumb is pointing along the axis (as in fig. 2.25). The force of the left person on the teeter totter causes a positive moment about the $\hat{\boldsymbol{k}}$ axis through the hinge.

So long as you interpret the quantities correctly, the freshman physics line

*"Moment is distance ($|\vec{r}^{\,r}|$) times force ($|\vec{F}^{\perp}|$)"*

perfectly defines moment about an axis.

Three dimensional geometry is difficult, so a formula for moment about an axis in terms of components would be most useful. The needed formula depends on the 3D moment vector defined by the 3D cross product which we introduce now.

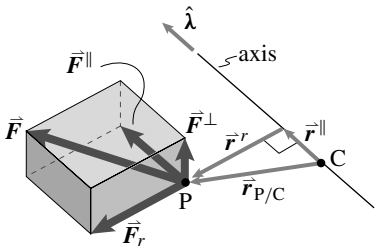



Figure 2.25: Moment about an axis

Filename:tfigure2-mom-axis

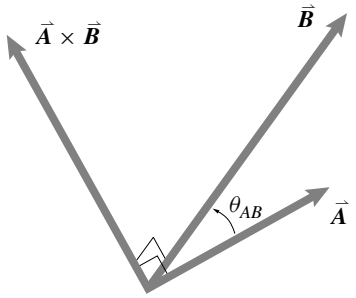



Figure 2.26: The cross product of $\vec{A}$ and $\vec{B}$ is perpendicular to $\vec{A}$ and $\vec{B}$ in the direction given by the right hand rule. The magnitude of $\vec{A} \times \vec{B}$ is $AB \sin\theta_{AB}$.

Filename:tfigure1-12

## The 3D cross product (or vector product)

The cross product of two vectors $\vec{A}$ and $\vec{B}$ is written $\vec{A} \times \vec{B}$ and pronounced 'A cross B.' In contrast to the dot product, which gives a scalar and measures

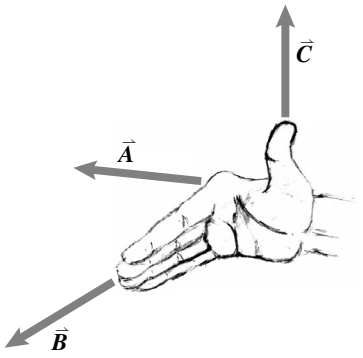



Figure 2.27: The right hand rule for determining the direction of the cross product of two vectors. $\vec{C} = \vec{A} \times \vec{B}$.

Filename:tfigure-rhr

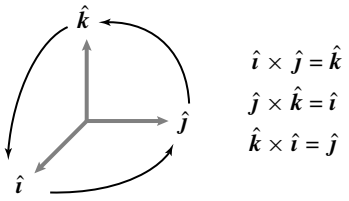



Figure 2.28: Mnemonic device to remember the cross product of the standard base unit vectors.

Filename:tfigure1-e

how much two vectors are parallel, the cross product is a vector and measures how much they are perpendicular. The cross product is also called *the vector product*.

The cross product is defined by:

$$\begin{aligned} \vec{A} \times \vec{B} \quad &\stackrel{def}{=} \quad |\vec{A}||\vec{B}| \sin\theta_{AB}\, \hat{n} \\ &\text{where} \quad |\hat{n}| = 1, \\ &\qquad \hat{n} \perp \vec{A}, \\ &\qquad \hat{n} \perp \vec{B}, \\ &\qquad 0 \le \theta_{AB} \le \pi, \text{ and} \\ &\qquad \hat{n} \text{ is in the direction given by the right hand rule, that is, in the direction of the right thumb when the fingers of the right hand are pointed in the direction of } \vec{A} \text{ and then wrapped towards the direction of } \vec{B}. \end{aligned}$$

If $\vec{A}$ and $\vec{B}$ are perpendicular then $\theta_{AB}$ is $\pi/2$, $\sin\theta_{AB} = 1$, and the magnitude of the cross product is $AB$. If $\vec{A}$ and $\vec{B}$ are parallel then $\theta_{AB}$ is 0, $\sin\theta_{AB} = 0$ and the cross product is $\vec{0}$ (the zero vector). This is why we say the cross product is a measure of the degree of orthogonalityof two vectors.

Using the definition above you should be able to verify to your own satisfaction that $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$. Applying the definition to the standard base unit vectors you can see that $\hat{\imath} \times \hat{\jmath} = \hat{k}$, $\hat{\jmath} \times \hat{k} = \hat{\imath}$, and $\hat{k} \times \hat{\imath} = \hat{\jmath}$ (figure 2.28).

The geometric definition above and the geometric (tip to tale) definition of vector addition imply that the cross product follows the distributive rule (see box 2.4 on page 56).

$$\vec{A} \times \left(\vec{B} + \vec{C}\right) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}.$$

Applying the distributive rule to the cross products of $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$ leads to the algebraic formula for the Cartesian components of the cross product.

$$\begin{aligned} \vec{A} \times \vec{B} \quad = \quad & [A_yB_z - A_zB_y]\hat{\imath} \\ & +[A_zB_x - A_xB_z]\hat{\jmath} \\ & +[A_xB_y - A_yB_x]\hat{k} \end{aligned}$$

There are various mnemonics for remembering the component formula for cross products. The most common is to calculate a 'determinant' of the $3 \times 3$ matrix with one row given by $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ and the other two rows the components

of $\vec{A}$ and $\vec{B}$.

$$\vec{A} \times \vec{B} = \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

The following identities and special cases of cross products are worth knowing well:

- $(a\vec{A}) \times \vec{B} = \vec{A} \times (a\vec{B}) = a(\vec{A} \times \vec{B})$ (a distributive law)
- $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ (the cross product is not commutative!)
- $\vec{A} \times \vec{B} = \vec{0}$ if $\vec{A} \parallel \vec{B}$ (parallel vectors have zero cross product)
- $|\vec{A} \times \vec{B}| = AB$ if $\vec{A} \perp \vec{B}$
- $\hat{\imath} \times \hat{\jmath} = \hat{k}$, $\hat{\jmath} \times \hat{k} = \hat{\imath}$, $\hat{k} \times \hat{\imath} = \hat{\jmath}$ (assuming the $x$, $y$, $z$ coordinate system is right handed — if you use your right hand and point your fingers along the positive $x$ axis, then curl them towards the positive $y$ axis, your thumb will point in the same direction as the positive $z$ axis. )
- $\hat{\imath}' \times \hat{\jmath}' = \hat{k}'$, $\hat{\jmath}' \times \hat{k}' = \hat{\imath}'$, $\hat{k}' \times \hat{\imath}' = \hat{\jmath}'$ (assuming the $x'y'z'$ coordinate system is also right handed.)
- $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$, $\hat{\imath}' \times \hat{\imath}' = \hat{\jmath}' \times \hat{\jmath}' = \hat{k}' \times \hat{k}' = \vec{0}$

## The moment vector

We now define the moment of a force $\vec{F}$ applied at P, relative to point C as

$$\vec{M}_{\mathrm{C}} = \vec{r}_{\mathrm{P/C}} \times \vec{F}$$

which we read in short as 'M is r cross F.' The moment vector is admittedly a difficult idea to intuit. A look at its components is helpful.

$$\vec{M}_{\mathrm{C}} = (r_y F_z - r_z F_y)\hat{\imath} + (r_z F_x - r_x F_z)\hat{\jmath} + (r_x F_y - r_y F_x)\hat{k}$$

You can recognize the $z$ component of the moment vector as the moment of the force about the $\hat{k}$ axis through C (eqn. 2.8). Similarly the $x$ and $y$ components of $\vec{M}_{\mathrm{C}}$ are the moments about the $\hat{\imath}$ and $\hat{\jmath}$ axis through C. So at least the components of $\vec{M}_{\mathrm{C}}$ have intuitive meaning. They are the moments around the positive $x$, $y$, and $z$ axes respectively.

Starting with this moment-about-the-coordinate-axes interpretation of the moment vector, each of the three components can be deduced graphically by the moves shown in fig. 2.30. The force is first broken into components. The components are then moved along their lines of action to the coordinate planes. From the resulting picture you can see, say, that the moment about the positive $y$ axis gets a positive contribution from $F_x$ with lever arm $r_z$ and a negative contribution from $F_z$ with lever arm $r_x$. Thus the $y$ component of $\vec{M}$ is $r_z F_x - r_x F_z$.

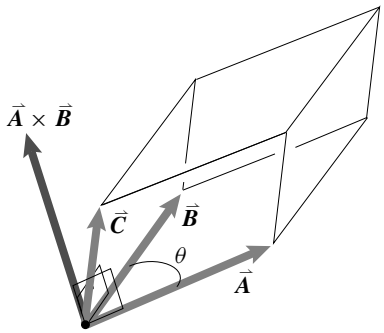



Figure 2.29: One interpretation of the mixed triple product of $\vec{A} \times \vec{B} \cdot \vec{C}$ is as the volume (a scalar) of a parallelepiped with $\vec{A}$, $\vec{B}$, and $\vec{C}$ as the three edges emanating from one corner. This interpretation only works if $\vec{A}$, $\vec{B}$, and $\vec{C}$ are taken in the appropriate order, otherwise $\vec{A} \times \vec{B} \cdot \vec{C}$ is minus the volume which is calculated.

Filename:tfigure1-13

*In the language of linear algebra, the mixed triple product of three vectors is zero if the vectors are linearly dependent.

## The mixed triple product

The 'mixed triple product' of $\vec{A}$, $\vec{B}$, and $\vec{C}$ is so called because it mixes both the dot product and cross product in a single expression. The mixed triple product is also sometimes called the scalar triple product because its value is a scalar. The mixed triple product is useful for calculating the moment of a force about an axis and for related dynamics quantities.The *mixed triple product* of $\vec{A}$, $\vec{B}$, and $\vec{C}$ is defined by and written as

$$\vec{A} \cdot \left( \vec{B} \times \vec{C} \right)$$

and pronounced 'A dot B cross C.' The parentheses () are sometimes omitted (*i.e.*, ., $\vec{A} \cdot \vec{B} \times \vec{C}$) because the wrong grouping $(\vec{A} \cdot \vec{B}) \times \vec{C}$ is nonsense (you can't take the cross product of a scalar with a vector) . It is apparent that one way of calculating the mixed triple product is to calculate the cross product of $\vec{B}$ and $\vec{C}$ and then to take the dot product of that result with $\vec{A}$. Some people use the notation $\left( \vec{A}, \vec{B}, \vec{C} \right)$ for the mixed triple product but it will not occur again in this book.

The mixed triple product has the same value if one takes the cross product of $\vec{A}$ and $\vec{B}$ and then the dot product of the result with $\vec{C}$. That is $\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$. This identity can be verified using the geometric description below, or by looking at the (complicated) expression for the mixed triple product of three general vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ in terms of their components as calculated the two different ways. One thus obtains the string of results

$$\vec{A} \cdot \vec{B} \times \vec{C} = \vec{A} \times \vec{B} \cdot \vec{C} = -\vec{B} \times \vec{A} \cdot \vec{C} = -\vec{B} \cdot \vec{A} \times \vec{C} = \ldots$$

The minus signs in the above expressions follow from the cross product identity that $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$.

The mixed triple product has various geometric interpretations, one of them is that $\vec{A} \cdot \vec{B} \times \vec{C}$ is (plus or minus) the volume of the parallelepiped, the crooked shoe box, edged by $\vec{A}$, $\vec{B}$ and $\vec{C}$ as shown in figure 2.29.

Another way of calculating the value of the mixed triple product is with the determinant of a matrix whose rows are the components of the vectors.

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \begin{array}{r} A_x(B_y C_Z - B_z C_y) \\ +A_y(B_z C_x - B_x C_z) \\ +A_z(B_x C_y - B_y C_x) \end{array}$$

The mixed triple product of three vectors is zero if*

- any two of them are parallel, or
- if all three of the vectors have one common plane.

A different triple product, sometimes called the vector triple product, is defined by $\vec{A} \times (\vec{B} \times \vec{C})$. It is discussed later in the text when it is needed (see box **??** in section **??** on page **??**).

## More on moment about an axis

We defined moment about an axis geometrically using fig. 2.25 on page 49 as $M_{\hat{\lambda}} = r^r F^\perp$. We can now verify that the mixed triple product gives the desired result by guessing the formula and seeing that it agrees with the geometric definition.

$$M_{\lambda C} = \hat{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{M}}_{/C} \quad \text{(An inspired guess...)} \tag{2.9}$$

We break both $\vec{r}$ and $\vec{F}$ into sums indicated in the figure, use the distributive law, and note that the mixed triple product gives zero if any two of the vectors are parallel. Thus,

$$\begin{aligned}
\hat{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{M}}_{/C} &= \hat{\boldsymbol{\lambda}} \cdot \vec{r}_{P/C} \times \vec{\boldsymbol{F}} \\
&= \hat{\boldsymbol{\lambda}} \cdot (\vec{r}^{\,r} + \vec{r}^{\,\parallel}) \times (\vec{F}^{\perp} + \vec{F}^{\parallel} + \vec{F}^{\,r}) \\
&= \hat{\boldsymbol{\lambda}} \cdot \vec{r}^{\,r} \times \vec{F}^{\perp} + \hat{\boldsymbol{\lambda}} \cdot \vec{r}^{\,r} \times \vec{F}^{\parallel} + \hat{\boldsymbol{\lambda}} \cdot \vec{r}^{\,r} \times \vec{F}^{\,r} \ldots \\
&\quad + \hat{\boldsymbol{\lambda}} \cdot \vec{r}^{\,\parallel} \times \vec{F}^{\perp} + \hat{\boldsymbol{\lambda}} \cdot \vec{r}^{\,\parallel} \times \vec{F}^{\parallel} + \hat{\boldsymbol{\lambda}} \cdot \vec{r}^{\,\parallel} \times \vec{F}^{\,r} \\
&= r^r F^\perp + 0 + 0 + 0 + 0 + 0 \\
&= r^r F^\perp. \quad (\ldots \text{and a good guess too.})
\end{aligned}$$

We can calculate the cross and dot product in any convenient way, say using vector components.

Example: **Moment about an axis**

Given a force, $\vec{F}_1 = (5\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 4\hat{\boldsymbol{k}})$ N acting at a point $P$ whose position is given by $\vec{r}_{P/O} = (3\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} - 2\hat{\boldsymbol{k}})$ m, what is the moment about an axis through the origin $O$ with direction $\hat{\boldsymbol{\lambda}} = \frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}} + \frac{1}{\sqrt{2}}\hat{\boldsymbol{k}}$?

$$\begin{aligned}
M_{\hat{\lambda}} &= (\vec{r}_{P/O} \times \vec{F}_1) \cdot \hat{\boldsymbol{\lambda}} \\
&= [(3\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} - 2\hat{\boldsymbol{k}})\,\text{m} \times (5\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 4\hat{\boldsymbol{k}})\,\text{N}] \cdot (\frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}} + \frac{1}{\sqrt{2}}\hat{\boldsymbol{k}}) \\
&= -\frac{17}{\sqrt{2}}\,\text{m N}.
\end{aligned}$$

The power of our abstract reasoning is apparent when we consider calculating the moment of a force about an axis with two different coordinate systems. Each of the vectors in eqn. 2.3 will have different components in the different systems. Yet the resulting scalar, after all the arithmetic, will be the same no matter what the coordinate system.

Finally, the moment about an axis gives us an interpretation of the moment vector. The direction of the moment vector $\vec{\boldsymbol{M}}_C$ is the direction of the axis through C about which $\vec{\boldsymbol{F}}$ has the greatest moment. The magnitude of $\vec{\boldsymbol{M}}_C$ is the moment of $\vec{\boldsymbol{F}}$ about that axis.

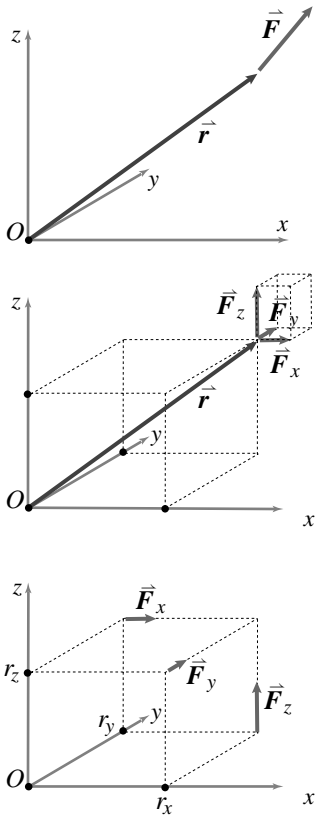



Figure 2.30: The three components of the 3D moment vector are the moments about the three axis. These can be found by sequentially breaking the force into components, sliding each component along its line of action to the coordinate planes, and noting the contribution of each component to moment about each axis. See Fig. 2.23 on page 48 for the 2D version of this construction.

Filename:tfigure1-3Dcrosscomps

### Special optional ways to draw moment vectors

Neither of the special rotation notations below is needed because moment is a vector like any other. The same is true for the angular velocity vector and the angular momentum vector in dynamics. None-the-less, sometimes people like to use a notation that suggests the rotational nature of these quantities.

**Arced arrow for 2-D moment and angular velocity.** In 2D problems in the $xy$ plane, the relevant moment, angular velocity, and angular momentum point straight out or into the plane in the $z$ ($\hat{\boldsymbol{k}}$) direction. A way of drawing this is to use an arced arrow. Wrap the fingers of your right hand in the direction of the arc and your thumb points in the direction of the unit vector that the scalar multiplies. The three representations in Fig. 2.31a indicate the same moment vector.

**Double headed arrow for 3-D rotations and moments.** Some people like to distinguish vectors for rotational motion and torque from other vectors. Two ways of making this distinction are to use double-headed arrows or to use an arrow with an arced arrow around it as shown in Fig. 2.31b.

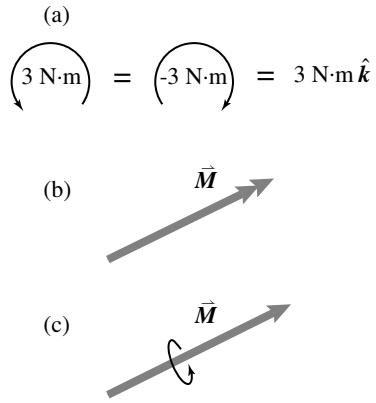



Figure 2.31: Optional drawing methods for moment vectors. (a) shows an arced arrow to represent vectors having to do with rotation in 2 dimensions. Such vectors point directly out of, or into, the page so are indicated with an arc in the direction of the rotation. (b) shows a double-headed arrow for torque or rotation quantity in three dimensions. (c) is another notation for a 3D vector quantity that uses rotations. But a vector is a vector and no special notation is usually used for vectors that concern turning or rotation.

Filename:tfigure1rot-d

## Cross products and computers

The components of the cross product can be calculated with computer code that may look something like this.

```
A = [  1  2  5 ]
B = [ -2  4 19 ]
C = [ ( A(2)*B(3) - A(3)*B(2) ) ...
      ( A(3)*B(1) - A(1)*B(3) ) ...
      ( A(1)*B(2) - A(2)*B(1) ) ]
```

giving the result `C=[18 -29 8]`. Many computer languages have a shorter way to write the cross product like `cross(A,B)`. The mixed triple product might be calculated by assembling a $3 \times 3$ matrix of rows and then taking a determinant like this:

```
      A   = [  1  2  5 ]
      B   = [ -2  4 19 ]
      C   = [ 32  4  5 ]
  matrix  = [A ;  B ; C]
mixedprod = det(matrix)
```

giving the result `mixedprod = 500`. A versatile language might well allow the command `dot( A, cross(B,C) )` to calculate the mixed triple product.

**SAMPLE 2.18 Cross product in 2-D:** Two vectors $\vec{a}$ and $\vec{b}$ of length 10 ft and 6 ft, respectively, are shown in the figure. The angle between the two vectors is $\theta = 60°$. Find the cross product of the two vectors.

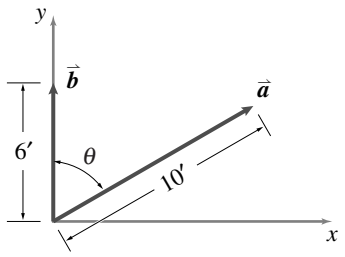



Figure 2.32:
Filename:sfig2-vec2-cross1

**Solution** Both vectors $\vec{a}$ and $\vec{b}$ are in the $xy$ plane. Therefore, their cross product is,

$$\begin{aligned}\vec{a}\times\vec{b} &= |\vec{a}||\vec{b}|\sin\theta\hat{n}\\ &= (10\,\text{ft})\cdot(6\,\text{ft})\cdot\sin 60°\hat{k}\\ &= 60\,\text{ft}^2\cdot\frac{\sqrt{3}}{2}\hat{k}\\ &= 30\sqrt{3}\,\text{ft}^2\hat{k}.\end{aligned}$$

$$\boxed{\vec{a}\times\vec{b} = 30\sqrt{3}\,\text{ft}^2\hat{k}}$$

**SAMPLE 2.19 Computing 2-D cross product in different ways:** The two vectors shown in the figure are $\vec{a} = 2\hat{\imath} - \hat{\jmath}$ and $\vec{b} = 4\hat{\imath} + 2\hat{\jmath}$. The angle between the two vectors is $\theta = \sin^{-1}(4/5)$ (this information can be found out from the given vectors). Find the cross product of the two vectors

1. using the angle $\theta$, and
2. using the components of the vectors.

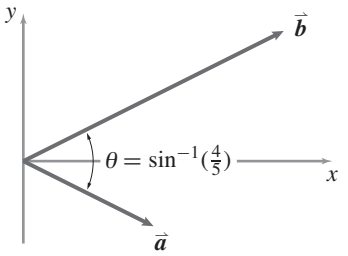



Figure 2.33:
Filename:sfig2-vec2-cross2

**Solution**

1. Cross product using the angle $\theta$:

$$\begin{aligned}\vec{a}\times\vec{b} &= |\vec{a}||\vec{b}|\sin\theta\hat{n}\\ &= |2\hat{\imath}-\hat{\jmath}||4\hat{\imath}+2\hat{\jmath}|\cdot\sin(\sin^{-1}\frac{4}{5})\hat{k}\\ &= (\sqrt{2^2+1^2})(\sqrt{4^2+2^2})\cdot\frac{4}{5}\hat{k}\\ &= \sqrt{5}\cdot\sqrt{20}\cdot\frac{4}{5}\hat{k} = 10\cdot\frac{4}{5}\hat{k}\\ &= 8\hat{k}.\end{aligned}$$

2. Cross product using components:

$$\begin{aligned}\vec{a}\times\vec{b} &= (2\hat{\imath}-\hat{\jmath})\times(4\hat{\imath}+2\hat{\jmath})\\ &= 2\hat{\imath}\times(4\hat{\imath}+2\hat{\jmath}) - \hat{\jmath}\times(4\hat{\imath}+2\hat{\jmath})\\ &= 8\underbrace{\hat{\imath}\times\hat{\imath}}_{\vec{0}} + 4\underbrace{\hat{\imath}\times\hat{\jmath}}_{\hat{k}} - 4\underbrace{\hat{\jmath}\times\hat{\imath}}_{-\hat{k}} - 2\underbrace{\hat{\jmath}\times\hat{\jmath}}_{\vec{0}}\\ &= 4\hat{k}+4\hat{k}\\ &= 8\hat{k}.\end{aligned}$$

The answers obtained from the two methods are, of course, the same as they must be.

$$\boxed{\vec{a}\times\vec{b} = 8\hat{k}}$$

## 2.4 *THEORY*

*The 3D cross product is distributive over sums; this allows calculation with components.*

In 3D the component formula for the cross product

$$[\vec{A}\times\vec{B}]_{xyz} = [(A_yB_z - A_zB_y),\ (A_zB_x - A_xB_z),\ (A_xB_y - A_yB_x)]$$

is much more often used than the geometric definition that

$$\vec{A}\times\vec{B} \equiv |\vec{A}||\vec{B}|\sin\theta_{AB} * \hat{\boldsymbol{n}}_{\perp AB},$$

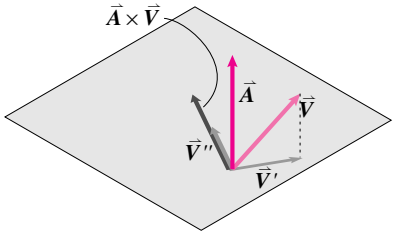
where $\theta_{AB} < \pi$ is the angle between $\vec{A}$ and $\vec{B}$ and $\hat{\boldsymbol{n}}_{\perp AB}$ is the unit vector orthogonal to $\vec{A}$ and $\vec{B}$ in the direction given by the right hand rule, rotating from $\vec{A}$ to $\vec{B}$. Why do these two different-looking formulas above describe the same vector? Starting with the geometric definition can we derive the component formula? Obviously the answer is yes, we wouldn't use two formulas for the same concept if they didn't agree. Why?

First we will show that the geometric definitions of the vector cross product and vector addition obey the distributive rule

$$\vec{A} \times \left(\vec{B}+\vec{C}\right) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}.$$

in which multiplacation (vector cross product) gets distributed over (to) the terms in the sum, one by one. Then we apply the distributive rule to the component reprentation of the vectors in a cross product. Thus yielding our component formula. Here goes. Hold on to your hat.

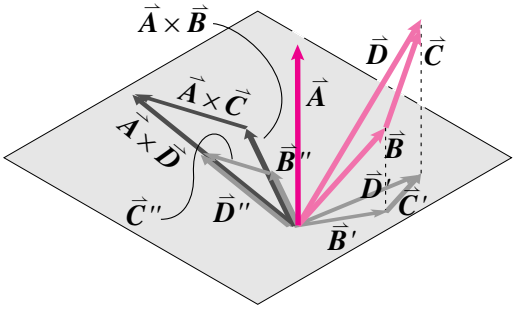
**The 'project, then rotate, then stretch' definition of cross product.** First lets find an equivalent geometric definition for the cross product of vectors $\vec{A}$ and $\vec{V}$. Look at a plane P that is perpendicular to $\vec{A}$.




Now look at $\vec{V}'$, the projection of $\vec{V}$ on to P. The right-hand-rule normal of $\vec{A}$ and $\vec{V}$ is the same as the common normal of $\vec{A}$ and $\vec{V}'$. The magnitude of the projection is $|\vec{V}'| = |\vec{V}|\sin\theta_{AV}$. So the cross product of $\vec{A}$ with $\vec{V}$ is the same as with $\vec{V}'$: $\vec{A}\times\vec{V} = \vec{A}\times\vec{V}'$. Now consider $\vec{V}''$ which is the rotation of $\vec{V}'$ by $90^\circ$ around $\vec{A}$. Note that $\vec{V}''$ is still in the plane P. Now stretch $\vec{V}''$ by $|\vec{A}|$. The result is a vector, its in the same plane as $\vec{A}\times\vec{V}$, has the same magnitude as $\vec{A}\times\vec{V}$ and it has the same direction as $\vec{A}\times\vec{V}$. So it is $\vec{A}\times\vec{V}$. As infamously reasoned by Joseph McCarthy,"If it looks like a duck, walks like a duck, and quacks like a duck, its a duck."

Thus the cross product $\vec{A}\times\vec{V}$ can be defined by by projecting $\vec{V}$ onto P, rotating that projection by $90^\circ$ about $\vec{A}$, and stretching that by $|\vec{A}|$.

**Apply the project-rotate-stretch definition to $\vec{A}\times\vec{D}$ with $\vec{D} \equiv \vec{B}+\vec{C}$.** The figure below shows the definition above appied to thecross products $\vec{A}\times\vec{D}$, $\vec{A}\times\vec{B}$, and $\vec{A}\times\vec{C}$.




Each of the operations (project, rotate, stretch) is distributive,

- the projection of a sum is the sum of the projections ($\vec{D}' = \vec{B}' + \vec{C}'$);
- the sum of two $90^\circ$ rotated vectors is the rotation of the sum ($\vec{D}'' = \vec{B}'' + \vec{C}''$); and
- (stretched $\vec{D}''$) = (stretched $\vec{B}''$) + (stretched $\vec{C}''$), scalar multiplication is distributive.

Thus the act of taking the cross product of $\vec{A}$ with $\vec{B}$ and adding that to the cross product of $\vec{A}$ with $\vec{C}$ gives the same result as taking the cross product of $\vec{A}$ with $\vec{D} \equiv \vec{B}+\vec{C}$. That's the distributive law for vector cross products over vector addition. That the distributive rule works when the sum is on the left follows by similar reasoning.

**Calculation of the cross product with components.** Now express the vectors in terms of components and apply the distributive rule. First in 2D, to better show the patterns in the algebra:

$$\begin{aligned}
\vec{A}\times\vec{B} &= [A_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\jmath}}]\times[B_x\hat{\boldsymbol{\imath}} + B_y\hat{\boldsymbol{\jmath}}] \\
&= [A_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\jmath}}]\times B_x\hat{\boldsymbol{\imath}} + [A_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\jmath}}]\times B_y\hat{\boldsymbol{\jmath}} \\
&= A_xB_x\hat{\boldsymbol{\imath}}\times\hat{\boldsymbol{\imath}} + A_xB_y\hat{\boldsymbol{\imath}}\times\hat{\boldsymbol{\jmath}} \\
&\quad + A_yB_x\hat{\boldsymbol{\jmath}}\times\hat{\boldsymbol{\imath}} + A_yB_y\hat{\boldsymbol{\jmath}}\times\hat{\boldsymbol{\jmath}} \\
&= A_xB_x\vec{\boldsymbol{0}} + A_xB_y\hat{\boldsymbol{k}} - A_yB_x\hat{\boldsymbol{k}} + A_yB_y\vec{\boldsymbol{0}} \\
&= [A_xB_y - A_yB_x]\hat{\boldsymbol{k}}.
\end{aligned}$$

Now in 3D, carrying out the distributive rule multiple times in the first step,

$$\begin{aligned}
\vec{A}\times\vec{B} &= [A_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\jmath}} + A_z\hat{\boldsymbol{k}}]\times[B_x\hat{\boldsymbol{\imath}} + B_y\hat{\boldsymbol{\jmath}} + B_z\hat{\boldsymbol{k}}] \\
&= A_xB_x\hat{\boldsymbol{\imath}}\times\hat{\boldsymbol{\imath}} + A_xB_y\hat{\boldsymbol{\imath}}\times\hat{\boldsymbol{\jmath}} + A_xB_z\hat{\boldsymbol{\imath}}\times\hat{\boldsymbol{k}} \\
&\quad + A_yB_x\hat{\boldsymbol{\jmath}}\times\hat{\boldsymbol{\imath}} + A_yB_y\hat{\boldsymbol{\jmath}}\times\hat{\boldsymbol{\jmath}} + A_yB_z\hat{\boldsymbol{\jmath}}\times\hat{\boldsymbol{k}} \\
&\quad + A_zB_x\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\imath}} + A_zB_y\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\jmath}} + A_zB_z\hat{\boldsymbol{k}}\times\hat{\boldsymbol{k}} \\
&= A_xB_x\vec{\boldsymbol{0}} + A_xB_y\hat{\boldsymbol{k}} - A_xB_z\hat{\boldsymbol{\jmath}} \\
&\quad - A_yB_x\hat{\boldsymbol{k}} + A_yB_y\vec{\boldsymbol{0}} + A_yB_z\hat{\boldsymbol{\imath}} \\
&\quad + A_zB_x\hat{\boldsymbol{\jmath}} - A_zB_y\hat{\boldsymbol{\imath}} + A_zB_z\vec{\boldsymbol{0}} \\
&= [A_yB_z - A_zB_y]\hat{\boldsymbol{\imath}} + [A_zB_x - A_xB_z]\hat{\boldsymbol{\jmath}} \\
&\quad + [A_xB_y - A_yB_x]\hat{\boldsymbol{k}},
\end{aligned}$$

from which you can pick out the familiar $xyz$ components of the cross product. We have derived the component formula for the cross product from the geometric definition of cross product.

**SAMPLE 2.20 Finding the minimum distance from a point to a line:** A straight line passes through two points, A (-1,4) and B (2,2), in the $xy$ plane. Find the shortest distance from the origin to the line.

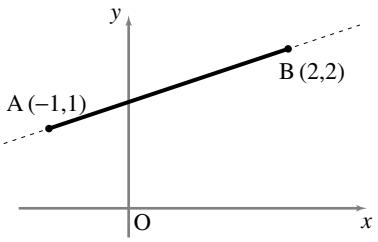



Figure 2.34:
Filename:sfig2-vec2-perp2D

**Solution** Let $\hat{\boldsymbol{\lambda}}_{\rm AB}$ be a unit vector along line AB. Then,

$$\hat{\boldsymbol{\lambda}}_{\rm AB} \times \vec{\boldsymbol{r}}_{\rm B} = \underbrace{|\hat{\boldsymbol{\lambda}}_{\rm AB}|}_{1} |\vec{\boldsymbol{r}}_{\rm B}| \sin\theta \hat{\boldsymbol{n}} = |\vec{\boldsymbol{r}}_{\rm B}| \sin\theta \hat{\boldsymbol{k}}.$$

Now $|\vec{\boldsymbol{r}}_{\rm B}| \sin\theta$ is the component of $\vec{\boldsymbol{r}}_{\rm B}$ that is perpendicular to $\hat{\boldsymbol{\lambda}}_{\rm AB}$ or line AB, *i.e.*, it is the perpendicular, and hence the shortest, distance from the origin (the root of vector $\vec{\boldsymbol{r}}_{\rm B}$) to the line AB. Thus, the shortest distance $d$ from the origin to the line AB is computed from,

$$\begin{aligned} d &= |\hat{\boldsymbol{\lambda}}_{\rm AB} \times \vec{\boldsymbol{r}}_{\rm B}| \\ &= \left| \left(\frac{3\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}}{\sqrt{3^2+1^2}}\right) \times (2\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}) \right| = \left| \frac{6}{\sqrt{10}}\hat{\boldsymbol{k}} - \frac{2}{\sqrt{10}}\hat{\boldsymbol{k}} \right| = \left| \frac{4}{\sqrt{10}}\hat{\boldsymbol{k}} \right| \\ &= \frac{4}{\sqrt{10}}. \end{aligned}$$

$$\boxed{d = 4/\sqrt{10}}$$

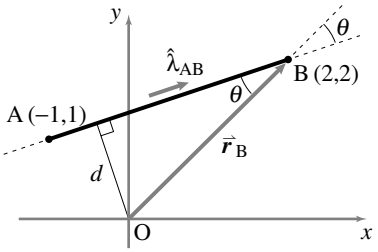



Figure 2.35:
Filename:sfig2-vec2-perp2Da

**Comments:** In this calculation, $\vec{\boldsymbol{r}}_{\rm B}$ is an arbitrary vector from the origin to some point on line AB. You can take any convenient vector. Since the shortest distance is unique, any such vector will give you the same answer. In fact, you can check your answer by selecting another vector and repeating the calculations, *e.g.*, vector $\vec{\boldsymbol{r}}_{\rm A}$.

---

**SAMPLE 2.21 Moment of a force:** Find the moment of force $\vec{\boldsymbol{F}} = 1\,{\rm N}\hat{\boldsymbol{\imath}} + 20\,{\rm N}\hat{\boldsymbol{\jmath}}$ shown in the figure about point O where OA = 2 m.

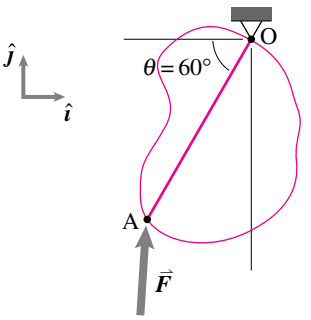



Figure 2.36:
Filename:sfig2-vec2-mom1

**Solution** The force acts through point A on the body. Therefore, we can compute its moment about O as follows.

$$\begin{aligned} \vec{\boldsymbol{M}}_{\rm O} &= \vec{\boldsymbol{r}}_{\rm OA} \times \vec{\boldsymbol{F}} \\ &= \underbrace{(-2\,{\rm m} \cdot \cos 60^\circ \hat{\boldsymbol{\imath}} - 2\,{\rm m} \cdot \sin 60^\circ \hat{\boldsymbol{\jmath}})}_{\vec{\boldsymbol{r}}_{\rm OA}} \times \underbrace{(1\,{\rm N}\hat{\boldsymbol{\imath}} + 20\,{\rm N}\hat{\boldsymbol{\jmath}})}_{\vec{\boldsymbol{F}}} \\ &= (-1\,{\rm m}\hat{\boldsymbol{\imath}} - \sqrt{3}\,{\rm m}\hat{\boldsymbol{\jmath}}) \times (1\,{\rm N}\hat{\boldsymbol{\imath}} + 20\,{\rm N}\hat{\boldsymbol{\jmath}}) \\ &= -20\,{\rm N{\cdot}m}\hat{\boldsymbol{k}} + 1.73\,{\rm N{\cdot}m}\hat{\boldsymbol{k}} \\ &= -18.27\,{\rm N{\cdot}m}\hat{\boldsymbol{k}}. \end{aligned}$$

$$\boxed{\vec{\boldsymbol{M}}_{\rm O} = -18.27\,{\rm N{\cdot}m}\hat{\boldsymbol{k}}}$$

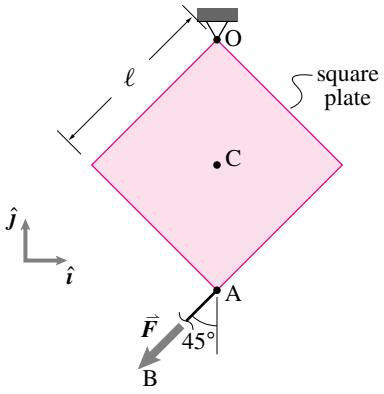



Figure 2.37:
Filename:sfig2-vec2-mom2

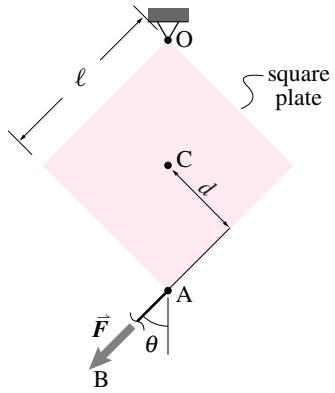



Figure 2.38: The moment of force $F$ about point C is $Fd$ and acts clockwise.
Filename:sfig2-vec2-mom2a

**SAMPLE 2.22** A 2 m×2 m square plate hangs from one of its corners as shown in the figure. At the diagonally opposite end, a force of 50 N is applied by pulling on the string AB. Find the moment of the applied force about the center C of the plate.

**Solution** The moment of $\vec{F}$ about point C is

$$\vec{M}_{\rm C} = \vec{r}_{\rm A/C} \times \vec{F}.$$

In 2D, the magnitude of this cross product is given by $M = Fd$ (*force times the lever arm*) and the direction is evident from the right hand rule. Thus,

$$\begin{aligned} \vec{M}_{\rm C} &= Fd(-\hat{k}) \\ &= -(50\,{\rm N}) \cdot (1\,{\rm m})\hat{k} \\ &= -50\,{\rm N{\cdot}m}\hat{k}. \end{aligned}$$

$$\boxed{\vec{M}_{\rm C} = -50\,{\rm N{\cdot}m}\hat{k}}$$

**Comments:** The moment calculation can, of course, be carried out by computing the cross product in a straight forward manner as shown below. We first need to find the vectors $\vec{r}_{\rm A/C}$ and $\vec{F}$:

$$\begin{aligned} \vec{r}_{\rm A/C} &= -CA\hat{j} = -\frac{\ell}{\sqrt{2}}\hat{j} \quad (\text{since OA} = 2\ \text{CA} = \sqrt{2}\ell) \\ \vec{F} &= F(-\cos\theta\hat{i} - \sin\theta\hat{j}) = -F(\cos\theta\hat{i} + \sin\theta\hat{j}). \end{aligned}$$

Hence,

$$\begin{aligned} \vec{M}_{\rm C} &= -\frac{\ell}{\sqrt{2}}\hat{j} \times [-F(\cos\theta\hat{i} + \sin\theta\hat{j})] \\ &= \frac{\ell}{\sqrt{2}}F(\cos\theta\underbrace{\hat{j}\times\hat{i}}_{-\hat{k}} + \sin\theta\underbrace{\hat{j}\times\hat{j}}_{\vec{0}}) \\ &= -\frac{\ell}{\sqrt{2}}F\cos\theta\hat{k} \\ &= -\frac{2\,{\rm m}}{\sqrt{2}} \cdot 50\,{\rm N} \cdot \cos 45^\circ\hat{k} = -50\,{\rm N{\cdot}m}\hat{k}. \end{aligned}$$

**SAMPLE 2.23 Computing cross product in 3-D:** Compute $\vec{a} \times \vec{b}$, where $\vec{a} = \hat{\imath} + \hat{\jmath} - 2\hat{k}$ and $\vec{b} = 3\hat{\imath} + -4\hat{\jmath} + \hat{k}$.

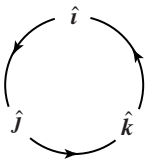



Figure 2.39: The cross product of any two basis vectors is positive in the direction of the arrow and negative if carried out backwards, e.g. $\hat{\imath} \times \hat{\jmath} = \hat{k}$ but $\hat{\jmath} \times \hat{\imath} = -\hat{k}$.
Filename:ijkcircle

**Solution** The calculation of a cross product between two 3-D vectors can be carried out by either using a determinant or the distributive rule. Usually, if the vectors involved have just one or two components, it is easier to use the distributive rule. We show you both methods here and encourage you to learn both. We are given two vectors:

$$\begin{aligned} \vec{a} &= a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k} = \hat{\imath} + \hat{\jmath} - 2\hat{k}, \\ \vec{b} &= b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k} = 3\hat{\imath} + -4\hat{\jmath} + \hat{k}. \end{aligned}$$

- **Calculation using the determinant formula:** In this method, we first write a $3 \times 3$ matrix whose first row has the basis vectors as its elements, the second row has the components of the first vector as its elements, and the third row has the components of the second vector as its elements. Thus,

$$\begin{aligned} \vec{a} \times \vec{b} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \hat{\imath}(a_2b_3 - a_3b_2) + \hat{\jmath}(a_3b_1 - a_1b_3) + \hat{k}(a_1b_2 - b_1a_2) \\ &= \hat{\imath}(1 - 8) + \hat{\jmath}(-6 - 1) + \hat{k}(-4 - 3) \\ &= -7(\hat{\imath} + \hat{\jmath} + \hat{k}). \end{aligned}$$

- **Calculation using the distributive rule:** In this method, we carry out the cross product by distributing the cross product properly over the three basis vectors. The steps involved are shown below.

$$\begin{aligned} \vec{a} \times \vec{b} &= (a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}) \times (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}) \\ &= a_1\hat{\imath} \times (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}) + \\ &\quad a_2\hat{\jmath} \times (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}) + \\ &\quad a_3\hat{k} \times (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}) \\ &= a_1b_1(\overbrace{\hat{\imath} \times \hat{\imath}}^{\mathbf{0}}) + a_1b_2(\overbrace{\hat{\imath} \times \hat{\jmath}}^{\hat{k}}) + a_1b_3(\overbrace{\hat{\imath} \times \hat{k}}^{-\hat{\jmath}}) + \\ &\quad a_2b_1(\overbrace{\hat{\jmath} \times \hat{\imath}}^{-\hat{k}}) + a_2b_2(\overbrace{\hat{\jmath} \times \hat{\jmath}}^{\mathbf{0}}) + a_2b_3(\overbrace{\hat{\jmath} \times \hat{k}}^{\hat{\imath}}) + \\ &\quad a_3b_1(\overbrace{\hat{k} \times \hat{\imath}}^{\hat{\jmath}}) + a_3b_2(\overbrace{\hat{k} \times \hat{\jmath}}^{-\hat{\imath}}) + a_3b_3(\overbrace{\hat{k} \times \hat{k}}^{\mathbf{0}}) \\ &= \hat{\imath}(a_2b_3 - a_3b_2) + \hat{\jmath}(a_3b_1 - a_1b_3) + \hat{k}(a_1b_2 - b_1a_2) \\ &= \hat{\imath}(1 - 8) + \hat{\jmath}(-6 - 1) + \hat{k}(-4 - 3) \\ &= -7(\hat{\imath} + \hat{\jmath} + \hat{k}) \end{aligned}$$

which, of course, is the same result as obtained above using the determinant. Making a sketch such as Fig. 2.39 is helpful while calculating cross products this way. The product of any two basis vectors is positive in the direction of the arrow and negative if carried out backwards, *e.g.*, $\hat{\imath} \times \hat{\jmath} = \hat{k}$ but $\hat{\jmath} \times \hat{\imath} = -\hat{k}$.

$$\boxed{\vec{a} \times \vec{b} = -7(\hat{\imath} + \hat{\jmath} + \hat{k})}$$

**SAMPLE 2.24 Finding a vector normal to two given vectors:** Find a unit vector perpendicular to the vectors $\vec{r}_A = \hat{\imath} - 2\hat{\jmath} + \hat{k}$ and $\vec{r}_b = 3\hat{\jmath} + 2\hat{k}$.

**Solution** The cross product between two vectors gives a vector perpendicular to the plane formed by the two vectors. The sense of direction is determined by the right hand rule.

Let $\vec{N} = N\hat{\lambda}$ be the perpendicular vector.

$$\begin{aligned} \vec{N} &= \vec{r}_A \times \vec{r}_B \\ &= (\hat{\imath} - 2\hat{\jmath} + \hat{k}) \times (3\hat{\jmath} + 2\hat{k}) \\ &= -7\hat{\imath} - 2\hat{\jmath} + 3\hat{k}. \end{aligned}$$

This calculation can be done in any of the two ways shown in the previous sample problem.

Therefore,

$$\begin{aligned} \hat{\lambda} &= \frac{\vec{N}}{N} \\ &= \frac{-7\hat{\imath} - 2\hat{\jmath} + 3\hat{k}}{\sqrt{7^2 + 2^2 + 3^2}} \\ &= -0.89\hat{\imath} - 0.25\hat{\jmath} + 0.38\hat{k} \end{aligned}$$

$$\boxed{\hat{\lambda} = -0.89\hat{\imath} - 0.25\hat{\jmath} + 0.38\hat{k}}$$

Check:

- $|\hat{\lambda}| = (0.89)^2 + (0.25)^2 + (0.38)^2 \overset{\surd}{=} 1$. (it is a unit vector)
- $\hat{\lambda} \cdot \vec{r}_A = 1(-0.89) - 2(-0.25) + 1(0.38) \overset{\surd}{=} 0$. $(\hat{\lambda} \perp \vec{r}_A)$.
- $\hat{\lambda} \cdot \vec{r}_B = 3(-0.25) + 2(0.38) \overset{\surd}{=} 0$. $(\hat{\lambda} \perp \vec{r}_B)$.

**Comments:** If $\hat{\lambda}$ is perpendicular to $\vec{r}_\mathrm{A}$ and $\vec{r}_\mathrm{B}$, then so is $-\hat{\lambda}$. The perpendicularity does not change by changing the *sense* of direction (from positive to negative) of the vector. In fact, if $\hat{\lambda}$ is perpendicular to a vector $\vec{r}$then any scalar multiple of $\hat{\lambda}$, *i.e.*, $\alpha\hat{\lambda}$, is also perpendicular to $\vec{r}$. This follows from the fact that

$$\alpha\hat{\lambda} \cdot \vec{r} = \alpha(\hat{\lambda} \cdot \vec{r}) = \alpha(0) = 0.$$

The case of $-\hat{\lambda}$ is just a particular instance of this rule with $\alpha = -1$.

**SAMPLE 2.25 Finding a vector normal to a plane:** Find a unit vector normal to the plane ABC shown in the figure.

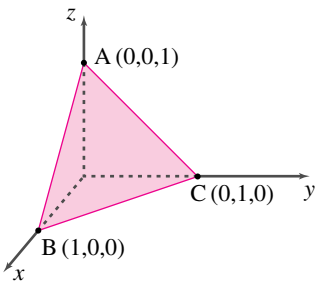



Figure 2.40:
Filename:sfig2-vec2-normal

**Solution** A vector normal to the plane ABC would be normal to any vector in that plane. In particular, if we take any two vectors, say $\vec{r}_{AB}$ and $\vec{r}_{AC}$, the normal to the plane would be perpendicular to both $\vec{r}_{AB}$ and $\vec{r}_{AC}$. Since the cross product of two vectors gives a vector perpendicular to both vectors, we can find the desired normal vector by taking the cross product of $\vec{r}_{AB}$ and $\vec{r}_{AC}$. Thus,

$$\begin{aligned}
\vec{N} &= \vec{r}_{AB} \times \vec{r}_{AC} \\
&= (\hat{\imath} - \hat{k}) \times (\hat{\jmath} - \hat{k}) \\
&= \underbrace{\hat{\imath} \times \hat{\jmath}}_{\hat{k}} - \underbrace{\hat{\imath} \times \hat{k}}_{-\hat{\jmath}} - \underbrace{\hat{k} \times \hat{\jmath}}_{-\hat{\imath}} + \underbrace{\hat{k} \times \hat{k}}_{\vec{0}} \\
&= \hat{\imath} + \hat{\jmath} + \hat{k} \\
\Rightarrow \quad \hat{n} &= \frac{\vec{N}}{|\vec{N}|} \\
&= \frac{1}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} + \hat{k}).
\end{aligned}$$

$$\boxed{\hat{n} = \tfrac{1}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} + \hat{k})}$$

Check: Now let us check if $\hat{n}$ is normal to any vector in the plane ABC. It is fairly easy to show that $\hat{n} \cdot \vec{r}_{AB} = \hat{n} \cdot \vec{r}_{AC} = 0$. It is, however, not a surprise; it better be since we found $\hat{n}$ from the cross product of $\vec{r}_{AB}$ and $\vec{r}_{AC}$. Let us check if $\hat{n}$ is normal to $\vec{r}_{BC}$:

$$\begin{aligned}
\hat{n} \cdot \vec{r}_{BC} &= \frac{1}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} + \hat{k}) \cdot (-\hat{\imath} + \hat{\jmath}) \\
&= \frac{1}{\sqrt{3}}(-\hat{\imath} \cdot \hat{\imath} + \hat{\jmath} \cdot \hat{\jmath}) \\
&= \frac{1}{\sqrt{3}}(-1 + 1) \overset{\surd}{=} 0.
\end{aligned}$$

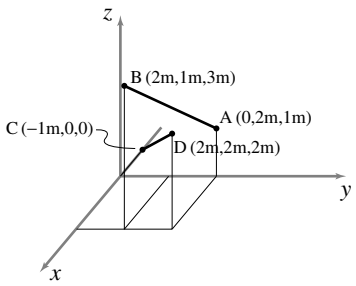



Figure 2.41:
Filename:sfig2-vec2-perp3D

**SAMPLE 2.26 The shortest distance between two lines:** Two lines, AB and CD, in 3-D space are defined by four specified points, A(0,2 m,1 m), B(2 m,1 m,3 m), C(-1 m,0,0), and D(2 m,2 m,2 m) as shown in the figure. Find the shortest distance between the two lines.

**Solution** The shortest distance between any pair of lines is the length of the line that is perpendicular to both the lines. We can find the shortest distance in three steps:

1. First find a vector that is perpendicular to both the lines. This is easy. Take two vectors $\vec{r}_1$ and $\vec{r}_2$, one along each of the two given lines. Take the cross product of the two unit vectors and make the resulting vector a unit vector $\hat{n}$.
2. Find a vector parallel to $\hat{n}$ that connects the two lines. This is a little tricky. We don't know where to start on any of the two lines. However, we can take any vector from one line to the other and then, take its component along $\hat{n}$.
3. Find the length (magnitude) of the vector just found (in the direction of $\hat{n}$). This is simply the component we find in step (b) devoid of its sign.

Now let us carry out these steps on the given problem.

1. Step-1: Find a unit vector $\hat{n}$ that is perpendicular to both the lines.

$$\begin{aligned}
\vec{r}_{\rm AB} &= 2\,{\rm m}\hat{\imath} - 1\,{\rm m}\hat{\jmath} + 2\,{\rm m}\hat{k} \\
\vec{r}_{\rm CD} &= 3\,{\rm m}\hat{\imath} + 2\,{\rm m}\hat{\jmath} + 2\,{\rm m}\hat{k} \\
\Rightarrow \quad \vec{r}_{\rm AB} \times \vec{r}_{\rm CD} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 2 \\ 3 & 2 & 2 \end{vmatrix} {\rm m}^2 \\
&= \hat{\imath}(-2-4)\,{\rm m}^2 + \hat{\jmath}(6-4)\,{\rm m}^2 + \hat{k}(4+3)\,{\rm m}^2 \\
&= (-6\hat{\imath} + 2\hat{\jmath} + 7\hat{k})\,{\rm m}^2
\end{aligned}$$

Therefore,

$$\begin{aligned}
\hat{n} &= \frac{\vec{r}_{\rm AB} \times \vec{r}_{\rm CD}}{|\vec{r}_{\rm AB} \times \vec{r}_{\rm CD}|} \\
&= \frac{1}{\sqrt{89}}(-6\hat{\imath} + 2\hat{\jmath} + 7\hat{k}).
\end{aligned}$$

2. Step-2: Find any vector from one line to the other line and find its component along $\hat{n}$.

$$\begin{aligned}
\vec{r}_{\rm AC} &= -1\,{\rm m}\hat{\imath} - 2\,{\rm m}\hat{\jmath} - 1\,{\rm m}\hat{k} \\
\vec{r}_{\rm AC} \cdot \hat{n} &= -(\hat{\imath} + 2\hat{\jmath} + \hat{k})\,{\rm m} \cdot \frac{1}{\sqrt{89}}(-6\hat{\imath} + 2\hat{\jmath} + 7\hat{k}) \\
&= \frac{1}{\sqrt{89}}(6-4-7)\,{\rm m} = -\frac{5}{\sqrt{89}}\,{\rm m}.
\end{aligned}$$

3. Step-3: Find the required distance $d$ by taking the magnitude of the component along $\hat{n}$.

$$d = |\vec{r}_{\rm AC} \cdot \hat{n}| = \left| -\frac{5}{\sqrt{89}}\,{\rm m} \right| = 0.53\,{\rm m}$$

$$\boxed{d = 0.53\,{\rm m}}$$

**SAMPLE 2.27 The mixed triple product:** Calculate the mixed triple product $\hat{\boldsymbol{\lambda}} \cdot (\vec{a} \times \vec{b})$ for $\hat{\boldsymbol{\lambda}} = \frac{1}{\sqrt{2}}(\hat{\imath} + \hat{\jmath})$, $\vec{a} = 3\hat{\imath}$, and $\vec{b} = \hat{\imath} + \hat{\jmath} + 3\hat{k}$.

**Solution** We compute the given mixed triple product in two ways here:

- Method-1: Straight calculation using cross product and dot product.

$$\begin{aligned} \text{Let} \quad \vec{c} &= \vec{a} \times \vec{b} \\ &= (3\hat{\imath}) \times (\hat{\imath} + \hat{\jmath} + 3\hat{k}) \\ &= 3\underbrace{\hat{\imath} \times \hat{\imath}}_{\vec{0}} + 3\underbrace{\hat{\imath} \times \hat{\jmath}}_{\hat{k}} + 9\underbrace{\hat{\imath} \times \hat{k}}_{-\hat{\jmath}} = -9\hat{\jmath} + 3\hat{k} \\ \text{So,} \quad \hat{\boldsymbol{\lambda}} \cdot (\vec{a} \times \vec{b}) &= \hat{\boldsymbol{\lambda}} \cdot \vec{c} \\ &= \frac{1}{\sqrt{2}}(\hat{\imath} + \hat{\jmath}) \cdot (-9\hat{\jmath} + 3\hat{k}) = -\frac{9}{\sqrt{2}}. \end{aligned}$$

- Method-2: Using the determinant formula for mixed product.

$$\begin{aligned} \hat{\boldsymbol{\lambda}} \cdot (\vec{a} \times \vec{b}) &= \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 3 & 0 & 0 \\ 1 & 1 & 3 \end{vmatrix} \\ &= \frac{1}{\sqrt{2}}(0-0) + \frac{1}{\sqrt{2}}(0-9) + 0 = -\frac{9}{\sqrt{2}}. \end{aligned}$$

$$\boxed{\hat{\boldsymbol{\lambda}} \cdot (\vec{a} \times \vec{b}) = -\frac{9}{\sqrt{2}}}$$

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**SAMPLE 2.28 Moment about an axis:** A vertical force of unknown magnitude $F$ acts at point B of a triangular plate ABC shown in the figure. Find the moment of the force about edge CA of the plate.

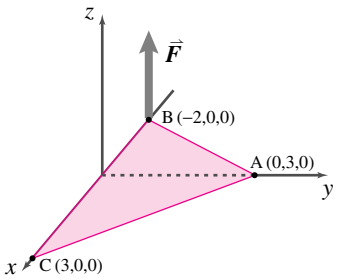



Figure 2.42:
Filename:sfig2-vec2-momxx

**Solution** The moment of a force $\vec{F}$ about an axis x-x is given by

$$M_{xx} = \hat{\boldsymbol{\lambda}}_{\text{xx}} \cdot (\vec{r} \times \vec{F})$$

where $\hat{\boldsymbol{\lambda}}_{\text{xx}}$ is a unit vector along the axis x-x, $\vec{r}$ is a position vector from any point on the axis to the applied force. In this problem, the given axis is CA. Therefore, we can take $\vec{r}$ to be $\vec{r}_{\text{AB}}$ or $\vec{r}_{\text{CB}}$. Here,

$$\hat{\boldsymbol{\lambda}}_{\text{CA}} = \frac{\vec{r}_{\text{CA}}}{|\vec{r}_{\text{CA}}|} = \frac{3(-\hat{\imath} + \hat{\jmath})}{\sqrt{9+9}} = -\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}.$$

Now, moment about point A is

$$\begin{aligned} \vec{M}_{\text{A}} &= \vec{r}_{\text{AB}} \times \vec{F} \\ &= (-2\hat{\imath} - 3\hat{\jmath}) \times F\hat{k} = 2F\hat{\jmath} - 3F\hat{\imath}. \end{aligned}$$

Therefore, the moment about CA is

$$\begin{aligned} M_{CA} &= \hat{\boldsymbol{\lambda}}_{\text{CA}} \cdot (\vec{r}_{\text{AB}} \times \vec{F}) = \hat{\boldsymbol{\lambda}}_{\text{CA}} \cdot \vec{M}_{\text{A}} \\ &= (-\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}) \cdot (-3F\hat{\imath} + 2F\hat{\jmath}) \\ &= (\frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}})F = \frac{5}{\sqrt{2}}F. \end{aligned}$$

$$\boxed{M_{CA} = \frac{5}{\sqrt{2}}F}$$

## 2.4 Solving vector equations

If as an engineer you knew all quantities of interest you would not need to calculate. But as a rule in life you know less than you would like to know. And you naturally try to figure out more. In engineering mechanics analysis you find more quantities of interest from others that you already know (or assume) using the laws of mechanics (including geometry and kinematics). Because many of these laws are vector equations, engineering analysis often requires the solving of vector equations.

The methods involved are much the same whether the problems are in geometry, kinematics, statics, dynamics or a combination of these. In this section we will show a few methods for solving some of the more common vector equations. In a sense there are no new concepts in this section; if you are already adept at vector manipulations you will find yourself reading quickly.

### Vector algebra

We will be concerned with manipulating equations that involve vectors (like $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{0}$) and scalars (like $a$, $b$, $c$, and 0). Without knowing anything about mechanics or the geometric meaning of vectors, one can learn to do correct vector algebra by just following the manipulation rules below, these are elaborations of elementary scalar algebra to accommodate vectors and the three new kinds of multiplication (scalar times vector, dot product, and cross product). Here is a summary.

Addition and all three kinds of multiplication (scalar multiplication, dot product, cross product) all follow the usual commutative, associative, and distributive laws of scalar addition and multiplication with the following exceptions:

- $a + \vec{A}$ is nonsense,
- $a/\vec{A}$ is nonsense,
- $\vec{A}/\vec{B}$ is nonsense,
- $a \cdot \vec{A}$ is nonsense (unless you mean by it $a\vec{A}$),
- $a \times \vec{A}$ is nonsense,
- $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$,

and the following extra simplification rules:

- $a\vec{A}$ is a vector,
- $\vec{A}\cdot\vec{B}$ is a scalar,
- $\vec{A} \times \vec{B}$ is a vector,
- $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ (so $\vec{A} \times \vec{A} = \vec{0}$)
- $\vec{A}\cdot(\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B})\cdot\vec{C}$.

Following these rules automatically enforces correct manipulations. Armed with insight you can direct these manipulations towards a desired end.

**Example.** Say you know $\vec{A}$, $\vec{B}$, $\vec{C}$ and $\vec{D}$ and you know that

$$a\vec{A} + b\vec{B} + c\vec{C} = \vec{D}$$

but you don't know $a$, $b$, and $c$. How could you find $a$? First dot both sides with $\vec{B} \times \vec{C}$ and then blindly follow the rules:

$$\begin{aligned} \left\{a\vec{A} + b\vec{B} + c\vec{C} \right. &= \left. \vec{D}\right\} \cdot \left(\vec{B} \times \vec{C}\right) \\ a\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) + b\underbrace{\vec{B} \cdot \left(\vec{B} \times \vec{C}\right)}_{0} + c\underbrace{\vec{C} \cdot \left(\vec{B} \times \vec{C}\right)}_{0} &= \vec{D} \cdot \left(\vec{B} \times \vec{C}\right) \\ \Rightarrow \quad a = \frac{\vec{D} \cdot \left(\vec{B} \times \vec{C}\right)}{\vec{A} \cdot \left(\vec{B} \times \vec{C}\right)}. & \end{aligned} \tag{2.10}$$

The two zeros followed from the general rules that $\vec{D} \cdot (\vec{V} \times \vec{W}) = (\vec{D} \times \vec{V}) \cdot \vec{W})$ and $\vec{D} \times \vec{D} = \vec{0}$. Note the derivation above breaks down if the vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ are co-planar and the last line of the calculation would have $\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = 0$ in the denominator*.

The point of the example above was to show the vector algebra rules at work. However, to get to the end took the first 'move' of dotting the equation with the appropriate vector. That move could be motivated this way. We are trying to find $a$ and not $b$ or $c$. We can get rid of the terms in the equation that contain $b$ and $c$ if we can dot $\vec{B}$ and $\vec{C}$ with a vector perpendicular to both of them. $\vec{B} \times \vec{C}$ is perpendicular to both $\vec{B}$ and $\vec{C}$ so can be used to kill them off with a dot product. The 0s in the example calculation were thus expected for geometric reasons.

A simpler two-dimensional example with the same spirit as the example above, using a judiciously chosen dot product, is on page 69

## Count equations and unknowns.

One cannot (usually) find more unknowns than one has scalar equations*. Before you do lots of algebra, you should check that you have as many equations as unknowns. If not, you probably can't find all the unknowns. How do you count vector equations and vector unknowns? A two- dimensional vector is fully described by two numbers. For example, a $2D$ vector is described by its $x$ and $y$ components or its magnitude and the angle it makes with the positive $x$ axis. A three-dimensional vector is described by three numbers. So a vector equation counts as 2 or 3 equations in 2 or 3 dimensional problems, respectively. And an unknown vector counts as 2 or 3 unknowns in 2 or 3 dimensions, respectively. If the direction of a vector is known but its magnitude is not, then the magnitude is the only unknown. Magnitude is a scalar, so it counts as one unknown.

Example: **Counting equations**
Say you are doing a 2-D problem where you already know the vector $\hat{\boldsymbol{\lambda}} = \sqrt{2}\hat{\boldsymbol{\imath}} + \sqrt{2}\hat{\boldsymbol{\jmath}}$ and you are given the vector equation

$$C\hat{\boldsymbol{\lambda}} = \vec{a}.$$

* The linear-algebra savvy reader may recognize the manipulation leading to eqn. (2.10) as a derivation of Cramer's rule for $3 \times 3$ matrices with the columns of the matrix being the components of the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$. See box 2.6 on page **??**box:geometryuniqueness for more discussion of when equations do and do not have solutions.)

* There are famous counter-examples where you can solve for more variables than you have equations. The simplest example, $x^2 + y^2 = 0$, is one equation which can be solved for both $x$ and $y$ to get $x = 0$ and $y = 0$. Although such examples seem to be mathematical trickery they do show up sometimes, but they are always nonlinear. The simultaneous equations in mechanics are most often linear equations (so you can safely ignore this margin comment).

You then have two equations (a vector equation in 2-D ) and three unknowns (the scalar $C$ and the vector $\vec{a}$). There are more unknowns than equations so this vector equation is not sufficient for finding $C$ and $\vec{a}$.

Most often when you have as many equations as unknowns the equations have a unique solution. When you have more equations than unknowns there is most often no solution to the equations. When you have more unknowns than equations most often you have a whole family of solutions.

However these are only guidelines, no matter how many equations and unknowns you have, you could have no solutions, many solutions or a unique solution. The geometric significance of some cases that satisfy and that don't satisfy these guidelines is given in box 2.6 on page 78.

## Vector triangles

In 2D one often wants to know all three vectors in a vector triangle, the diagram for expressions like

$$\vec{A} + \vec{B} = \vec{C} \quad \text{or} \quad \vec{A} - \vec{C} = \vec{B} \quad \text{or} \quad \vec{A} + \vec{B} + \vec{C} = \vec{0} \quad etc.$$

Usually at least one vector is given and some information is given about the others. The situation is much like the geometry problem of drawing a triangle given various bits of information about the lengths of its sides and its interior angles. If enough information is given to prove triangle congruence, then enough information is given to determine all angles and sides. A difference between vector triangles and proofs of triangle congruence is that triangle congruence does not depend on the overall orientation, whereas vector triangles need to have the correct orientation. Nonetheless, the tools used to solve triangles are useful for solving vector equations.

## Vector addition

We start with a problem that is in some sense solved at the start. Say $\vec{A}$ and $\vec{B}$ are known and you want to find $\vec{C}$ given that

$$\vec{C} = \vec{A} + \vec{B}.$$

The obvious and correct answer is that you find $\vec{C}$ by vector addition. You could do this addition graphically by drawing a scale picture, or by adding corresponding vector components. Suppose now that $\vec{A}$ and $\vec{B}$ are given to you in terms of magnitude and direction and that you are interested in the direction of $\vec{C}$.

Example: **adding vectors defined by magnitude and direction**

Say direction is indicated by angle measured counterclockwise form the positive $x$ axis and that $A = 5\sqrt{2}$, $\theta_A = \pi/4$, $B = 4$, and $\theta_B = 2\pi/3$. So

$$\begin{aligned}
\vec{A} &= A\left(\cos\theta_A\hat{\imath} + \sin\theta_A\hat{\jmath}\right) \\
&= 5\sqrt{2}\left(\cos(\pi/4)\hat{\imath} + \sin(\pi/4)\hat{\jmath}\right) = 5\hat{\imath} + 5\hat{\jmath} \\
\vec{B} &= B\left(\cos\theta_B\hat{\imath} + \sin\theta_B\hat{\jmath}\right) \\
&= 4\left(\cos(2\pi/3)\hat{\imath} + \sin(2\pi/3)\hat{\jmath}\right) = -2\hat{\imath} + 2\sqrt{3}\hat{\jmath} \\
\vec{C} &= \vec{A} + \vec{B} = \left(5\hat{\imath} + 5\hat{\jmath}\right) + \left(-2\hat{\imath} + 2\sqrt{3}\hat{\jmath}\right) \\
&= 3\hat{\imath} + (5 + 2\sqrt{3})\hat{\jmath} \\
\Rightarrow \quad \theta_C &= \tan^{-1}\left(C_y/C_x\right) = \tan^{-1}\left(\left(5 + 2\sqrt{3}\right)/3\right) \approx 1.23 \approx 70.5^\circ \\
\text{and} \quad C &= \sqrt{3^2 + (5 + 2\sqrt{3})^2} \approx 8.98
\end{aligned}$$

To find $\theta_C$ we used the arctan (or $\tan^{-1}$) function which can be off by $\pi$ *.To find the angle of $\vec{C}$ we had to convert $\vec{A}$ and $\vec{B}$ to coordinate form, add components, and then convert back to find the angle of $\vec{C}$. That is, even though the desired answer is given by a sum, carrying out the sum takes a bit of effort. An alternative approach avoids some work.

Example: **Same as above, different method**
Start with picture of the situation, Fig. 2.43. By adding angles,

$$\theta_2 = \pi/4 + \pi/3 = 7\pi/12.$$

From the law of cosines (see box 2.5 on page 77),

$$\begin{aligned}
C^2 &= A^2 + B^2 - 2AB\cos\theta_2 \\
\Rightarrow \quad C &= \sqrt{(5\sqrt{2})^2 + 4^2 - 2(5\sqrt{2})\cdot 4\cdot\cos(7\pi/12)} \\
&\approx 8.98 \quad \text{(as before)}
\end{aligned}$$

And from the law of sines (see box 2.5),

$$\begin{aligned}
\frac{\sin\theta_1}{B} &= \frac{\sin\theta_2}{C} \\
\Rightarrow \quad \theta_1 &= \sin^{-1}\left(\frac{B\sin\theta_2}{C}\right) \approx \sin^{-1}\left(\frac{4\sin(7\pi/12)}{8.98}\right) \\
&\approx .445 \\
\Rightarrow \quad \theta_C &= \theta_A + \theta_2 \approx \pi/4 + .445 \approx 1.23 \quad \text{(as before).}
\end{aligned}$$

This second approach is somewhat more direct in some situations.

The determination of a third vector by vector addition is analogous to the determination of a triangle in geometry by "side-angle-side".

## Vector subtraction

Say you want to find $\vec{C}$ given $\vec{A}$ and $\vec{B}$ and that $\vec{A}$, $\vec{B}$ and $\vec{C}$ add to zero. So, subtracting $\vec{C}$ from both sides and multiplying through by -1 we get

$$\begin{aligned}
\vec{A} + \vec{B} + \vec{C} &= \vec{0} \\
\Rightarrow \quad \vec{C} &= -\vec{A} - \vec{B}.
\end{aligned}$$

The problem has now been reduced to one of addition which can be done by drawing, components, or trig as shown above.

* The problem is that, measuring angles between 0 and $2\pi$ (or equivalently between $-\pi$ and $\pi$) there are always two different angles that have the same tangent. The inverse tangent function picks one. Some computers or calculators always pick an angle between 0 and $\pi$ and some always pick a value between $-\pi/2$ and $\pi/2$. Both of these could be the wrong answer. So you need to check and possibly add $\pi$ to your answer, or, alternatively use one of these two commands: 1) the two-argument inverse tangent (arctan$(x, y)$) or 2) rectangular-to-polar coordinate conversion, using the angle as the desired arctangent.

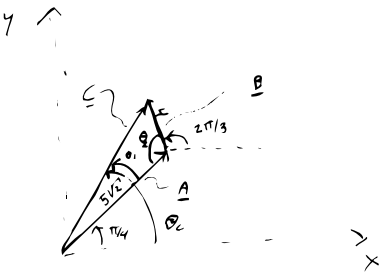



Figure 2.43: Using trig to solve vector triangles

Filename:tfigure-cosinetriangle

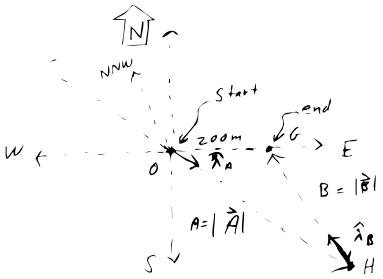



Figure 2.44: An indirect walk from O to D via C.

Filename:tfigure-2-walk

## Find the magnitude of two vectors given their directions and their sum (2D)

Often one knows that 2 vectors $\vec{A}$ and $\vec{B}$ add to a given third vector $\vec{C}$. The directions of $\vec{A}$ and $\vec{B}$ are known but not their magnitudes. That is, given $\hat{\boldsymbol{\lambda}}_A$, $\hat{\boldsymbol{\lambda}}_B$ and $\vec{C}$ and that

$$\begin{aligned} \vec{A} \quad + \quad \vec{B} \quad &= \quad \vec{C} \\ A\hat{\boldsymbol{\lambda}}_A \quad + \quad B\hat{\boldsymbol{\lambda}}_B \quad &= \quad \vec{C} \end{aligned} \tag{2.11}$$

you would like to find $\vec{A}$ and $\vec{B}$ (which you will know if you find $A$ and $B$).

Example: **A walk**
You walked SE (half way between South and East) for a while and NNW (half way between North and NorthWest, 22.5° West of North) for a while and ended up going a net distance of 200 m East. $\vec{A}$ and $\vec{B}$ are your displacements on the first and second parts of your walk.
So, taking $xy$ axes aligned with East and North, the directions are

$$\hat{\boldsymbol{\lambda}}_A = \frac{\sqrt{2}}{2}\hat{\boldsymbol{\imath}} - \frac{\sqrt{2}}{2}\hat{\boldsymbol{\jmath}} \text{ and } \hat{\boldsymbol{\lambda}}_B = -\sin(\tfrac{\pi}{8})\hat{\boldsymbol{\imath}} + \cos(\tfrac{\pi}{8})\hat{\boldsymbol{\jmath}}$$

and the given sum is $\vec{C} = 200\,\text{m}\hat{\boldsymbol{\imath}}$. Still unknown are the distances $A$ and $B$.

In statics problems of this type or frequent with $A$ and $B$ representing the unknown magnitudes of forces $\vec{A}$ and $\vec{B}$ and $\hat{\boldsymbol{\lambda}}_A$ and $B\hat{\boldsymbol{\lambda}}_B$ their known directions. Here are four ways to solve eqn. (2.11) which will be illustrated with "**a walk**".

### Method I: Use dot products with $\hat{\imath}$ and $\hat{\jmath}$

If we take the dot product of both sides of eqn. (2.11) with $\hat{\boldsymbol{\imath}}$ and then again with $\hat{\boldsymbol{\jmath}}$ we get:

$$\begin{aligned} \hat{\boldsymbol{\imath}} \cdot \{\text{eqn. (2.11)}\} \quad &\Rightarrow \quad A\lambda_{Ax} + B\lambda_{Bx} = C_x, \text{ and} \\ \hat{\boldsymbol{\jmath}} \cdot \{\text{eqn. (2.11)}\} \quad &\Rightarrow \quad A\lambda_{Ay} + B\lambda_{By} = C_y \end{aligned} \tag{2.12}$$

where the components of the vectors $\hat{\boldsymbol{\lambda}}_A$, $\hat{\boldsymbol{\lambda}}_B$, and $\vec{C}$ are known, or easily determined, because the vectors are known (however they are represented). Eqns. 2.12 are two scalar equations in the unknowns $A$ and $B$. You can solve these any way that pleases you. One method would be to write the equations in matrix form

$$\begin{bmatrix} \lambda_{Ax} & \lambda_{Bx} \\ \lambda_{Ay} & \lambda_{By} \end{bmatrix} \cdot \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} C_x \\ C_y \end{bmatrix} \tag{2.13}$$

Example: **Solving "A walk": method I, simultaneous equations**
For the walk example above we would have

$$\begin{bmatrix} \sqrt{2}/2 & -\sin(\frac{\pi}{8}) \\ -\sqrt{2}/2 & \cos(\frac{\pi}{8}) \end{bmatrix} \cdot \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 200\,\text{m} \\ 0 \end{bmatrix}$$

which solves (on a computer or calculator) to $A \approx 483$ m and $B \approx 370$ (with the total walked distance being about 852 m).

Taking dot products of a vector equation with $\hat{\imath}$ and $\hat{\jmath}$ is equivalent to extracting the $x$ and $y$ components of the equation. But we use the dot product notation to highlight that you could dot both sides of the vector equation with any vector that pleases you and you would get a legitimate scalar equation. Use any other vector that pleases you (not parallel with the first) and you will get a second independent equation. And the two resulting equations will have the same solution for $A$ and $B$ as the $x$ and $y$ (or "$\hat{\imath}$" and "$\hat{\jmath}$") equations above.

## Method II: pick a vector for a dot product that gets rid of terms you don't know.

Pretend for a paragraph that you only want to find $A$ in eqn. (2.11), for example that you only wanted to know the distance walked on the first leg of the indirect walk in the example above. It would be nice to reduce eqn. (2.11) to a single scalar equation in the single unknown $A$. We'd like to get rid of the term with $B$, a quantity that we do not know. Suppose we knew a vector $\hat{\boldsymbol{n}}_B$ that was perpendicular to $\hat{\boldsymbol{\lambda}}_B$. If we dotted both sides of eqn. (2.11) we'd get:

$$\begin{aligned}
\hat{\boldsymbol{n}} \cdot \{\text{eqn. (2.11)}\} \quad &\Rightarrow \quad \hat{\boldsymbol{n}}_B \cdot \left(A\hat{\boldsymbol{\lambda}}_A + B\hat{\boldsymbol{\lambda}}_B\right) = \hat{\boldsymbol{n}}_B \cdot \vec{\boldsymbol{C}} \\
&\Rightarrow \quad \hat{\boldsymbol{n}}_B \cdot \left(A\hat{\boldsymbol{\lambda}}_A\right) + \hat{\boldsymbol{n}}_B \cdot \left(B\hat{\boldsymbol{\lambda}}_B\right) = \hat{\boldsymbol{n}}_B \cdot \vec{\boldsymbol{C}} \\
\hat{\boldsymbol{n}}_B \perp \hat{\boldsymbol{\lambda}}_B \text{ so } \hat{\boldsymbol{n}}_B \cdot \hat{\boldsymbol{\lambda}}_B = 0 \quad &\Rightarrow \quad \left(\hat{\boldsymbol{n}}_B \cdot \hat{\boldsymbol{\lambda}}_A\right) A = \hat{\boldsymbol{n}}_B \cdot \vec{\boldsymbol{C}} \\
&\Rightarrow \quad A = \frac{\hat{\boldsymbol{n}}_B \cdot \vec{\boldsymbol{C}}}{\hat{\boldsymbol{n}}_B \cdot \hat{\boldsymbol{\lambda}}_A}.
\end{aligned}$$

To make use of this method we have to cook up a vector $\hat{\boldsymbol{n}}_B$ that is perpendicular to $\hat{\boldsymbol{\lambda}}_B$ *.Crossing $\hat{\boldsymbol{\lambda}}_B$ with $\hat{\boldsymbol{k}}$ serves the purpose:

$$\hat{\boldsymbol{n}}_B = \hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}_B = \hat{\boldsymbol{k}} \times \left(\lambda_{Bx}\hat{\boldsymbol{\imath}} + \lambda_{By}\hat{\boldsymbol{\jmath}}\right) = -\lambda_{By}\hat{\boldsymbol{\imath}} + \lambda_{Bx}\hat{\boldsymbol{\jmath}}.$$

Without doing the cross product explicitly you can remember that a vector orthogonal to a 2D vector $\hat{\boldsymbol{\lambda}}_B$ has the $x$ and $y$ components switched and the sign of first component then changed. So we get

$$A = \frac{(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}_B) \cdot \vec{\boldsymbol{C}}}{(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}_B) \cdot \hat{\boldsymbol{\lambda}}_A} = \frac{\lambda_{By}C_x - \lambda_{Bx}C_y}{\lambda_{By}\lambda_{Ax} - \lambda_{Bx}\lambda_{Ay}}$$

which is a direct formula for the desired answer *. You could use this formula by substituting in numbers, but that requires memorization or look up. Rather, if you like this short cut, you should remember the idea and reproduce the steps with the symbols or numbers in your problem. Summarizing,

To reduce eqn. (2.11) to one scalar equation in the one unknown $A$, use a judiciously chosen dot product. For example, get rid of the $\hat{\boldsymbol{\lambda}}_B$ or $\vec{\boldsymbol{B}}$ term by dotting both sides of with $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}_B$ (or, to save the trouble of finding the unit vector $\hat{\boldsymbol{\lambda}}_B$ just dot with $\hat{\boldsymbol{k}} \times \vec{\boldsymbol{B}}$). ①.

* The vector $\hat{\boldsymbol{k}}$ (the unit vector out of the page) is perpendicular to $\hat{\boldsymbol{\lambda}}_B$ but is unfortunately not suitable because it is also perpendicular to $\hat{\boldsymbol{\lambda}}_A$ and $\vec{\boldsymbol{C}}$ so only yields the equation $0 + 0 = 0$ or the nonsense that $A = 0/0$.

* This solution is identical to the Cramer's rule solution of eqn. (2.19) on page 73. That is, we have used dot products to derive Cramer's rule for $2 \times 2$ matrices.

① Here is a way to think about these judiciously chosen dot products: Get rid of things you don't know and don't care about. First take an interest in $A$. You don't know anything about $B$, and for a moment you don't care about it. So get rid of it. The two ways to get rid of a vector you don't know and don't care about are 1) dotting the force-balance equation with a vector orthogonal to the vector of disinterest, and 2) using moment balance about a point or axis about which the force of disinterest has no moment. A minute later you can do a replay and take an interest in $B$ instead.
(Whether you think of this as killing terms you don't like or as gently putting aside terms until you can deal with them kindly is a matter of personal disposition.)

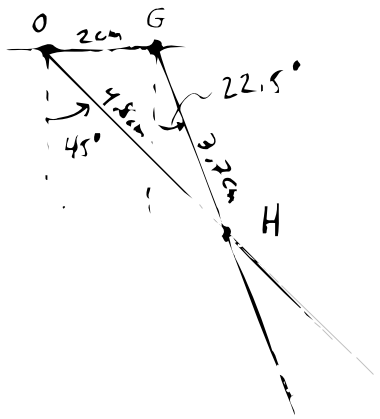



Figure 2.45: Method III: Find point H as the intersection of two known lines.
Filename:tfigure-2-walkIII

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Altogether you can think of this method as something like the "component" method. But we are taking components of the vectors in the direction perpendicular to $\vec{\boldsymbol{B}}$. Alternatively you can think of this method as taking the projection of the vector equation onto a line perpendicular to $\vec{\boldsymbol{B}}$.

Similarly dotting both sides of eqn. (2.11) with $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}_A$ gives

$$B = \frac{(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}_A) \cdot \vec{\boldsymbol{C}}}{(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}_A) \cdot \hat{\boldsymbol{\lambda}}_B}.$$

Example: **Solving "A walk": method II, judicious dot products**
You should be able to derive the formulas above as needed. Dotting, for example, both sides of eqn. (2.11) with $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}_B$ and plugging in the known components yields

$$\begin{aligned} A = \frac{(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}_B) \cdot \vec{\boldsymbol{C}}}{(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}_B) \cdot \hat{\boldsymbol{\lambda}}_A} &= \frac{\lambda_{By} C_x - \lambda_{Bx} C_y}{\lambda_{By}\lambda_{Ax} - \lambda_{Bx}\lambda_{Ay}} \\ &= \frac{\cos(\pi/8) \cdot 200\,\text{m} - (-\sin(\pi/8)) \cdot 0}{\cos(\pi/8) \cdot (\sqrt{2}/2) - (-\sin(\pi/8)) \cdot (-\sqrt{2}/2)} \\ &\approx 483\,\text{m} \qquad \text{(as before)} \end{aligned}$$

## Method III, graphical solution

On the vector triangle defined by $\vec{\boldsymbol{A}} + \vec{\boldsymbol{B}} = \vec{\boldsymbol{C}}$ we call O the tail end of $\vec{\boldsymbol{A}}$. The location of the tip of $\vec{\boldsymbol{C}}$ at G can be drawn to scale. Then the point H can be located as at the intersection of two lines: one emanating from O and in the direction of $\hat{\boldsymbol{\lambda}}_A$ and one emanating from H and in the direction of $\hat{\boldsymbol{\lambda}}_B$. Once the point H is located, the lengths $A$ and $B$ can be measured.

Example: **Solving "A walk": method III, graphing**
Taking 100 m as drawn to scale as, say 1 cm, point G is drawn 2 cm to the right of O. The location of the point H is found as the intersection of two lines: one emanating from O and pointing $45^\circ$ counterclockwise from the $-\hat{\boldsymbol{\jmath}}$ axis, and the other emanating from G and pointing $22.5^\circ$ counterclockwise from the $-\hat{\boldsymbol{\jmath}}$ axis. The distance from O to H can be measured as about 4.8 cm yielding $A \approx 480$ m.
This construction can be done with pencil and paper or with a computer drawing program.

## Method IV, trigonometry

The final method, the classical method used predominantly before vector notation was well accepted, is to treat the vector triangle as a triangle with some known sides and some known angles, and to use the *law of sines* (discussed in box 2.5 ).

Because $\vec{\boldsymbol{C}}$ and the directions of $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ are assumed known, the angles $a$ (opposite side $A$) and $b$ (opposite side $B$) are known. Because the sum of interior angles in a triangle is $\pi$ we know the angle $c = \pi - a - b$. The law of sines tells us that

$$\frac{\sin a}{A} = \frac{\sin c}{C} \quad \text{and} \quad \frac{\sin b}{B} = \frac{\sin c}{C}$$

which we can rewrite as

$$A = \frac{C \sin a}{\sin c} \quad \text{and} \quad B = \frac{C \sin b}{\sin c}.$$

Example: **Solving "A walk": method IV, the law of sines**
Referring to Fig. 2.46 we get

$$A = \frac{C \sin a}{\sin c} = \frac{200\,\text{m} \cdot \sin(5\pi/8)}{\sin(\pi/8)} \approx 483\,\text{m}$$

$$\text{and} \quad B = \frac{C \sin b}{\sin c} = \frac{200\,\text{m} \cdot \sin(\pi/4)}{\sin(\pi/8)} \approx 370\,\text{m}$$

as we have found three times already.

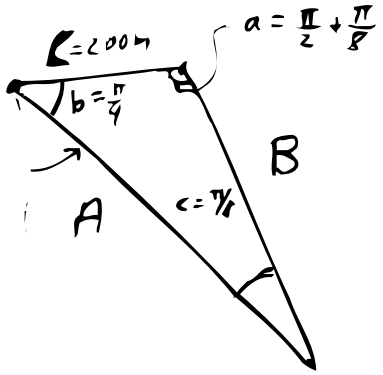



Figure 2.46: Solving "A walk" using the law of sines.
Filename:tfigure-2-walkIV

The determination of two vectors by knowing their directions and their sum is analogous to determination of a triangle by "angle-side-angle".

## The magnitudes and sum of two vectors are known (2D)

Two vectors $\vec{A}$ and $\vec{B}$ in the plane have known magnitudes $A$ and $B$ but unknown directions $\hat{\boldsymbol{\lambda}}_A$ and $\hat{\boldsymbol{\lambda}}_B$. Their sum $\vec{C}$ is known. So, measuring angles counterclockwise relative to the positive $x$ axis, we have:

$$\begin{aligned} \vec{A} + \vec{B} &= \vec{C} \\ A\hat{\boldsymbol{\lambda}}_A + B\hat{\boldsymbol{\lambda}}_B &= \vec{C} \\ A\left(\cos\theta_A \hat{\boldsymbol{\imath}} + \sin\theta_A \hat{\boldsymbol{\jmath}}\right) + B\left(\cos\theta_B \hat{\boldsymbol{\imath}} + \sin\theta_B \hat{\boldsymbol{\jmath}}\right) &= \vec{C} \end{aligned} \qquad (2.14)$$

where eqn. (2.14) is one 2D vector equation in 2 unknowns: $\theta_A$ and $\theta_B$.

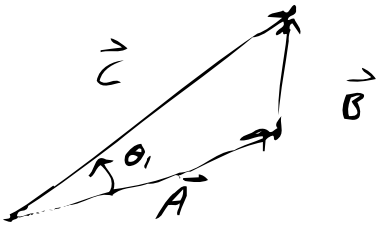



Figure 2.47: For use in using the law of cosines to solve a vector triangle.
Filename:tfigure-cosinestriangle

### Method 1: using an appropriate dot product

This problem is really best solved with trig (see below) and getting it right with component method is a matter of hindsight. Eqn. 2.14 can be rewritten as

$$C\left(\cos\theta_C \hat{\boldsymbol{\imath}} + \sin\theta_C \hat{\boldsymbol{\jmath}}\right) - A\left(\cos\theta_A \hat{\boldsymbol{\imath}} + \sin\theta_A \hat{\boldsymbol{\jmath}}\right) = B\left(\cos\theta_B \hat{\boldsymbol{\imath}} + \sin\theta_B \hat{\boldsymbol{\jmath}}\right)$$

Taking the dot product of each side with itself gives

$$C^2 + A^2 - 2AC \underbrace{(\cos\theta_C \cos\theta_A + \sin\theta_C \sin\theta_A)}_{\cos(\theta_C - \theta_A)} = B^2$$

so

$$\theta_A = \theta_C - \arccos\left(\frac{C^2 + A^2 - B^2}{2AC}\right).$$

Now $\vec{A}$ is fully determined and $\vec{B}$ can be found by vector subtraction. Note that the arccos function is always double valued (the negative of any arccos is also a legitimate arccos), so that the solution of this problem is not unique. Also, if the argument of the arccos function is greater than 1 in magnitude, there is no solution; this happens if any two of $A$, $B$, and $C$ is greater than the third (that is, if the so-called "triangle inequality" is violated) and there is no way of making a triangle with the given lengths.)

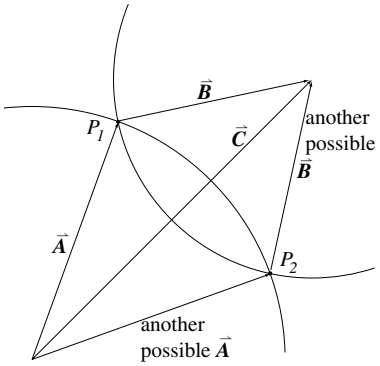



Figure 2.48: Solving a vector triangle where vectors $\vec{A}$ and $\vec{B}$ have known magnitude but unknown direction.

Filename:tfigure-2circs

### Method II: The law of cosines

Referring to Fig. 2.47, we can apply the law of cosines directly to get

$$B^2 = A^2 + C^2 - 2AB\cos\theta_B \quad (2.15)$$

which we can solve to get

$$\theta_1 = \arccos\left(\frac{C^2 + A^2 - B^2}{2AC}\right) \quad (2.16)$$

Thus the orientation of $\vec{A}$ is determined in relation to $\vec{C}$. This method is a bit quicker than the component method above because it skips the steps where, in effect, the component method derives the law of cosines.

### Method III: graphical construction

From the tail of $\vec{C}$ draw a circle with radius $A$ (see Fig. 2.48). From the tip of $\vec{C}$ draw a circle with radius $B$. For each of the two points of intersection, $P_1$ and $P_2$, a solution has been found. Vector $\vec{A}$ goes from the tail of $\vec{C}$ to, say, $P_1$, and $\vec{B}$ goes from $P_1$ to the tip of $\vec{C}$. An $\vec{A}$ and $\vec{B}$ based on $P_2$ is also a legitimate solution. Each pair is a legitimate solution to the problem. To get a unique solution set other information would have to be provided.

Determining a vector triangle when one vector is known and only the magnitudes of the other two are known is analogous to determining a triangle from "side-side-side" in geometry. It is interesting that this, the most elementary of all geometric constructions does not have an equally simple analytic representation.

## Find the magnitude of three vectors given their directions and their sum (3D)

This problem is close in approach to its junior 2D cousin on page 68 and to the example on page 65. It is the most common of the 3D vector equation problems. Assume that you know the directions of three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ (given, say, as the unit vectors $\hat{\boldsymbol{\lambda}}_A$, $\hat{\boldsymbol{\lambda}}_B$, and $\hat{\boldsymbol{\lambda}}_C$), as well as their sum $\vec{D}$. So we have

$$\begin{aligned} \vec{A} + \vec{B} + \vec{C} &= \vec{D} \\ A\hat{\boldsymbol{\lambda}}_A + B\hat{\boldsymbol{\lambda}}_B + C\hat{\boldsymbol{\lambda}}_C &= \vec{D} \end{aligned} \quad (2.17)$$

and we want to find $A$, $B$, and $C$ from which we can find $\vec{A}$, $\vec{B}$, and $\vec{C}$ ($e.g.$, $\vec{A} = A\hat{\boldsymbol{\lambda}}_A$). We can think of the last of eqn. (2.17) as one 3D vector equation in three unknowns.

In three dimensions the graphical approach is essentially impossible. And the trigonometric approach is awkward to say the least, and probably only generally practical for people with British accents who are long dead. The general ideas in the first two methods still stand, however. Thus the use of vector concepts is basically unavoidable in 3D problems.

### Method I: doting with $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.

We can dot the left and right sides of eqn. (2.17) with $\hat{\imath}$ or $\hat{\jmath}$ or $\hat{k}$. This is equivalent to taking the $x$, $y$ and $z$ components of the equation. We get then

$$\begin{aligned}
\hat{\imath}\cdot\{\text{eqn. (2.17)}\} \quad &\Rightarrow \quad A\lambda_{Ax}+B\lambda_{Bx}+C\lambda_{Cx}=D_x,\\
\hat{\jmath}\cdot\{\text{eqn. (2.17)}\} \quad &\Rightarrow \quad A\lambda_{Ay}+B\lambda_{By}+C\lambda_{Cy}=D_y, \text{ and}\\
\hat{k}\cdot\{\text{eqn. (2.17)}\} \quad &\Rightarrow \quad A\lambda_{Az}+B\lambda_{Bz}+C\lambda_{Cz}=D_z
\end{aligned} \tag{2.18}$$

which can be written in matrix form as

$$\begin{bmatrix} \lambda_{Ax} & \lambda_{Bx} & \lambda_{Cx} \\ \lambda_{Ay} & \lambda_{By} & \lambda_{Cy} \\ \lambda_{Az} & \lambda_{Bz} & \lambda_{Cz} \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix}. \tag{2.19}$$

Unless the matrix is sparse (has a lot of zeros as entries) it is probably best to solve such a set of equations for $A$, $B$ and $C$ on a computer or calculator.

### Method II: pick a vector for dot product that kills terms you don't know.

The philosophy here is the same as for method II in 2D (page 69). Pretend for a paragraph that you only want to find $A$ in eqn. (2.17). We can kill the terms involving the unknowns $B$ and $C$ by dotting both sides of the equation with a vector perpendicular to $\hat{\boldsymbol{\lambda}}_B$ and $\hat{\boldsymbol{\lambda}}_C$. Such a vector is $\hat{\boldsymbol{\lambda}}_B \times \hat{\boldsymbol{\lambda}}_C$. Thus

$$\begin{aligned}
&(\hat{\boldsymbol{\lambda}}_B \times \hat{\boldsymbol{\lambda}}_C)\cdot\{\text{eqn. (2.11)}\}\\
&\Rightarrow \quad (\hat{\boldsymbol{\lambda}}_B \times \hat{\boldsymbol{\lambda}}_C)\cdot\left(A\hat{\boldsymbol{\lambda}}_A + B\hat{\boldsymbol{\lambda}}_B + C\hat{\boldsymbol{\lambda}}_C\right) &&= (\hat{\boldsymbol{\lambda}}_B \times \hat{\boldsymbol{\lambda}}_C)\cdot\vec{\boldsymbol{D}}\\
&\Rightarrow \quad (\hat{\boldsymbol{\lambda}}_B \times \hat{\boldsymbol{\lambda}}_C)\cdot\left(A\hat{\boldsymbol{\lambda}}_A\right) + \vec{\mathbf{0}} + \vec{\mathbf{0}} &&= (\hat{\boldsymbol{\lambda}}_B \times \hat{\boldsymbol{\lambda}}_C)\cdot\vec{\boldsymbol{D}}\\
&\Rightarrow \quad A &&= \frac{\vec{\boldsymbol{D}}\cdot(\hat{\boldsymbol{\lambda}}_B \times \hat{\boldsymbol{\lambda}}_C)}{\hat{\boldsymbol{\lambda}}_A\cdot(\hat{\boldsymbol{\lambda}}_B \times \hat{\boldsymbol{\lambda}}_C).}
\end{aligned}$$

If you use a matrix determinant to evaluate the mixed triple product you can recognize this formula (like the formula solving the example on 65) as Cramer's rule. By a judicious dot product we have reduced the vector equation to a scalar equation in one unknown. Similarly we could get one equation in one unknown for $B$ or for $C$ by doting eqn. (2.17) with $\hat{\boldsymbol{\lambda}}_A \times \hat{\boldsymbol{\lambda}}_C$ and $\hat{\boldsymbol{\lambda}}_A \times \hat{\boldsymbol{\lambda}}_B$, respectively*.

## Parametric equations for lines and planes

### A line in 2D

In geometry a line on a plane is often describe as the set of $x$ and $y$ points that satisfy an equation like

$$Ax + By = D \qquad \text{or} \qquad y = mx + b$$

* Note that the key to the method was dotting with a vector in an appropriate direction, the magnitude of the vector did not matter. So if, for example, you knew any vector $\vec{\boldsymbol{v}}_B$ in the direction of $\hat{\boldsymbol{\lambda}}_B$ and any vector $\vec{\boldsymbol{v}}_C$ in the direction of $\hat{\boldsymbol{\lambda}}_C$ you could dot both sides of eqn. (2.17) with $\vec{\boldsymbol{v}}_B \times \vec{\boldsymbol{v}}_C$ to get one scalar equation for $A$. This can simplify calculations by avoiding the square roots (which cancel in the end) that you calculate to find unit vectors.

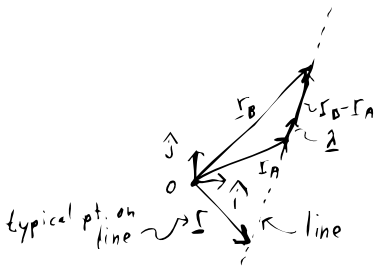



Figure 2.49: Parametric description of a line using vectors.

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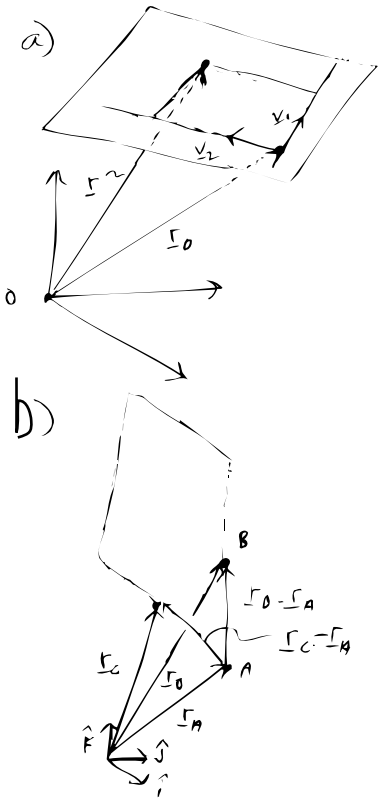



Figure 2.50: a) Parametric equation of a plane. b) the plane through the points A,B, and C

Filename:tfigure-parametricplane

for given $A$, $B$, and $D$ or $m$ and $b$. However a line is a "one dimensional" object and it is nice to describe it that way. The parametric form that is often useful is:

$$\vec{r} = \vec{r}_0 + s\vec{v} \tag{2.20}$$

where $\vec{r}$ are the position vectors of set of points on the line, one point for each value of the scalar parameter $s$. $\vec{r}_0$ is the position vector of one given reference point on the line and $\vec{v}$ is a vector parallel to the line. In the special case that $\vec{v}$ is a unit vector, $s$ is the distance from the point at $\vec{r}_0$ to the point at $\vec{r}$. If the vector $\vec{v}$ was the velocity of a point moving on the line then $s|\vec{v}|$ would be the distance of the point from the point at $\vec{r}_0$.

Example: **Parametric equation of a line**
A parametric equation for the line going through the points with position vectors $\vec{r}_\mathrm{A}$ and $\vec{r}_\mathrm{B}$ is

$$\vec{r} = \vec{r}_\mathrm{A} + s\left(\underbrace{\vec{r}_\mathrm{B} - \vec{r}_\mathrm{A}}_{\vec{v}}\right) \qquad \text{or better} \qquad \vec{r} = \vec{r}_\mathrm{A} + s\hat{\boldsymbol{\lambda}}_A$$

where $\hat{\boldsymbol{\lambda}}_A = \left(\vec{r}_\mathrm{B} - \vec{r}_\mathrm{A}\right) / \left|\vec{r}_\mathrm{B} - \vec{r}_\mathrm{A}\right|$

## A line in 3D

In three dimensions a line is often described geometrically as the intersection of two planes. But a line in three dimensions is still a one dimensional object so the parametric form eqn. (2.20), applicable in three dimensions as well as two, is nice.

## A plane

A plane in three dimensions can be described as the set of points $x$,$y$, and $z$ that satisfy an equation like:

$$Ax + By + Cz = D$$

for a given A,B,C, and D. The parametric description of a plane uses two parameters $s_1$ and $s_2$ and is

$$\vec{r} = \vec{r}_\mathrm{O} + s_1\vec{v}_1 + s_2\vec{v}_2 \tag{2.21}$$

where $\vec{r}$ is a typical point on the plane, $\vec{v}_1$ and $\vec{v}_2$ are any two non-parallel vectors that lie in the plane and $s_1$ and $s_2$ are any two real numbers. Each pair $(s_1, s_2)$ corresponds to one point in the plane and vice versa. The numbers $s_1$ and $s_2$ can be thought of as in-plane distance coordinates if the vectors $\vec{v}_1$ and $\vec{v}_2$ are mutually orthogonal unit vectors.

Example: **A plane**
A parametric equation for the plane going through the three points with position vectors $\vec{r}_\mathrm{A}$, $\vec{r}_\mathrm{B}$, and $\vec{r}_\mathrm{C}$ is

$$\vec{r} = \underbrace{\vec{r}_\mathrm{A}}_{\vec{r}_0} + s_1\underbrace{\left(\vec{r}_\mathrm{B} - \vec{r}_\mathrm{A}\right)}_{\vec{v}_1} + s_2\underbrace{\left(\vec{r}_\mathrm{C} - \vec{r}_\mathrm{A}\right)}_{\vec{v}_2}$$

You can check that when $s_1 = s_2 = 0$ the point on the plane $\vec{r}_\mathrm{A}$ is given. And when one of the $s$ values is one and the other zero the points $\vec{r}_\mathrm{B}$ and $\vec{r}_\mathrm{C}$ are given.

## Vectors, matrices, and linear algebraic equations

Once one has drawn a free body diagram and written the force and moment balance equations one is left with vector equations to solve for various unknowns. The vector equations of mechanics can be reduced to scalar equations by using dot products. The simplest dot product to use is with the unit vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$. This use of dot products is equivalent to taking the $x$, $y$, and $z$ components of the vector equation. The two vector equations

$$\begin{aligned} a\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}} &= (c-5)\hat{\boldsymbol{\imath}} + (d+7)\hat{\boldsymbol{\jmath}} \\ (a-c)\hat{\boldsymbol{\imath}} + (a+b)\hat{\boldsymbol{\jmath}} &= (c+b)\hat{\boldsymbol{\imath}} + (2a+c)\hat{\boldsymbol{\jmath}} \end{aligned}$$

with four scalar unknowns $a, b, c$, and $d$, can be rewritten as four scalar equations, two from each two-dimensional vector equation. Taking the dot product of the first equation with $\hat{\boldsymbol{\imath}}$ gives $a = c - 5$. Similarly dotting with $\hat{\boldsymbol{\jmath}}$ gives $b = d + 7$. Repeating the procedure with the second equation gives 4 scalar equations:

$$\begin{aligned} a &= c-5 \\ b &= d+7 \\ a-c &= c+b \\ a+b &= 2a+c. \end{aligned}$$

These equations can be re-arranged putting unknowns on the left side and knowns on the right side:

$$\begin{aligned} 1a + 0b + -1c + 0d &= -5 \\ 0a + 1b + 0c + -1d &= 7 \\ 1a + -1b + -2c + 0d &= 0 \\ -1a + 1b + -1c + 0d &= 0 \end{aligned}$$

These equations can in turn be written in standard matrix form. The standard matrix form is a short hand notation for writing (linear) equations, such as the equations above:

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & -1 & -2 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix}}_{[\boldsymbol{A}]} \cdot \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{[\boldsymbol{x}]} = \underbrace{\begin{bmatrix} -5 \\ 7 \\ 0 \\ 0 \end{bmatrix}}_{[\boldsymbol{y}]}$$

$$\Rightarrow \quad [\boldsymbol{A}] \cdot [\boldsymbol{x}] = [\boldsymbol{y}] .$$

The matrix equation $[\boldsymbol{A}] \cdot [\boldsymbol{x}] = [\boldsymbol{y}]$ is in a form that is easy to input to any of several programs that solve linear equations. The computer (or a do-able but probably untrustworthy hand calculation) should return the following solution for $[\boldsymbol{x}]$ ($a$, $b$, $c$, and $d$).

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \\ 0 \\ -12 \end{bmatrix} .$$

That is, $a = -5$, $b = -5$, $c = 0$, and $d = -12$. If you doubt the solution, check it. To check the answer, plug it back into the original matrix equation and note the equality (or lack thereof!). In this case, we have done our calculations correctly and

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & -1 & -2 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -5 \\ 0 \\ -12 \end{bmatrix} = \begin{bmatrix} -5 \\ 7 \\ 0 \\ 0 \end{bmatrix}.$$

Going back to the original vector equations we can also check that

$$\begin{aligned} -5\hat{\boldsymbol{\imath}} + -5\hat{\boldsymbol{\jmath}} &= (0-5)\hat{\boldsymbol{\imath}} + (-12+7)\hat{\boldsymbol{\jmath}} \\ (-5-0)\hat{\boldsymbol{\imath}} + (-5+-5)\hat{\boldsymbol{\jmath}} &= (0+-5)\hat{\boldsymbol{\imath}} + (2\cdot -5+0)\hat{\boldsymbol{\jmath}}. \end{aligned}$$

## Computer solution of simultaneous equations

Depending on your computer package you might solve the equations above like this

```
eqset  =  {  a     -  c      = -5
                b       - d  =  7
             a - b - 2c      =  0
            -a + b -  c      =  0}
Solve eqset for a,b,c,d.
```

Or, if your computer package is set up especially for linear algebra then you could write something analogous to this:

```
 M =  [  1  0 -1  0
         0  1  0 -1
         1 -1 -2  0
        -1  1 -1  0]
 b = [-5 7 0 0]'
Solve M*z = b for z % the elements of z are a,b,c,d
```

## 'Physical' vectors and row or column vectors

The word 'vector' has two related but subtly different meanings. One is a physical vector like $\vec{\boldsymbol{F}} = F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}} + F_z\hat{\boldsymbol{k}}$, a quantity with magnitude and direction. The other meaning is a list of numbers like the row vector

$$[\boldsymbol{x}] = [x_1, x_2, x_3]$$

or the column vector

$$[\boldsymbol{y}] = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

Once you have picked a basis, like $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$, you can represent a physical vector $\vec{\boldsymbol{F}}$ as a row vector $\begin{bmatrix} F_x, F_y, F_z \end{bmatrix}$ or a column vector $\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$. But the

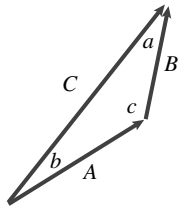
components of a given vector depend on the base coordinate system (or base vectors) that are used. For clarity it is best to distinguish a physical vector from a list of components using a notation like the following:

$$[\vec{\boldsymbol{F}}]_{XYZ} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$$

The square brackets around $\vec{\boldsymbol{F}}$ indicate that we are looking at its components. The subscript $XYZ$ identifies what coordinate system or base vectors are being used. The right side is a list of three numbers (in this case arranged as a column, the default arrangement in linear algebra).

## **2.5** *THEORY*
### *Vector triangles and the laws of sines and cosines*

The tip to tail rule of vector addition defines a triangle. Knowing something about the vectors in this triangle how can we find more? One approach is to use the laws of sines and cosines.

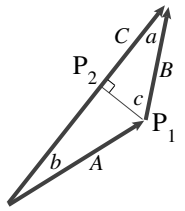



Consider the vector sum $\vec{\boldsymbol{A}} + \vec{\boldsymbol{B}} = \vec{\boldsymbol{C}}$ represented by the triangle shown with traditionally labeled sides $A$, $B$, and $C$ and internal angles $a$, $b$, and $c$. The sides and angles are related by

$$\frac{\sin a}{A} = \frac{\sin b}{B} = \frac{\sin c}{C} \qquad \text{the law of sines, and}$$

$$C^2 = A^2 + B^2 - 2AB\cos c \qquad \text{the law of cosines.}$$

**Proof of the law of sines** The first equality in the law of sines can be proved by calculating the altitude from $c$ two ways.

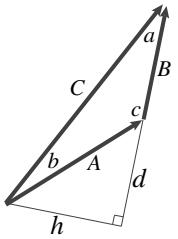



On the one hand length $P_1P_2$ is given by $P_1P_2 = B\sin a$ and on the other hand by $P_1P_2 = A\sin b$

so $B\sin a = A\sin b \quad \Rightarrow \quad \dfrac{\sin a}{A} = \dfrac{\sin b}{B}$.

We can do likewise with all three altitudes thus proving the triple equality.

**Proof of the law of cosines.** Look at altitude $h$ of the triangle.




This is the base of two different right triangles. So by the pythagorean theorem we have on the one hand that

$$h^2 = A^2 - d^2 \quad \text{and on the other that} \quad h^2 = C^2 - (B+d)^2.$$

Equating these expressions and expanding the square we get

$$\begin{aligned} A^2 - d^2 &= C^2 - (B^2 + d^2 - 2dB) \\ \Rightarrow \quad A^2 + B^2 + 2dB &= C^2 \end{aligned} \tag{2.22}$$

But $d = -A\cos c$ so

$$C^2 = A^2 + B^2 - 2AB\cos c.$$

Sometimes the angle we call $c$ is called $\theta$.

**Applications.** These laws are useful when you want to figure out the shape and size of a triangle when, of the six triangle quantities (thee sides and three angles), only 3 are given. At least one of these three has to be a length.

As noted, it is possible to give problems of this type that have no solutions. And it is possible to give problems that have either 1 or 2 solutions.

In this era where vector algebra is popular as is the representation of vectors in terms of their components, the laws of sines and cosines are used little. But sometimes they are the easiest approach.

## 2.6 *THEORY*
### *Existence, uniqueness, and geometry*

Sometimes there is a *unique* solution set to a set of simultaneous solutions. Sometimes its impossible to solve a set of vector equations; no solutions *exist*. And sometimes there are lots of solutions; solutions *exist* but are not *unique*. These cases sometimes have simple geometric interpretations.

**Example 1.** Consider a very simple equation

$$a\,\vec{\boldsymbol{v}}_1 = \vec{\boldsymbol{w}}$$

where $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{w}}$ are given and you are to find $a$. The left hand side is a parametric expression for points on a line through the origin in the direction $\vec{\boldsymbol{v}}_1$.

- if $\vec{\boldsymbol{w}}$ is parallel to $\vec{\boldsymbol{v}}$ then the equation has exactly one solution for $a$;
- if $\vec{\boldsymbol{w}}$ is not parallel to $\vec{\boldsymbol{v}}$ then there is no possible $a$ that could make the equation true. The equation has no solutions.

This vector equation is equivalent to 2 scalar equations (3 in 3D) with one scalar unknown and we expect generally to find no solution. That is, two random vectors $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{w}}$ are unlikely to be parallel either in 2D or 3D.

**Example 2.** Now consider this 2D vector equation in two unknown scalars $a$ and $b$:

$$a\,\vec{\boldsymbol{v}}_1 + b\,\vec{\boldsymbol{v}}_2 = \vec{\boldsymbol{w}}.$$

- If $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$ are not parallel $a\,\vec{\boldsymbol{v}}_1 + b\,\vec{\boldsymbol{v}}_2$ could be, with appropriate choice of $a$ and $b$, any 2D vector. There would be a unique solution for every possible $\vec{\boldsymbol{w}}$.
- But if $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$ are parallel then the expression $a\,\vec{\boldsymbol{v}}_1 + b\,\vec{\boldsymbol{v}}_2$ describes a line.
  - If $\vec{\boldsymbol{w}}$ is on this line there are many solutions for $a$ and $b$ because the two vectors $a\,\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$ can be added various ways that partially cancel.
  - If $\vec{\boldsymbol{w}}$ is off the line then there are no combinations of $a$ and $b$ that get vectors off the line, there are no solutions.

In 2D a test to see if two vectors are parallel is to take their cross product. So, if

$$(\vec{\boldsymbol{v}}_1 \times \vec{\boldsymbol{v}}_2)\cdot\hat{\boldsymbol{k}} = v_{1x}v_{2y} - v_{1y}v_{2x} = \det\begin{bmatrix} v_{1x} & v_{2x} \\ v_{1y} & v_{2y} \end{bmatrix} = 0$$

then $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$ are parallel and there are either many solutions or no solutions depending on whether or not $\vec{\boldsymbol{w}}$ is also parallel to $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$.

**Example 3.** Consider the same example as above but now in 3D.

$$a\,\vec{\boldsymbol{v}}_1 + b\,\vec{\boldsymbol{v}}_2 = \vec{\boldsymbol{w}}.$$

Now the question is whether the vector $\vec{\boldsymbol{w}}$ is in the plane described parametrically by $a\,\vec{\boldsymbol{v}}_1 + b\,\vec{\boldsymbol{v}}_2$. We have more equations than unknowns, $3 > 2$ so solution should be unlikely. Given 3 random vectors in 3D $\vec{\boldsymbol{v}}_1$, $\vec{\boldsymbol{v}}_2$ and $\vec{\boldsymbol{w}}$, it is unlikely that $\vec{\boldsymbol{w}}$ would be in the plane determined by $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$. If $\vec{\boldsymbol{w}}$ is in that plane, we get again the three possibilities from the previous example.

**Example 4.** Finally consider this common equation in 3D.

$$a\,\vec{\boldsymbol{v}}_1 + b\,\vec{\boldsymbol{v}}_2 + c\,\vec{\boldsymbol{v}}_3 = \vec{\boldsymbol{w}}. \tag{2.23}$$

where $\vec{\boldsymbol{v}}_1$, $\vec{\boldsymbol{v}}_2$, $\vec{\boldsymbol{v}}_3$, and $\vec{\boldsymbol{w}}$ are given vectors and $a$, $b$ and $c$ are unknowns.

- If $\vec{\boldsymbol{v}}_1$, $\vec{\boldsymbol{v}}_2$, and $\vec{\boldsymbol{v}}_3$ are not co-planar, then by imagining flying in through space in each of three directions, you can see that you can get to any point in space $\vec{\boldsymbol{w}}$ by using one and only one set of multiples $a$, $b$ and $c$ of the three vectors.
- On the other hand, if $\vec{\boldsymbol{v}}_1$, $\vec{\boldsymbol{v}}_2$, and $\vec{\boldsymbol{v}}_3$ are co-planar, they are redundant, and
  - there can only be a solution if $\vec{\boldsymbol{w}}$ is on the plane and, assuming the three vectors are not also colinear, there are many solutions. There are various ways for combinations of $\vec{\boldsymbol{v}}_1$, $\vec{\boldsymbol{v}}_2$, and $\vec{\boldsymbol{v}}_3$ to cancel each other out.
  - if $\vec{\boldsymbol{w}}$ is off this plane there are no solutions.

If the vectors $\vec{\boldsymbol{v}}_1$, $\vec{\boldsymbol{v}}_2$, and $\vec{\boldsymbol{v}}_3$ are coplanar then there are either no solutions for $a$,$b$ and $c$ or many solutions. We can test for coplanarity of $\vec{\boldsymbol{v}}_1$, $\vec{\boldsymbol{v}}_2$, and $\vec{\boldsymbol{v}}_3$ with geometric reasoning and cross products. The vector $\vec{\boldsymbol{v}}_1 \times \vec{\boldsymbol{v}}_2$ is orthogonal to the plane of $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$. So, if $\vec{\boldsymbol{v}}_3$ is in the plane defined by $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$ it will be orthogonal to $\vec{\boldsymbol{v}}_1 \times \vec{\boldsymbol{v}}_2$. Thus if

$$(\vec{\boldsymbol{v}}_1 \times \vec{\boldsymbol{v}}_2)\cdot\vec{\boldsymbol{v}}_3 = 0$$

the three vectors are co-planar and . This test can also be written as

$$\det\begin{bmatrix} v_{1x} & v_{2x} & v_{3x} \\ v_{1y} & v_{2y} & v_{3y} \\ v_{1z} & v_{2z} & v_{3z} \end{bmatrix} = 0$$

which is what we would expect from considering the matrix form of eqn. (2.23)

$$\begin{bmatrix} v_{1x} & v_{2x} & v_{3x} \\ v_{1y} & v_{2y} & v_{3y} \\ v_{1z} & v_{2z} & v_{3z} \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$$

and checking to see if the $3\times 3$ matrix is "singular" (a linear algebra word meaning that the determinant is zero).

**Relation to more general linear algebra.** For systems of equations in 4 or more dimensions we can't use our geometric intuition quite so directly. But the cases above are analagous to what one always finds. The geometric interpretations are helpful for gaining an intuition, even in higher than 3 dimensions when they don't strictly hold. Consider the matrix equation

$$M * v = b$$

with the square matrix $M$ and the column vector $b$ given.

- If the columns of $M$ are not redundant (e.e., they are linearly independent) then there exists a unique $v$ for any $b$. This is like having $\vec{\boldsymbol{v}}_1$, $\vec{\boldsymbol{v}}_2$, $\vec{\boldsymbol{v}}_3$ not coplanar in 3D.
- If the columns of $M$ are redundant (e.g., they are linearly dependent) this is like having coplanar $\vec{\boldsymbol{v}}_1$, $\vec{\boldsymbol{v}}_2$, $\vec{\boldsymbol{v}}_3$ and
  - if $b$ is in the span of the columns of $M$, like $\vec{\boldsymbol{w}}$ being in the plane, there are many solutions, and
  - if $b$ is not in the span of the columns of $M$, like $\vec{\boldsymbol{W}}$ being off the plane, there are no solutions.

**SAMPLE 2.29 Plain vanilla vector equation in 2-D:** Three forces act on a particle as shown in the figure. The equilibrium condition of the particle requires that $\vec{F}_1 + \vec{F}_2 + \vec{W} = \vec{0}$. It is given that $\vec{W} = -20\,\text{N}\,\hat{j}$. Find the magnitudes of forces $\vec{F}_1$ and $\vec{F}_2$.

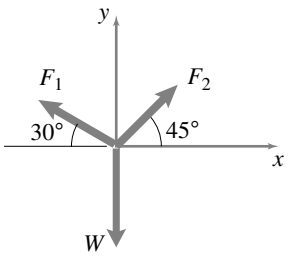



Figure 2.51:
Filename:sfig2-4-veceqn-1

**Solution** We are given a vector equation, $\vec{F}_1 + \vec{F}_2 + \vec{W} = \vec{0}$, in which one vector $\vec{W}$ is completely known and the directions of the other two vectors are given. We need to find their magnitudes. Let us write the vectors as

$$\vec{F}_1 = F_1\hat{\lambda}_1, \quad \vec{F}_2 = F_2\hat{\lambda}_2, \quad \vec{W} = -W\,\hat{j},$$

where $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are unit vectors along $\vec{F}_1$ and $\vec{F}_2$, respectively (their directions are specified by the given angles in the figure), and $W = 20\,\text{N}$ as given. We can write the unit vectors in component form as

$$\hat{\lambda}_1 = \lambda_{1_x}\hat{\imath} + \lambda_{1_y}\hat{\jmath} \quad \text{and} \quad \hat{\lambda}_2 = \lambda_{2_x}\hat{\imath} + \lambda_{2_y}\hat{\jmath}.$$

Now we can write the given vector equation as

$$F_1(\lambda_{1_x}\hat{\imath} + \lambda_{1_y}\hat{\jmath}) + F_2(\lambda_{2_x}\hat{\imath} + \lambda_{2_y}\hat{\jmath}) = W\,\hat{\jmath}. \tag{2.24}$$

Dotting both sides of eqn. (2.24) with $\hat{\imath}$ and $\hat{\jmath}$ respectively, we get

$$\lambda_{1_x}F_1 + \lambda_{2_x}F_2 = 0 \tag{2.25}$$
$$\lambda_{1_y}F_1 + \lambda_{2_y}F_2 = W. \tag{2.26}$$

Here, we have two equations in two unknowns ($F_1$ and $F_2$). We can solve these equations for the unknowns. Let us solve these two linear equations by first putting them into a matrix form and then solving the matrix equation. The matrix equation is

$$\begin{bmatrix} \lambda_{1_x} & \lambda_{2_x} \\ \lambda_{1_y} & \lambda_{2_y} \end{bmatrix}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} 0 \\ W \end{pmatrix}.$$

Using Cramer's rule for matrix inversion, we get

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \frac{1}{\lambda_{1_x}\lambda_{2_y} - \lambda_{2_x}\lambda_{1_y}}\begin{bmatrix} \lambda_{2_y} & -\lambda_{2_x} \\ -\lambda_{1_y} & \lambda_{1_x} \end{bmatrix}\begin{pmatrix} 0 \\ W \end{pmatrix}.$$

Substituting the numerical values of $\lambda_{1_x} = -\cos 30^\circ = -\sqrt{3}/2$, $\lambda_{1_y} = \sin 30^\circ = 1/2$ and similarly, $\lambda_{2_x} = 1/\sqrt{2}$, $\lambda_{2_y} = 1/\sqrt{2}$, and $W = 20\,\text{N}$, we get

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} 14.64 \\ 17.93 \end{pmatrix}\text{N}.$$

$$\boxed{F_1 = 14.64\,\text{N}, \quad F_2 = 17.93\,\text{N}}$$

**Check:** We can easily check if the values we have got are correct. For example, substituting the numerical values in eqn. (2.25), we get

$$14.64\,\text{N}\cdot\left(-\frac{\sqrt{3}}{2}\right) + 17.93\,\text{N}\cdot\frac{1}{\sqrt{2}} \overset{\surd}{=} 0.$$

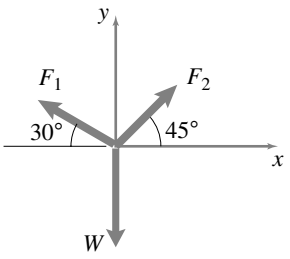



Figure 2.52:
Filename:sfig2-4-veceqn-2

**SAMPLE 2.30 Solving for a single unknown from a 2-D vector equation:** Consider the same problem as in Sample 2.29. That is, you are given that $\vec{F}_1+\vec{F}_2+\vec{W}=\vec{0}$ where $\vec{W}=-20\,\text{N}\,\hat{\jmath}$ and $\vec{F}_1$ and $\vec{F}_2$ act along the directions shown in the figure. Find the magnitude of $\vec{F}_2$.

**Solution** Once again, we write the given vector equation as

$$F_1\hat{\boldsymbol{\lambda}}_1 + F_2\hat{\boldsymbol{\lambda}}_2 = W\hat{\jmath},$$

where

$$\begin{aligned} W &= 20\,\text{N}, \\ \hat{\boldsymbol{\lambda}}_1 &= \lambda_{1_x}\hat{\imath} + \lambda_{1_y}\hat{\jmath} \\ &= -\sqrt{3}/2\hat{\imath} + 1/2\hat{\jmath}, \quad \text{and} \\ \hat{\boldsymbol{\lambda}}_2 &= \lambda_{2_x}\hat{\imath} + \lambda_{2_y}\hat{\jmath} \\ &= 1/\sqrt{2}(\hat{\imath}+\hat{\jmath}). \end{aligned}$$

We are interested in finding $F_2$ only. So, let us take a dot product of this equation with a vector that gets rid of the $F_1$ term. Any such vector would have to be perpendicular to $\hat{\boldsymbol{\lambda}}_1$. One such vector is $\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\lambda}}_1$. Let us call this vector $\hat{\boldsymbol{n}}_1$, that is,

$$\hat{\boldsymbol{n}}_1 = \hat{\boldsymbol{k}}\times(\lambda_{1_x}\hat{\imath}+\lambda_{1_y}\hat{\jmath}) = \lambda_{1_x}\hat{\jmath} - \lambda_{1_y}\hat{\imath}$$

. Now, dotting the given vector equation with $\hat{\boldsymbol{n}}_1$, we get

$$\begin{aligned} F_1\overbrace{(\hat{\boldsymbol{n}}_1\cdot\hat{\boldsymbol{\lambda}}_1)}^{0} + F_2(\hat{\boldsymbol{n}}_1\cdot\hat{\boldsymbol{\lambda}}_2) &= W(\hat{\boldsymbol{n}}_1\cdot\hat{\jmath}) \\ \Rightarrow \quad F_2 &= W\,\frac{\hat{\boldsymbol{n}}_1\cdot\hat{\jmath}}{\hat{\boldsymbol{n}}_1\cdot\hat{\boldsymbol{\lambda}}_2} \\ &= W\,\frac{(\lambda_{1_x}\hat{\jmath}-\lambda_{1_y}\hat{\imath})\cdot\hat{\jmath}}{(\lambda_{1_x}\hat{\jmath}-\lambda_{1_y}\hat{\imath})\cdot(\lambda_{2_x}\hat{\imath}+\lambda_{2_y}\hat{\jmath})} \\ &= W\,\frac{\lambda_{1_x}}{\lambda_{1_x}\lambda_{2_y}-\lambda_{1_y}\lambda_{2_x}} \\ &= 20\,\text{N}\,\frac{-\sqrt{3}/2}{-\sqrt{3}/2\cdot 1/\sqrt{2}-1/2\cdot 1/\sqrt{2}} \\ &= 20\,\text{N}\,\frac{\sqrt{6}}{\sqrt{3}+1} = 17.93\,\text{N} \end{aligned}$$

which, of course, is the same value we got in Sample 2.29. Note that here we obtained one scalar equation in one unknown by dotting the 2-D vector equation with an appropriate vector to get rid of the other unknown $F_1$.

$$\boxed{F_2 = 17.93\,\text{N}}$$

**SAMPLE 2.31 Solving a 3-D vector equation on a computer:** Four forces, $\vec{F}_1$, $\vec{F}_2$, $\vec{F}_3$ and $\vec{N}$ are in equilibrium, that is, $\vec{F}_1 + \vec{F}_2 + +\vec{F}_3 + \vec{N} = \vec{0}$ where $\vec{N} = -100\,\text{kN}\hat{k}$ is known and the directions of the other three forces are known. $\vec{F}_1$ is directed from (0,0,0) to (1,-1,1), $\vec{F}_2$ from (0,0,0) to (-1,-1,1), and $\vec{F}_3$ from (0,0,0) to (0,1,1). Find the magnitudes of $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$.

**Solution** Let $\vec{F}_1 = F_1\hat{\lambda}_1$, $\vec{F}_2 = F_2\hat{\lambda}_2$, and $\vec{F}_1 = F_3\hat{\lambda}_3$, where $\hat{\lambda}_1$, $\hat{\lambda}_2$ and $\hat{\lambda}_3$ are unit vectors in the directions of $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$, respectively. Then the given vector equation can be written as

$$F_1\hat{\lambda}_1 + F_2\hat{\lambda}_2 + F_3\hat{\lambda}_3 = -\vec{N} = -N\hat{k}$$

where $N = -100\,\text{kN}$. Dotting this equation with $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ respectively, and realizing that $\hat{\imath}\cdot\hat{\lambda}_1 = \lambda_{1_x}$, $\hat{\jmath}\cdot\hat{\lambda}_1 = \lambda_{1_y}$, etc., we get the following three scalar equations.

$$\begin{aligned} \lambda_{1_x}F_1 + \lambda_{2_x}F_2 + \lambda_{3_x}F_3 &= 0 \\ \lambda_{1_y}F_1 + \lambda_{2_y}F_2 + \lambda_{3_x}F_3 &= 0 \\ \lambda_{1_z}F_1 + \lambda_{2_z}F_2 + \lambda_{3_z}F_3 &= -N. \end{aligned}$$

Thus we get a system of three linear equations in three unknowns. To solve for the unknowns, we set up these equations as a matrix equation and then use a computer to solve it. In matrix form these equations are

$$\begin{bmatrix} \lambda_{1_x} & \lambda_{2_x} & \lambda_{3_x} \\ \lambda_{1_y} & \lambda_{2_y} & \lambda_{3_y} \\ \lambda_{1_z} & \lambda_{2_z} & \lambda_{3_z} \end{bmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -N \end{pmatrix}.$$

To solve this equation on a computer, we need to input the matrix of unit vector components and the known vector on the right hand side. From the given coordinates for the directions of forces, we have $\hat{\lambda}_1 = (\hat{\imath} - \hat{\jmath} + \hat{k})/\sqrt{3}$, $\hat{\lambda}_2 = (-\hat{\imath} - \hat{\jmath} + \hat{k})/\sqrt{3}$, and $\hat{\lambda}_3 = (\hat{\jmath} + \hat{k})/\sqrt{2}$. * We are also given that $N = -100\,\text{kN}$. Now, we use the following pseudo-code to find the solution on a computer.

```
Let s2 = sqrt(2),  s3 = sqrt(3)
    A = [ 1/s3  -1/s3  0
         -1/s3  -1/s3  1/s2
          1/s3   1/s3  1/s2 ]
    b = [ 0   0   100]'
    solve A*F = b for F
```

Using this pseudo-code we find the solution to be

```
    F = [ 43.3013
          43.3013
          70.7107 ]
```

That is, $F_1 = F_2 = 43.3\,\text{kN}$ and $F_3 = 70.7\,\text{kN}$.

$$\boxed{F_1 = 43.3\,\text{kN},\ F_2 = 43.3\,\text{kN},\ F_3 = 70.7\,\text{kN}}$$

* These unit vectors are computed by taking a vector from one end point to the other end point (as given) and then dividing by its magnitude. For example, we find $\hat{\lambda}_1$ by first finding $\vec{r}_1 = (1)\hat{\imath} + (-1)\hat{\jmath} + (1)\hat{k}$, a vector from (0,0,0) to (1,-1,1), and then $\hat{\lambda}_1 = \frac{\vec{r}_1}{|\vec{r}_1|}$.

**SAMPLE 2.32 Vector operations on a computer:** Consider the problem of Sample 2.31 again. That is, you are given the vector equation $\vec{F}_1 + \vec{F}_2 + +\vec{F}_3 + \vec{N} = \vec{0}$ where $\vec{N} = -100\,\text{kN}\hat{k}$ and the directions of $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$ are given by the unit vectors $\hat{\lambda}_1 = (\hat{\imath} - \hat{\jmath} + \hat{k})/\sqrt{3}$, $\hat{\lambda}_2 = (-\hat{\imath} - \hat{\jmath} + \hat{k})/\sqrt{3}$, and $\hat{\lambda}_3 = (\hat{\jmath} + \hat{k})/\sqrt{2}$, respectively. Find $F_1$.

**Solution** We can, of course, solve the problem as we did in Sample 2.31 and we get the answer as a part of the unknown forces we solved for. However, we would like to show here that we can extract one scalar equation in just one unknown ($F_3$) from the given 3-D vector equation and solve for the unknown without solving a matrix equation. Although we can carry out all required calculations by hand, we will show how we can use a computer to do these operations.

We can write the given vector equation as

$$F_1\hat{\lambda}_1 + F_2\hat{\lambda}_2 + F_3\hat{\lambda}_3 = -\vec{N}. \tag{2.27}$$

We want to find $F_1$. Therefore, we should dot this equation with a vector that gets rid of both $F_2$ and $F_3$, i.e., with a vector which is perpendicular to both $\vec{F}_2$ and $\vec{F}_3$. One such vector is $\vec{F}_2 \times \vec{F}_3$ or $\hat{\lambda}_2 \times \hat{\lambda}_3$. Let $\hat{n} = \hat{\lambda}_2 \times \hat{\lambda}_3$. Now, dotting both sides of eqn. (2.27) with $\hat{n}$, we get

$$F_1(\hat{\lambda}_1 \cdot \hat{n}) + F_2(\hat{\lambda}_2 \cdot \hat{n}) + F_3(\hat{\lambda}_3 \cdot \hat{n}) = -\vec{N} \cdot \hat{n}$$

Since $\hat{\lambda}_2 \cdot \hat{n} = 0$ and $\hat{\lambda}_3 \cdot \hat{n} = 0$ ($\hat{n}$ is normal to both $\hat{\lambda}_2$ and $\hat{\lambda}_3$), we get

$$\begin{aligned} F_1(\hat{\lambda}_1 \cdot \hat{n}) &= -\vec{N} \cdot \hat{n} \\ \Rightarrow \quad F_1 &= \frac{-\vec{N} \cdot \hat{n}}{\hat{\lambda}_1 \cdot \hat{n}}. \end{aligned}$$

Thus we have found the solution. To compute the expression on the right hand side of the above equation we use the following pseudo-code which assumes that you have written (or have access to) two functions, `dot` and `cross`, that compute the dot and cross product of two given vectors.

```
lambda_1 = 1/sqrt(3)*[1  -1  1]';
lambda_2 = 1/sqrt(3)*[-1  -1  1]';
lambda_3 = 1/sqrt(2)*[0  1  1]';
       N = [0  0  -100]';
       n = cross(lambda_2, lambda_3);
      F1 = - dot(N, n)/dot(lambda_1, n)
```

By following these steps on a computer, we get the output `F1 = 43.3013`, that is, $F_1 = 43.3\,\text{kN}$, which, of course, is the same answer we obtained in Sample 2.31.

$\boxed{F_1 = 43.3\,\text{kN}}$

## 2.5 Equivalent force systems

Most often one does not want to know the complete details of all the forces acting on a system. When you think of the force of the ground on your bare foot you do not think of the thousands of little forces at each micro-asperity or the billions and billions of molecular interactions between the wood (say) and your skin. Instead you think of some kind of equivalent force. In what way equivalent? Well, because all that the equations of mechanics know about forces is their net force and net moment, you have a criterion. You replace the actual force system with a simpler force system, possibly just a single well-placed force, that has the same total force and same total moment with respect to a reference point C.

The replacement of one system with an equivalent system is often used to help simplify or solve mechanics problems. Further, the concept of equivalent force systems allows us to define a *couple*, a concept we will use throughout the book. Here is the definition of the word *equivalent*[*] when applied to force systems in mechanics.

Two force systems are said to be *equivalent* if they have the same sum (the same *resultant*) and the same net moment about any one point C.

[*] Other phrases used to describe the same concept include: *statically equivalent*, *mechanically equivalent*, and *equipollent*.

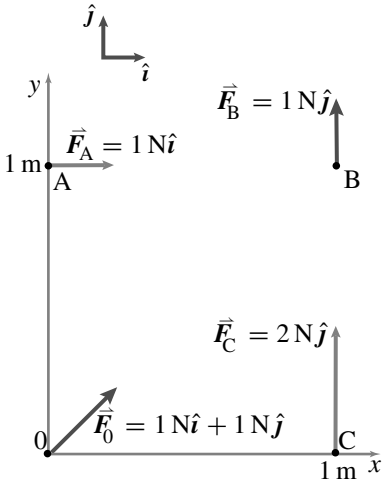



Figure 2.53: The force system $\vec{F}_A, \vec{F}_C$ is equivalent to the force system $\vec{F}_0, \vec{F}_B$.

Filename:tfigure-equivforcepair

We have already discussed two important cases of equivalent force systems. On page 20 we stated the mechanics assumption that a set of forces applied at one point is equivalent to a single resultant force, their sum, applied at that point. Thus when doing a mechanics analysis you can replace a collection of forces at a point with their sum. If you think of your whole foot as a 'point' this justifies the replacement of the billions of little atomic ground contact forces with a single force.

On page 46 we discovered that a force applied at a different point is equivalent to the same force applied at a point displaced in the direction of the force. You can thus harmlessly slide the point of force application along the line of the force.

More generally, we can compare two sets of forces. The first set consists of $\vec{F}_1^{(1)}, \vec{F}_2^{(1)}, \vec{F}_3^{(1)}, etc.$ applied at positions $\vec{r}_{1/C}^{(1)}, \vec{r}_{2/C}^{(1)}, \vec{r}_{3/C}^{(1)}, etc.$ In short hand, these forces are $\vec{F}_i^{(1)}$ applied at positions $\vec{r}_{i/C}^{(1)}$, where each value of $i$ describes a different force ($i = 7$ refers to the seventh force in the set). The second set of forces consists of $\vec{F}_j^{(2)}$ applied at positions $\vec{r}_{j/C}^{(2)}$ where each value of $j$ describes a different force in the second set.

Now we compare the net (resultant) force and net moment of the two sets. If

$$\vec{F}_{\text{tot}}^{(1)} = \vec{F}_{\text{tot}}^{(2)} \qquad \text{and} \qquad \vec{M}_C^{(1)} = \vec{M}_C^{(2)} \tag{2.28}$$

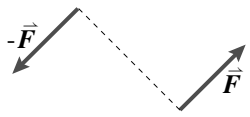



Figure 2.55: One couple. The forces add to zero. The net moment they cause does not.

Filename:tfigure-onecouple

then the two sets are *equivalent*. Here we have defined the net forces and net moments by

$$\vec{F}^{(1)}_{\text{tot}} = \sum_{\text{all forces } i} \vec{F}^{(1)}_i, \qquad \vec{M}^{(1)}_{\text{C}} = \sum_{\text{all forces } i} \vec{r}^{(1)}_{i/C} \times \vec{F}^{(1)}_i,$$
$$\vec{F}^{(2)}_{\text{tot}} = \sum_{\text{all forces } j} \vec{F}^{(2)}_j, \quad \text{and} \quad \vec{M}^{(2)}_{\text{C}} = \sum_{\text{all forces } j} \vec{r}^{(2)}_{j/C} \times \vec{F}^{(2)}_j.$$

If you find the $\sum$ (sum) symbol intimidating see box 2.5 on page 85.

Example:
Consider force system (1) with forces $\vec{F}_{\text{A}}$ and $\vec{F}_{\text{C}}$ and force system (2) with forces $\vec{F}_0$ and $\vec{F}_{\text{B}}$ as shown in fig. 2.53. Are the systems equivalent? First check the sum of forces.

$$\begin{aligned}
\vec{F}^{(1)}_{\text{tot}} &\stackrel{?}{=} \vec{F}^{(2)}_{\text{tot}} \\
\sum \vec{F}^{(1)}_i &\stackrel{?}{=} \sum \vec{F}^{(2)}_j \\
\vec{F}_{\text{A}} + \vec{F}_{\text{C}} &\stackrel{?}{=} \vec{F}_0 + \vec{F}_{\text{B}} \\
1\,\text{N}\hat{\boldsymbol{\imath}} + 2\,\text{N}\hat{\boldsymbol{\jmath}} &\stackrel{\surd}{=} (1\,\text{N}\hat{\boldsymbol{\imath}} + 1\,\text{N}\hat{\boldsymbol{\jmath}}) + 1\,\text{N}\hat{\boldsymbol{\jmath}}
\end{aligned}$$

Then check the sum of moments about C.

$$\begin{aligned}
\vec{M}^{(1)}_{\text{C}} &\stackrel{?}{=} \vec{M}^{(2)}_{\text{C}} \\
\sum \vec{r}^{(1)}_{i/C} \times \vec{F}^{(1)}_i &\stackrel{?}{=} \sum \vec{r}^{(2)}_{j/C} \times \vec{F}^{(2)}_j \\
\vec{r}_{\text{A/C}} \times \vec{F}_{\text{A}} + \vec{r}_{\text{C/C}} \times \vec{F}_{\text{C}} &\stackrel{?}{=} \vec{r}_{0/\text{C}} \times \vec{F}_0 + \vec{r}_{\text{B/C}} \times \vec{F}_{\text{B}} \\
(-1\,\text{m}\hat{\boldsymbol{\imath}} + 1\,\text{m}\hat{\boldsymbol{\jmath}}) \times 1\,\text{N}\hat{\boldsymbol{\imath}} + \vec{\mathbf{0}} \times 2\,\text{N}\hat{\boldsymbol{\jmath}} &\stackrel{?}{=} (-1\,\text{m}\hat{\boldsymbol{\imath}}) \times (1\,\text{N}\hat{\boldsymbol{\imath}} + 1\,\text{N}\hat{\boldsymbol{\jmath}}) + 1\,\text{m}\hat{\boldsymbol{\jmath}} \times 1\,\text{N}\hat{\boldsymbol{\jmath}} \\
-1\,\text{m N}\hat{\boldsymbol{k}} &\stackrel{\surd}{=} -1\,\text{m N}\hat{\boldsymbol{k}}
\end{aligned}$$

So the two force systems are indeed equivalent.

What is so special about the point C in the example above? Nothing.

If two force systems are equivalent with respect to some point C, they are equivalent with respect to *any* point.

For example, both of the force systems in the example above have the same moment of $2\,\text{N m}\hat{\boldsymbol{k}}$ about the point A. See box 2.5 for the proof of the general case.

Example: **Frictionless wheel bearing**
If the contact of an axle with a bearing housing is perfectly frictionless then each of the contact forces has no moment about the center of the wheel. Thus the whole force system is equivalent to a single force at the center of the wheel.

## Couples

Consider a pair of equal and opposite forces that are not colinear. Such a pair is called a *couple*.The net moment caused by a couple is the size of the force

times the perpendicular distance between the two lines of action and doesn't depend on the reference point. In fact, any force system that has $\vec{F}_{\text{tot}} = \vec{0}$ causes the same moment about all different reference points (as shown at the end of box 2.5).So, in modern usage, any force system with any number of forces and with $\vec{F}_{\text{tot}} = \vec{0}$ is called a couple. A couple is described by its net moment.*

*People who have been in difficult long term relationships don't need a mechanics text to know that a couple is a pair of equal and opposite forces that push each other around.

A *couple* is any force system that has a total force of $\vec{0}$. It is described by the net moment $\vec{M}$ that it causes.

We then think of $\vec{M}$ as representing an equivalent force system that contributes $\vec{0}$ to the net force and $\vec{M}$ to the net moment with respect to every reference point.

The concept of a couple (also called an applied moment or an applied torque) is especially useful for representing the net effect of a complicated collection of forces that causes some turning. The complicated set of electromagnetic forces turning a motor shaft can be replaced by a couple.

## 2.7 $\sum$ means add

In mechanics we often need to add up lots of things: all the forces on a body, all the moments they cause, all the mass of a system, *etc*. One notation for adding up all 14 forces on some body is

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 + \vec{F}_5 + \vec{F}_6 + \vec{F}_7 \\ &\quad + \vec{F}_8 + \vec{F}_9 + \vec{F}_{10} + \vec{F}_{11} + \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14}.\end{aligned}$$

which is a bit long, so we might abbreviate it as

$$\vec{F}_{\text{net}} = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_{14}.$$

But this is definition by pattern recognition. A more explicit statement would be

$$\vec{F}_{\text{net}} = \text{The sum of all 14 forces} \vec{F}_{\text{i}} \text{ where } i = 1 \ldots 14$$

which is too space consuming. This kind of summing is so important that mathematicians use up a whole letter of the greek alphabet as a short hand for 'the sum of all'. They use the capital greek 'S' (for Sum) called sigma which looks like this:

$$\sum.$$

When you read $\sum$ aloud you don't say 'S' or'sigma' but rather 'the sum of.' The $\sum$ (sum) notation may remind you of infinite series, and convergence thereof. We will rarely be concerned with infinite sums in this book and never with convergence issues. So panic on those grounds is unjustified. We just want to easily write about adding things. For example we use the $\sum$ (sum) to write the sum of 14 forces $\vec{F}_{\text{i}}$ explicitly and concisely as

$$\sum_{i=1}^{14} \vec{F}_{\text{i}}$$

and say 'the sum of F sub i where i goes from one to fourteen'. Sometimes we don't know, say, how many forces are being added. We just want to add all of them so we write (a little informally)

$$\sum \vec{F}_{\text{i}} \quad \text{meaning} \quad \vec{F}_1 + \vec{F}_2 + etc.,$$

where the subscript $i$ lets us know that the forces are numbered.

Rather than panic when you see something like $\sum_{i=1}^{14}$, just relax and think: oh, we want to add up a bunch of things all of which look like the next thing written. In general,

$$\sum (\text{thing})_i \quad \text{translates to} \quad (\text{thing})_1 + (\text{thing})_2 + (\text{thing})_3 + etc.$$

no matter how intimidating the 'thing' is. In time you can skip writing out the translation and will enjoy the concise notation.

## Every system of forces is equivalent to a force and a couple

Given any point C, we can calculate the net moment of a system of forces relative to C. We then can replace the sum of forces with a single force at C and the net moment with a couple at C and we have an equivalent force system.

A force system is equivalent to a force $\vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}_{\text{tot}}$ acting at C and a couple $M$ equal to the net moment of the forces about C, *i.e.*, $\vec{\boldsymbol{M}} = \vec{\boldsymbol{M}}_{\text{C}}$.

If instead we want a force system at D we could recalculate the net moment about D or just use the translation formula (see box 2.5 on page 86).

### 2.8 *THEORY*

*Two force systems that are equivalent for one reference point are equivalent for all reference points.*

Consider two sets of forces $\vec{\boldsymbol{F}}_i^{(1)}$ and $\vec{\boldsymbol{F}}_j^{(2)}$ with corresponding points of application $\text{P}_i^{(1)}$ and $\text{P}_j^{(2)}$ at positions relative to the origin of $\vec{\boldsymbol{r}}_i^{(1)}$ and $\vec{\boldsymbol{r}}_j^{(2)}$. To simplify the discussion let's define the net forces of the two systems as

$$\vec{\boldsymbol{F}}_{\text{tot}}^{(1)} \equiv \sum \vec{\boldsymbol{F}}_i^{(1)} \quad \text{and} \quad \vec{\boldsymbol{F}}_{\text{tot}}^{(2)} \equiv \sum \vec{\boldsymbol{F}}_j^{(2)},$$

and the net moments about the origin as

$$\vec{\boldsymbol{M}}_0^{(1)} \equiv \sum \vec{\boldsymbol{r}}_i^{(1)} \times \vec{\boldsymbol{F}}_i^{(1)} \quad \text{and} \quad \vec{\boldsymbol{M}}_0^{(2)} \equiv \sum \vec{\boldsymbol{r}}_j^{(2)} \times \vec{\boldsymbol{F}}_j^{(2)}.$$

Using point 0 as a reference, the statement that the two systems are equivalent is then $\vec{\boldsymbol{F}}_{\text{tot}}^{(1)} = \vec{\boldsymbol{F}}_{\text{tot}}^{(2)}$ and $\vec{\boldsymbol{M}}_0^{(1)} = \vec{\boldsymbol{M}}_0^{(2)}$. Now consider point C with position $\vec{\boldsymbol{r}}_{\text{C}} = \vec{\boldsymbol{r}}_{\text{C}/0} = -\vec{\boldsymbol{r}}_{0/\text{C}}$. What is the net moment of force system (1) about point C?

$$\begin{aligned}
\vec{\boldsymbol{M}}_{\text{C}}^{(1)} &\equiv \sum \vec{\boldsymbol{r}}_{i/\text{C}}^{(1)} \times \vec{\boldsymbol{F}}_i^{(1)} \\
&= \sum \left( \vec{\boldsymbol{r}}_i^{(1)} - \vec{\boldsymbol{r}}_{\text{C}} \right) \times \vec{\boldsymbol{F}}_i^{(1)} \\
&= \sum \left( \vec{\boldsymbol{r}}_i^{(1)} \times \vec{\boldsymbol{F}}_i^{(1)} - \vec{\boldsymbol{r}}_{\text{C}} \times \vec{\boldsymbol{F}}_i^{(1)} \right) \\
&= \sum \vec{\boldsymbol{r}}_i^{(1)} \times \vec{\boldsymbol{F}}_i^{(1)} - \sum \vec{\boldsymbol{r}}_{\text{C}} \times \vec{\boldsymbol{F}}_i^{(1)} \\
&= \sum \vec{\boldsymbol{r}}_i^{(1)} \times \vec{\boldsymbol{F}}_i^{(1)} - \vec{\boldsymbol{r}}_{\text{C}} \times \left( \sum \vec{\boldsymbol{F}}_i^{(1)} \right) \\
&= \vec{\boldsymbol{M}}_0^{(1)} - \vec{\boldsymbol{r}}_{\text{C}} \times \vec{\boldsymbol{F}}_{\text{tot}}^{(1)}. \\
&= \vec{\boldsymbol{M}}_0^{(1)} + \vec{\boldsymbol{r}}_{0/\text{C}} \times \vec{\boldsymbol{F}}_{\text{tot}}^{(1)}.
\end{aligned}$$

[ **Note.** The calculation above uses the 'move' of factoring a constant vector out of a sum. This mathematical move will be used again and again in the development of the theory of mechanics. ] c

Similarly, for force system (2)

$$\vec{\boldsymbol{M}}_{\text{C}}^{(2)} = \vec{\boldsymbol{M}}_0^{(2)} + \vec{\boldsymbol{r}}_{0/\text{C}} \times \vec{\boldsymbol{F}}_{\text{tot}}^{(2)}.$$

If the two force systems are equivalent for reference point 0 then $\vec{\boldsymbol{F}}_{\text{tot}}^{(1)} = \vec{\boldsymbol{F}}_{\text{tot}}^{(2)}$ and $\vec{\boldsymbol{M}}_0^{(1)} = \vec{\boldsymbol{M}}_0^{(2)}$ and the expressions above imply that $\vec{\boldsymbol{M}}_{\text{C}}^{(1)} = \vec{\boldsymbol{M}}_{\text{C}}^{(2)}$. Because we specified nothing special about the point C, the systems are equivalent for *any* reference point. Thus, to demonstrate equivalence we need to use a reference point, but once equivalence is demonstrated we need not name the point since the equivalence holds for all points.

By the same reasoning we find that once we know the net force and net moment of a force system ($\vec{\boldsymbol{F}}_{\text{tot}}$) relative to some point C (call it $\vec{\boldsymbol{M}}_{\text{C}}$), we know the net moment relative to point D as

$$\vec{\boldsymbol{M}}_{\text{D}} = \vec{\boldsymbol{M}}_{\text{C}} + \vec{\boldsymbol{r}}_{\text{C/D}} \times \vec{\boldsymbol{F}}_{\text{tot}}.$$

Note that if the net force is $\vec{\boldsymbol{0}}$ (and the force system is then called a couple) that $\vec{\boldsymbol{M}}_{\text{D}} = \vec{\boldsymbol{M}}_{\text{C}}$ so the net moment is the same for all reference points.

$$\begin{aligned} \vec{\boldsymbol{F}}_{\text{tot}} &= \vec{\boldsymbol{F}}_{\text{tot}}, \quad \text{and} \\ \vec{\boldsymbol{M}}_{\text{D}} &= \vec{\boldsymbol{M}}_{\text{C}} + \vec{\boldsymbol{r}}_{\text{C/D}} \times \vec{\boldsymbol{F}}_{\text{tot}}. \end{aligned}$$

The total force $\vec{\boldsymbol{F}}_{\text{net}}$ stays the same and the moment at D is the moment at C plus the moment caused by $\vec{\boldsymbol{F}}_{\text{net}}$ acting at position C relative to D. The net effect of the forces of the ground on a tree, for example, is of a force and a couple acting on the base of the tree.

## Equivalent does not mean equivalent for all purposes

We have perhaps oversimplified.

Imagine you stayed up late studying and overslept. Your roommate was not so diligent; woke up on time and went to wake you by gently shaking you. Having read this chapter so far and no further, and being rather literal, your roommate gets down on the floor and presses on the linoleum underneath your bed applying a force that is *equivalent* to pressing on you. Obviously this is not equivalent in the ordinary sense of the word. It isn't even equivalent in all of its mechanics effects. One force moves you even if you don't wake up, and the other doesn't.

Any two force systems that are 'equivalent' but different *do* have different mechanical effects. In what sense are two force systems that have the same net force and the same net moment really equivalent?

'Equivalent' force systems are equivalent in their contributions to the equations of mechanics (equations 0-II on the inside cover) for any system to which they are both applied.

But full mechanical analysis of a situation requires looking at the mechanics equations of many subsystems. In the mechanics equations for each subsys-

**2.9** *THEORY*

*The tidiest representation of a force system: a "wrench"*

Any force system can be represented by an equivalent force and a couple at any point. But force systems can be reduced to simpler forms. That this is so is of more theoretical than practical import. We state the results here without proof.

In 2D one of these two things is true:

- The system is equivalent to a couple, or most often
- There is a line parallel to the force which the system can be described by an equivalent force with no couple.

In 3D one of these three things is true:

- The system is equivalent to a couple (applied anywhere), or
- The system is equivalent to a force (applied on a given line parallel to the force), or most often
- There is a line for which the system can be reduced to a force and a couple where the force, couple, and line are all parallel. The representation of the system of forces as a force and a parallel moment is called a *wrench*.

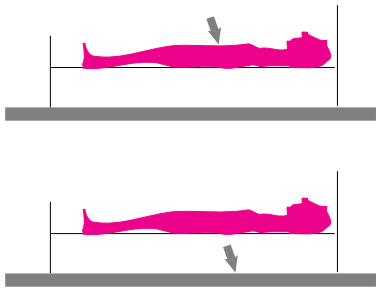


Figure 2.56: It feels different if someone presses on you or presses on the floor underneath you with an 'equivalent' force. The equivalence of 'equivalent' force systems depends on them both being applied to the same system.

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tem, two 'equivalent' force systems are equivalent if they are both applied to that subsystem.

For the analysis of the subsystem that is you sleeping, the force of your roommate's hand on the floor isn't applied to you, so it doesn't show up in the mechanics equations for you, and doesn't have the same effect as a force on you.

**SAMPLE 2.33 Equivalent force on a particle:** Four forces $\vec{F}_1 = 2\,\text{N}\hat{\imath} - 1\,\text{N}\hat{\jmath}$, $\vec{F}_2 = -5\,\text{N}\hat{\jmath}$, $\vec{F}_3 = 3\,\text{N}\hat{\imath} + 12\,\text{N}\hat{\jmath}$, and $\vec{F}_4 = 1\,\text{N}\hat{\imath}$ act on a particle. Find the equivalent force on the particle.

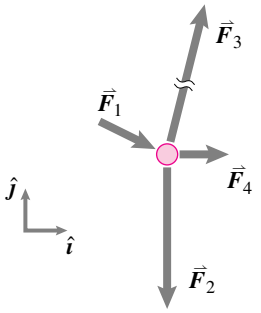



Figure 2.57:

Filename:sfig2-vec3-particle

**Solution** The equivalent force on the particle is the net force, *i.e.*, the vector sum of all forces acting on the particle. Thus,

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 \\ &= (2\,\text{N}\hat{\imath} - 1\,\text{N}\hat{\jmath}) + (-5\,\text{N}\hat{\jmath}) + (3\,\text{N}\hat{\imath} + 12\,\text{N}\hat{\jmath}) + (1\,\text{N}\hat{\imath}) \\ &= 6\,\text{N}\hat{\imath} + 6\,\text{N}\hat{\jmath}.\end{aligned}$$

$$\boxed{\vec{F}_{\text{net}} = 6\,\text{N}(\hat{\imath} + \hat{\jmath})}$$

Note that there is no net couple since all the four forces act at the same point. This is always true for particles. Thus, the equivalent force-couple system for particles consists of only the net force.

---

**SAMPLE 2.34 Equivalent force with no net moment:** In the figure shown, $F_1 = 50\,\text{N}$, $F_2 = 10\,\text{N}$, $F_3 = 30\,\text{N}$, and $\theta = 60°$. Find the equivalent force-couple system about point D of the structure.

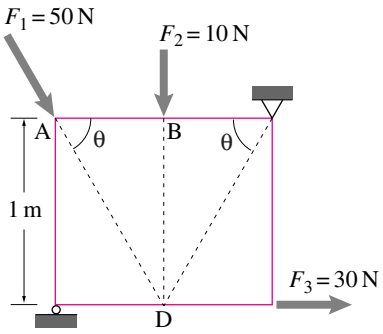



Figure 2.58:

Filename:sfig2-vec3-plate

**Solution** From the given geometry, we see that the three forces $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$ pass through point D. Thus they are *concurrent* forces. Since point D is on the line of action of these forces, we can simply slide the three forces to point D without altering their mechanical effect on the structure. Then the equivalent force-couple system at point D consists of only the net force, $\vec{F}_{\text{net}}$, with no couple (the three forces passing through point D produce no moment about D). This is true for all concurrent forces. Thus,

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \\ &= F_1(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) - F_2\hat{\jmath} + F_3\hat{\imath} \\ &= (F_1\cos\theta + F_3)\hat{\imath} - (F_1\sin\theta + F_2)\hat{\jmath} \\ &= (50\,\text{N}\cdot\frac{1}{2} + 30\,\text{N})\hat{\imath} - (50\,\text{N}\cdot\frac{\sqrt{3}}{2} + 10\,\text{N})\hat{\jmath} \\ &= 50\,\text{N}\hat{\imath} - 53.3\,\text{N}\hat{\jmath},\end{aligned}$$

and $\vec{M}_{\text{D}} = \vec{0}$.

$$\boxed{\vec{F}_{\text{net}} = 50\,\text{N}\hat{\imath} - 53.3\,\text{N}\hat{\jmath},\ \vec{M}_{\text{D}} = \vec{0}}$$

Graphically, the solution is shown in Fig. 2.59

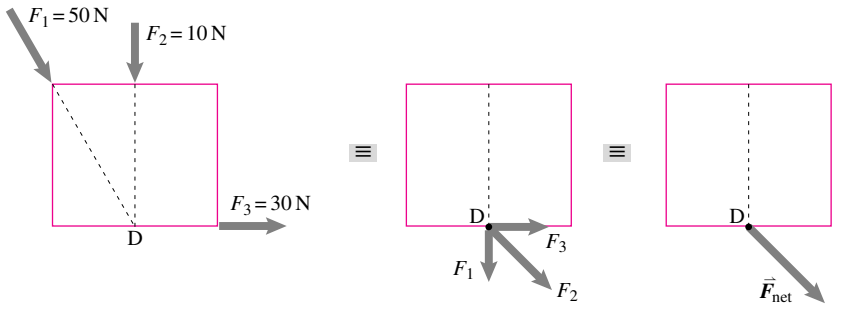



Figure 2.59:

Filename:sfig2-vec3-plate-a

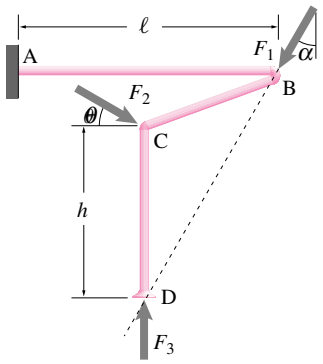



Figure 2.60:
Filename:sfig2-vec3-bar

**SAMPLE 2.35 An equivalent force-couple system:** Three forces $F_1 = 100\,\text{N}$, $F_2 = 50\,\text{N}$, and $F_3 = 30\,\text{N}$ act on a structure as shown in the figure where $\alpha = 30°$, $\theta = 60°$, $\ell = 1\,\text{m}$ and $h = 0.5\,\text{m}$. Find the equivalent force-couple system about point D.

**Solution** The net force is the sum of all applied forces, *i.e.*,

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \\ &= F_1(-\sin\alpha\hat{\imath} - \cos\alpha\hat{\jmath}) + F_2(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) + F_3\hat{\jmath} \\ &= (-F_1\sin\alpha + F_2\cos\theta)\hat{\imath} + (-F_1\cos\alpha - F_2\sin\theta + F_3)\hat{\jmath} \\ &= (-100\,\text{N}\cdot\frac{1}{2} + 50\,\text{N}\cdot\frac{1}{2})\hat{\imath} + (-100\,\text{N}\cdot\frac{\sqrt{3}}{2} - 50\,\text{N}\cdot\frac{\sqrt{3}}{2} + 30\,\text{N})\hat{\jmath} \\ &= -25\,\text{N}\hat{\imath} - 99.9\,\text{N}\hat{\jmath}.\end{aligned}$$

Forces $\vec{F}_1$ and $\vec{F}_3$ pass through point D. Therefore, they do not produce any moment about D. So, the net moment about D is the moment caused by force $\vec{F}_2$:

$$\begin{aligned}\vec{M}_{\text{D}} &= \vec{r}_{\text{C/D}} \times \vec{F}_2 \\ &= h\hat{\jmath} \times F_2(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) \\ &= -F_2 h\cos\theta\hat{k} \\ &= -50\,\text{N}\cdot 0.5\,\text{m}\frac{1}{2}\hat{k} = -12.5\,\text{N·m}\hat{k}.\end{aligned}$$

The equivalent force-couple system is shown in Fig. 2.61

$$\boxed{\vec{F}_{\text{net}} = -25\,\text{N}\hat{\imath} - 99.9\,\text{N}\hat{\jmath} \text{ and } \vec{M}_{\text{D}} = -12.5\,\text{N·m}\hat{k}}$$

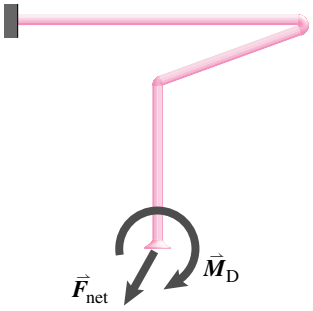



Figure 2.61:
Filename:sfig2-vec3-bar-a

**SAMPLE 2.36 Translating a force-couple system:** The net force and couple acting about point O on the 'L' shaped bar shown in the figure are 100 N and 20 N·m, respectively. Find the net force and moment about point G.

**Solution** The net force on a structure is the same about any point since it is just the vector sum of all the forces acting on the structure and is independent of their point of application. Therefore,

$$\vec{F}_{\text{net}} = \vec{F} = -100\,\text{N}\hat{\jmath}.$$

The net moment about a point, however, depends on the location of points of application of the forces with respect to that point. Thus,

$$\begin{aligned}\vec{M}_{\text{G}} &= \vec{M}_{\text{O}} + \vec{r}_{\text{O/G}} \times \vec{F} \\ &= M\hat{k} + (-\ell\hat{\imath} + h\hat{\jmath}) \times (-F\hat{\jmath}) \\ &= (M + F\ell)\hat{k} \\ &= (20\,\text{N·m} + 100\,\text{N}\cdot 1\,\text{m})\hat{k} = 120\,\text{N·m}\hat{k}.\end{aligned}$$

$$\boxed{\vec{F}_{\text{net}} = -100\,\text{N}\hat{\jmath}, \text{ and } \vec{M}_{\text{G}} = 120\,\text{N·m}\hat{k}}$$

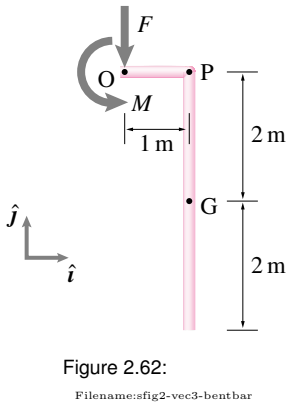



Figure 2.62:
Filename:sfig2-vec3-bentbar

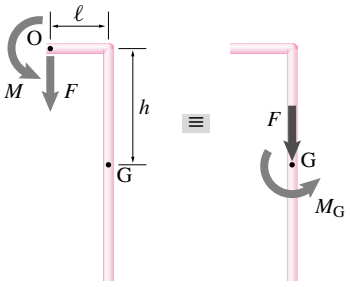



Figure 2.63:
Filename:sfig2-vec3-bentbar-a

**SAMPLE 2.37 Checking equivalence of force-couple systems:** In the figure shown below, which of the force-couple systems shown in (b), (c), and (d) are equivalent to the force system shown in (a)?

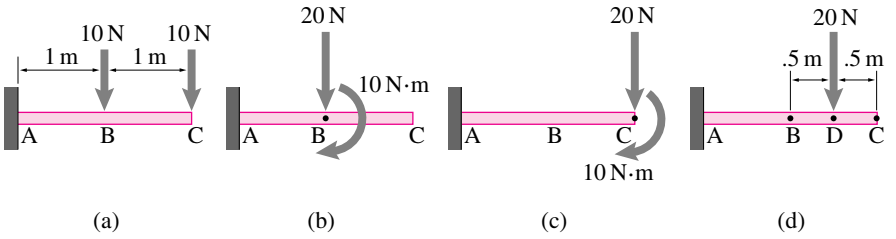



Figure 2.64:
Filename:sfig2-vec3-beam

**Solution** The equivalence of force-couple systems require that (i) the net force be the same, and (ii) the net moment about any reference point be the same. For the given systems, let us choose point B as our reference point for comparing their equivalence. For the force system shown in Fig. 2.64(a), we have,

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 = -10\,\text{N}\hat{\jmath} - 10\,\text{N}\hat{\jmath} = -20\,\text{N}\hat{\jmath}\\ \vec{M}_{\text{B}_{\text{net}}} &= \vec{r}_{\text{C/B}} \times \vec{F}_2 = 1\,\text{m}\hat{\imath} \times (-10\,\text{N}\hat{\jmath}) = -10\,\text{N·m}\hat{k}.\end{aligned}$$

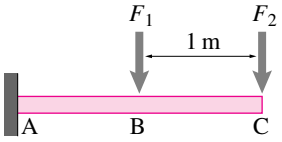



Figure 2.65:
Filename:sfig2-vec3-beam-a

Now, we can compare the systems shown in (b), (c), and (d) against the computed equivalent force-couple system, $\vec{F}_{\text{net}}$ and $\vec{M}_{\text{D}}$.

- Figure (b) shows exactly the system we calculated. Therefore, it represents an equivalent force-couple system.
- Figure (c): Let us calculate the net force and moment about point B for this system.

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_{\text{C}} \overset{\surd}{=} -20\,\text{N}\hat{\jmath}\\ \vec{M}_{\text{B}} &= \vec{M}_{\text{C}} + \vec{r}_{\text{C/B}} \times \vec{F}_{\text{C}}\\ &= -10\,\text{N·m}\hat{k} + 1\,\text{m}\hat{\imath} \times (-20\,\text{N}\hat{\jmath}) = -30\,\text{N·m}\hat{k} \neq \vec{M}_{\text{B}_{\text{net}}}.\end{aligned}$$

  Thus, the given force-couple system in this case is not equivalent to the force system in (a).

- Figure (d): Again, we compute the net force and the net couple about point B:

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_{\text{D}} \overset{\surd}{=} -20\,\text{N}\hat{\jmath}\\ \vec{M}_{\text{B}} &= \vec{r}_{\text{D/B}} \times \vec{F}_{\text{D}}\\ &= 0.5\,\text{m}\hat{\imath} \times (-20\,\text{N}\hat{\jmath}) = -10\,\text{N·m}\hat{k} \overset{\surd}{=} \vec{M}_{\text{B}_{\text{net}}}.\end{aligned}$$

  Thus, the given force-couple system (with zero couple) at D is equivalent to the force system in (a).

(b) and (d) are equivalent to (a); (c) is not.

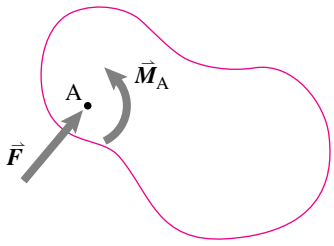



Figure 2.66:
Filename:sfig2-vec3-body

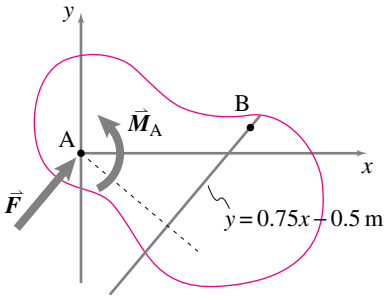



Figure 2.67:
Filename:sfig2-vec3-body-a

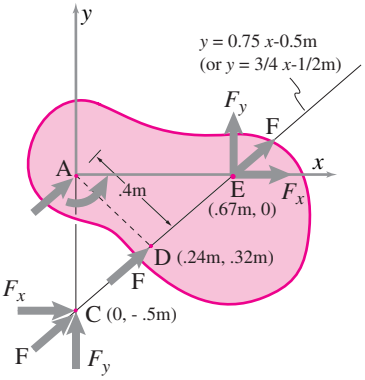



Figure 2.68:
Filename:sfig2-vec3-body-b

**SAMPLE 2.38 Equivalent force with no couple:** For a body, an equivalent force-couple system at point A consists of a force $\vec{F} = 20\,\text{N}\hat{\imath} + 15\,\text{N}\hat{\jmath}$ and a couple $\vec{M}_\text{A} = 10\,\text{N·m}\hat{k}$. Find a point on the body such that the equivalent force-couple system at that point consists of only a force (zero couple).

**Solution** The net force in the two equivalent force-couple systems has to be the same. Therefore, for the new system, $\vec{F}_\text{net} = \vec{F} = 20\,\text{N}\hat{\imath} + 15\,\text{N}\hat{\jmath}$. Let B be the point at which the equivalent force-couple system consists of only the net force, with zero couple. We need to find the location of point B. Let A be the origin of a $xy$ coordinate system in which the coordinates of B are $(x, y)$. Then, the moment about point B is,

$$\begin{aligned}\vec{M}_\text{B} &= \vec{M}_\text{A} + \vec{r}_{\text{A/B}} \times \vec{F} \\ &= M_A\hat{k} + (-x\hat{\imath} - y\hat{\jmath}) \times (F_x\hat{\imath} + F_y\hat{\jmath}) \\ &= M_A\hat{k} + (-F_y x + F_x y)\hat{k}.\end{aligned}$$

Since we require that $\vec{M}_\text{B}$ be zero, we must have

$$\begin{aligned}F_y x - F_x y &= M_A \\ \Rightarrow \quad y &= \frac{F_y}{F_x}x - \frac{M_A}{F_x} \\ &= \frac{15\,\text{N}}{20\,\text{N}}x - \frac{10\,\text{N·m}}{20\,\text{N}} \\ &= 0.75x - 0.5\,\text{m}.\end{aligned}$$

This is the equation of a line. Thus, we can select any point on this line and apply the force $\vec{F} = 20\,\text{N}\hat{\imath} + 15\,\text{N}\hat{\jmath}$ with zero couple as an equivalent force-couple system.

Any point on the line $y = 0.75x - 0.5\,\text{m}$.

So, how or why does it work? The line we obtained is shown in gray in Fig. 2.67. Note that this line has the same slope as that of the given force vector (slope $= 0.75 = F_y/F_x$) and the offset is such that shifting the force $\vec{F}$ to this line counter balances the given couple at A. To see this clearly, let us select three points C, D, and E on the line as shown in Fig. 2.68. From the equation of the line, we find the coordinates of C(0,-.5m), D(.24m,.32m) and E(.67m,0). Now imagine moving the force $\vec{F}$ to C, D, or E. In each case, it must produce the same moment $\vec{M}_\text{A}$ about point A. Let us do a quick check.

- $\vec{F}$ at point C: The moment about point A is due to the horizontal component $F_x = 20\,\text{N}$, since $F_y$ passes through point A. The moment is $F_x \cdot AC = 20\,\text{N} \cdot 0.5\,\text{m} = 10\,\text{N·m}$, same as $M_A$. The direction is counterclockwise as required.
- $\vec{F}$ at point D: The moment about point A is $|\vec{F}| \cdot AD = 25\,\text{N} \cdot 0.4\,\text{m} = 10\,\text{N·m}$, same as $M_A$. The direction is counterclockwise as required.
- $\vec{F}$ at point E: The moment about point A is due to the vertical component $F_y$, since $F_x$ passes through point A. The moment is $F_y \cdot AE = 15\,\text{N} \cdot 0.67\,\text{m} = 10\,\text{N·m}$, same as $M_A$. The direction here too is counterclockwise as required.

Once we check the calculation for one point on the line, we should not have to do any more checks since we know that sliding the force along its line of action (line CB) produces no couple and thus preserves the equivalence.

## 2.6 Center of mass and gravity

For every system and at every instant in time, there is a unique location in space that is the average position of the system's mass. This place is called the *center of mass*, commonly designated by cm, c.o.m., COM, G, c.g., or ◒.

One of the routine but important tasks of many real engineers is to find the center-of-mass of a complex machine*. Just knowing the location of the center-of-mass of a car, for example, is enough to estimate whether it can be tipped over by maneuvers on level ground. The center-of-mass of a boat must be low enough for the boat to be stable. Any propulsive force on a space craft must be directed towards the center-of-mass in order to not induce rotations. Tracking the trajectory of the center-of-mass of an exploding plane can determine whether or not it was hit by a massive object. Any rotating piece of machinery must have its center-of-mass on the axis of rotation if it is not to cause much vibration.

Also, many calculations in mechanics are greatly simplified by making use of a system's center-of-mass. In particular, the whole complicated distribution of near-earth gravity forces on a body is equivalent to a single force at the body's center-of-mass. Many of the important quantities in dynamics are similarly simplified using the center-of-mass.

The center-of-mass of a system is the point at the position $\vec{r}_{\text{cm}}$ defined by

$$\begin{aligned} \vec{r}_{\text{cm}} &= \frac{\sum \vec{r}_{\text{i}} m_i}{m_{\text{tot}}} \quad \text{for discrete systems} \\ &= \frac{\int \vec{r}\, dm}{m_{\text{tot}}} \quad \text{for continuous systems} \end{aligned} \tag{2.29}$$

where $m_{\text{tot}} = \sum m_i$ for discrete systems and $m_{\text{tot}} = \int dm$ for continuous systems (see boxes 2.5 and 2.6 on pages 85 and 94 for a discussion of the $\sum$ and $\int$ sum notations).

Often it is convenient to remember the rearranged definition of center of mass as

$$m_{\text{tot}}\vec{r}_{\text{cm}} = \sum m_i \vec{r}_{\text{i}} \quad \text{or} \quad m_{\text{tot}}\vec{r}_{\text{cm}} = \int \vec{r}\, dm.$$

For theoretical purposes we rarely need to evaluate these sums and integrals, and for simple problems there are sometimes shortcuts that reduce the calculation to a matter of observation. For complex machines one or both of the formulas 2.29 must be evaluated in detail.

Example: **System of two point masses**
Intuitively, the center-of-mass of the two masses shown in figure 2.69 is between the two

* Nowadays this routine work is often done with CAD (computer aided design) software. But an engineer still needs to know the basic calculation skills, to make sanity checks on computer calculations if nothing else.

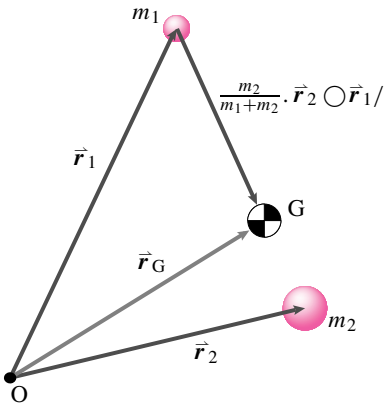



Figure 2.69: Center of mass of a system consisting of two points.

Filename:tfigure3-com-twomass

masses and closer to the larger one. Referring to equation 2.29,

$$\begin{aligned}\vec{r}_{\text{cm}} &= \frac{\sum \vec{r}_i m_i}{m_{tot}} \\ &= \frac{\vec{r}_1 m_1 + \vec{r}_2 m_2}{m_1 + m_2} \\ &= \frac{\vec{r}_1(m_1+m_2) - \vec{r}_1 m_2 + \vec{r}_2 m_2}{m_1+m_2} \\ &= \vec{r}_1 + \underbrace{\left(\frac{m_2}{m_1+m_2}\right)}_{\text{the fraction of the distance that the cm is from } \vec{r}_1 \text{ to } \vec{r}_2} \underbrace{\left(\vec{r}_2 - \vec{r}_1\right)}_{\text{the vector from } \vec{r}_1 \text{ to } \vec{r}_2}.\end{aligned}$$

so that the math agrees with common sense — the center-of-mass is on the line connecting the masses. If $m_2 \gg m_1$, then the center-of-mass is near $m_2$. If $m_1 \gg m_2$, then the center-of-mass is near $m_1$. If $m_1 = m_2$ the center-of-mass is right in the middle at $(\vec{r}_1 + \vec{r}_2)/2$.

## 2.10 Like $\sum$, the symbol $\int$ also means add

We often add things up in mechanics. For example, the total mass of some particles is

$$m_{\text{tot}} = m_1 + m_2 + m_3 + \cdots = \sum m_i$$

or more specifically the mass of 137 particles is, say, $m_{\text{tot}} = \sum_{i=1}^{137} m_i$.

And the total mass of a bicycle is:

$$m_{\text{bike}} = \sum_{i=1}^{100,000,000,000,000,000,000,000} m_i$$

where $m_i$ are the masses of each of the $10^{23}$ (or so) atoms of metal, rubber, plastic, cotton, and paint. But atoms are so small and there are so many of them. Instead we often think of a bike as built of macroscopic parts. The total mass of the bike is then the sum of the masses of the tires, the tubes, the wheel rims, the spokes and nipples, the ball bearings, the chain pins, and so on. And we would write:

$$m_{\text{bike}} = \sum_{i=1}^{2,000} m_i$$

where now the $m_i$ are the masses of the 2,000 or so bike parts. This sum is more manageable but still too detailed in concept for some purposes.

An approach that avoids attending to atoms or ball bearings, is to think of sending the bike to a big shredding machine that cuts it up into very small bits. Now we write

$$m_{\text{bike}} = \sum m_i$$

where the $m_i$ are the masses of the very small bits. We don't fuss over whether one bit is a piece of ball bearing or fragment of cotton from the tire walls. We just chop the bike into bits and add up the contribution of each bit. If you take the letter S, as in SUM, and distort it and you get a big old fashioned German 'S' used in calculus as the integral sign

$$\text{S} \quad \int$$

So we write

$$m_{\text{bike}} = \int dm$$

to mean the ∫um of all the teeny bits of mass. More formally we mean the value of that sum in the limit that all the bits are infinitesimal (not minding the technical fine point that its hard to chop atoms into infinitesimal pieces).

The mass is one of many things we would like to add up, though many of the others also involve mass. In center-of-mass calculations, for example, we add up the positions 'weighted' by mass.

$$\int \vec{r}\, dm \quad \text{which means} \quad \sum_{\lim m_i \to 0} \vec{r}_i m_i.$$

That is, you take your object of interest and chop it into a billion pieces and then re-assemble it. For each piece you make the vector which is the position vector of the piece multiplied by ('weighted by') its mass and then add up the billion vectors. Well really you chop the thing into a trillion trillion . . . pieces, but a billion gives the idea.

## Continuous systems

How do we evaluate integrals like $\int (\text{something})\, dm$? In center-of-mass calculations, (something) is position, but we will evaluate similar integrals where (something) is some other scalar or vector function of position. Most often we label the material by its spatial position, and evaluate $dm$ in terms of increments of position. For 3D solids $dm = \rho dV$ where $\rho$ is density (mass per unit volume). So $\int (\text{something}) dm$ turns into a standard volume integral $\int_V (\text{something})\, \rho\, dV$*.For thin flat things like metal sheets we often take $\rho$ to mean mass per unit area $A$ so then $dm = \rho dA$ and $\int (\text{something}) dm = \int_A (\text{something})\, \rho\, dA$. For mass distributed along a line or curve we take $\rho$ to be the mass per unit length or arc length $s$ and so $dm = \rho ds$ and $\int (\text{something}) dm = \int_{curve} (\text{something})\, \rho\, ds$.

**Example.** The center-of-mass of a uniform rod is naturally in the middle, as the calculations here show (see fig. 2.70a). Assume the rod has length $L = 3$ m and mass $m = 7$ kg.

$$\vec{r}_{\text{cm}} = \frac{\int \vec{r}\, dm}{m_{\text{tot}}} = \frac{\int_0^L x\hat{\imath}\, \overbrace{\rho dx}^{dm}}{\int_0^L \rho\, dx} = \frac{\rho(x^2/2)|_0^L}{\rho(1)|_1^L}\hat{\imath} = \frac{\rho(L^2/2)}{\rho L}\hat{\imath} = (L/2)\hat{\imath}$$

So $\vec{r}_{\text{cm}} = (L/2)\hat{\imath}$, or by dotting with $\hat{\imath}$ (taking the x component) we get that the center-of-mass is on the rod a distance $d = L/2 = 1.5$ m from the end.

The center-of-mass calculation is *objective*. It describes something about the object that does not depend on the coordinate system. In different coordinate systems the center-of-mass for the rod above will have different coordinates, but it will always be at the middle of the rod.

**Example.** Find the center-of-mass using the coordinate system with $s$ & $\hat{\boldsymbol{\lambda}}$ in fig. 2.70b:

$$\vec{r}_{\text{cm}} = \frac{\int \vec{r}\, dm}{m_{\text{tot}}} = \frac{\int_0^L s\hat{\boldsymbol{\lambda}}\, \rho ds}{\int_0^L \rho\, ds}\hat{\boldsymbol{\lambda}} = \frac{\rho(s^2/2)|_0^L}{\rho(1)|_0^L}\hat{\boldsymbol{\lambda}} = \frac{\rho(L^2/2)}{\rho L}\hat{\boldsymbol{\lambda}} = (L/2)\hat{\boldsymbol{\lambda}},$$

again showing that the center-of-mass is in the middle.

Note, one can treat the center-of-mass vector calculations as separate scalar equations, one for each component. For example:

$$\hat{\imath} \cdot \left\{ \vec{r}_{\text{cm}} = \frac{\int \vec{r}\, dm}{m_{\text{tot}}} \right\} \quad \Rightarrow \quad r_{x\text{cm}} = x_{\text{cm}} = \frac{\int x\, dm}{m_{\text{tot}}}.$$

Finally, there is no law that says you have to use the best coordinate system. One is free to make trouble for oneself and use an inconvenient coordinate system.

**Example.** Use the $xy$ coordinates of fig. 2.70c to find the center-of-mass of the rod.

$$x_{\text{cm}} = \frac{\int x dm}{m_{\text{tot}}} = \frac{\int_{-\ell_1}^{\ell_2} \overbrace{s\cos\theta}^{x}\, \rho ds}{\int_0^L \rho\, ds} = \frac{\rho\cos\theta \frac{s^2}{2}|_{-\ell_1}^{\ell_2}}{\rho(1)|_{-\ell_1}^{\ell_2}} = \frac{\rho\cos\theta\frac{(\ell_2^2-\ell_1^2)}{2}}{\rho(\ell_1+\ell_2)} = \frac{\cos\theta(\ell_2-\ell_1)}{2}$$

Similarly $y_{\text{cm}} = \sin\theta(\ell_2 - \ell_1)/2$ so

$$\vec{r}_{\text{cm}} = \frac{\ell_2 - \ell_1}{2}(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath})$$

which still describes the point at the middle of the rod.

*Note: writing $\int_{m_1}^{m_2} (\text{something})\, dm$ is nonsense because $m$ is not a scalar parameter which labels points in a material (there is no point at $m = 3$ kg).

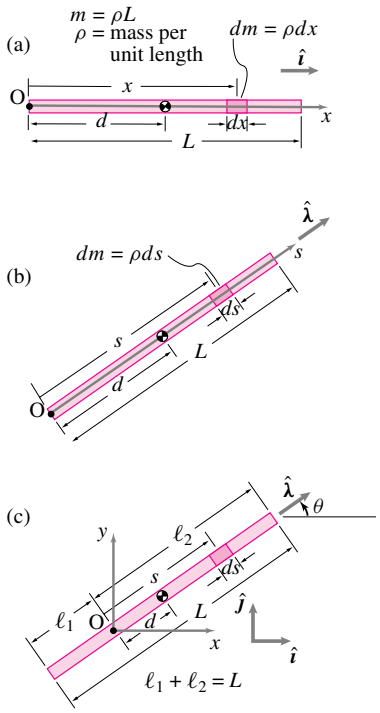



Figure 2.70: Where is the center-of-mass of a uniform rod? In the middle, as you can find calculating a few ways or by symmetry.

Filename:tfigure1-rodcm

The most commonly needed center-of-mass that can be found analytically but not directly from symmetry is that of a triangle (see box 2.6 on page 102). In your calculus text you will find more examples of finding the center-of-mass using integration.

## Center of mass and centroid

For objects with uniform material density we have

$$\vec{r}_{\text{cm}} = \frac{\int \vec{r}\,dm}{m_{\text{tot}}} = \frac{\int_V \vec{r}\,\rho dV}{\int_V \rho dV} = \frac{\rho \int_V \vec{r}\,dV}{\rho \int_V dV} = \frac{\int_V \vec{r}\,dV}{V}$$

where the last expression is just the formula for geometric centroid. Analogous calculations hold for 2D and 1D geometric objects.

For objects with density that does not vary from point to point, the geometric centroid and the center-of-mass coincide.

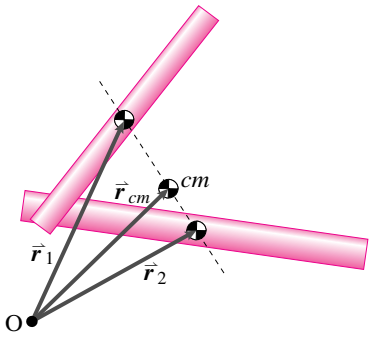



Figure 2.72: Center of mass of two rods

Filename:tfigure3-com-tworods

## Center of mass and symmetry

The center-of-mass respects any symmetry in the mass distribution of a system. If the word 'middle' has unambiguous meaning in English then that is the location of the center-of-mass, as for the rod of fig. 2.70 and the other examples in fig. 2.71.

## Systems of systems and composite objects

Another way of interpreting the formula

$$\vec{r}_{\text{cm}} = \frac{\vec{r}_1 m_1 + \vec{r}_2 m_2 + \cdots}{m_1 + m_2 + \cdots}$$

is that the $m$'s are the masses of subsystems, not just points, and that the $\vec{r}_i$ are the positions of the centers of mass of these systems. This subdivision is justified in box 2.11 on page 101. The center-of-mass of a single complex shaped object can be found by treating it as an assembly of simpler objects.

Example: **Two rods**
The center-of-mass of two rods shown in figure 2.72 can be found as

$$\vec{r}_{\text{cm}} = \frac{\vec{r}_1 m_1 + \vec{r}_2 m_2}{m_1 + m_2}$$

where $\vec{r}_1$ and $\vec{r}_2$ are the positions of the centers of mass of each rod and $m_1$ and $m_2$ are the masses.

Example: **'L' shaped plate**

Consider the plate with uniform mass per unit area $\rho$.

$$\begin{aligned}\vec{r}_{\mathrm{G}} &= \frac{\vec{r}_{\mathrm{I}} m_I + \vec{r}_{\mathrm{II}} m_{II}}{m_I + m_{II}} \\ &= \frac{(\frac{a}{2}\hat{\imath} + a\hat{\jmath})(2\rho a^2) + (\frac{3}{2}a\hat{\imath} + \frac{a}{2}\hat{\jmath})(\rho a^2)}{(2\rho a^2) + (\rho a^2)} \\ &= \frac{5}{6}a(\hat{\imath} + \hat{\jmath}).\end{aligned}$$

## Composite objects using subtraction

It is sometimes useful to think of an object as composed of pieces, some of which have negative mass.

Example: **'L' shaped plate, again**
Reconsider the plate from the previous example.

$$\begin{aligned}\vec{r}_{\mathrm{G}} &= \frac{\vec{r}_{\mathrm{I}} m_I + \vec{r}_{\mathrm{II}} m_{II}}{m_I + m_{II}} \\ &= \frac{(a\hat{\imath} + a\hat{\jmath})(\rho(2a)^2) + (\frac{3}{2}a\hat{\imath} + \frac{3}{2}a\hat{\jmath})\overbrace{(-\rho a^2)}^{m_{II}}}{(\rho(2a)^2) + \underbrace{(-\rho a^2)}_{m_{II}}} \\ &= \frac{5}{6}a(\hat{\imath} + \hat{\jmath}).\end{aligned}$$

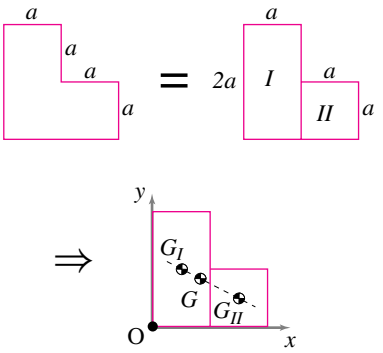



Figure 2.73: The center-of-mass of the 'L' shaped object can be found by thinking of it as a rectangle plus a square.

Filename:tfigure3-1-Lshaped

## Center of gravity

The force of gravity on each little bit of an object is $gm_i$ where $g$ is the local gravitational 'constant' and $m_i$ is the mass of the bit. For objects that are small compared to the radius of the earth (a reasonable assumption for all but a few special engineering calculations) the gravity constant is indeed constant from one point on the object to another (see box A on page A for a discussion of the meaning and history of $g$.)

Not only that, all the gravity forces point in the same direction, down. For engineering purposes, the two intersecting lines that go from your two hands to the center of the earth are parallel.) Lets call this the $-\hat{\boldsymbol{k}}$ direction. So the net force of gravity on an object is:

$$\begin{aligned}\vec{\boldsymbol{F}}_{\mathrm{net}} &= \sum \vec{\boldsymbol{F}}_{\mathrm{i}} = \sum m_i g(-\hat{\boldsymbol{k}}) = -mg\hat{\boldsymbol{k}} \quad \text{for discrete systems, and} \\ &= \int d\vec{\boldsymbol{F}} = \int \underbrace{-g\hat{\boldsymbol{k}}\, dm}_{d\vec{\boldsymbol{F}}} = -mg\hat{\boldsymbol{k}} \quad \text{for continuous systems.}\end{aligned}$$

That's easy, the billions of gravity forces on an objects microscopic constituents add up to $mg$ pointed down. What about the net moment of the gravity forces? The answer turns out to be simple. The top line of the calcu-

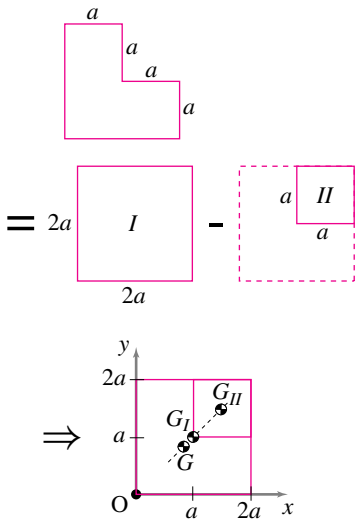



Figure 2.74: Another way of looking at the 'L' shaped object is as a square minus a smaller square in its upper right-hand corner.

Filename:tfigure3-1-Lshaped-a

lation below poses the question, the last line gives the lucky answer.*

$$
\begin{aligned}
\vec{\boldsymbol{M}}_{\mathrm{C}} &= \int \vec{\boldsymbol{r}} \times d\vec{\boldsymbol{F}} && \text{The net moment with respect to C.} \\
&= \int \vec{\boldsymbol{r}}_{/\mathrm{C}} \times \left(-g\hat{\boldsymbol{k}}\, dm\right) && \text{A force bit is gravity acting on a mass bit.} \\
&= \left(\int \vec{\boldsymbol{r}}_{/\mathrm{C}} dm\right) \times \left(-g\hat{\boldsymbol{k}}\right) && \text{Cross product distributive law } (g,\ \hat{\boldsymbol{k}} \text{ are constants}) \\
&= (\vec{\boldsymbol{r}}_{\mathrm{cm/C}} m) \times \left(-g\hat{\boldsymbol{k}}\right) && \text{Definition of center-of-mass.} \\
&= \vec{\boldsymbol{r}}_{\mathrm{cm/C}} \times \left(-mg\hat{\boldsymbol{k}}\right) && \text{Re-arranging terms.} \\
&= \vec{\boldsymbol{r}}_{\mathrm{cm/C}} \times \vec{\boldsymbol{F}}_{\mathrm{net}} && \text{Express in terms of net gravity force.}
\end{aligned}
$$

Thus the net moment is the same as for the total gravity force acting at the center-of-mass.

The near-earth gravity forces acting on a system are *equivalent* to a single force, $mg$, acting at the system's center-of-mass.

For the purposes of calculating the net force and moment from near-earth (constant g) gravity forces, a system can be replaced by a point mass at the center of gravity. The words 'center-of-mass' and 'center of gravity' both describe the same point in space.

Although the result we have just found seems plain enough, here are two things to ponder about gravity when viewed as an inverse square law (and thus not constant like we have assumed) that may make the result above seem less obvious.

- The net gravity force on a sphere is indeed equivalent to the force of a point mass at the center of the sphere. It took the genius Isaac Newton 3 years to deduce this result and the reasoning involved is too advanced for this book.
- The net gravity force on systems that are not spheres is generally *not* equivalent to a force acting at the center-of-mass (this is important for the understanding of tides as well as the orientational stability of satellites).

## A recipe for finding the center-of-mass of a complex system

You find the center-of-mass of a complex system by knowing the masses and mass centers of its components. You find each of these centers of mass by

- Treating it as a point mass, or
- Treating it as a symmetric body and locating the center-of-mass in the middle, or

*We do the calculation here using the $\int$ notation for sums. But it could be done just as well using $\sum$.

- Using integration, or
- Using the result of an experiment (which we will discuss in statics), or
- Treating the component as a complex system itself and applying this very recipe.

The recipe is just an application of the basic definition of center-of-mass (eqn. 2.29) but with our accumulated wisdom that the locations and masses in that sum can be the centers of mass and total masses of complex subsystems.

One way to arrange one's data is in a table or spreadsheet, like below.

| Subsys# | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| Subsys 1 | $x_1$ | $y_1$ | $z_1$ | $m_1$ | $m_1 x_1$ | $m_1 y_1$ | $m_1 z_1$ |
| Subsys 2 | $x_2$ | $y_2$ | $z_2$ | $m_2$ | $m_2 x_2$ | $m_2 y_2$ | $m_2 z_2$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| Subsys N | $x_N$ | $y_N$ | $z_N$ | $m_N$ | $m_N x_N$ | $m_N y_N$ | $m_N z_N$ |
| Row N+1 sums | | | | $m_{\text{tot}} = \sum m_i$ | $\sum m_i x_i$ | $\sum m_i y_i$ | $\sum m_i z_i$ |

| | | | | | $x_{\text{cm}}$ | $y_{\text{cm}}$ | $z_{\text{cm}}$ |
|---|---|---|---|---|---|---|---|
| Result | | | | | $\frac{\sum m_i x_i}{m_{\text{tot}}}$ | $\frac{\sum m_i y_i}{m_{\text{tot}}}$ | $\frac{\sum m_i z_i}{m_{\text{tot}}}$ |

1. The first four columns are the basic data. They are the $x$, $y$, and $z$ coordinates of the subsystem center-of-mass locations (relative to some clear reference point), and the masses of the subsystems, one row for each of the $N$ subsystems.
2. One next calculates three new columns (5,6, and 7) which come from each coordinate multiplied by its mass. For example the entry in the 6th row and 7th column is the $z$ component of the 6th subsystem's center-of-mass multiplied by the mass of the 6th subsystem.
3. Then one sums columns 4 through 7. The sum of column 4 is the total mass, the sums of columns 5 through 7 are the total mass-weighted positions.
4. Finally the result, the system center of mass coordinates, are found by dividing columns 5-7 of row N+1 by column 4 of row N+1.

Of course, there are multiple ways of systematically representing the data. The spreadsheet-like calculation above is just one oragnization scheme.

## Summary of center-of-mass

All discussions in mechanics make frequent reference to the concept of center of mass

$$\begin{aligned} \vec{\boldsymbol{r}}_{\text{cm}} &= \frac{\sum \vec{\boldsymbol{r}}_{\text{i}} m_i}{m_{tot}} && \text{for discrete systems or systems of systems} \\ &= \frac{\int \vec{\boldsymbol{r}}\, dm}{m_{tot}} && \text{for continuous systems} \end{aligned}$$

where

$$\begin{aligned} m_{\text{tot}} &= \sum \text{m}_i && \text{for discrete systems or systems of systems} \\ &= \int dm && \text{for continuous systems.} \end{aligned}$$

Who cares about the center of mass? We have demonstrated that the gravity moment is calculated correctly by applying the net gravity force at the center-of-mass. These other useful facts about center-of-mass will come later in the book.

*For non-point-mass systems, the expressions for gravitational moment, linear momentum, angular momentum, and energy are all simplified by using the center-of-mass.*

Simple center-of-mass calculations also can serve as a check of a more complicated analysis. For example, after a computer simulation of a system with many moving parts is complete, one way of checking the calculation is to see if the whole system's center of mass moves as would be expected by applying the net external force to the system.

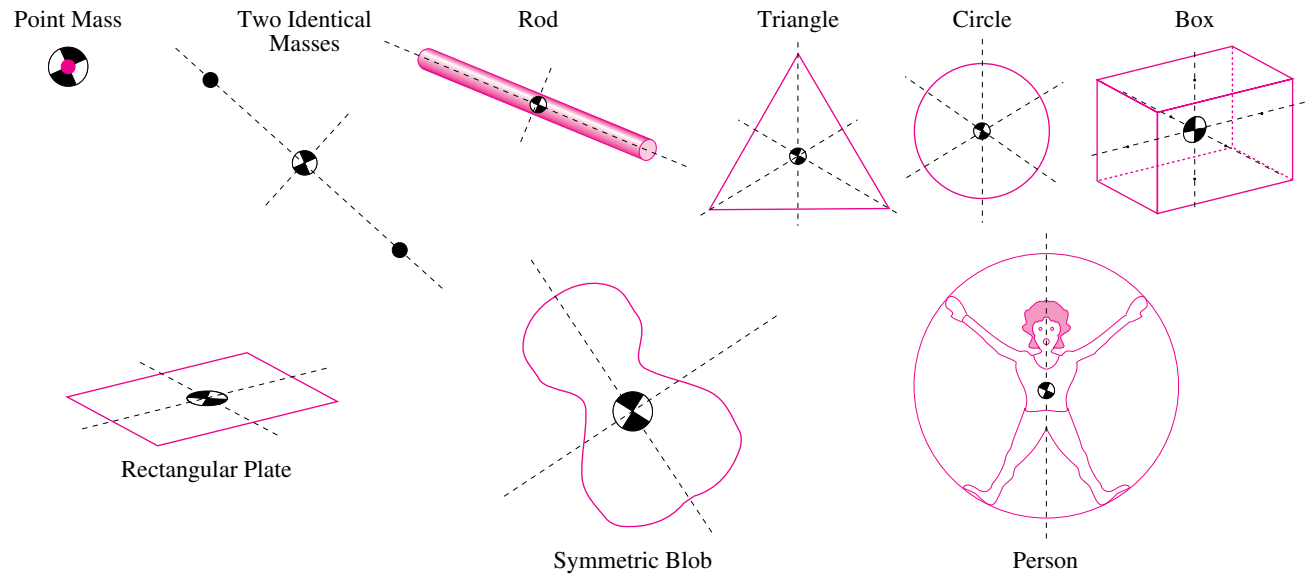
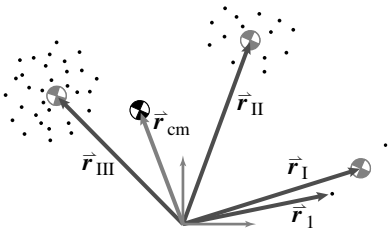



Figure 2.71: The center of mass and the geometric centroid share the symmetries of the object.

Filename:tfigure3-com-symm

## 2.11 *THEORY*

### *Why can subsystems be treated like particles when finding the center-of-mass?*




Lets look at the collection of 47 particles above and then think of it as a set of three subsystems: I, II, and III with 2, 14, and 31 particles respectively. We treat masses 1 and 2 as subsystem $I$ with center-of-mass $\vec{r}_{\rm I}$ and total mass $m_I$. Similarly, we call subsystem $II$ masses $m_3$ to $m_{16}$, and subsystem $III$, masses $m_{17}$ to $m_{47}$. We can calculate the center-of-mass of the system by treating it as 47 particles, or we can re-arrange the sum as follows:

$$\begin{aligned}
\vec{r}_{\rm cm} &= \frac{\vec{r}_1 m_1 + \vec{r}_2 m_2 + \cdots + \vec{r}_{46} m_{46} + \vec{r}_{47} m_{47}}{m_1 + m_2 + \cdots + m_{47}} \\
&= \frac{\frac{\vec{r}_1 m_1 + \vec{r}_2 m_2}{m_1 + m_2}(m_1 + m_2)}{m_1 + m_2 + \cdots + m_{47}} \\
&\quad + \frac{\frac{\vec{r}_3 m_3 + \cdots + \vec{r}_{16} m_{16}}{m_3 + \cdots + m_{16}}(m_3 + \cdots + m_{16})}{m_1 + m_2 + \cdots + m_{47}} \\
&\quad + \frac{\frac{\vec{r}_{17} m_{17} + \cdots \vec{r}_{47} m_{47}}{m_{17} + \cdots + m_{47}}(m_{17} + \cdots + m_{47})}{m_1 + m_2 + \cdots + m_{47}} \\
&= \frac{\vec{r}_{\rm I} m_I + \vec{r}_{\rm II} m_{II} + \vec{r}_{\rm III} m_{III}}{m_I + m_{II} + m_{III}}, \text{ where}
\end{aligned}$$

$$\begin{aligned}
\vec{r}_{\rm I} &= \frac{\vec{r}_1 m_1 + \vec{r}_2 m_2}{m_1 + m_2}, \\
m_I &= m_1 + m_2 \\
\vec{r}_{\rm II} &\quad etc.
\end{aligned}$$

The formula for the center-of-mass of the whole system reduces to one that looks like a sum over three (aggregate) particles.

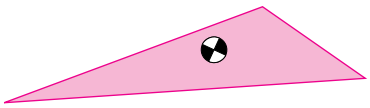
This idea is easily generalized to the integral formulae as well like this.

$$\begin{aligned}
\vec{r}_{\rm cm} &= \frac{\int \vec{r}\, dm}{\int dm} \\
&= \frac{\int_{\text{region 1}} \vec{r}\, dm + \int_{\text{region 2}} \vec{r}\, dm + \int_{\text{region 3}} \vec{r}\, dm + \cdots}{\int_{\text{region 1}} dm + \int_{\text{region 2}} dm + \int_{\text{region 1}} dm + \cdots} \\
&= \frac{\vec{r}_{\rm I} m_I + \vec{r}_{\rm II} m_{II} + \vec{r}_{\rm III} m_{III} + \cdots}{m_I + m_{II} + m_{III} + \cdots}.
\end{aligned}$$

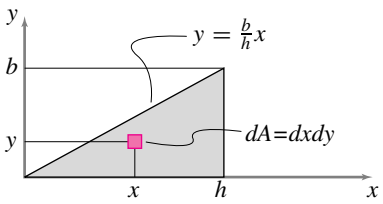
The general idea of the calculations above is that center-of-mass calculations are basically big sums (addition), and addition is 'associative.'

## 2.12 The center-of-mass of a uniform triangle is a third of the way up from the base

The center-of-mass of a 2D uniform triangular region is the centroid of the area.



First we consider a right triangle with perpendicular sides $b$ and $h$



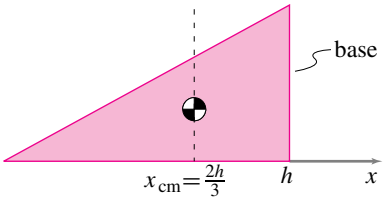

and find the $x$ coordinate of the centroid as

$$x_{\text{cm}} A = \int x\, dA$$

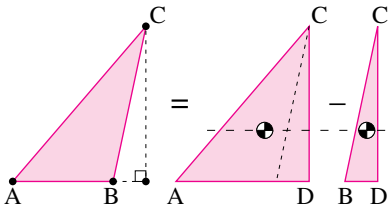
$$= \int_0^h \left[ \int_0^{\frac{b}{h}x} x\, dy \right] dx = \int_0^h [xy] \Big|_{y=0}^{y=\frac{b}{h}x} dx$$

$$x_{\text{cm}} \left( \frac{bh}{2} \right) = \int_0^h x \left( \frac{b}{h} x \right) dx = \frac{b}{h} \frac{x^3}{3} \Big|_0^h = \frac{bh^2}{3}$$

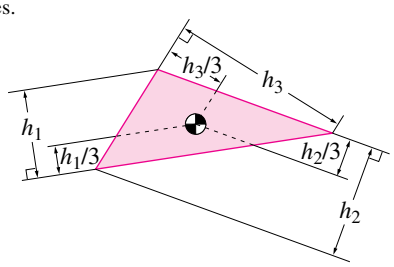
$$\Rightarrow \quad x_{\text{cm}} = \frac{2h}{3},$$ a third of the way to the left of the vertical base on the right. By similar reasoning, but in the $y$ direction, the centroid is a third of the way up from the base.




The center-of-mass of an arbitrary triangle can be found by treating it as the sum of two right triangles

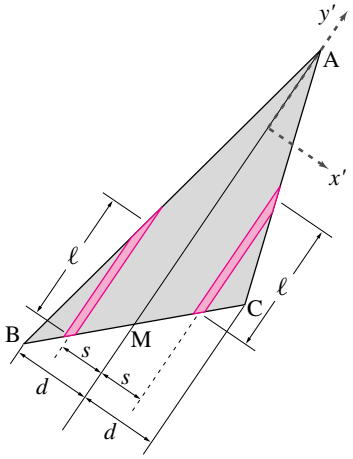



so the centroid is a third of the way up from the base of any triangle. Finally, the result holds for all three bases. Summarizing, the centroid of a triangle is at the point one third up from each of the bases.



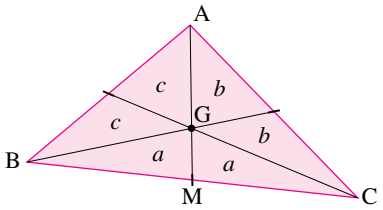

## Non-calculus approach

Consider the line segment from A to the midpoint M of side BC.




We can divide triangle ABC into equal width strips that are parallel to AM. We can group these strips into pairs, each a distance $s$ from AM. Because M is the midpoint of BC, by proportions each of these strips has the same length $\ell$. Now in trying to find the distance of the center-of-mass from the line AM we notice that all contributions to the sum come in canceling pairs because the strips are of equal area and equal distance from AM but on opposite sides. Thus the centroid is on AM. Likewise for all three sides. Thus the centroid is at the point of intersection of the three side bisectors.

That the three side bisectors intersect a third of the way up from the three bases can be reasoned by looking at the 6 triangles formed by the side bisectors.




The two triangles marked $a$ and $a$ have the same area (lets call it $a$) because they have the same height and bases of equal length (BM and CM). Similar reasoning with the other side bisectors shows that the pairs marked $b$ have equal area and so have the pairs marked $c$. But the triangle ABM has the same base and height and thus the same area as the triangle ACM. So $a + b + b = a + c + c$. Thus $b = c$ and by similar reasoning $a = b$ and all six little triangles have the same area. Thus the area of big triangle ABC is 3 times the area of GBC. Because ABC and GBC share the base BC, ABC must have 3 times the height as GBC, and point G is thus a third of the way up from the base.

## Where is the middle of a triangle?

We have shown that the centroid of a triangle is at the point that is at the intersection of: the three side bisectors; the three area bisectors (which are the side bisectors); and the three lines one third of the way up from the three bases.

If the triangle only had three equal point masses on its vertices the center of mass lands on the same place. Thus the 'middle' of a triangle seems pretty well defined. *But*, there is some ambiguity. If the triangle were made of bars along each edge, each with equal cross sections, the center-of-mass would be in a different location for all but equilateral triangles. Also, the three angle bisectors of a triangle do not intersect at the centroid. Unless we define middle to mean centroid, the "middle" of a triangle is not well defined.

**SAMPLE 2.39 Center of mass in 1-D:** Three particles (point masses) of mass 2 kg, 3 kg, and 3 kg, are welded to a straight massless rod as shown in the figure. Find the location of the center-of-mass of the assembly.

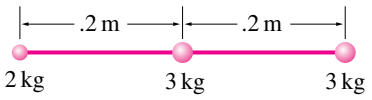



Figure 2.75:
Filename:sfig2-cm-1D

**Solution** Let us select the first mass, $m_1 = 2\,\text{kg}$, to be at the origin of our co-ordinate system with the $x$-axis along the rod. Since all the three masses lie on the $x$-axis, the center-of-mass will also lie on this axis. Let the center-of-mass be located at $x_{cm}$ on the $x$-axis. Then,

$$\begin{aligned} m_{\text{tot}} x_{cm} &= \sum_{i=1}^{3} m_i x_i = m_1 x_1 + m_2 x_2 + m_3 x_3 \\ &= m_1(0) + m_2(\ell) + m_3(2\ell) \\ \Rightarrow \quad x_{cm} &= \frac{m_2\ell + m_3 2\ell}{m_1 + m_2 + m_3} \\ &= \frac{3\,\text{kg} \cdot 0.2\,\text{m} + 3\,\text{kg} \cdot 0.4\,\text{m}}{(2+3+3)\,\text{kg}} \\ &= \frac{1.8\,\text{m}}{8} = 0.225\,\text{m}. \end{aligned}$$

$$\boxed{x_{cm} = 0.225\,\text{m}}$$

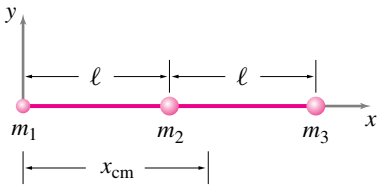



Figure 2.76:
Filename:sfig2-cm-1Da

**Alternatively**, we could find the center-of-mass by first replacing the two 3 kg masses with a single 6 kg mass located in the middle of the two masses (the center-of-mass of the two equal masses) and then calculate the value of $x_{cm}$ for a two particle system consisting of the 2 kg mass and the 6 kg mass (see Fig. 2.77):

$$x_{cm} = \frac{6\,\text{kg} \cdot 0.3\,\text{m}}{8\,\text{kg}} = \frac{1.8\,\text{m}}{8} = 0.225\,\text{m}.$$

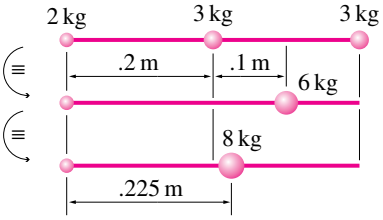



Figure 2.77:
Filename:sfig2-cm-1Db

**SAMPLE 2.40 Center of mass in 2-D:** Two particles of mass $m_1 = 1\,\text{kg}$ and $m_2 = 2\,\text{kg}$ are located at coordinates (1m, 2m) and (-2m, 5m), respectively, in the $xy$-plane. Find the location of their center-of-mass.

**Solution** Let $\vec{r}_{\text{cm}}$ be the position vector of the center-of-mass. Then,

$$\begin{aligned} m_{\text{tot}}\vec{r}_{\text{cm}} &= m_1\vec{r}_1 + m_2\vec{r}_2 \\ \Rightarrow \quad \vec{r}_{\text{cm}} &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_{\text{tot}}} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \\ &= \frac{1\,\text{kg}(1\,\text{m}\hat{\imath} + 2\,\text{m}\hat{\jmath}) + 2\,\text{kg}(-2\,\text{m}\hat{\imath} + 5\,\text{m}\hat{\jmath})}{3\,\text{kg}} \\ &= \frac{(1\,\text{m} - 4\,\text{m})\hat{\imath} + (2\,\text{m} + 10\,\text{m})\hat{\jmath}}{3} = -1\,\text{m}\hat{\imath} + 4\,\text{m}\hat{\jmath}. \end{aligned}$$

Thus the center-of-mass is located at the coordinates(-1m, 4m).

$$\boxed{(x_{cm}, y_{cm}) = (-1\,\text{m}, 4\,\text{m})}$$

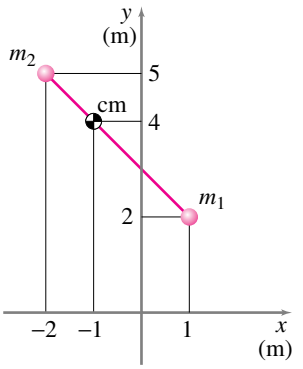



Figure 2.78:
Filename:sfig2-cm-2Da

Geometrically, this is just a 1-D problem like the previous sample. The center-of-mass has to be located on the straight line joining the two masses. Since the center-of-mass is a point about which the distribution of mass is *balanced*, it is easy to see (see Fig. 2.78) that the center-of-mass must lie one-third way from $m_2$ on the line joining the two masses so that $2\,\text{kg} \cdot (d/3) = 1\,\text{kg} \cdot (2d/3)$.

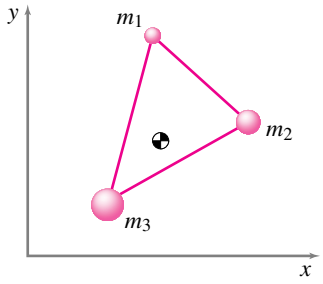



Figure 2.79:

Filename:sfig2-4-2

**SAMPLE 2.41 Location of the center-of-mass.** A structure is made up of three point masses, $m_1 = 1\,\text{kg}$, $m_2 = 2\,\text{kg}$ and $m_3 = 3\,\text{kg}$, connected rigidly by massless rods. At the moment of interest, the coordinates of the three masses are (1.25 m, 3 m), (2 m, 2 m), and (0.75 m, 0.5 m), respectively. At the same instant, the velocities of the three masses are $2\,\text{m/s}\hat{\boldsymbol{\imath}}$, $2\,\text{m/s}(\hat{\boldsymbol{\imath}} - 1.5\hat{\boldsymbol{\jmath}})$ and $1\,\text{m/s}\hat{\boldsymbol{\jmath}}$, respectively. Find the coordinates of the center-of-mass of the structure.

**Solution** Just for fun, let us do this problem two ways — first using scalar equations for the coordinates of the center-of-mass, and second, using vector equations for the position of the center-of-mass.

1. **Scalar calculations:** Let $(x_{cm}, y_{cm})$ be the coordinates of the mass-center. Then from the definition of mass-center,

$$\begin{aligned} x_{cm} &= \frac{\sum m_i x_i}{\sum m_i} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} \\ &= \frac{1\,\text{kg}\cdot 1.25\,\text{m} + 2\,\text{kg}\cdot 2\,\text{m} + 3\,\text{kg}\cdot 0.75\,\text{m}}{1\,\text{kg} + 2\,\text{kg} + 3\,\text{kg}} \\ &= \frac{7.5\,\cancel{\text{kg}}\cdot\,\text{m}}{6\,\cancel{\text{kg}}} = 1.25\,\text{m}. \end{aligned}$$

Similarly,

$$\begin{aligned} y_{cm} &= \frac{\sum m_i y_i}{\sum m_i} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} \\ &= \frac{1\,\text{kg}\cdot 3\,\text{m} + 2\,\text{kg}\cdot 2\,\text{m} + 3\,\text{kg}\cdot 0.5\,\text{m}}{1\,\text{kg} + 2\,\text{kg} + 3\,\text{kg}} \\ &= \frac{8.5\,\cancel{\text{kg}}\cdot\,\text{m}}{6\,\cancel{\text{kg}}} = 1.42\,\text{m}. \end{aligned}$$

Thus the center-of-mass is located at the coordinates (1.25 m, 1.42 m).

$$\boxed{(1.25\,\text{m},\ 1.42\,\text{m})}$$

2. **Vector calculations:** Let $\vec{r}_{\text{cm}}$ be the position vector of the mass-center. Then,

$$\begin{aligned} m_{\text{tot}}\vec{r}_{\text{cm}} &= \sum_{i=1}^{3} m_i \vec{r}_i = m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 \\ \Rightarrow \quad \vec{r}_{\text{cm}} &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1 + m_2 + m_3} \end{aligned}$$

Substituting the values of $m_1$, $m_2$, and $m_3$, and $\vec{r}_1 = 1.25\,\text{m}\hat{\boldsymbol{\imath}} + 3\,\text{m}\hat{\boldsymbol{\jmath}}$, $\vec{r}_2 = 2\,\text{m}\hat{\boldsymbol{\imath}} + 2\,\text{m}\hat{\boldsymbol{\jmath}}$, and $\vec{r}_3 = 0.75\,\text{m}\hat{\boldsymbol{\imath}} + 0.5\,\text{m}\hat{\boldsymbol{\jmath}}$, we get,

$$\begin{aligned} \vec{r}_{\text{cm}} &= \frac{1\,\text{kg}\cdot(1.25\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}})\,\text{m} + 2\,\text{kg}\cdot(2\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}})\,\text{m} + 3\,\text{kg}\cdot(0.75\hat{\boldsymbol{\imath}} + 0.5\hat{\boldsymbol{\jmath}})\,\text{m}}{(1+2+3)\,\text{kg}} \\ &= \frac{(7.5\hat{\boldsymbol{\imath}} + 8.5\hat{\boldsymbol{\jmath}})\,\cancel{\text{kg}}\cdot\,\text{m}}{6\,\cancel{\text{kg}}} \\ &= 1.25\,\text{m}\hat{\boldsymbol{\imath}} + 1.42\,\text{m}\hat{\boldsymbol{\jmath}} \end{aligned}$$

which, of course, gives the same location of the mass-center as above.

$$\boxed{\vec{r}_{\text{cm}} = 1.25\,\text{m}\hat{\boldsymbol{\imath}} + 1.42\,\text{m}\hat{\boldsymbol{\jmath}}}$$

**SAMPLE 2.42 Center of mass of a bent bar:** A uniform bar of mass 4 kg is bent in the shape of an asymmetric 'Z' as shown in the figure. Locate the center-of-mass of the bar.

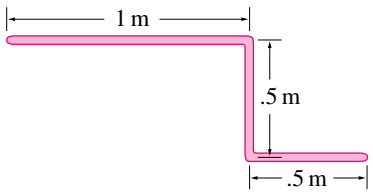



Figure 2.80:
Filename:sfig2-cm-wire

**Solution** Since the bar is uniform along its length, we can divide it into three straight segments and use their individual mass-centers (located at the geometric centers of each segment) to locate the center-of-mass of the entire bar. The mass of each segment is proportional to its length. Therefore, if we let $m_2 = m_3 = m$, then $m_1 = 2m$; and $m_1 + m_2 + m_3 = 4m = 4\,\text{kg}$ which gives $m = 1\,\text{kg}$. Now, from Fig. 2.81,

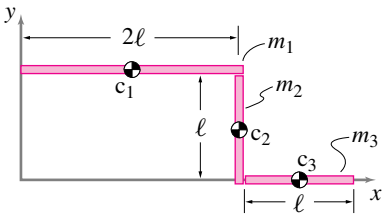



Figure 2.81:
Filename:sfig2-cm-wire-a

$$\begin{aligned}
\vec{r}_1 &= \ell\hat{\imath} + \ell\hat{\jmath} \\
\vec{r}_2 &= 2\ell\hat{\imath} + \frac{\ell}{2}\hat{\jmath} \\
\vec{r}_3 &= (2\ell + \frac{\ell}{2})\hat{\imath} = \frac{5\ell}{2}\hat{\imath}
\end{aligned}$$

So,

$$\begin{aligned}
\vec{r}_{\text{cm}} &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_{\text{tot}}} \\
&= \frac{2m(\ell\hat{\imath} + \ell\hat{\jmath}) + m(2\ell\hat{\imath} + \frac{\ell}{2}\hat{\jmath}) + m(\frac{5\ell}{2}\hat{\imath})}{4m} \\
&= \frac{\cancel{m}\ell(2\hat{\imath} + 2\hat{\jmath} + 2\hat{\imath} + \frac{1}{2}\hat{\jmath} + \frac{5}{2}\hat{\imath})}{4\cancel{m}} \\
&= \frac{\ell}{8}(13\hat{\imath} + 5\hat{\jmath}) \\
&= \frac{0.5\,\text{m}}{8}(13\hat{\imath} + 5\hat{\jmath}) \\
&= 0.812\,\text{m}\hat{\imath} + 0.312\,\text{m}\hat{\jmath}.
\end{aligned}$$

$$\boxed{\vec{r}_{\text{cm}} = 0.812\,\text{m}\hat{\imath} + 0.312\,\text{m}\hat{\jmath}}$$

Geometrically, we could find the center-of-mass by considering two masses at a time, connecting them by a line and locating their mass-center on that line, and then repeating the process as shown in Fig. 2.82.

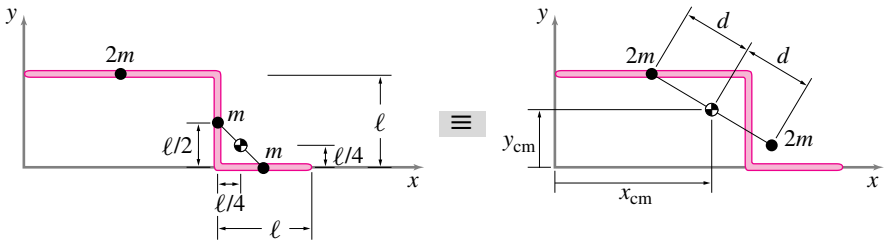



Figure 2.82:
Filename:sfig2-cm-wire-b

The center-of-mass of $m_2$ and $m_3$ (each of mass $m$) is at the mid-point of the line connecting the two masses. Now, we replace these two masses with a single mass $2m$ at their mass-center. Next, we connect this mass-center and $m_1$ with a line and find their combined mass-center at the mid-point of this line. The mass-center just found is the center-of-mass of the entire bar.

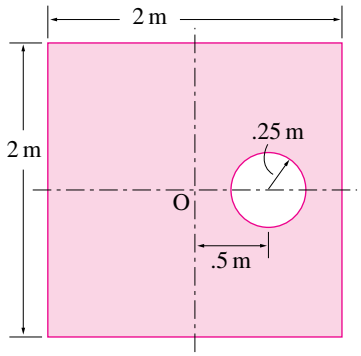



Figure 2.83:
Filename:sfig2-cm-plate

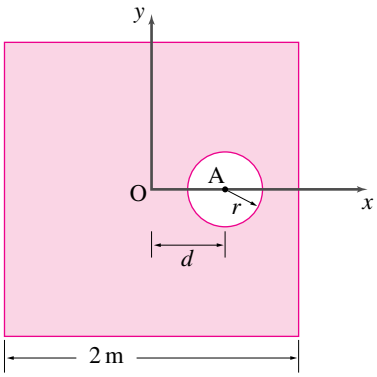



Figure 2.84:
Filename:sfig2-cm-plate-a

**SAMPLE 2.43 Shift of mass-center due to cut-outs:** A 2 m × 2 m uniform square plate has mass $m = 4\,\text{kg}$. A circular section of radius 250 mm is cut out from the plate as shown in the figure. Find the center-of-mass of the plate.

**Solution** Let us use an $xy$-coordinate system with its origin at the geometric center of the plate and the $x$-axis passing through the center of the cut-out. Since the plate and the cut-out are symmetric about the $x$-axis, the new center-of-mass must lie somewhere on the $x$-axis. Thus, we only need to find $x_{cm}$ (since $y_{cm} = 0$). Let $m_1$ be the mass of the plate with the hole, and $m_2$ be the mass of the circular cut-out. Clearly, $m_1 + m_2 = m = 4\,\text{kg}$. The center-of-mass of the circular cut-out is at A, the center of the circle. The center-of-mass of the intact square plate (without the cut-out) must be at O, the middle of the square. Then,

$$\begin{aligned} m_1 x_{cm} + m_2 x_A &= m x_O = 0 \\ \Rightarrow \quad x_{cm} &= -\frac{m_2}{m_1} x_A . \end{aligned}$$

Now, since the plate is uniform, the masses $m_1$ and $m_2$ are proportional to the surface areas of the geometric objects they represent, *i.e.*,

$$\frac{m_2}{m_1} = \frac{\pi r^2}{\ell^2 - \pi r^2} = \frac{\pi}{\left(\frac{\ell}{r}\right)^2 - \pi} .$$

Therefore,

$$\begin{aligned} x_{cm} &= -\frac{m_2}{m_1} d = -\frac{\pi}{\left(\frac{\ell}{r}\right)^2 - \pi} d \qquad (2.30) \\ &= -\frac{\pi}{\left(\frac{2\,\text{m}}{.25\,\text{m}}\right)^2 - \pi} \cdot 0.5\,\text{m} \\ &= -25.81 \times 10^{-3}\,\text{m} = -25.81\,\text{mm} \end{aligned}$$

Thus the center-of-mass shifts to the left by about 26 mm because of the circular cut-out of the given size.

$$\boxed{x_{cm} = -25.81\,\text{mm}}$$

**Comments:** The advantage of finding the expression for $x_{cm}$ in terms of $r$ and $\ell$ as in eqn. (2.30) is that you can easily find the center-of-mass of any size circular cut-out located at any distance $d$ on the $x$-axis. This is useful in design where you like to select the size or location of the cut-out to have the center-of-mass at a particular location.

**SAMPLE 2.44 Center of mass of two objects:** A square block of side 0.1 m and mass 2 kg sits on the side of a triangular wedge of mass 6 kg as shown in the figure. Locate the center-of-mass of the combined system.

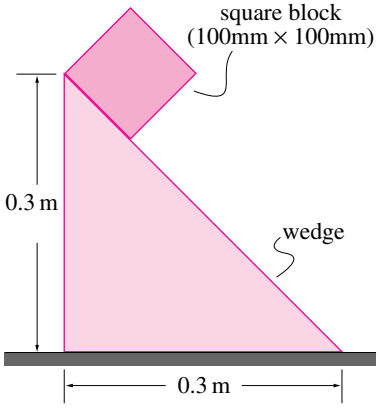



Figure 2.85:
Filename:sfig2-cm-2blocks

**Solution** The center-of-mass of the triangular wedge is located at $h/3$ above the base and $\ell/3$ to the right of the vertical side. Let $m_1$ be the mass of the wedge and $\vec{r}_1$ be the position vector of its mass-center. Then, referring to Fig. 2.86,

$$\vec{r}_1 = \frac{\ell}{3}\hat{\imath} + \frac{h}{3}\hat{\jmath}.$$

The center-of-mass of the square block is located at its geometric center $C_2$. From geometry, we can see that the line AE that passes through $C_2$ is horizontal since $\angle OAB = 45°$ ($h = \ell = 0.3\,\text{m}$) and $\angle DAE = 45°$. Therefore, the coordinates of $C_2$ are $(d/\sqrt{2}, h)$. Let $m_2$ and $\vec{r}_2$ be the mass and the position vector of the mass-center of the block, respectively. Then,

$$\vec{r}_2 = \frac{d}{\sqrt{2}}\hat{\imath} + h\hat{\jmath}.$$

Now, noting that $m_1 = 3m_2$ or $m_1 = 3m$, and $m_2 = m$ where $m = 2\,\text{kg}$, we find the center-of-mass of the combined system:

$$\begin{aligned}
\vec{r}_{\text{cm}} &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{(m_1 + m_2)} \\
&= \frac{3m(\frac{\ell}{3}\hat{\imath} + \frac{h}{3}\hat{\jmath}) + m(\frac{d}{\sqrt{2}}\hat{\imath} + h\hat{\jmath})}{3m + m} \\
&= \frac{\cancel{m}h[(\ell + \frac{d}{\sqrt{2}})\hat{\imath} + 2h\hat{\jmath}]}{4\cancel{m}} \\
&= \frac{1}{4}(\frac{d}{\sqrt{2}} + \ell)\hat{\imath} + \frac{h}{2}\hat{\jmath} \\
&= \frac{1}{4}(\frac{0.1\,\text{m}}{\sqrt{2}} + 0.3\,\text{m})\hat{\imath} + \frac{0.3\,\text{m}}{2}\hat{\jmath} \\
&= 0.093\,\text{m}\hat{\imath} + 0.150\,\text{m}\hat{\jmath}.
\end{aligned}$$

$$\boxed{\vec{r}_{\text{cm}} = 0.093\,\text{m}\hat{\imath} + 0.150\,\text{m}\hat{\jmath}}$$

Thus, the center-of-mass of the wedge and the block together is slightly closer to the side OA and higher up from the bottom OB than $C_1(0.1\,\text{m}, 0.1\,\text{m})$. This is what we should expect from the placement of the square block.

Note that we could have, again, used a 1-D calculation by placing a point mass $3m$ at $C_1$ and $m$ at $C_2$, connected the two points by a straight line, and located the center-of-mass $C$ on that line such that $CC_2 = 3CC_1$. You can verify that the distance from $C_1(0.1\,\text{m}, 0.1\,\text{m})$ to $C(.093\,\text{m}, 0.15\,\text{m})$ is one third the distance from $C$ to $C_2(.071\,\text{m}, 0.3\,\text{m})$.

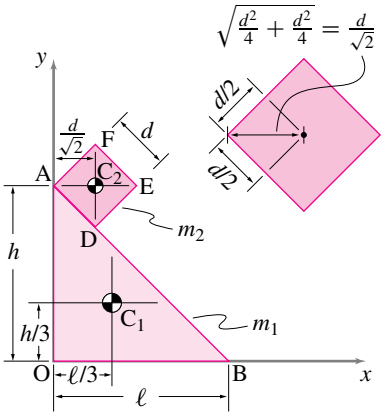



Figure 2.86:
Filename:sfig2-cm-2blocks-a

# Problems for Chapter 2

Vector skills for mechanics

## 2.1 Vector notation and vector addition

**2.1** Draw the vector $\vec{r} = (5\,\text{m})\hat{\imath} + (5\,\text{m})\hat{\jmath}$.

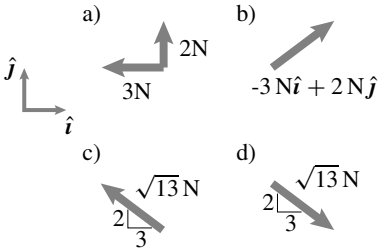
**2.2** A vector $\vec{a}$ is 2 m long and points northwest at an angle $60^\circ$ from the north. Draw the vector.

**2.3** The position vector of a point B measures 3 m and is directed at $40^\circ$ CCW from the negative $x$-axis. Show the position vector vector.

**2.4** Draw a force vector that is given as $\vec{F} = 2\,\text{N}\hat{\imath} + 2\,\text{N}\hat{\jmath} + 1\,\text{N}uk$.

**2.5** Represent the vector $\vec{r} = 5\,\text{m}\hat{\imath} - 2\,\text{m}\hat{\jmath}$ in three different ways.

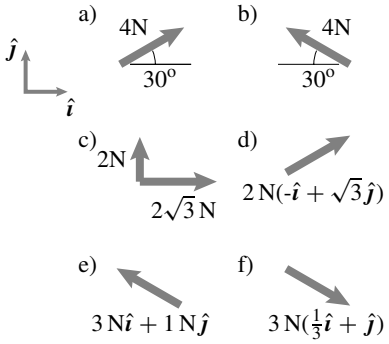
**2.6** Which one of the following representations of the same vector $\vec{F}$ is wrong and why?




**problem 2.6:**

Filename:pfigure2-vec1-2

**2.7** There are exactly two representations that describe the same vector in the following pictures. Match the correct pictures into pairs.

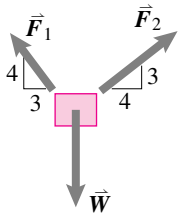



**problem 2.7:**

Filename:pfigure2-vec1-3

**2.8** Find the sum of forces $\vec{F}_1 = 20\,\text{N}\hat{\imath} - 2\,\text{N}\hat{\jmath}$, $\vec{F}_2 = 30\,\text{N}(\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath})$, and $\vec{F}_3 = -20\,\text{N}(-\hat{\imath} + \sqrt{3}\hat{\jmath})$.

**2.9** The forces acting on a block of mass $m = 5\,\text{kg}$ are shown in the figure, where $F_1 = 20\,\text{N}$, $F_2 = 50\,\text{N}$, and $W = mg$. Find the sum $\vec{F}$ $(= \vec{F}_1 + \vec{F}_2 + \vec{W})$?




**problem 2.9:**

Filename:pfigure2-vec1-6

**2.10** Given that the sum of four vectors $\vec{F}_i, i = 1$ to 4, is zero, where $\vec{F}_1 = 20\,\text{N}\hat{\imath}$, $\vec{F}_2 = -50\,\text{N}\hat{\jmath}$, $\vec{F}_3 = 10\,\text{N}(-\hat{\imath}+\hat{\jmath})$, find $\vec{F}_4$.

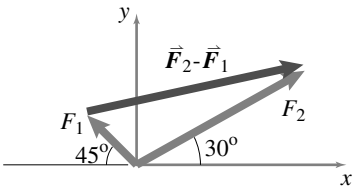
**2.11** Three forces $\vec{F} = 2\,\text{N}\hat{\imath} - 5\,\text{N}\hat{\jmath}$, $\vec{R} = 10\,\text{N}(\cos\theta\hat{\imath}+\sin\theta\hat{\jmath})$ and $\vec{W} = W\,\text{N}\hat{\jmath}$ with $W > 0$, sum up to zero. Determine $\theta$ and $W$ and draw the force vector $\vec{R}$ clearly showing its direction.

**2.12** Given that $\vec{R}_1 = 1\,\text{N}\hat{\imath} + 1.5\,\text{N}\hat{\jmath}$ and $\vec{R}_2 = 3.2\,\text{N}\hat{\imath} - 0.4\,\text{N}\hat{\jmath}$, find $2\vec{R}_1 + 5\vec{R}_2$.

**2.13** Find the magnitudes of the forces $\vec{F}_1 = 30\,\text{N}\hat{\imath} - 40\,\text{N}\hat{\jmath}$ and $\vec{F}_2 = 30\,\text{N}\hat{\imath} + 40\,\text{N}\hat{\jmath}$. Draw the two forces, representing them with their magnitudes.

**2.14** Two forces $\vec{R} = 2\,\text{N}(0.16\hat{\imath} + 0.80\hat{\jmath})$ and $\vec{W} = -36\,\text{N}\hat{\jmath}$ act on a particle. Find the magnitude of the net force. What is the direction of this force?

**2.15** In the figure shown, $F_1 = 100\,\text{N}$ and $F_2 = 300\,\text{N}$. Find the magnitude and direction of $\vec{F}_2 - \vec{F}_1$.



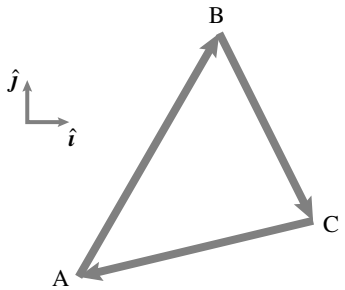

**problem 2.15:**

Filename:pfigure2-vec1-17

**2.16** Two points A and B are located in the $xy$ plane. The coordinates of A and B are (4 mm, 8 mm) and (90 mm, 6 mm), respectively.

1. Draw position vectors $\vec{r}_A$ and $\vec{r}_B$.
2. Find the magnitude of $\vec{r}_A$ and $\vec{r}_B$.
3. How far is A from B?

**2.17** Three position vectors are shown in the figure below. Given that $\vec{r}_{B/A} = 3\,\text{m}(\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath})$ and $\vec{r}_{C/B} = 1\,\text{m}\hat{\imath} - 2\,\text{m}\hat{\jmath}$, find $\vec{r}_{A/C}$.

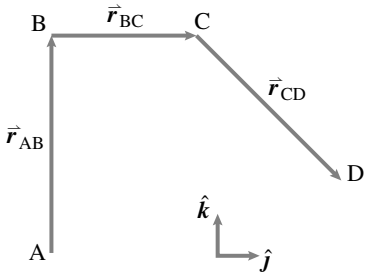



**problem 2.17:**

Filename:pfigure2-vec1-7

**2.18** In the figure shown below, the position vectors are $\vec{r}_{AB} = 3\,\text{ft}\hat{k}$, $\vec{r}_{BC} = 2\,\text{ft}\hat{\jmath}$,

and $\vec{r}_{CD} = 2(\hat{j} + \hat{k})$ ft. Find the position vector $\vec{r}_{AD}$.

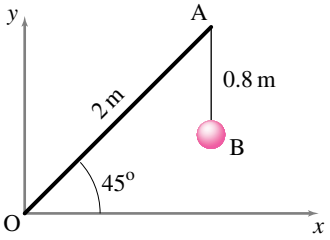



**problem 2.18:**

Filename:pfigure2-vec1-5

**2.19** In the figure shown, a ball is suspended with a 0.8 m long cord from a 2 m long hoist OA.

1. Find the position vector $\vec{r}_B$ of the ball.
2. Find the distance of the ball from the origin.

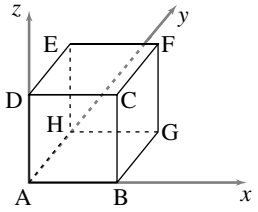



**problem 2.19:**

Filename:pfigure2-vec1-20

**2.20** A cube of side 6 in is shown in the figure.

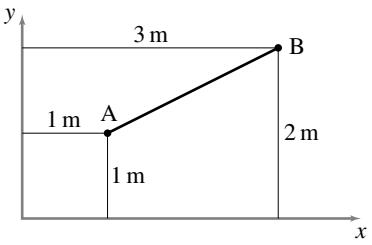
1. Find the position vector of point F, $\vec{r}_F$, from the vector sum $\vec{r}_F = \vec{r}_D + \vec{r}_{C/D} + \vec{r}_{F/c}$.
2. Calculate $|\vec{r}_F|$.
3. Find $\vec{r}_G$ using $\vec{r}_F$.




**problem 2.20:**

Filename:pfigure2-vec1-23

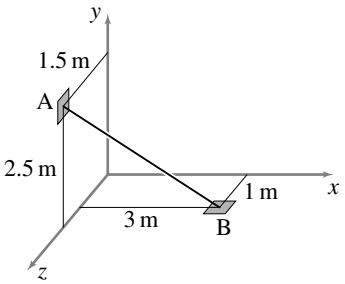
**2.21** Find the unit vector $\hat{\lambda}_{AB}$, directed from point A to point B shown in the figure.




**problem 2.21:**

Filename:pfigure2-vec1-25

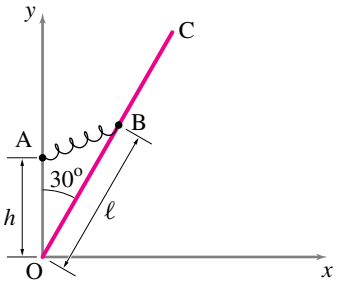
**2.22** Find a unit vector along string BA and express the position vector of A with respect to B, $\vec{r}_{A/B}$, in terms of the unit vector.




**problem 2.22:**

Filename:pfigure2-vec1-26

**2.23** In the structure shown in the figure, $\ell = 2$ ft, $h = 1.5$ ft. The force in the spring is $\vec{F} = k\vec{r}_{AB}$, where $k = 100$ lbf/ ft. Find a unit vector $\hat{\lambda}_{AB}$ along AB and calculate the spring force $\vec{F} = F\hat{\lambda}_{AB}$.




**problem 2.23:**

Filename:pfigure2-vec1-27

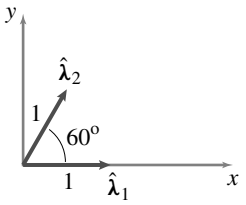
**2.24** Express the vector $\vec{r}_A = 2\,\text{m}\hat{\imath} - 3\,\text{m}\hat{\jmath} + 5\,\text{m}\hat{k}$ in terms of its magnitude and a unit vector indicating its direction.

**2.25** Let $\vec{F} = 10\,\text{lbf}\hat{\imath} + 30\,\text{lbf}\hat{\jmath}$ and $\vec{W} = -20\,\text{lbf}\hat{\jmath}$. Find a unit vector in the direction of the net force $\vec{F} + \vec{W}$, and express the the net force in terms of the unit vector.

**2.26** Let $\hat{\lambda}_1 = 0.80\hat{\imath} + 0.60\hat{\jmath}$ and $\hat{\lambda}_2 = 0.5\hat{\imath} + 0.866\hat{\jmath}$.

1. Show that $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are unit vectors.
2. Is the sum of these two unit vectors also a unit vector? If not, find a unit vector along the sum of $\hat{\lambda}_1$ and $\hat{\lambda}_2$.

**2.27** For the unit vectors $\hat{\lambda}_1$ and $\hat{\lambda}_2$ shown below, find the scalars $\alpha$ and $\beta$ such that $\alpha\hat{\lambda}_2 - 3\hat{\lambda}_2 = \beta\hat{\jmath}$.

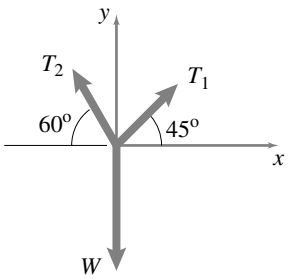



**problem 2.27:**

Filename:pfigure2-vec1-11

**2.28** If a mass slides from point A towards point B along a straight path and the coordinates of points A and B are (0 in, 5 in, 0 in) and (10 in, 0 in, 10 in), respectively, find the unit vector $\hat{\lambda}_{AB}$ directed from A to B along the path.

**2.29** In the figure shown, $T_1 = 20\sqrt{2}$ N, $T_2 = 40$ N, and $W$ is such that the sum of the three forces equals zero. If $W$ is doubled, find $\alpha$ and $\beta$ such that $\alpha\vec{T}_1$, $\beta\vec{T}_2$, and $2\vec{W}$ still sum up to zero.




**problem 2.29:**

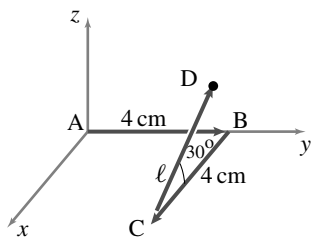
Filename:pfigure2-vec1-12

**2.30** In the figure shown, rods AB and BC are each 4 cm long and lie along $y$ and $x$ axes, respectively. Rod CD is in the $xz$ plane and makes an angle $\theta = 30^\circ$ with the $x$-axis.

1. Find $\vec{r}_{AD}$ in terms of the variable length $\ell$.

2. Find $\ell$ and $\alpha$ such that

$$\vec{r}_{\rm AD} = \vec{r}_{\rm AB} - \vec{r}_{\rm BC} + \alpha\hat{k}.$$

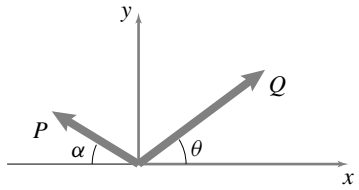



**problem 2.30:**

Filename:pfigure2-vec1-13

**2.31** In Problem 2.30, find $\ell$ such that the length of the position vector $\vec{r}_{\rm AD}$ is 6 cm.

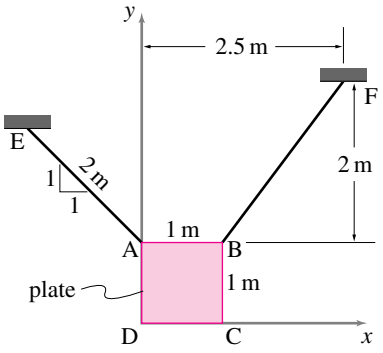
**2.32** Let two forces $\vec{P}$ and $\vec{Q}$ act in the directions shown in the figure. You are allowed to change the direction of the forces by changing the angles $\alpha$ and $\theta$ while keeping the magnitudes fixed. What should be the values of $\alpha$ and $\theta$ if the magnitude of $\vec{P} + \vec{Q}$ is to be maximum?




**problem 2.32:**

Filename:pfigure2-vec1-18

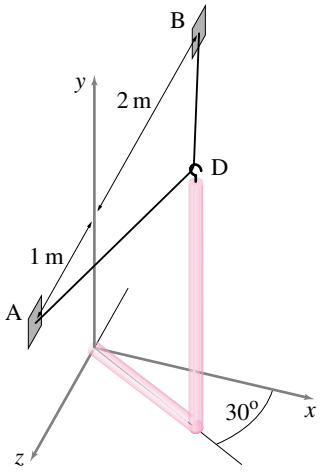
**2.33** A 1 m × 1 m square board is supported by two strings AE and BF. The tension in the string BF is 20 N. Express this tension as a vector.




**problem 2.33:**

Filename:pfigure2-vec1-21

**2.34** The top of an L-shaped bar, shown in the figure, is to be tied by strings AD and BD to the points A and B in the $yz$ plane. Find the length of the strings AD and BD using vectors $\vec{r}_{\rm AD}$ and $\vec{r}_{\rm BD}$.

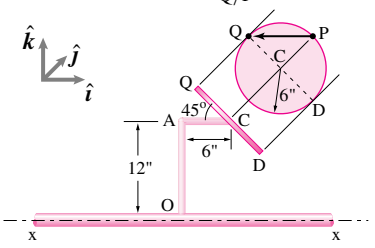



**problem 2.34:**

Filename:pfigure2-vec1-22

**2.35** A circular disk of radius 6 in is mounted on axle x-x at the end an L-shaped bar as shown in the figure. The disk is tipped 45° with respect to the horizontal bar AC. Two points, P and Q, are marked on the rim of the disk; with CP directly into the page, and Q at the highest point above the center C. Taking the base vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ as shown in the figure ($\hat{\jmath}$ into the page), find

1. the relative position vector $\vec{r}_{\rm Q/P}$,
2. the magnitude $|\vec{r}_{\rm Q/P}|$.




**problem 2.35:**

Filename:pfigure2-vec1-24

**2.36** Write the vectors $\vec{F}_1 = 30\,{\rm N}\hat{\imath} + 40\,{\rm N}\hat{\jmath} - 10\,{\rm N}\hat{k}$, $\vec{F}_2 = -20\,{\rm N}\hat{\jmath} + 2\,{\rm N}\hat{k}$, and $\vec{F}_3 = -10\,{\rm N}\hat{\imath} - 100\,{\rm N}\hat{k}$ as a list of numbers (rows or columns). Find the sum of the forces using a computer.

**2.37** Let $\alpha\vec{F}_1 + \beta\vec{F}_2 + \gamma\vec{F}_3 = \vec{0}$, where $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$ are as given in Problem 2.36. Solve for $\alpha$, $\beta$, and $\gamma$ using a computer.

**2.38** Let $\vec{r}_{\rm n} = 1\,{\rm m}(\cos\theta_n\hat{\imath} + \sin\theta_n\hat{\jmath})$, where $\theta_n = \theta_0 - n\Delta\theta$. Using a computer generate the required vectors and find the sum

$$\sum_{n=0}^{44} \vec{r}_{\rm i}, \quad \text{with } \Delta\theta = 1^\circ \text{ and } \theta_0 = 45^\circ.$$

## 2.2 The dot product of two vectors

**2.39** Find the dot product of $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} - \hat{k}$ and $\vec{b} = 2\hat{\imath} + \hat{\jmath} + 2\hat{k}$.

**2.40** Find the dot product of $\vec{F} = 0.5\,{\rm N}\hat{\imath} + 1.2\,{\rm N}\hat{\jmath} + 1.5\,{\rm N}\hat{k}$ and $\hat{\lambda} = -0.8\hat{\imath} + 0.6\hat{\jmath}$.

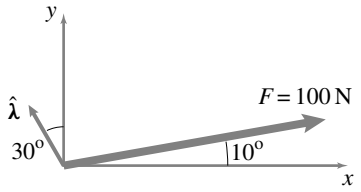
**2.41** Find the dot product $\vec{F}\cdot\vec{r}$ where $\vec{F} = (5\hat{\imath} + 4\hat{\jmath})\,{\rm N}$ and $\vec{r} = (-0.8\hat{\imath} + \hat{\jmath})\,{\rm m}$. Draw the two vectors and justify your answer for the dot product.

**2.42** Two vectors, $\vec{a} = -4\sqrt{3}\hat{\imath} + 12\hat{\jmath}$ and $\vec{b} = \hat{\imath} - \sqrt{3}\hat{\jmath}$ are given. Find the dot product of the two vectors. How is $\vec{a}\cdot\vec{b}$ related to $|\vec{a}||\vec{b}|$ in this case?

**2.43** Find the dot product of two vectors $\vec{F} = 10\,{\rm lbf}\hat{\imath} - 20\,{\rm lbf}\hat{\jmath}$ and $\hat{\lambda} = 0.8\hat{\imath} + 0.6\hat{\jmath}$. Sketch $\vec{F}$ and $\hat{\lambda}$ and show what their dot product represents.

**2.44** The position vector of a point A is $\vec{r}_{\rm A} = 30\,{\rm cm}\hat{\imath}$. Find the dot product of $\vec{r}_{\rm A}$ with $\hat{\lambda} = \frac{\sqrt{3}}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath}$.

**2.45** From the figure below, find the component of force $\vec{F}$ in the direction of $\hat{\boldsymbol{\lambda}}$.




**problem 2.45:**

Filename:pfigure2-vec1-33

**2.46** Find the angle between $\vec{F}_1 = 2\,\text{N}\hat{\imath} + 5\,\text{N}\hat{\jmath}$ and $\vec{F}_2 = -2\,\text{N}\hat{\imath} + 6\,\text{N}\hat{\jmath}$.

**2.47** Given $\vec{\omega} = 2\,\text{rad/s}\hat{\imath} + 3\,\text{rad/s}\hat{\jmath}$, $\vec{H}_1 = (20\hat{\imath} + 30\hat{\jmath})\,\text{kg m}^2/\text{s}$ and $\vec{H}_2 = (10\hat{\imath} + 15\hat{\jmath} + 6\hat{k})\,\text{kg m}^2/\text{s}$, find (a) the angle between $\vec{\omega}$ and $\vec{H}_1$ and (b) the angle between $\vec{\omega}$ and $\vec{H}_2$.

**2.48** The unit normal to a surface is given as $\hat{n} = 0.74\hat{\imath} + 0.67\hat{\jmath}$. If the weight of a block on this surface acts in the $-\hat{\jmath}$ direction, find the angle that a 1000 N normal force makes with the direction of weight of the block.

**2.49 Vector algebra.** For each equation below state whether:

1. The equation is nonsense. If so, why?
2. Is always true. Why? Give an example.
3. Is never true. Why? Give an example.
4. Is sometimes true. Give examples both ways.

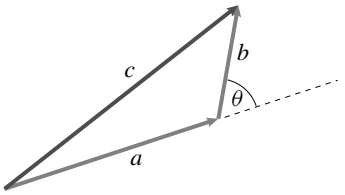
You may use trivial examples.

a) $\vec{A} + \vec{B} = \vec{B} + \vec{A}$
b) $\vec{A} + b = b + \vec{A}$
c) $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
d) $\vec{B}/\vec{C} = B/C$
e) $b/\vec{A} = b/A$
f) $\vec{A} = (\vec{A}\cdot\vec{B})\vec{B} + (\vec{A}\cdot\vec{C})\vec{C} + (\vec{A}\cdot\vec{D})\vec{D}$

**2.50** Use the dot product to show 'the law of cosines'; i. e.,

$$c^2 = a^2 + b^2 + 2ab\cos\theta.$$

(Hint: $\vec{c} = \vec{a} + \vec{b}$; also, $\vec{c}\cdot\vec{c} = \vec{c}\cdot\vec{c}$)




**problem 2.50:**

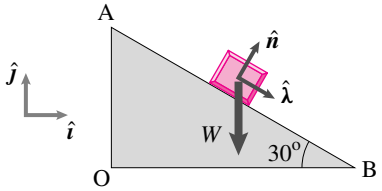
Filename:pfigure-blue-2-1

**2.51** Find the direction cosines of $\vec{F} = 3\,\text{N}\hat{\imath} - 4\,\text{N}\hat{\jmath} + 5\,\text{N}\hat{k}$.

**2.52** A force acting on a bead of mass $m$ is given as $\vec{F} = -20\,\text{lbf}\hat{\imath} + 22\,\text{lbf}\hat{\jmath} + 12\,\text{lbf}\hat{k}$. What is the angle between the force and the $z$-axis?

**2.53** (a) Draw the vector $\vec{r} = 3.5\,\text{in}\hat{\imath} + 3.5\,\text{in}\hat{\jmath} - 4.95\,\text{in}\hat{k}$. (b) Find the angle this vector makes with the $z$-axis. (c) Find the angle this vector makes with the $x$-$y$ plane.

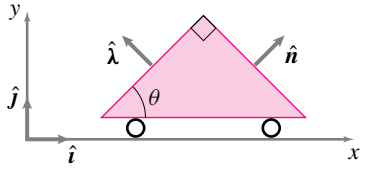
**2.54** In the figure shown, $\hat{\boldsymbol{\lambda}}$ and $\hat{\boldsymbol{n}}$ are unit vectors parallel and perpendicular to the surface AB, respectively. A force $\vec{W} = -50\,\text{N}\hat{\jmath}$ acts on the block. Find the components of $\vec{W}$ along $\hat{\boldsymbol{\lambda}}$ and $\hat{\boldsymbol{n}}$.




**problem 2.54:**

Filename:pfigure2-vec1-41

**2.55** Express the unit vectors $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{\lambda}}$ in terms of $\hat{\imath}$ and $\hat{\jmath}$ shown in the figure. What are the $x$ and $y$ components of $\vec{r} = 3.0\,\text{ft}\hat{\boldsymbol{n}} - 1.5\,\text{ft}\hat{\boldsymbol{\lambda}}$?

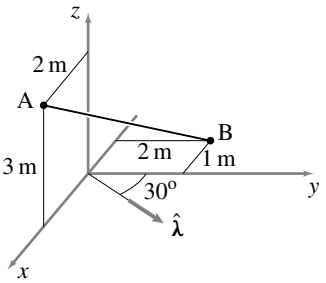



**problem 2.55:**

Filename:efig1-2-27

**2.56** From the figure shown, find the components of vector $\vec{r}_{\text{AB}}$ (you have to first find this position vector) along

1. the $y$-axis, and
2. along $\hat{\boldsymbol{\lambda}}$.

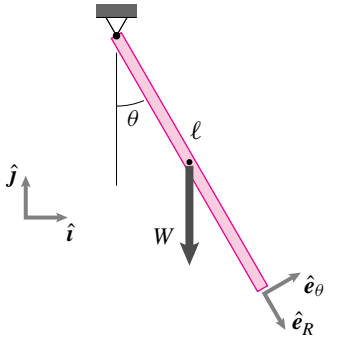



**problem 2.56:**

Filename:pfigure2-vec1-42

**2.57** The net force acting on a particle is $\vec{F} = 2\,\text{N}\hat{\imath} + 10\,\text{N}\hat{\jmath}$. Find the components of this force in another coordinate system with basis vectors $\hat{\imath}' = -\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$ and $\hat{\jmath}' = -\sin\theta\hat{\imath} - \cos\theta\hat{\jmath}$. For $\theta = 30^\circ$, sketch the vector $\vec{F}$ and show its components in the two coordinate systems.

**2.58** Find the unit vectors $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ in terms of $\hat{\imath}$ and $\hat{\jmath}$ with the geometry shown in the figure. What are the components of $\vec{W}$ along $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$?

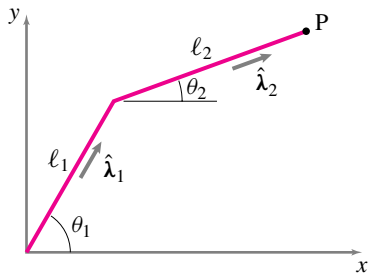



**problem 2.58:**

Filename:pfigure2-vec1-44

**2.59** Write the position vector of point P in terms of $\hat{\boldsymbol{\lambda}}_1$ and $\hat{\boldsymbol{\lambda}}_2$ and

1. find the $y$-component of $\vec{r}_{\text{P}}$,
2. find the component of $\vec{r}_{\text{P}}$ along $\hat{\boldsymbol{\lambda}}_1$.




**problem 2.59:**

Filename:pfigure2-vec1-45

**2.60** Let $\vec{F}_1 = 30\,\text{N}\hat{\imath} + 40\,\text{N}\hat{\jmath} - 10\,\text{N}\hat{k}$, $\vec{F}_2 = -20\,\text{N}\hat{\jmath} + 2\,\text{N}\hat{k}$, and $\vec{F}_3 = F_{3_x}\hat{\imath} + F_{3_y}\hat{\jmath} - F_{3_z}\hat{k}$. If the sum of all these forces must equal zero, find the required scalar equations to solve for the components of $\vec{F}_3$.

**2.61** A force $\vec{F}$ is directed from point A(3,2,0) to point B(0,2,4). If the $x$-component of the force is 120 N, find the $y$- and $z$-components of $\vec{F}$.

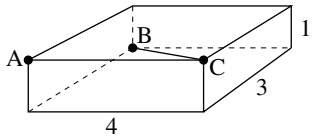
**2.62** A vector equation for the sum of forces results into the following equation:

$$\frac{F}{2}(\hat{\imath} - \sqrt{3}\hat{\jmath}) + \frac{R}{5}(3\hat{\imath} + 6\hat{\jmath}) = 25\,\text{N}\hat{\boldsymbol{\lambda}}$$

where $\hat{\boldsymbol{\lambda}} = 0.30\hat{\imath} - 0.954\hat{\jmath}$. Find two scalar equations by dotting both sides of the equation first with $\hat{\boldsymbol{\lambda}}$ and then with a vector orthogonal to $\hat{\boldsymbol{\lambda}}$.

**2.63** Write a computer program (or use a canned program) to find the dot product of two 3-D vectors. Test the program by computing the dot products $\hat{\imath}\cdot\hat{\imath}$, $\hat{\imath}\cdot\hat{\jmath}$, and $\hat{\jmath}\cdot\hat{k}$. Now use the program to find the components of $\vec{F} = (2\hat{\imath} + 2\hat{\jmath} - 3\hat{k})$ N along the line $\vec{r}_{\text{AB}} = (0.5\hat{\imath} - 0.2\hat{\jmath} + 0.1\hat{k})$ m.

**2.64** What is the shortest distance between the point A and the diagonal BC of the parallelepiped shown? (Use vector methods.)

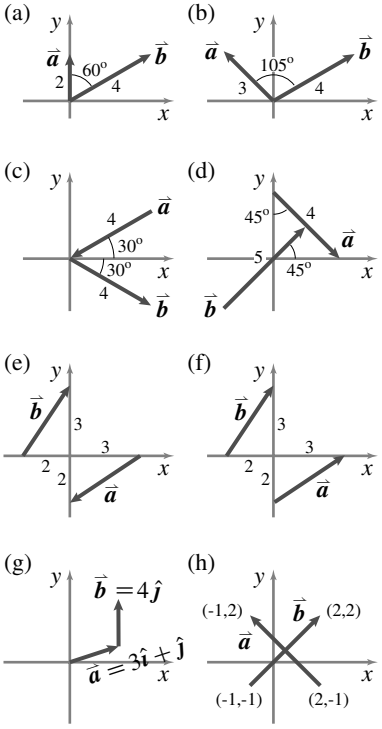



**problem 2.64:**

Filename:pfigure-blue-2-3

## 2.3 Cross product, moment, and moment about an axis

**2.65** Find the cross product of the two vectors shown in the figures below from the information given in the figures.




**problem 2.65:**

Filename:pfigure2-vec2-1

**2.66 Vector algebra.** For each equation below state whether:

1. The equation is nonsense. If so, why?
2. Is always true. Why? Give an example.
3. Is never true. Why? Give an example.
4. Is sometimes true. Give examples both ways.

You may use trivial examples.

a) $\vec{B}\times\vec{C} = \vec{C}\times\vec{B}$

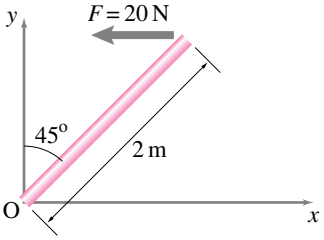
b) $\vec{B}\times\vec{C} = \vec{C}\cdot\vec{B}$

c) $\vec{C}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{C}\times\vec{A})$

d) $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$

**2.67** What is the moment $\vec{M}$ produced by a 20 N force $F$ acting in the $x$ direction with a lever arm of $\vec{r} = (16\,\text{mm})\hat{\jmath}$?

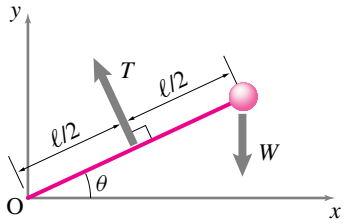
**2.68** Find the moment of the force shown on the rod about point O.




**problem 2.68:**

Filename:pfigure2-vec2-2

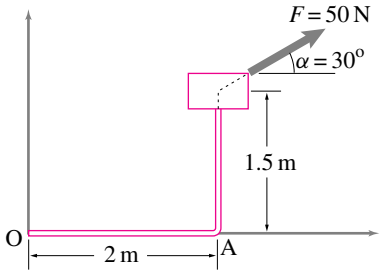
**2.69** Find the sum of moments of forces $\vec{W}$ and $\vec{T}$ about the origin, given that $W = 100\,\text{N}$, $T = 120\,\text{N}$, $\ell = 4\,\text{m}$, and $\theta = 30°$.




**problem 2.69:**

Filename:pfigure2-vec2-3

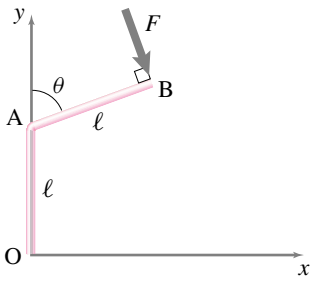
**2.70** Find the moment of the force
a) about point A
b) about point O.




**problem 2.70:**

Filename:pfigure2-vec2-4

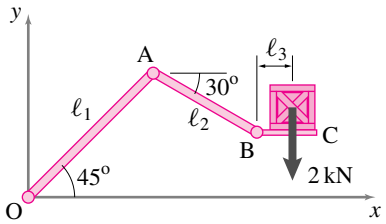
**2.71** In the figure shown, OA = AB = 2 m. The force $F = 40\,\text{N}$ acts perpendicular to the arm AB. Find the moment of $\vec{F}$ about O, given that $\theta = 45°$. If $\vec{F}$ always acts normal to the arm AB, would increasing $\theta$ increase the magnitude of the moment? In particular, what value of $\theta$ will give the largest moment?




**problem 2.71:**

Filename:pfigure2-vec2-5

**2.72** Calculate the moment of the 2 kNpayload on the robot arm about (i) joint A, and (ii) joint B, if $\ell_1 = 0.8\,\text{m}$, $\ell_2 = 0.4\,\text{m}$, and $\ell_3 = 0.1\,\text{m}$.

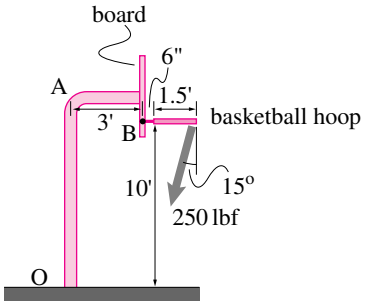



**problem 2.72:**

Filename:pfigure2-vec2-6

**2.73** During a slam-dunk, a basketball player pulls on the hoop with a 250 lbf at point C of the ring as shown in the figure. Find the moment of the force about

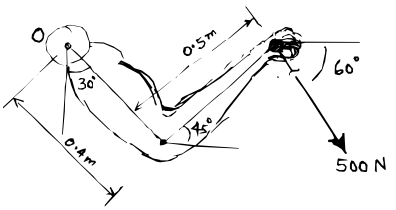
a) the point of the ring attachment to the board (point B), and
b) the root of the pole, point O.




**problem 2.73:**

Filename:pfigure2-vec2-7

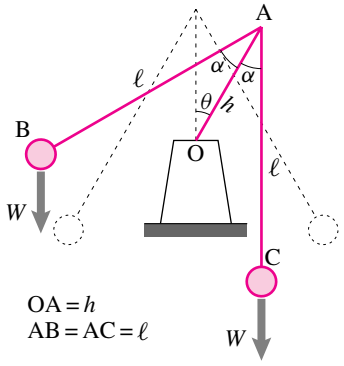
**2.74** During weight training, an athlete pulls a weight of 500 N with his arms pulling on a handlebar connected to a universal machine by a cable. Find the moment of the force about the shoulder joint O in the configuration shown.




**problem 2.74:**

Filename:pfigure2-vec2-8

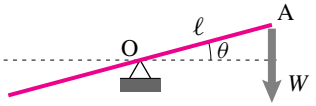
**2.75** Find the sum of moments due to the two weights of the teeter-totter when the teeter-totter is tipped at an angle $\theta$ from its vertical position. Give your answer in terms of the variables shown in the figure.




**problem 2.75:**

Filename:pfigure2-vec2-9

**2.76** Find the percentage error in computing the moment of $\vec{W}$ about the pivot point O as a function of $\theta$, if the weight is assumed to act normal to the arm OA (a good approximation when $\theta$ is very small).




**problem 2.76:**

Filename:pfigure2-vec2-10

**2.77** What do you get when you cross a vector and a scalar?

**2.78** Why did the chicken cross the road?

**2.79** Carry out the following cross products in different ways and determine which method takes the least amount of time for you.

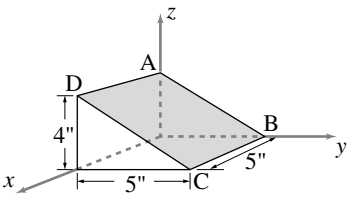
a) $\vec{r} = 2.0\,\text{ft}\hat{\imath} + 3.0\,\text{ft}\hat{\jmath} - 1.5\,\text{ft}\hat{k}$; $\vec{F} = -0.3\,\text{lbf}\hat{\imath} - 1.0\,\text{lbf}\hat{k}$; $\vec{r} \times \vec{F} = ?$
b) $\vec{r} = (-\hat{\imath} + 2.0\hat{\jmath} + 0.4\hat{k})\,\text{m}$; $\vec{L} = (3.5\hat{\jmath} - 2.0\hat{k})\,\text{kg m/s}$; $\vec{r} \times \vec{L} = ?$
c) $\vec{\omega} = (\hat{\imath} - 1.5\hat{\jmath})\,\text{rad/s}$; $\vec{r} = (10\hat{\imath} - 2\hat{\jmath} + 3\hat{k})\,\text{in}$; $\vec{\omega} \times \vec{r} = ?$

**2.80** The line of action of a force $\vec{F} = 20\,\text{N}\hat{\jmath} - 5\,\text{N}\hat{k}$ passes through a point A with coordinates (200 mm, 300 mm, -100 mm). What is the moment $\vec{M}(= \vec{r} \times \vec{F})$ of the force about the origin?

**2.81 Cross Product program** Write a program that will calculate cross products. The input to the function should be the components of the two vectors and the output should be the components of the cross product. As a model, here is a function file that calculates dot products in pseudo code.

```
%program definition
z(1)=a(1)*b(1);
z(2)=a(2)*b(2);
z(3)=a(3)*b(3);
w=z(1)+z(2)+z(3);
```

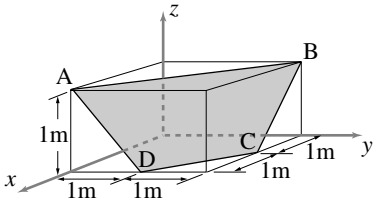
**2.82** Find a unit vector normal to the surface ABCD shown in the figure.




**problem 2.82:**

Filename:efig1-2-11

**2.83** If the magnitude of a force $\vec{N}$ normal to the surface ABCD in the figure is 1000 N, write $\vec{N}$ as a vector.

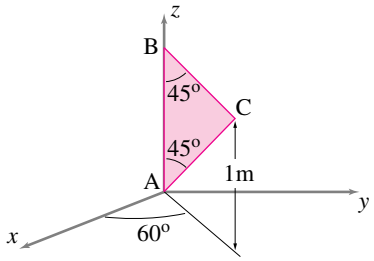



**problem 2.83:**

Filename:efig1-2-12

**2.84** The equation of a surface is given as $z = 2x - y$. Find a unit vector $\hat{\boldsymbol{n}}$ normal to the surface.

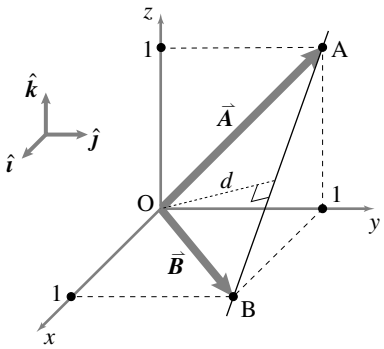
**2.85** In the figure, a triangular plate ACB, attached to rod AB, rotates about the $z$-axis. At the instant shown, the plate makes an angle of $60^\circ$ with the $x$-axis. Find and draw a vector normal to the surface ACB.




**problem 2.85:**

Filename:efig1-2-14

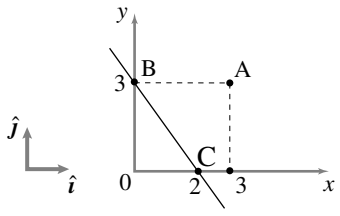
**2.86** What is the distance $d$ between the origin and the line $AB$ shown? (You may write your solution in terms of $\vec{A}$ and $\vec{B}$ before doing any arithmetic).




**problem 2.86:**

Filename:pfigure-blue-1-3

**2.87** What is the perpendicular distance between point A and line BC shown? (There are at least 3 ways to do this using various vector products, how many ways can you find?)




**problem 2.87:**

Filename:pfigure-blue-2-2

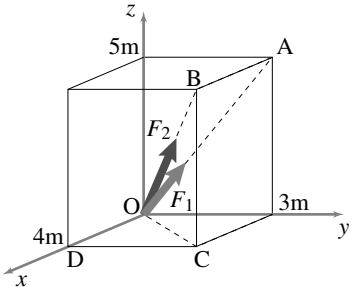
**2.88** Given a force, $\vec{F}_1 = (-3\hat{\imath} + 2\hat{\jmath} + 5\hat{k})$ N acting at a point $P$ whose position is given by $\vec{r}_{P/O} = (4\hat{\imath} - 2\hat{\jmath} + 7\hat{k})$ m, what is the moment about an axis through the origin $O$ with direction $\hat{\boldsymbol{\lambda}} = \frac{2}{\sqrt{5}}\hat{\jmath} + \frac{1}{\sqrt{5}}\hat{\jmath}$?

**2.89 Drawing vectors and computing with vectors.** The point O is the origin. Point A has $xyz$ coordinates (0, 5, 12)m. Point B has $xyz$ coordinates (4, 5, 12)m.

a) Make a neat sketch of the vectors OA, OB, and AB.
b) Find a unit vector in the direction of OA, call it $\hat{\boldsymbol{\lambda}}_{OA}$.
c) Find the force $\vec{F}$ which is 5N in size and is in the direction of OA.
d) What is the angle between OA and OB?
e) What is $\vec{r}_{BO} \times \vec{F}$?
f) What is the moment of $\vec{F}$ about a line parallel to the $z$ axis that goes through the point B?

**2.90 Vector Calculations and Geometry.** The 5 N force $\vec{F}_1$ is along the line OA. The 7 N force $\vec{F}_2$ is along the line OB.

a) Find a unit vector in the direction OB.
b) Find a unit vector in the direction OA.
c) Write both $\vec{F}_1$ and $\vec{F}_2$ as the product of their magnitudes and unit vectors in their directions.
d) What is the angle AOB?
e) What is the component of $\vec{F}_1$ in the $x$-direction?
f) What is $\vec{r}_{DO} \times \vec{F}_1$? ($\vec{r}_{DO} \equiv \vec{r}_{O/D}$ is the position of O relative to D.)
g) What is the moment of $\vec{F}_2$ about the axis DC?
h) Repeat the last problem using either a different reference point on the axis DC or the line of action OB. Does the solution agree? [Hint: it should.]




**problem 2.90:**

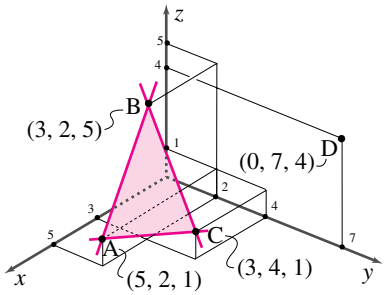
Filename:p1sp92

**2.91** $A$, $B$, and $C$ are located by position vectors $\vec{r}_A = (1, 2, 3)$, $\vec{r}_B = (4, 5, 6)$, and $\vec{r}_C = (7, 8, 9)$.

a) Use the vector dot product to find the angle $BAC$ ($A$ is at the vertex of this angle).
b) Use the vector cross product to find the angle $BCA$ ($C$ is at the vertex of this angle).
c) Find a unit vector perpendicular to the plane $ABC$.
d) How far is the infinite line defined by $AB$ from the origin? (That is, how close is the closest point on this line to the origin?)
e) Is the origin co-planar with the points $A$, $B$, and $C$?

**2.92** Points A, B, and C in the figure define a plane.

a) Find a unit normal vector to the plane.
b) Find the distance from perpendicular distance from point D to this infinite plane.
c) What are the coordinates of the point on the plane closest to point D?
d) Is this point on or off the triangle used to define the plane?




**problem 2.92:**

Filename:pfigure-s95q2

**2.93** What point on the line that goes through the points (1,2,3) and (7,12,15) is closest to the origin?

**2.94** A regular tetrahedron is a triangular-based pyramid where all 6 edges have the same length $\ell$. What is the perpendicular distance between a pair of non-touching edges? (There are many ways to solve this problem).

## 2.4 Solving vector equations

**2.95** Consider the vector equation

$$a\vec{A} + b\vec{B} = \vec{C}$$

with $\vec{A}$, $\vec{B}$, and $\vec{C}$ given. For the cases below find $a$ and $b$ if possible. If there are multiple solutions give at least 2. If there are no solutions explain why.

a) $\vec{A} = \hat{\imath}$, $\vec{B} = \hat{\jmath}$, $\vec{C} = 3\hat{\imath} + 4\hat{\jmath}$

b) $\vec{A} = \hat{\imath}$, $\vec{B} = 2\hat{\imath}$, $\vec{C} = 3\hat{\imath}$

c) $\vec{A} = \hat{\jmath}$, $\vec{B} = 2\hat{\jmath}$, $\vec{C} = 3\hat{\imath}$

d) $\vec{A} = \hat{\imath} + \hat{\jmath}$, $\vec{B} = -\hat{\imath} + \hat{\jmath}$, $\vec{C} = 2\hat{\jmath}$

e) $\vec{A} = \hat{\imath} + 2\hat{\jmath}$, $\vec{B} = 2\hat{\imath} + 3\hat{\jmath}$, $\vec{C} = 3\hat{\imath} + 4\hat{\jmath}$

f) $\vec{A} = \pi\hat{\imath} + e\hat{\jmath}$, $\vec{B} = \sqrt{2}\hat{\imath} + \sqrt{3}\hat{\jmath}$, $\vec{C} = \hat{\imath}$

**2.96** Consider the vector equation

$$a\vec{A} + b\vec{B} + c\vec{C} = \vec{D}$$

with $\vec{A}$, $\vec{B}$, $\vec{C}$ and $\vec{D}$ given. For the cases below find $a$ if possible, there is no need to find $b$ and $c$.

a) $\vec{A} = \hat{\imath}$, $\vec{C} = \hat{k}$, $\vec{B} = \hat{\jmath}$, $\vec{D} = 3\hat{\imath} + 4\hat{\jmath} + 19\hat{k}$

b) $\vec{A} = 2\hat{\imath}$, $\vec{C} = 15\hat{\jmath} + 360\hat{k}$, $\vec{B} = 3\hat{\imath} + 4\hat{\jmath}$, $\vec{D} = 2\hat{\imath} + 17\hat{\jmath} + 37\hat{k}$

c) $\vec{A} = \hat{k}$, $\vec{C} = \hat{\imath} + \hat{\jmath} + \hat{k}$, $\vec{B} = \hat{\jmath} + \hat{k}$, $\vec{D} = \hat{\jmath}$

d) $\vec{A} = \hat{\imath} + \hat{\jmath} + \hat{k}$, $\vec{C} = 3\hat{\imath} + 4\hat{\jmath} + 5\hat{k}$, $\vec{B} = 2\hat{\imath}+3\hat{\jmath}+4\hat{k}$, $\vec{D} = 4\hat{\imath}+5\hat{\jmath}+7\hat{k}$

e) $\vec{A} = \sqrt{1}\hat{\imath} + \sqrt{2}\hat{\jmath} + \sqrt{3}\hat{k}$, $\vec{C} = \sqrt{7}\hat{\imath} + \sqrt{8}\hat{\jmath} + \sqrt{9}\hat{k}$, $\vec{B} = \sqrt{4}\hat{\imath} + \sqrt{5}\hat{\jmath} + \sqrt{6}\hat{k}$, $\vec{D} = -\hat{\imath} + \pi\hat{\jmath} + e\hat{k}$

**2.97** In the problems below use matrix algebra on a computer to find $a$, $b$ and $c$ uniquely if possible. If not possible explain why not. You are given that

$$a\vec{A} + b\vec{B} + c\vec{C} = \vec{D}$$

and that

a) $\vec{A} = \hat{\imath}$, $\vec{B} = \hat{\jmath}$, $\vec{C} = \hat{k}$, $\vec{D} = -2\hat{\imath} + 5\hat{\jmath} + 10\hat{k}$

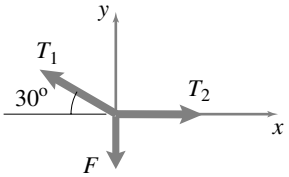
b) $\vec{A} = \hat{\imath} + \hat{\jmath}$, $\vec{B} = -\hat{\imath} + \hat{\jmath}$, $\vec{C} = \hat{k}$, $\vec{D} = 2\hat{\imath}$

c) $\vec{A} = \hat{\imath} + \hat{\jmath} + \hat{k}$, $\vec{B} = 2\hat{\imath} + \hat{\jmath} + \hat{k}$, $\vec{C} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$, $\vec{D} = 2\hat{\imath} + \hat{\jmath} + \hat{k}$

d) $\vec{A} = 1\hat{\imath}+2\hat{\jmath}+3\hat{k}$, $\vec{B} = 4\hat{\imath}+5\hat{\jmath}+6\hat{k}$, $\vec{C} = 7\hat{\imath} + 8\hat{\jmath} + 9\hat{k}$, $\vec{D} = 10\hat{k}$

e) $\vec{A} = \sqrt{1}\hat{\imath} + \sqrt{2}\hat{\jmath} + \sqrt{3}\hat{k}$, $\vec{B} = \sqrt{4}\hat{\imath} + \sqrt{5}\hat{\jmath} + \sqrt{6}\hat{k}$, $\vec{C} = \sqrt{7}\hat{\imath} + \sqrt{8}\hat{\jmath} + \sqrt{9}\hat{k}$, $\vec{D} = \sqrt{10}\hat{k}$

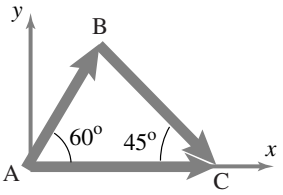
**2.98** The three forces shown in the figure are in equilibrium, *i.e.*, $\vec{T}_1 + \vec{T}_2 + \vec{F} = \vec{0}$. If $|\vec{F}| = 10\,\text{N}$, find tensions $T_1$ and $T_2$ (magnitudes of $\vec{T}_1$ and $\vec{T}_2$.




**problem 2.98:**

Filename:pfigure2-4-2Dp1

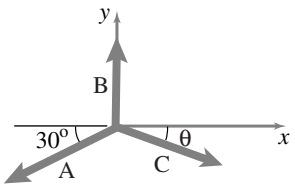
**2.99** Points A, B, and C are located in the $xy$ plane as shown in the figure. For position vectors, we can write, $\vec{r}_B + \vec{r}_{C/B} = \vec{r}_C$. Find $|\vec{r}_B|$ and $|\vec{r}_{C/B}|$ if $\vec{r}_C = 10\,\text{m}\,\hat{\imath}$.




**problem 2.99:**

Filename:pfigure2-4-2Dp2

**2.100** Three vectors, $\vec{A}$, $\vec{B}$, and $\vec{C}$, (shown in the figure) are such that $\vec{A} + \vec{B} + \vec{C} = \vec{0}$. You are given that $A = |\vec{A}| = 8$ and $C = |\vec{C}| = 5$. Find $\theta$.




**problem 2.100:**

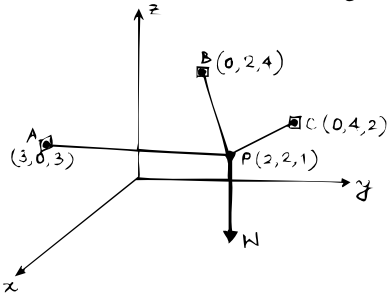
Filename:pfigure2-4-2Dp3

**2.101** Let $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \vec{0}$, where $\vec{F}_3 = 10\,\text{N}(\hat{\imath} - \hat{\jmath})$, and $\vec{F}_1/|\vec{F}_1| = 0.250\hat{\imath} + 0.968\hat{\jmath}$ and $\vec{F}_2/|\vec{F}_2| = -0.425\hat{\imath} - 0.905\hat{\jmath}$. Find $|\vec{F}_1|$ from a single scalar equation.

**2.102** To evaluate the equation $\sum \vec{F} = m\vec{a}$ for some problem, a student writes $\sum \vec{F} = F_x\hat{\imath} - (F_y - 30\,\text{N})\hat{\jmath} + 50\,\text{N}\hat{k}$ in the $xyz$ coordinate system, but $\vec{a} = 2.5\,\text{m/s}^2\hat{\imath}' + 1.8\,\text{m/s}^2\hat{\jmath}' - a_z\hat{k}'$ in a rotated $x'y'z'$ coordinate system. If $\hat{\imath}' = \cos 60^\circ\hat{\imath} + \sin 60^\circ\hat{\jmath}$, $\hat{\jmath}' = -\sin 60^\circ\hat{\imath} + \cos 60^\circ\hat{\jmath}$ and $\hat{k}' = \hat{k}$, find the scalar equations for the $x'$, $y'$, and $z'$ directions.

**2.103** A car travels straight north-east for a while on a dirt road that leads to a north-south highway. The car travels on the highway due north for a while. When the driver stops, the GPS system indicates that the car is 60 miles north and 30 miles east from the starting point. Find the distance travelled on the dirt road.

**2.104** A particle is held at point P with the help of three strings PA, PB, and PC. Let the tensions in the three strings be $\vec{T}_A$, $\vec{T}_B$, and $\vec{T}_C$, respectively (so that $\vec{T}_A$ acts along line PA and so on). The equilibrium of the particle requires that $\vec{T}_A + \vec{T}_B + \vec{T}_C + \vec{W} = \vec{0}$ where $\vec{W} = -10\,\text{N}\,\hat{k}$ is the weight of the particle. Find the magnitudes of tensions in the three strings.




**problem 2.104:**

Filename:pfigure2-4-3D1

**2.105** You are given that $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 5\,\text{kN}\ \hat{\jmath}$ where $\vec{F}_1 = (2\hat{\imath} - 3\hat{\jmath} + 4\hat{k})\,\text{kN}$, $\vec{F}_2 = (\hat{\imath} + 5\hat{k})\,\text{kN}$. Find the direction of $\vec{F}_3$ (An angle measured CCW from the $+x$ axis to the direction of positive $\vec{F}_3$.

**2.106** A plane intersects the $x$, $y$, and $z$ axis at 3,4, and 5 respectively. What point on the plane is in the direction $\hat{\imath} + 2\hat{\jmath} + 3\hat{\jmath}$ from the point (10,10,10)? (Find the $x$, $y$ and $z$ components of the point.).

These problems concern the solution of simultaneous equations. These could come from various vector equations.

**2.107** Write the following equations in matrix form to solve for x, y, and z:

$$2x - 3y + 5 = 0,$$
$$y + 2\pi z = 21,$$
$$\frac{1}{3}x - 2y + \pi z - 11 = 0.$$

**2.108** Are the following equations linearly independent?

a) $x_1 + 2x_2 + x_3 = 30$
b) $3x_1 + 6x_2 + 9x_3 = 4.5$
c) $2x_1 + 4x_2 + 15x_3 = 7.5$.

**2.109** Write computer commands (or a program) to solve for $x$, $y$ and $z$ from the following equations with $r$ as an input variable. Your program should display an error message if, for a particular $r$, the equations are not linearly independent.

a) $5x + 2r\,y + z = 2$
b) $3x + 6y + (2r - 1)z = 3$
c) $2x + (r - 1)y + 3r\,z = 5$.

Find the solutions for $r = 3$, 4.99, and 5.

**2.110** An exam problem in statics has three unknown forces. A student writes the following three equations (he knows that he needs three equations for three unknowns!) — one for the force balance in the $x$-direction and the other two for the moment balance about two different points.

a) $F_1 - \frac{1}{2}F_2 + \frac{1}{\sqrt{2}}F_3 = 0$
b) $2F_1 + \frac{3}{2}F_2 = 0$
c) $\frac{5}{2}F_2 + \sqrt{2}F_3 = 0$.

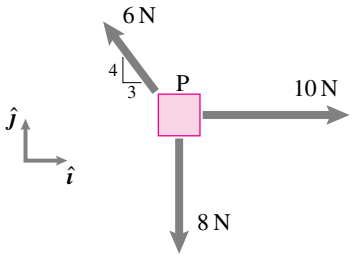
Can the student solve for $F_1$, $F_2$, and $F_3$ uniquely from these equations?

**2.111** What is the solution to the set of equations:

$$\begin{aligned} x + y + z + w &= 0 \\ x - y + z - w &= 0 \\ x + y - z - w &= 0 \\ x + y + z - w &= 2? \end{aligned}$$

## 2.5 Equivalent force systems and couples

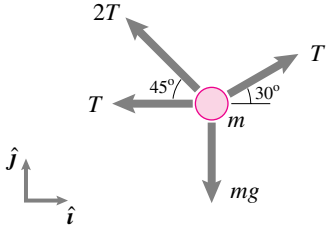
**2.112** Find the net force on the particle shown in the figure.




**problem 2.112:**

Filename:pfigure2-3-rp1

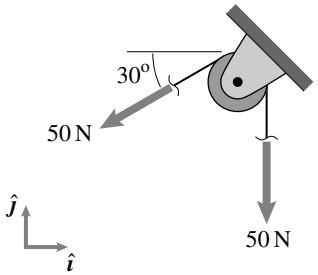
**2.113** Replace the forces acting on the particle of mass $m$ shown in the figure by a single equivalent force.




**problem 2.113:**

Filename:pfigure2-3-rp2

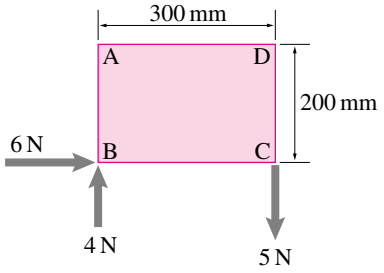
**2.114** Find the net force on the pulley due to the belt tensions shown in the figure.




**problem 2.114:**

Filename:pfigure2-3-rp3

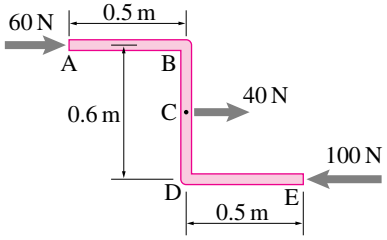
**2.115** Replace the forces shown on the rectangular plate by a single equivalent force. Where should this equivalent force act on the plate and why?




**problem 2.115:**

Filename:pfigure2-3-rp4

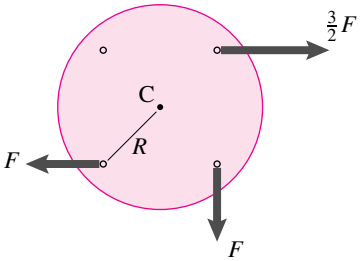
**2.116** Three forces act on a Z-section ABCDE as shown in the figure. Point C lies in the middle of the vertical section BD. Find an equivalent force-couple system acting on the structure and make a sketch to show where it acts.




**problem 2.116:**

Filename:pfigure2-3-rp5

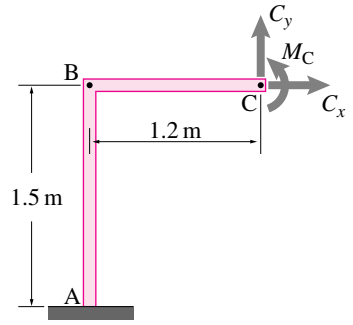
**2.117** The three forces acting on the circular plate shown in the figure are equidistant from the center C. Find an equivalent force-couple system acting at point C.




**problem 2.117:**

Filename:pfigure2-3-rp6

**2.118** The forces and the moment acting on point C of the frame ABC shown in the figure are $C_x = 48\,\text{N}$, $C_y = 40\,\text{N}$, and $M_c = 20\,\text{N·m}$. Find an equivalent force couple system at point B.

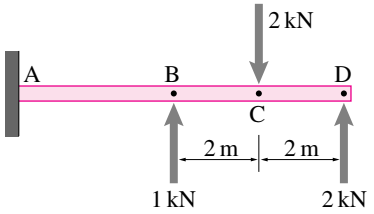



**problem 2.118:**

Filename:pfigure2-3-rp7

**2.119** Find an equivalent force-couple system for the forces acting on the beam shown in the figure, if the equivalent system is to act at

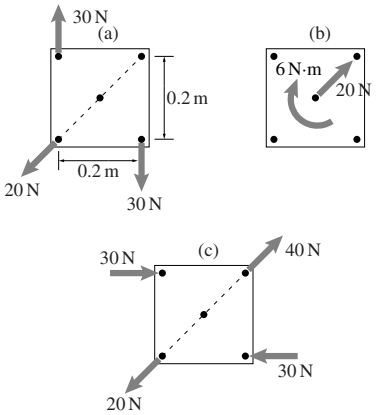
a) point B,
b) point D.




**problem 2.119:**

Filename:pfigure2-3-rp8

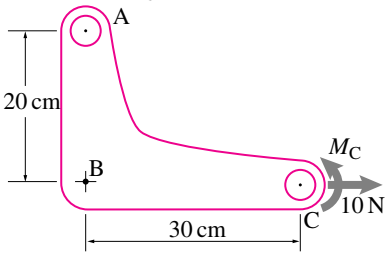
**2.120** The figure shows three different force-couple systems acting on a square plate. Identify which force-couple systems are equivalent.




**problem 2.120:**

Filename:pfigure2-3-rp9

**2.121** The force and moment acting at point C of a machine part are shown in the figure where $M_c$ is not known. It is found that if the given force-couple system is replaced by a single horizontal force of magnitude 10 N acting at point A then the net effect on the machine part is the same. What is the magnitude of the moment $M_c$?




**problem 2.121:**

Filename:pfigure2-3-rp10

**2.122** 2D. Assume a force system is equivalent to a force $\vec{F}_1 \neq \vec{0}$ and couple $\vec{M}_1 = M_1\hat{k}$ acting at point $\vec{r}_1$.

a) Find some point $\vec{r}_2$, and force $\vec{F}_2$ so that $\vec{F}_2$ acting at $\vec{r}_2$ is equivalent to $\vec{F}_1$ and $\vec{M}_1$ acting at $\vec{r}_1$.
b) Find all possible wrenches (combinations of point location, force and moment) equivalent to the system with $\vec{F}_1$ and $\vec{M}_1$ acting at the point with position vector $\vec{r}_1$.
c) Describe the situation in the special case when $\vec{F}_1 = \vec{0}$.

**2.123** 3D. Assume a force system is equivalent to a force $\vec{F}_1$ and couple $\vec{M}_1$ acting at point with position vector $\vec{r}_1$.

a) Find a point P with position vector $\vec{r}_2$, so that an equivalent force system $\vec{F}_2$ and $\vec{M}_2$ acting at acting at P has $\vec{F}_2$ is parallel to $\vec{M}_2$. (Finding such a point, force and moment is called "reducing the force system to a *wrench*").
b) Find all possible wrenches (combinations of point location, force and moment) equivalent to the system with $\vec{F}_1$ and $\vec{M}_1$ acting at $\vec{r}_1$.

Note, one special case with a slightly different result than the other cases is if $\vec{F}_1 = \vec{0}$, so it should be treated separately.

## 2.6 Center of mass and center of gravity

**2.124** An otherwise massless structure is made of four point masses, $m$, $2m$, $3m$ and $4m$, located at coordinates (0, 1 m), (1 m, 1 m), (1 m, −1 m), and (0, −1 m), respectively. Locate the center of mass of the structure.

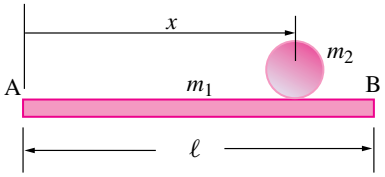
**2.125 -3-D:** The following data is given for a structural system modeled with five point masses in 3-D-space:

| mass | coordinates (in m) |
|---|---|
| 0.4 kg | (1,0,0) |
| 0.4 kg | (1,1,0) |
| 0.4 kg | (2,1,0) |
| 0.4 kg | (2,0,0) |
| 1.0 kg | (1.5,1.5,3) |

Locate the center of mass of the system.

**2.126** Write a computer program to find the center of mass of a point-mass-system. The input to the program should be a table (or matrix) containing individual masses and their coordinates. (It is possible to write a single program for both 2-D and 3-D cases, write separate programs for the two cases if that is easier for you.) Check your program on Problems 2.124 and 2.125.

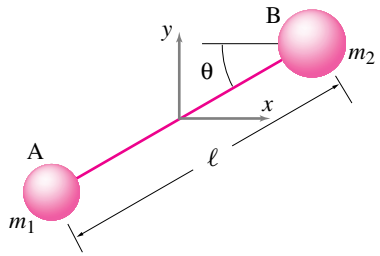
**2.127** A cylinder of mass $m_2$ and radius $R$ rolls on a flat circular plate of mass $m_1$ and length $\ell$. Let the position of the cylinder from the left edge of the plate be $x$. Find the horizontal position of the center of mass of the system as a function of $x$ and a non-dimensional mass parameter $M = m_1/m_2$.




**problem 2.127:**

Filename:pfigure2-5-baranddisk

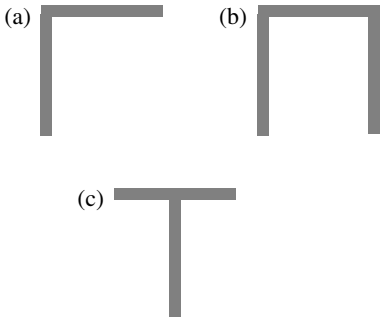
**2.128** Two masses $m_1$ and $m_2$ are connected by a massless rod AB of length $\ell$. In the position shown, the rod is inclined to the horizontal axis at an angle $\theta$. Find the position of the center of mass of the system as a function of angle $\theta$ and the other given variables. Check if your answer makes sense by setting appropriate values for $m_1$ and $m_2$.




**problem 2.128:**

Filename:pfigure2-5-dumbbell

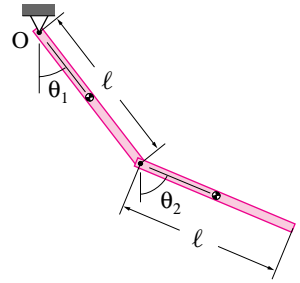
**2.129** Find the center of mass of the following composite bars. Each composite shape is made of two or more uniform bars of length 0.2 m and mass 0.5 kg.




**problem 2.129:**

Filename:pfigure3-cm-rp7

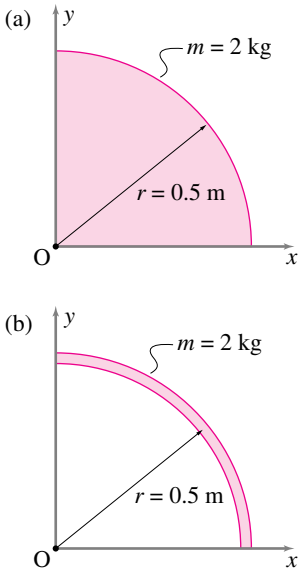
**2.130** A double pendulum consists of two uniform bars of length $\ell$ and mass $m$ each. The pendulum hangs in the vertical plane from a hinge at point O. Taking O as the origin of a $xy$ coordinate system, find the location of the center of gravity of the pendulum as a function of angles $\theta_1$ and $\theta_2$.




**problem 2.130:**

Filename:pfigure2-5-dpend

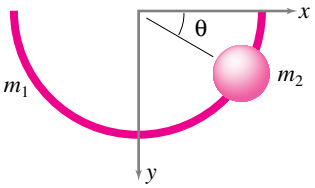
**2.131** Find the center of mass of the following two objects [Hint: set up and evaluate the needed integrals.]




**problem 2.131:**

Filename:pfigure3-cm-rp8

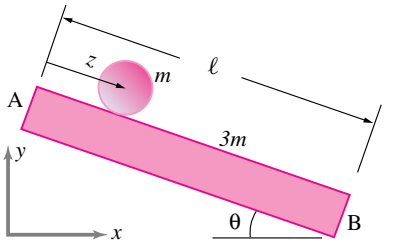
**2.132** A semicircular ring of radius $R = 1$ m and mass $m_1 = 0.1$ kg rests in the vertical plane. A bead of mass $m_2 = 0.25$ kg slides on the ring. Find the position of the center of mass of the ring-bead-system at an instant when $\theta = 30^\circ$. How does the center of mass position change as $\theta$ changes?




**problem 2.132:**

Filename:pfigure2-5-halfring

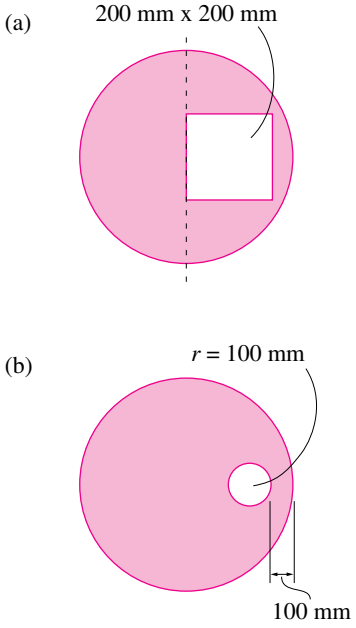
**2.133** A uniform circular disk of mass $m$ and radius $R$ rolls on an inclined rectangular plate of mass $3m$ and dimensions $2R \times \ell$. When the plate is horizontal ($\theta = 0$), the left lower corner of the plate is at the origin of a fixed $xy$ coordinate system. Find the coordinates of the center of mass of the system for $m = 1$ kg, $\ell = 1$ m, $z = 0.2$ m, and $R = 0.1$ m.




**problem 2.133:**

Filename:pfigure2-5-baranddisk2

**2.134** Find the center of mass of the following plates obtained from cutting out a small section from a uniform circular plate of mass 1 kg (prior to removing the cutout) and radius 1/4 m.




**problem 2.134:**

Filename:pfigure3-cm-rp9

CHAPTER 3

# Free-Body Diagrams

*A free-body diagram is a sketch of the system to which you will apply the laws of mechanics, and all the non-negligible external forces and couples which act on it. The diagram indicates what material is in the system. The diagram shows what is, and what is not, known about the forces. Generally there is a force or moment component associated with any connection that causes or prevents a motion. Conversely, there is no force or moment component associated with motions that are freely allowed. Mechanics reasoning entirely rests on free body diagrams. Many student errors in problem solving are due to problems with their free body diagrams, so we give tips about how to avoid various common free-body diagram mistakes.*

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# The zeroth laws of mechanics

One way to understand something is to isolate it, see how it behaves on its own, and see how it responds to various stimuli. Then, when the thing is not isolated, you still think of it as isolated, but think of the effects of all its surroundings as stimuli. Reversing the point of view, we can also see the system's behavior as causing stimulus to other things around it, which themselves can be thought of as isolated and stimulating back, and so on.

This reductionist approach is used throughout the physical and social sciences. A tobacco plant is understood in terms of its response to light, heat flow, the chemical environment, insects, and viruses. The economy of Singapore is understood in terms of the flow of money and goods in and out of the country. And social behavior is regarded as being a result of individuals reacting to the sights, sounds, smells, and touch of other individuals and thus causing sights, sounds, smell and touch that the others react to in turn, etc.

The isolated system approach to understanding is made most clear in thermodynamics courses. A system, usually a fluid, is isolated with rigid walls that allow no heat, motion or material to pass. Then, bit by bit, as the subject is developed, the response of the system to certain interactions across the boundaries is allowed. Eventually, enough interactions are understood that the system can be viewed as isolated even when in a useful context. The gas expanding in a refrigerator follows the same rules of heat-flow and work as when it was expanded in its 'isolated' container.

**The "*free-body*" is a closed system.** The subject of mechanics is also firmly rooted in the idea of an isolated system. As in elementary thermodynamics we will only be concerned with so-called *closed* systems. A (closed) system, in mechanics, is a fixed collection of material. You can draw an imaginary boundary around a system, then in your mind paint all the atoms inside the boundary red, and then define the system as being the red atoms, no matter whether they later cross the original boundary markers or not. Thus mechanics depends on bits of matter as being durable and non-ephemeral. We assume that*

> A given bit of matter in a system exists forever, has the same mass forever, and is always in that system.

* **Open Systems.** The mechanics of *open systems*, where material crosses the system boundaries, is important in fluid mechanics. Such open fluid systems are first seen in some elementary dynamics problems (like rockets), where material is allowed to cross the system boundaries. But the equations governing these *open* systems are deduced from careful application of the more fundamental governing mechanics equations of *closed* systems. So we have to master the mechanics of closed systems first.

Mechanics is based on the notion that any part of a system is itself a system and that all interactions between systems or subsystems have certain simple rules, most basically:

> *Force is the measure of mechanical interaction,*

and

> *The principle of "action and reaction": what one system does to another, the other does back to the first.*

Thus a person can be moved by forces, but not by the sight of a tree falling towards them or the attractive smell of a flower (these things may cause, by rules that fall outside of mechanics, forces that move a person). When a person moves towards a flower or away from a falling tree she is moved by the force of the ground on her feet. And she pushes back on the ground just as hard.

The two simple rules above, which we call the zeroth* laws of mechanics, imply that all the mechanical effects of interaction on a system can be represented by a sketch of the system with arrows showing the forces of interaction. If we want to know how the system, in turn, effects its surroundings we draw the opposite arrows on a sketch of the surroundings.

In mechanics a system is often called a *body* and when it is isolated it is *free*, as in *free* from its surroundings.

> A *free body diagram* is a *sketch of an isolated system* and the external forces which act on it.

The laws of mechanics are applied using the forces shown on a free body diagram and not using any other forces. Thus, as we say again and again, drawing good free-body diagrams is essential for both statics and dynamics. The skills for drawing these diagrams are presented in the following sections.

## 3.1 Free-body diagrams: interactions, representing forces and partial FBDs

A *free-body diagram* is a sketch of the system of interest and the forces that act on the system. A free body diagram precisely defines the system to which you are applying mechanics equations and the forces to be considered. Any

* Why do we awkwardly number the first law as zero? Because it is really more of an underlying assumption, a background concept, than a law. As a law it is a little imprecise since force has not yet been defined. You could take the zeroth law as an implicit and partial definition of force. The phrase "zeroth law" means "important implicit assumption". The second part of the zeroth law is usually called "Newton's third law."

reader of your calculations needs to see your free body diagrams. To put it directly, if you want to be right and be seen as right, then *

*Draw a free-body diagram!*

The concept of the free body diagram is simple. In practice, however, drawing useful free body diagrams takes some thought, even for those practiced at the art. Some basic tips are described below a few different ways.

## What shows on a free body diagram? What doesn't?

- **The system.** A free body diagram is a picture of the system for which you would like to apply linear or angular momentum balance (force and moment balance being special cases) or power balance. It shows the system isolated ('free') from its environment. That is, the free body diagram does *not* show things that are near or touching the system of interest. See figure 3.1.
- **The word 'body' means system.** A free body diagram may show one or more particles, rigid bodies, deformable bodies, or parts thereof such as a machine, a component of a machine, or a part of a component of a machine. You can draw a free body diagram of any collection of material that you can identify. The word *body* connotes a standard object in some people's minds. In the context of free-body diagrams, 'body' means system. The body in a free-body diagram may be a subsystem of the overall system of interest.
- **Forces fool the system.** The free body diagram of a system shows the forces and moments that the surroundings impose on the system. That is, since the only method of mechanical interaction that God has invented is force (and moment), the free body diagram shows what it would take to mechanically fool the system if it was literally cut free. That is, the motion of the system would be totally unchanged if it were cut free and the forces shown on the free body diagram were applied as a replacement for all external interactions.
- **Place forces at cuts.** The forces and moments are located on the free-body diagram at the points where they are applied. These places are where you made 'cuts' to *free* the *body*.
- **Motion is caused or prevented by forces.** At places where the outside environment causes or restricts translation of the isolated system, a contact force is drawn on the free body diagram.
- **Draw contact forces outside the body.** Draw the contact force outside the sketch of the system for viewing clarity. A block supported by a hinge with friction in figure 3.2 illustrates how the reaction force on the block due to the hinge is best shown outside the block.
- **Rotation is caused or prevented by torques.** At connections to the outside world that cause or restrict rotation of the system a contact

* **Free-body Perkins.** In the 1950's Cornell-professor Harold C. Perkins was nick named 'free-body Perkins'. Perkins would stop students in the hall and say "You! Come in my office! Draw a free body diagram of this!" Students drew free body diagrams to please persnickety Perkins. And so learned how to get mechanics problems right.

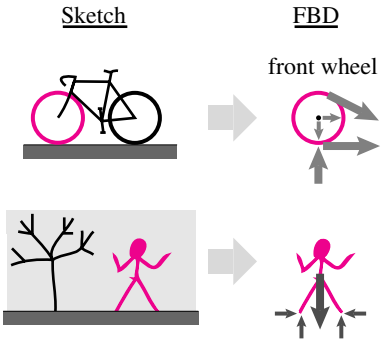



Figure 3.1: A sketch of a bicycle and a free body diagram of the braked front wheel. A sketch of a person and a free body diagram of a person.

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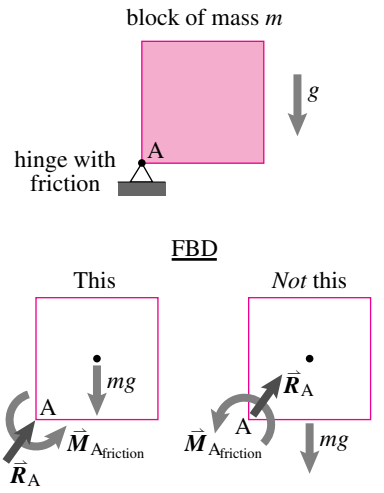



Figure 3.2: A uniform block of mass $m$ supported by a hinge with friction in the presence of gravity. The free body diagram on the right is correct, just less clear than the one on the left.

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torque (or couple or moment) is drawn. Draw this moment outside the system for viewing clarity. Refer again to figure 3.2 to see how the moment on the block due to the friction of the hinge is best shown outside the block.

- **Draw body forces (*e.g.*, gravity forces) inside the body.** The free body diagram shows the system cut free from the source of any *body forces* applied to the system. Body forces are forces that act on the inside of a body from objects outside the body. It is best to draw the body forces on the interior of the body, at the center-of-mass if that correctly represents the net effect of the body forces. Figure 3.2 shows the cleanest way to represent the gravity force on the uniform block acting at the center-of-mass. *
- **Internal forces are not drawn.** The free body diagram shows all external forces acting on the system but *no* internal forces — forces between objects within the body are not shown.
- **No velocity, no acceleration.** The free body diagram shows nothing about the motion. It shows: no "centrifugal force", no "acceleration force", and no "inertial force". (Of course for statics this is a non-issue because inertial terms are neglected for all purposes.) Repeating

*Velocities, accelerations and inertial forces do not show on a free body diagram.*

[Aside: The prescription that you *not show inertial forces* is based on a white lie. To be honest we admit that in the D'Alembert approach to dynamics, a legitimate and intuitive approach for experts, one does show inertial forces on the free body diagram. The D'Alembert approach is not followed in this book in any theory or examples because of the frequent sign errors and mind-confusions it causes in beginners (translation: "not allowed in homework or exams"). For those who are drawn to forbidden fruit, see box 9.2 on page 390.]

## How to draw a free body diagram

We suggest the following procedure for drawing a free body diagram, as shown schematically in fig. 3.3

1. **Define the system.** Define in your own mind what system or what collection of material, you would like to write momentum balance equations for. This subsystem may be part of your overall system of interest.
2. **Sketch the system.** Your sketch may include various cut marks to show how it is isolated from its environment. At each place the system has been cut free from its environment you imagine that you have cut the system free with a sharp scalpel or with a chain saw.

* In this book, the only body force we consider is gravity. For near-earth gravity, gravity forces show on the free body diagram as a single force at the center of gravity, or as a collection of forces at the center of gravity of each of the system parts. For parts of electric motors and generators, not covered here in detail, electrostatic or electro-dynamic body forces also need to be considered.

3. **Stare at each cut.** Look systematically at the picture at the places that the system interacts with material *not* shown in the picture, places where you made 'cuts'.
4. **Fool the body.** Use forces and torques to fool the system into thinking it has not been cut. For example, if the system is being pushed in a given direction at a given contact point where you have cut the system free, then show a force in that direction at that point. If a system is being prevented from rotating by a (cut) rod, then show a torque at that cut.
5. **Replace gravity with a force.** To show that you have cut the system from the earth's gravity force show the force of gravity on the system's center-of-mass or on the centers of mass of its parts.

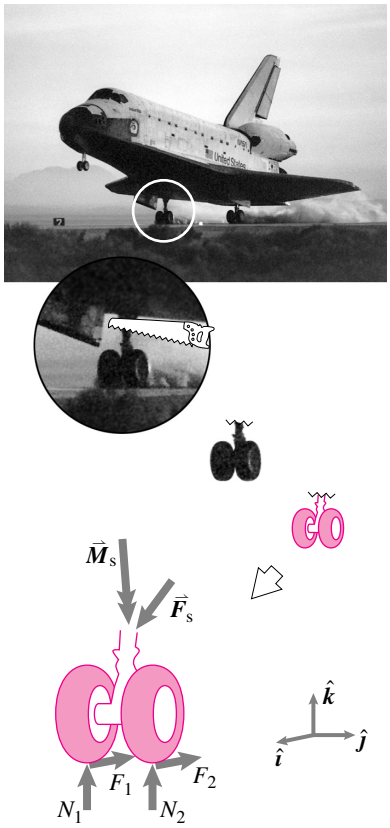



Figure 3.3: The process of drawing a FBD is illustrated by the sequence shown.

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## How to draw forces on free body diagrams

How you draw a force on a free body diagram depends on

- How much you know about the force when you draw the free body diagram. Do you know its direction? its magnitude?; and
- Your choice of notation (which may vary from vector to vector within one free body diagram). See page 23 for a description of the 'symbolic' and 'graphical' vector notations.

Some of the possibilities are shown in fig. 3.4 for three common notations for a 2D force in the cases when (a) any $\vec{F}$ possible, (b) the direction of $\vec{F}$ is known, and (c) everything about $\vec{F}$ is known.

## Simplify using equivalent force systems

The concept of 'fooling' a system with forces is somewhat subtle. If the free-body diagram involves 'cutting' a rope what force should one show? A rope is made of many fibers so cutting the rope means cutting all of the rope fibers. Should one show hundreds of force vectors, one for each fiber that is cut? The answer is: yes and no. You would be correct to draw all of these hundreds of forces at the fiber cuts. But, since the equations that are used with any free body diagram involve only the total force and total moment, you are also allowed to replace these forces with an equivalent force system (see section 2.5).

> *Any force system acting on a given free-body diagram can be replaced by an equivalent force and couple.*

In the case of a rope, a single force directed nearly parallel to the rope and acting at about the center of the rope's cross section is equivalent to the force system consisting of all the fiber forces. In the case of an ideal rope, the force is exactly parallel to the rope and acts exactly at its center.

Similarly the force of the net effect of the distributed ground forces on a shoe is often represented by a single force at “the center of pressure”.

## Action and reaction

For some systems you will want to draw free body diagrams of subsystems. For example, to study a machine, you may need to draw free body diagrams of several of its parts; for a building, you may draw free body diagrams of various structural components; and, for a biomechanics analysis, you may ‘cut up’ a human body (with your imagined scalpal or chainsaw). When separating a system into parts, you must take account of how the subsystems interact. Say the two touching parts of a machine, are called $\mathcal{A}$ and $\mathcal{B}$. We then have that

| | | | | | | |
|---|---|---|---|---|---|---|
| If | $\mathcal{A}$ | feels force | $\vec{F}$ | and couple | $\vec{M}$ | from $\mathcal{B}$, |
| then | $\mathcal{B}$ | feels force | $-\vec{F}$ | and couple | $-\vec{M}$ | from $\mathcal{A}$. |

To be precise we must make clear that $\vec{F}$ and $-\vec{F}$ have the same line of action.*

The principle of action and reaction doesn't say anything about what force or moment acts on one object. It only says that the actor of a force and

* The principle of action and reaction can be derived from the momentum balance laws by drawing free body diagrams of little slivers of material. Nonetheless, in practice you can think of the principle of action and reaction as a basic law of mechanics. Newton did. The principal of action and reaction is “Newton's third law”.

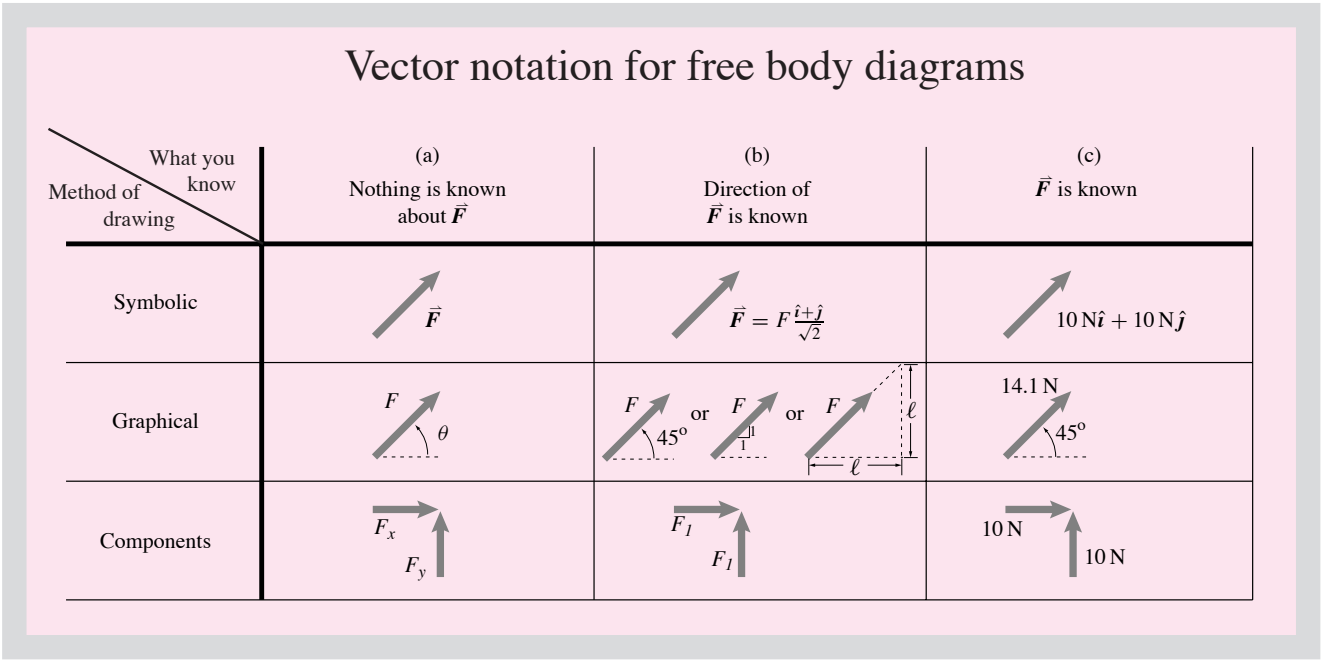



Figure 3.4: The various ways of notating a force on a free body diagram. In column (a) nothing is known and everything is variable. In column (b) the direction is known and the magnitude isn't. In column (c) Everything is known. In one free body diagram different notations can be used for different forces, as needed or convenient. Other unusual cases can be extrapolated, such as if the magnitude is known and the direction is unknown.

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moment gets back the opposite force and moment.

It is easy to make mistakes when drawing free body diagrams involving action and reaction. Box 3.2 on page 134 shows some correct and incorrect partial FBD's of interacting bodies $\mathcal{A}$ and $\mathcal{B}$. Use notation consistent with Fig. 3.4 on page 126 for the action and reaction vectors.

## Interactions

The way objects interact mechanically is by the transmission of a force or a set of forces. If you want to show the effect of body $\mathcal{B}$ on $\mathcal{A}$, in the most general case you can expect a force and a moment which are equivalent to the whole force system, however complex.

That is, the most general interaction of two bodies requires knowing

- six numbers in three dimensions (three force components and three moment components)
- and three numbers in two dimensions (two force components and one moment).

Many things often do not interact in this most general way so often fewer numbers are required. You will use what you know about the interaction of particular bodies to reduce the number of unknown quantities in your free body diagrams.

Some of the common ways in which mechanical things interact, or are assumed to interact, are described in the following sections. You can use these simplifications in your work.

## Constrained motion and free motion

One general principle of interaction forces and moments concerns 'geometric' constraints.

> Wherever a *motion* of $\mathcal{A}$ is either caused *or* prevented by $\mathcal{B}$ there is a corresponding *force* shown at the interaction point on the free body diagram of $\mathcal{A}$.

Similarly

> if $\mathcal{B}$ causes or prevents *rotation* there is a *moment* (or torque or couple) shown on the free body diagram of $\mathcal{A}$ at the place of interaction.

The converse is also true. Many kinds of mechanical attachment gadgets are specifically designed to allow motion.

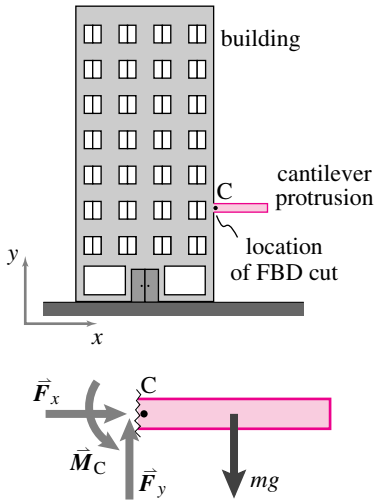



Figure 3.5: **A rigid connection: a cantilever structure on a building.** At the point C where the cantilever structure is connected to the building all motions are restricted so every possible force needs to be shown on the free body diagram cut at C.

Filename:tfigure2-rigid

If an attachment allows free motion in some direction the free body diagram shows no force in that direction. If the attachment allows free rotation about an axis then the free body diagram shows no moment (couple or torque) about that axis.

You can think of each attachment point as having a variety of jobs to do. For every possible direction of translation and rotation, the attachment has to either allow free motion or restrict the motion. In every way that motion is restricted (or caused) by the connection a force or moment is required. In every way that motion is free there is no force or couple. Motion of body $\mathcal{A}$ is caused and restricted by forces and couples which act on $\mathcal{A}$. Motion is freely allowed by the absence of such forces and couples.

Here, demononstrating the ideas above, are some of the common connections and the free-body-diagram forces and moments with which they are associated.

## Cuts at 'rigid' connections

Sometimes the body you draw in a free body diagram is firmly attached to another. Figure 3.5 shows a cantilever structure on a building. The free body diagram of the cantilever has to show all possible force and load components. Since we have used vector notation for the force $\vec{F}$ and the moment $\vec{M}_C$ we can be ambiguous about whether we are doing a two or three dimensional analysis.

**Gravity is pointing down, so why do we show a horizontal reaction force at C?** This is a common question by new mechanics students seeing a free body diagram like in figure 3.5. The question is reasonable because a quick statics analysis reveals, for an assumed stationary building and cantilever, that $\vec{F}_C$ must be vertical. But one must remember: this book is about statics *and* dynamics and in dynamics *the forces on a body do not add to zero*. In fact, we forgot to tell you, the building shown in figure 3.5 happens to be accelerating rapidly to the right due to the motions of a violent earthquake occurring at the instant pictured in the figure. Whether or not there is an earthquake, the attachment of the cantilever to the building at C in figure 3.5 is surely intended to be rigid and prevent the cantilever from moving up or down (falling), from moving sideways (and drifting into another building) or from rotating about point C. In most of the building's life, the horizontal reaction at C is small. But since the connection at C clearly prevents relative horizontal motion, it is probably best to draw a horizontal reaction force on the free body diagram. Then the same free-body diagram is good during earthquakes and during more boring times.

When you know a force is going to turn out to be zero, as for the sideways force in this example if treated as a statics problem. it is a matter of taste whether or not you show the sideways force on the free-body dia-

gram (see box 3.10 on page 132). But better safe then sorry; if you don't *know* that a force or moment is going to turn out to be zero, leave it in the free-body diagram. . The situation with rigid connections, like the cantilever above, is shown more abstractly in both 3D and 2D in figure 3.6.

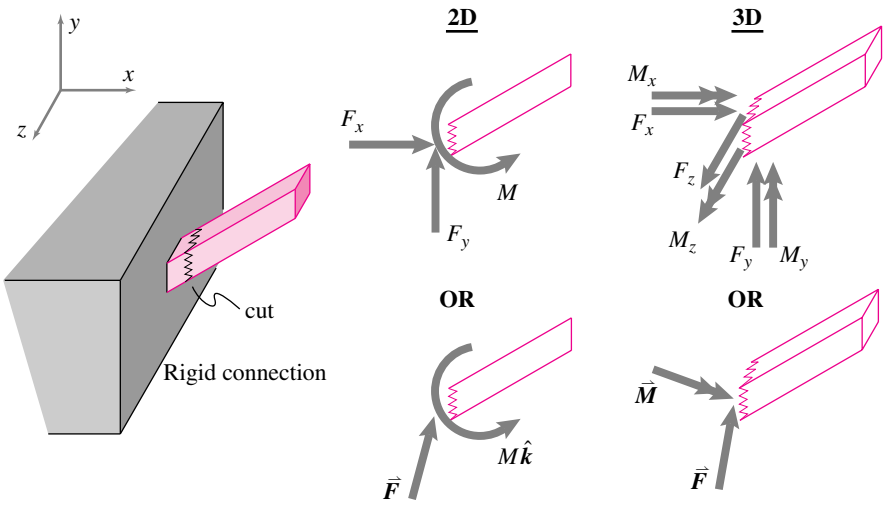



Figure 3.6: A rigid connection shown with partial free body diagrams in two and three dimensions. One has a choice between showing the separate force components (top) or using the vector notation for forces and moments (bottom). The double head on the moment vector is optional.

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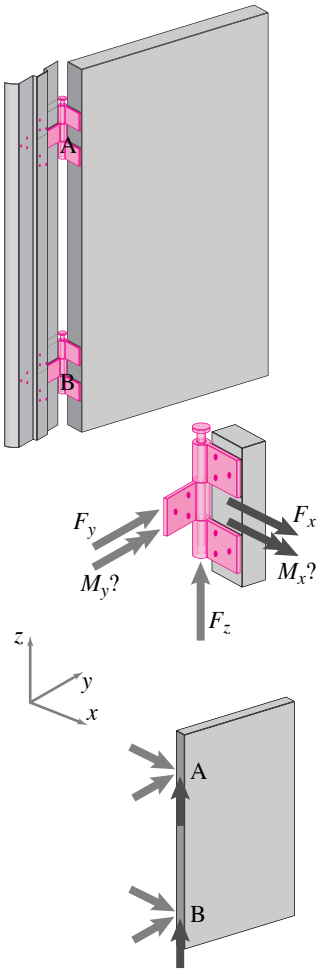



Figure 3.7: **A door held by hinges.** One must decide whether to model hinges as proper hinges or as ball-and-socket joints. The partial free body diagram of the door at the lower right neglects the couples at the hinges, effectively idealizing the hinges as ball-and-socket joints. This idealization is generally quite accurate since the rotations that each hinge might resist are already resisted by their being two connection points.

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## Cuts at hinges

A hinge, shown in figure 3.8, allows rotation and prevents translation. Thus, the free body diagram of an object cut at a hinge shows *no* torque about the hinge axis but does show the force or its components which prevent translation.

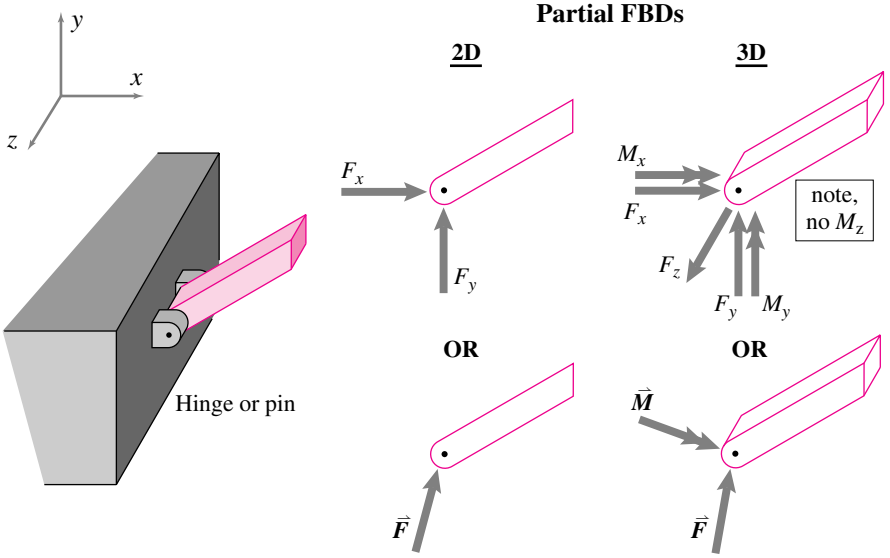


Figure 3.8: **A hinge with partial free body diagrams in 2D and 3-D.** A hinge joint is also called a pin joint because it is sometimes built by drilling a hole and inserting a pin.

Filename:tfigure2-hinge

There is some ambiguity about how to model pin joints (hinges) in three dimensions. The ambiguity is shown with reference to a hinged door (figure 3.7) and discussed in detail below. Clearly, one hinge, if the sole attachment, prevents rotation of the door about the $x$ and $y$ axes shown. So, it is natural to show a couple (torque or moment) in the $x$ direction, $M_x$, and in the $y$ direction, $M_y$. But, the hinge does not provide very stiff resistance to rotations in these directions compared to the resistance of the other hinge. That is, even if both hinges are modeled as ball-and-socket joints (see the next sub-section), offering no resistance to rotation, the door still cannot rotate about the $x$ and $y$ axes.

If a connection between objects prevents relative translation or rotation that is already prevented by another stiffer connection, then the more compliant connection reaction is often neglected. Even without rotational constraints, the translational constraints at the hinges A and B restrict rotation of the door shown in figure 3.7. Thus each of the two hinges are probably well modeled — that is, they will lead to reasonably accurate calculations of forces and motions — by ball-and-socket joints at A and B.

In 2-D a ball-and-socket joint is equivalent to a hinge or pin joint (with the axis of the hinge orthogonal to the page).

## Ball-and-socket joint

Sometimes one wishes to attach two objects in a way that allows no relative translation but for which all rotation is free. The device that is used for this purpose is called a 'ball-and-socket' joint. It is constructed by rigidly attaching a sphere (the ball) to one of the objects and rigidly attaching a partial spherical cavity (the socket) to the other object.

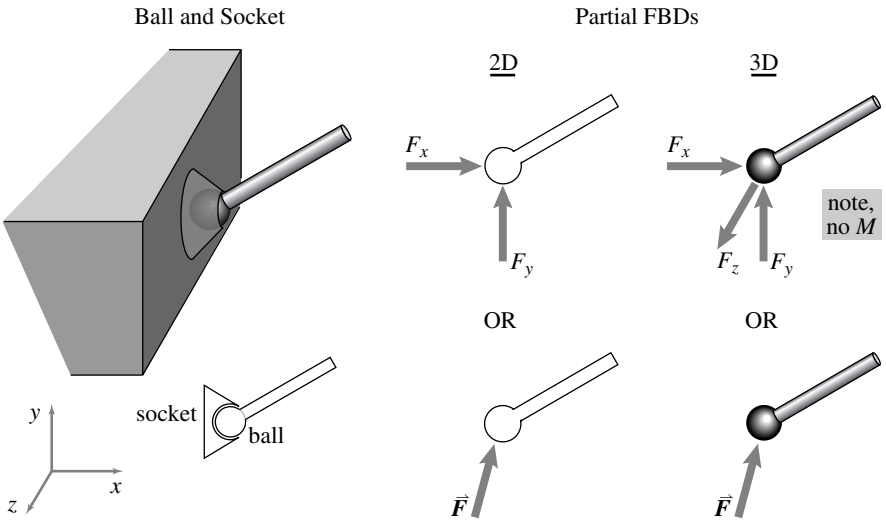



Figure 3.9: **A ball-and-socket joint** allows all relative rotations and no relative translations so reaction forces, but not moments, are shown on the partial free body diagrams. In two dimensions a ball-and-

socket joint is just like a pin joint. The top partial free body diagrams show the reaction in component form. The bottom illustrations show the reaction in vector form.

Filename:tfigure2-ballands

The human hip joint is a ball-and-socket joint. At the upper end of the femur bone is the femoral head, a sphere to within a few thousandths of an inch. The hip bone has a spherical cup that accurately fits the femoral head*.

Car suspensions are constructed from a three-dimensional truss-like mechanism. Some of the parts need free relative rotation in three dimensions and thus use a joint called a 'ball joint' or 'rod end' that is a ball-and-socket joint.

Since the ball-and-socket joint allows all rotations, no moment is shown at a cut ball-and-socket joint. Since a ball-and-socket joint prevents relative translation in all directions, the possibility of force in any direction is shown.

## String, rope, wires, and light chain

One way to keep a radio tower from falling over is with wire, as shown in figure 3.10.If the weight of the wires seems small, and the wind resistance is negligible, it is common to assume they can only transmit forces along the line connecting their end points. Moments are not shown because ropes, strings, and wires are generally assumed to be so compliant in bending that the bending moments are negligible. For wires

> *tension* is the force pulling away from a free body diagram cut(1).

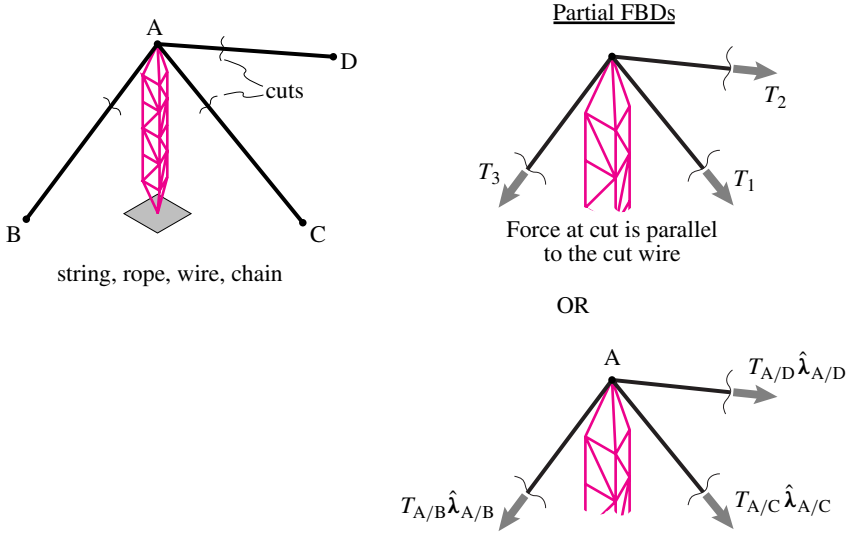



Figure 3.10: **A radio tower kept from falling with three wires.** A partial free body diagram of the tower is drawn two different ways. The upper figure shows three tensions that are parallel to the three wires. The lower partial free body diagram is more explicit, showing the forces to be in the directions of the $\hat{\boldsymbol{\lambda}}$s, unit vectors parallel to the wires.

Filename:tfigure2-string

* **True story.** The Mann biomechanics lab at MIT put strain gauges in artificial hip joints, then surgically implanted these in patients with bad bones, and measured the forces. Dicky at the MIT boat house mentioned that he wanted a ball-and-socket joint for the base of the mast of the sailboat he was building. "Oh" said Crispin of the Mann lab, "Mrs.

## All this talk about force, what is force?

Force is the measure of mechanical interaction. It is a vector. It obeys the principle of action and reaction. Using forces on free body diagrams, with constitutive laws (like $F = kx$) and mechanics laws (like $\sum \vec{F}_i = \vec{0}$ or $\vec{F} = m\vec{a}$) we make accurate predictions. What is force? Its that quantity, that miraculously, has all these properties. What is force really? Beyond this constellation of relations, we don't know.

Operationally, you can define force by how you can measure it. A force on a system can be measured by comparing its effect on the given system to

- a weight suspended by a string which goes over a pulley and is attached to the system of interest instead of the force.
- the effect of a calibrated spring on the system, or
- the effect, and this is tricky, of an accelerating mass connected by pul-

**3.1** *THEORY*

*How much mechanics reasoning should you use when you draw a free body diagram?*

The simple rules for drawing free body diagrams prescribe an unknown force every place a motion is prevented and an unknown torque where rotation is prevented. Consider the simple symmetric truss with a load $W$ in the middle. By this prescription the free body diagram to draw is shown as (a). There is an unknown force restricting both horizontal and vertical motion at the hinge at B.

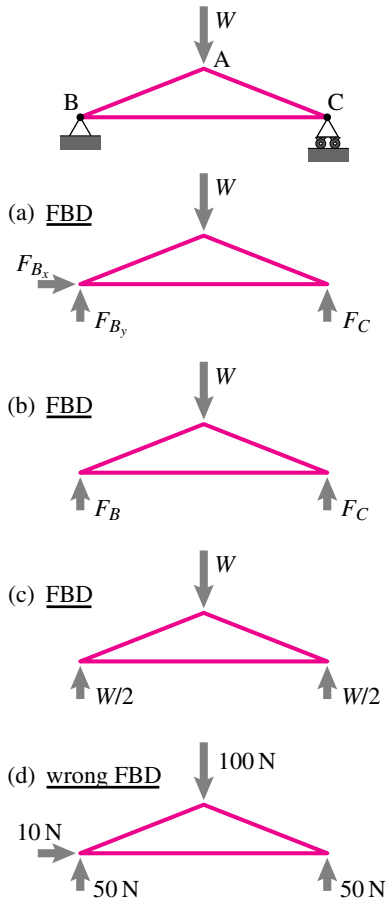
However, a person who knows some statics will quickly deduce that the horizontal force at B is zero and thus draw the free body diagram in figure (b). Or if they really think ahead they will draw the free body diagram in (c). All three free body diagrams are correct. In particular diagram (a) is correct even though $F_{Bx}$ turns out to be zero and (b) is correct even though $F_B$ turns out to be equal to $F_C$.

Someone thinking ahead might say that the free body diagram in (a) is wrong. But it is not wrong. The force $F_{Bx}$ is not specified because it is not known from just looking at the cut pin without using force-balance on the whole structure. That $F_{Bx}$ turns out to be zero is consistent with the picture where $F_{Bx}$ is not specified and thus could have any value, including zero. In contast, Free body diagram (d), on the other hand, explicitly and incorrectly assigns a non-zero value to $F_{Bx}$, so it is wrong.

A reasonable approach is to follow the naive rules, and then later use the force and momentum equations to find out more about the forces. That is use free body diagram (a) and then later use the laws of mechanics to discover FBD (c), and maybe never explitly draw it. If you are confident about the anticipated results, it might be a time saver to use diagrams analogous to (b) or (c) but beware of

- making assumptions that are not reasonable, and
- wasting time trying to think ahead when the force and momentum balance equations will tell all in the end anyway.

A common student error is to sloppily think through the mechanics laws and then incorrectly eliminate, or over-specify, forces from a FBD.



leys and strings to the system. Of course if you are sure its the same force, you can apply it to a mass and measure the acceleration it causes.
- other contraptions that somehow show the effect of the questionable force on a suspended weight, a stretched spring or an accelerated mass.

## Summary of free-body diagrams.

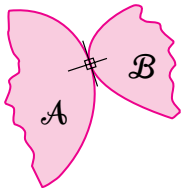
- Draw one or more clear free-body diagrams!
- Forces and moments on the free-body diagram show *all* mechanical interactions from outside the body.
- Every point on the boundary of a body has a force in every direction that motion is either being caused *or* prevented. Similarly with torques.
- If you do not know the direction of a force, use vector notation to show that the direction is yet to be determined.
- Leave off the free body diagram forces that you think are negligible such as, possibly:
  - The force of air on small slowly moving bodies;
  - Forces that prevent motion that is already prevented by a much stiffer means (as for the torques at each of a pair of hinges);

## Collisional free-body diagrams

There are special conventions for drawing free-body diagrams of objects that are in the process of colliding. These we treat in the relevant dynamics portions of the book.

## 3.2 Action and reaction on partial FBD's of interacting bodies

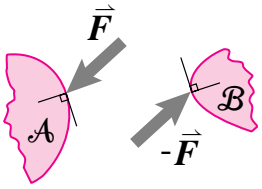
Imagine bodies $\mathcal{A}$ and $\mathcal{B}$ are interacting and that you want to draw separate free body diagrams (FBD's) of each.




Part of the FBD of each shows the interaction force. The FBD of $\mathcal{A}$ shows the force of $\mathcal{B}$ on $\mathcal{A}$ and the FBD of $\mathcal{B}$ shows the force of $\mathcal{A}$ on $\mathcal{B}$. To illustrate the concept, we show partial FBD's of both $\mathcal{A}$ and $\mathcal{B}$ using the principle of action and reaction. Items (a - d) are correct and items (e - g) are wrong. See sample 2.1 on page 28 for related comments on vector notation.

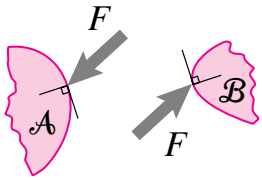
### Correct partial FBD's

(a)



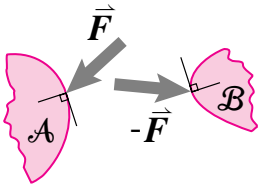

**(a)** These are good partial FBD's. the action and re-action vectors ($\vec{F}$ and $-\vec{F}$) are equal in magnitude, opposite in sign, and applied on the same line of action. Because the symbolic notation takes precedence (see page 23)the direction and length of the drawn arrows, although drawn nicely here, are irrelevant.

(b)



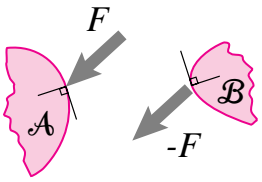

**(b)** These partial FBD's are also good since the opposite arrows multiplied by equal magnitude $F$ produce net vectors that are equal and opposite.

(c)




**(c)** The partial FBD's may *look* wrong, and they are impractically misleading and not advised. But technically they are okay because we take the vector notation to have precedence over the drawing inaccuracy.

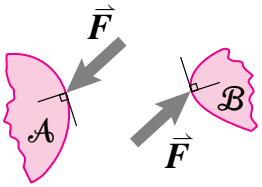
(d)




**(d)** The partial FBD's may look wrong but since no vector notation is used, the forces should be interpreted as in the direction of the drawn arrows and multiplied by the shown scalars. Since the same arrow is multiplied by $F$ and $-F$, the net vectors are actually equal and opposite.

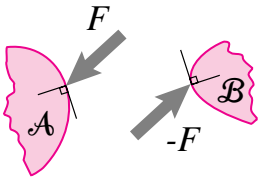
### Wrong partial FBD's

(e)



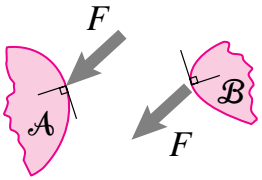

**(e)** These partial FBD's are wrong because the vector notation $\vec{F}$ takes precedence over the drawn arrows. So the drawing shows the *same* force $\vec{F}$ acting on both $\mathcal{A}$ and $\mathcal{B}$, rather than the opposite force.

(f)




**(f)** Because the opposite arrow is multiplied by the negative scalars, the partial FBD's here show the *same* force acting on both $\mathcal{A}$ and $\mathcal{B}$. Treating a double-negative as a negative is a common mistake.

(g)




**(g)** These partial FBD's are obviously wrong since they again show the same force acting on $\mathcal{A}$ and $\mathcal{B}$. These FBD's would represent *the principle of double action* which applies to laundry detergents but not to mechanics.

**SAMPLE 3.1 A mass and a pulley.** A block of mass $m$ is held up by applying a force $F$ through a massless pulley as shown in the figure. Assume the string to be massless. Draw free-body diagrams of the mass and the pulley separately and as one system.

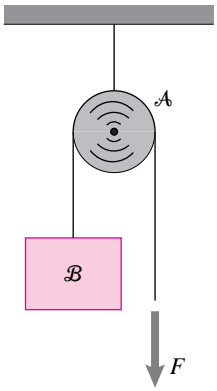



Figure 3.11:

Filename:sfig2-1-02

**Solution** The free-body diagrams of the block and the pulley are shown in Fig. 3.12. Since the string is massless and we assume an ideal massless pulley, the tension in the string is the same on both sides of the pulley. Therefore, the force applied by the string on the block is simply $F$. When the mass and the pulley are considered as one system, the force in the string on the left side of the pulley doesn't show because it is internal to the system.

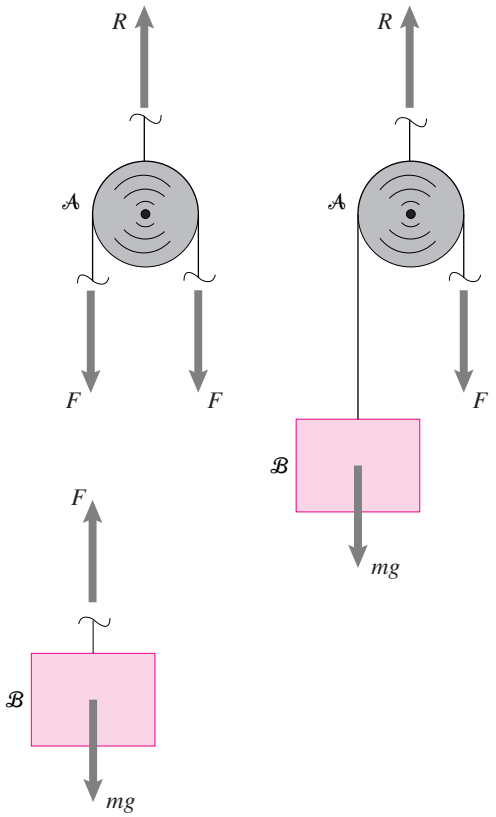



Figure 3.12: The free-body diagrams of the mass, the pulley, and the mass-pulley system. Note that for the purpose of drawing the free-body diagram we need not show that we know that $R = 2F$. Similarly, we could have chosen to show two different rope tensions on the sides of the pulley and reasoned that they are equal as is done in the text.

Filename:sfig2-1-02a

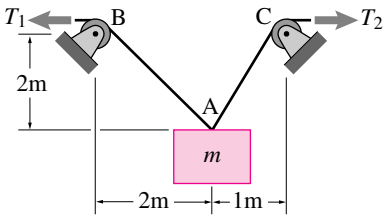



Figure 3.13:

Filename:sfig2-1-2a

**SAMPLE 3.2 Forces in strings.** A block of mass $m$ is held in position by strings $AB$ and $AC$ as shown in Fig. 3.13. Draw a free-body diagram of the block and write the vector sum of all the forces shown on the diagram. Use a suitable coordinate system.

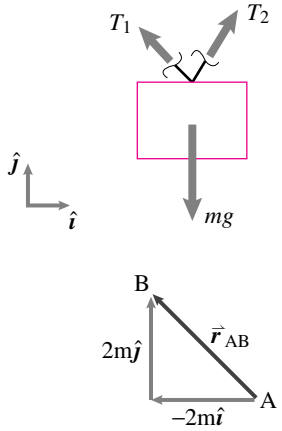



Figure 3.14: Free-body diagram of the block and a diagram of the vector $\vec{r}_{AB}$.

Filename:sfig2-1-2b

**Solution** To draw a free body diagram of the block, we first *free* the block. We cut strings $AB$ and $AC$ very close to point $A$ and show the forces applied by the cut strings on the block. We also isolate the block from the earth and show the force due to gravity. (See Fig. 3.14.)

To write the vector sum of all the forces, we need to write the forces as vectors. To write these vectors, we first choose an $xy$ coordinate system with basis vectors $\hat{\imath}$ and $\hat{\jmath}$ as shown in Fig. 3.14. Then, we express each force as a product of its magnitude and a unit vector in the direction of the force. So,

$$\vec{T}_1 = T_1\hat{\lambda}_{\rm AB} = T_1\frac{\vec{r}_{\rm AB}}{|\vec{r}_{\rm AB}|},$$

where $\vec{r}_{\rm AB}$ is a vector from $A$ to $B$ and $|\vec{r}_{\rm AB}|$ is its magnitude. From the given geometry,

$$\begin{aligned} \vec{r}_{AB} &= -2\,\text{m}\hat{\imath} + 2\,\text{m}\hat{\jmath} \\ \Rightarrow \quad \hat{\lambda}_{AB} &= \frac{2\,\text{m}(-\hat{\imath}+\hat{\jmath})}{\sqrt{2^2+2^2}\,\text{m}} = \frac{1}{\sqrt{2}}(-\hat{\imath}+\hat{\jmath}). \end{aligned}$$

Thus,

$$\vec{T}_1 = T_1\frac{1}{\sqrt{2}}(-\hat{\imath}+\hat{\jmath}).$$

Similarly,

$$\begin{aligned} \vec{T}_2 &= T_2\frac{1}{\sqrt{5}}(\hat{\imath}+2\hat{\jmath}) \\ m\vec{g} &= -mg\hat{\jmath}. \end{aligned}$$

Now, we write the sum of all the forces:

$$\begin{aligned} \sum\vec{F} &= \vec{T}_1 + \vec{T}_2 + m\vec{g} \\ &= \left(-\frac{T_1}{\sqrt{2}} + \frac{T_2}{\sqrt{5}}\right)\hat{\imath} + \left(\frac{T_1}{\sqrt{2}} + \frac{2T_2}{\sqrt{5}} - mg\right)\hat{\jmath}. \end{aligned}$$

$$\boxed{\sum\vec{F} = \left(-\frac{T_1}{\sqrt{2}} + \frac{T_2}{\sqrt{5}}\right)\hat{\imath} + \left(\frac{T_1}{\sqrt{2}} + \frac{2T_2}{\sqrt{5}} - mg\right)\hat{\jmath}}$$

**SAMPLE 3.3 Two bodies connected by a massless spring.** Two carts $A$ and $B$ are connected by a massless spring. The carts are pulled to the left with a force $F$ and to the right with a force $T$ as shown in Fig. 3.15. Assume the wheels of the carts to be massless and frictionless. Draw free body diagrams of

- cart $A$,
- cart $B$, and
- carts $A$ and $B$ together.

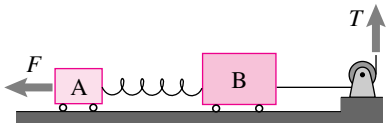



Figure 3.15: Two carts connected by a massless spring

Filename:sfig2-1-3a

**Solution** The three free body diagrams are shown in Fig. 3.16 (a) and (b). In Fig. 3.16 (a) the force $F_s$ is applied by the spring on the two carts. Why is this force the same on both carts? In Fig. 3.16(b) the spring is a part of the system. Therefore, the forces applied by the spring on the carts and the forces applied by the carts on the spring are internal to the system. Therefore these forces do not show on the free body diagram.

Note that the normal reaction of the ground can be shown either as separate forces on the two wheels of each cart or as a resultant reaction.

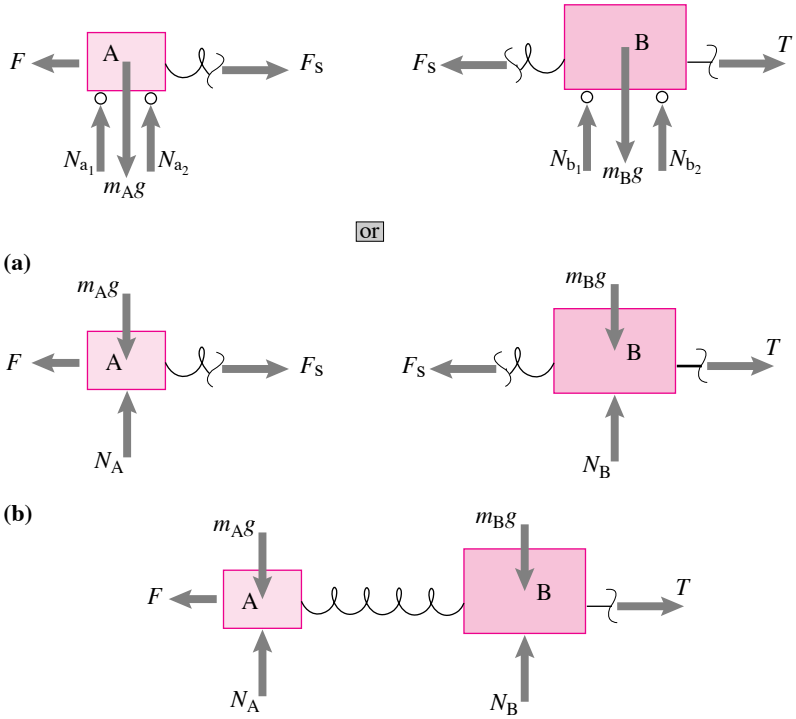



Figure 3.16: Free body diagrams of (a) cart A and cart B separately and (b) cart A and B together

Filename:sfig2-1-3b

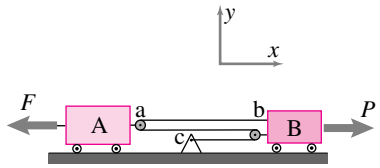



Figure 3.17: Two carts connected by massless pulleys.

Filename:sfig2-1-12

**SAMPLE 3.4 Two carts connected by pulleys.** The two masses shown in Fig. 3.17 have frictionless bases and round frictionless pulleys. The inextensible massless cord connecting them is always taut. Mass A is pulled to the left by force $F$ and mass B is pulled to the right by force $P$ as shown in the figure. Draw free body diagrams of each mass.

**Solution** Let the tension in the cord be $T$. Since the pulleys and the cord are massless, the tension is the same in each section of the cord. This equality is clearly shown in the Free body diagrams of the two masses below.

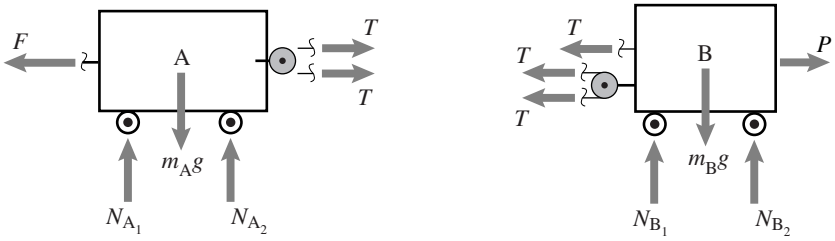



Figure 3.18: Free body diagrams of the two masses.

Filename:sfig2-1-12a

**Comments:** We have shown unequal normal reactions on the wheels of mass B. In fact, the two reactions would be equal only if the forces applied by the cord on mass B satisfy a particular condition. Can you see what condition must be satisfy for, say, $N_{A_1} = N_{A_2}$. [Hint: think about the moment balance about the center-of-mass A.]

**SAMPLE 3.5 Structures with pin connections.** A horizontal force $T$ is applied on the structure shown in the figure. The structure has pin connections at A and B and a roller support at C. Bars AB and BC are rigid. Draw free body diagrams of each bar and the structure including the spring.

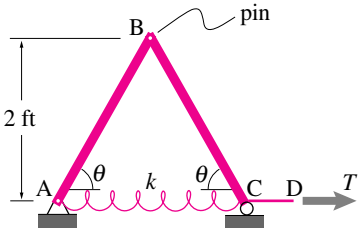



Figure 3.19:

Filename:sfig2-1-5

**Solution** The free body diagrams are shown in figure 3.20. Note that there are both vertical and horizontal forces at the pin connections because pins restrict translation in any direction. At the roller support at point C there is only vertical force from the support ($T$ is an externally applied force).

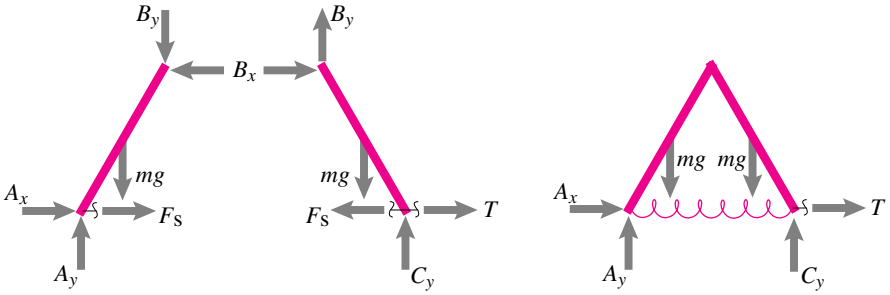



Figure 3.20: Free body diagrams of (a) the individual bars and (b) the structure as a whole.

Filename:sfig2-1-5a

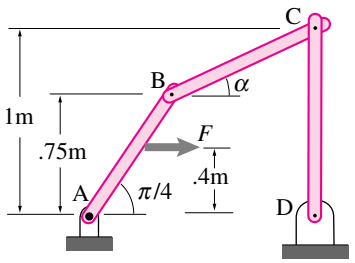



Figure 3.21: A four bar linkage.

Filename:sfig2-2-1

**SAMPLE 3.6** The four bar linkage shown in the figure is pushed to the right with a force $F$ as shown in the figure. Pins A, C & D are frictionless but joint B is rusty and has friction. Neglect gravity; and assume that bar AB is massless. Draw free body diagrams of each of the bars separately and of the whole structure. Use consistent notation for the interaction forces and moments. Clearly mark the action-reaction pairs.

**Solution** A 'good' pin resists any translation of the pinned body, but allows free rotation of the body about an axis through the pin. The body reacts with an equal and opposite force on the pin. When two bodies are connected by a pin, the pin exerts separate forces on the two bodies. Ideally, in the free-body diagram , we should show the pin, the first body, and the second body separately and draw the interaction forces. This process, however, results in too many free body diagrams. Therefore, usually, we let the pin be a part of one of the objects and draw the free body diagrams of the two objects.

Note that the pin at joint B is rusty, which means, it will resist a relative rotation of the two bars. Therefore, we show a moment, in addition to a force, at point B of each of the two rods AB and BC.

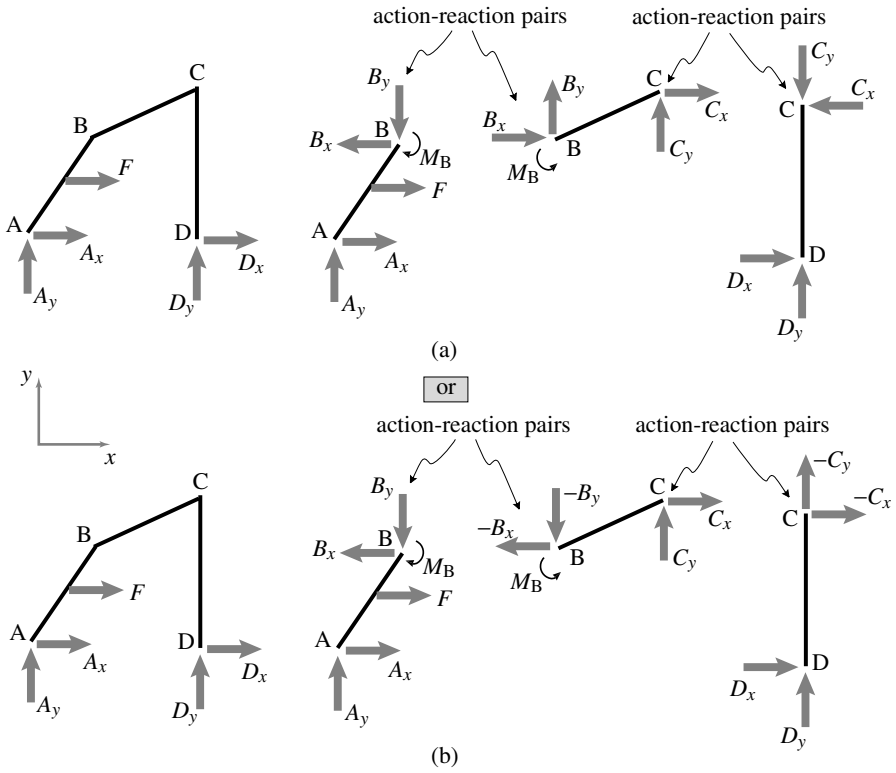



Figure 3.22: Style 1: Free body diagrams of the structure and the individual bars. The forces shown in (a) and (b) are the same.

Filename:sfig2-2-1b

Figure 3.22 shows the free body diagrams of the structure and the individual rods. In this figure, we show the forces in terms of their $x$- and $y$-components. The directions of the forces are shown by the arrows and the magnitude is labeled as $A_x$, $A_y$, etc. Therefore, a force, shown as an arrow in the positive x-direction with 'magnitude' $A_x$, is the same as that shown as an arrow in the negative x-direction with magnitude $-A_x$. Thus, the free body diagrams in Fig. 3.22(a) show exactly the same forces as in Fig. 3.22(b).

In Fig. 3.23, we show the forces by an arrow in an arbitrary direction. The corresponding labels represent their magnitudes. The angles represent the unknown directions of the forces.

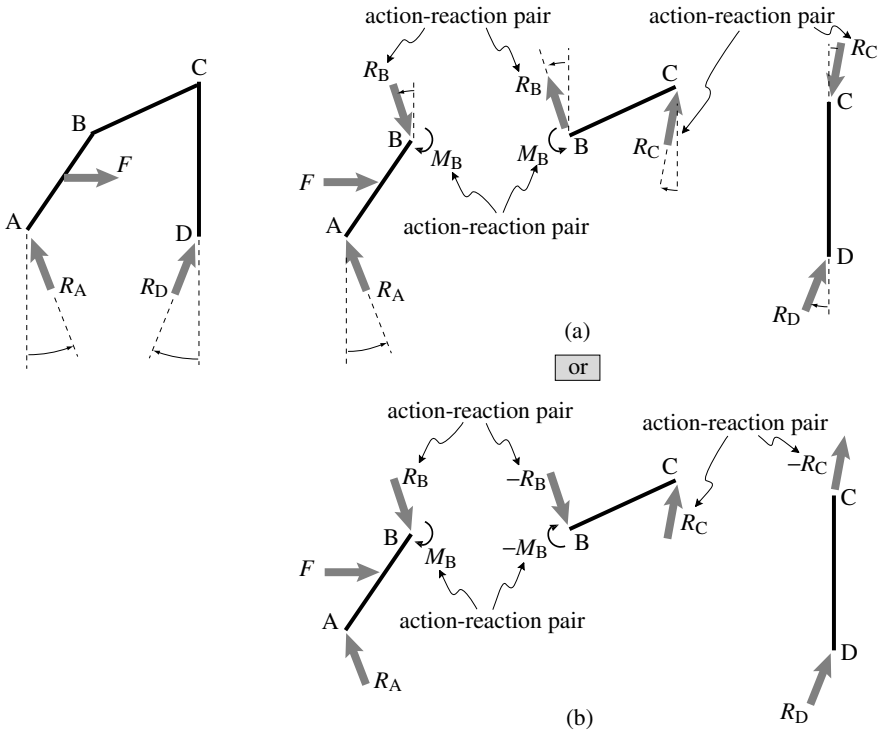



Figure 3.23: Style 2: Free body diagrams of the structure and the individual bars. The forces shown in (a) and (b) are the same.

Filename:sfig2-2-1c

In Fig. 3.24, we show yet another way of drawing and labeling the free body diagrams, where the forces are labeled as vectors.

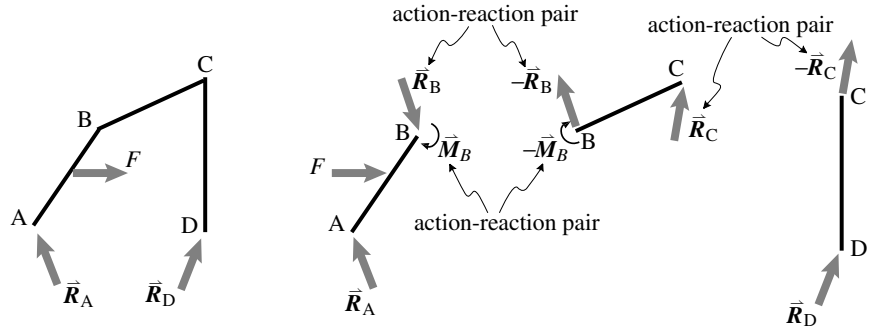



Figure 3.24: Style 3: Free body diagrams of the structure and the individual bars. The label of a force indicates both its magnitude and direction. The arrows are arbitrary and merely indicate that a force or a moment acts on those locations.

Filename:sfig2-2-1d

Note: There are *no* two-force bodies in this problem. Bar AB is massless but is not a two-force member because it has a couple at its end.

## 3.2 Contact: Sliding, friction, and rolling

The primary mechanical interaction between intermediate-sized objects, say much smaller than the earth and much larger than an atom, is through contact *. Things cause contact forces on each other when and where they touch. Contact between two bodies restricts their possible motions and causes forces on the bodies. Some contact situations are modeled in standard ways that we have discussed, including contact at a pin, ball-and-socket, hinge, weld, and tied string.

Here we consider objects that press against each other in ways not necessarily well-idealized with one of the standard mechanical connections. We need models for the relation between the contact forces and the motions between the contacting objects. Of special concern are sliding and rolling. That is, we need *constitutive* laws for sliding and rolling contact. The behaviors of real contacting objects are complex and not well understood. There are many candidate constitutive laws for friction and rolling. They vary in their conceptual simplicity, their ease of use in analytical or numerical calculations, and their accuracy and applicability. We will present the simplest rules, describe some of the short-comings and then give some guidance towards more sophisticated rules.

* The most-often encountered *non*-contact force is gravity, especially the gravity force from the earth on terrestrial objects like cars, people, rocks, and ants.

### Contact laws are all rough approximations

Unfortunately, there are no simple and accurate general rules for describing contact forces. When we study the dynamics of a system that involves the interaction of bodies we are forced to use one or another approximate description for finding the forces of interaction in terms of the bodies positions and velocities. Such a description is called, as mentioned in Chapter 1, a *constitutive law* or *constitutive relation*. Generally people write separate constitutive laws after categorizing the motion into being one of the three major types of contact interaction: friction, rolling, or collision. * Because collisions are only relevant to dynamics, discussion of collisional free body diagrams is deferred until the dynamics portion of the book.

We must emphasize at the outset:

> Constitutive laws for contact interaction are generally only rough approximations, with theory and practice differing by 5-50% for at least some of the quantities of interest.

Equations for forces of contact are of a lower class than the fundamental equations in mechanics. At the scale of most engineering, the momentum balance equations are extremely accurate, with error of well less than a part per billion. Newton's law of gravitational attraction is a similarly accurate law. And the laws of Euclidean (non-Riemanian) geometry and calculus (the kinds you studied) are also extremely accurate. Less accurate are the laws

* In practice it is not always clear how to make the distinctions between sliding and rolling or between sliding and collision. But at least for a first pass it is a useful conceptual distinction to think of sliding, rolling, and collisions as three different kinds of contact.

for spring's and dashpots. But still, accuracies of one part per thousand are possible for measuring spring stiffness, say, and perhaps parts per hundred for dashpot constants.

But the laws for the contact interactions of solids are much less accurate. Not only is it difficult to know the coefficient of friction between two pieces of steel with any certainty, you also can't trust even the *concept* of a coefficient of friction to have any great accuracy. It is easy to forget this inaccuracy in contact laws because you will see contact-force equations in books. Once we see an equation in print, we are too-easily tempted into believing it is 'true.' So a common mistake amongst beginning engineers is to use contact constitutive equations with confidence, as if accurate, when the best you can get is only a rough approximation at best.

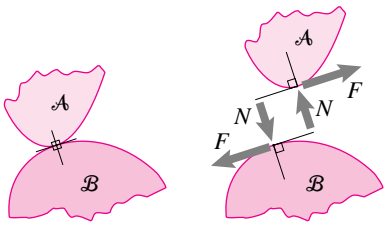



Figure 3.25: Two bodies in contact. The forces between them satisfy the law of action and re-action. It is often convenient to decompose the force of interaction into a part $F$ tangent to the surface of interaction and a part $N$ perpendicular to the surface of interaction.

Filename:tfigure11-contact

## Friction

When two objects are in contact and one is sliding with respect to the other, we call the force which resists this sliding *friction*. Frictional contact is usually assumed to be either 'lubricated' or 'dry.' When bodies are in lubricated contact they are not in real contact at all, a thin layer of liquid or gas separates them. Most of the metal to metal contact in a car engine is so lubricated. The contact of the car tires with the road is 'dry' unless the car is 'hydroplaning' on worn-smooth tires on a very wet road. The friction forces in lubricated contact are very small compared forces of unlubricated contact. There is no quick way to estimate these small lubricated slip forces. The accurate estimation of lubricated friction forces requires use of lubrication theory, a part of fluid mechanics. For many purposes lubricated friction forces are neglected. We now drop discussion of lubricated friction forces because they are often negligible and because estimating them is a more advanced topic.

Dry friction forces are not small and thus cannot be sensibly neglected in mechanics problems involving sliding contact. The simplest model for friction forces is called *Coulomb's law of friction* or just *Coulomb friction*. But, use of even this law is full of subtleties.

### 'Smooth' and 'Rough', common misnomers for low-friction and high-friction

As a modeling simplification when we would like to neglect friction we sometimes assume frictionless contact and thus set $\mu = \phi = 0$. In many books the phrase "perfectly smooth" is used to describe this assumed neglect of friction. It is true that when separated by a little fluid (say water between your feet and the bathroom tile, or oil between pieces of a bearing) that smooth surfaces slide easily by each other. And even without a lubricant sometimes slipping can be reduced by roughening a surface. But making a surface progressively smoother does not diminish the friction to zero. In fact, extremely smooth surfaces sometimes have anomalously high friction. In general, there is no reliable correlation smoothness and low friction.

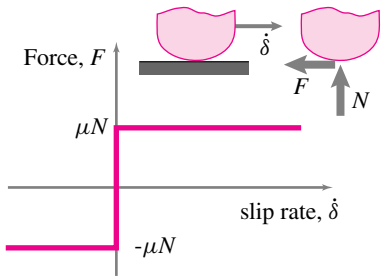



Figure 3.26: **Coulomb friction.** The relation between friction force $F$ and relative slip rate $\dot{\delta}$ is described by the dark line. Since there is a jump from $-\mu N$ to $\mu N$ in the friction force when the slip rate goes from negative to positive the relation is not a proper mathematical function between $F$ and $\dot{\delta}$. Instead the relation is a curve in the $F, \dot{\delta}$ plane.

Filename:tfigure11-coulomb

Similarly many books use the phrase "perfectly rough" to mean perfectly high friction ($\mu \to \infty$ and $\phi \to 90^\circ$) and hence that no slip is allowed. This is misleading twice over. First, as just stated, rougher surfaces do not reliably have more friction than smooth ones. Second, even when $\mu \to \infty$ slip can proceed in some situations (see, for example, box 4.3 on page 187).

We use the phrase *frictionless* or *negligible friction* to mean that there is no tangential force component. We use the phrase *no slip* to mean that no tangential motion is allowed and that there is some unknown tangential force. So

We do not use the words smooth and rough in this book to indicate low and high friction.

## Coulomb friction

Coulomb's law of friction, also attributed sometimes to Amonton and sometimes to DaVinci, is summarized by the simple equation:

$$F = \mu N. \tag{3.1}$$

This equation, like many other simple equations, is not really a complete description of Coulomb's law of friction. Some words are required.

First of all the direction of the force $F$ on body $\mathcal{A}$ is in the opposite direction of the slip velocity of $\mathcal{A}$ relative to $\mathcal{B}$. By the principle of action and reaction we deduce that the force on body $\mathcal{B}$ is in the opposite direction. This force is also opposite to the relative slip velocity of $\mathcal{B}$ relative to $\mathcal{A}$. That is, $F$ resists relative motion of $\mathcal{A}$ and $\mathcal{B}$.

The friction force $F$ is proportional to the normal force $N$ with the proportionality constant $\mu$. The constant $\mu$ is assumed to be independent of the area of contact between bodies $\mathcal{A}$ and $\mathcal{B}$. In the simplest renditions of Coulomb's law $\mu$ is assumed to be independent of slip distance, slip velocity, time of contact, etc. When contacting bodies are not sliding the role of friction changes somewhat. In some sense the friction still resists slip, in fact it is the presence of the friction force that prevents slip. But another way to think of friction is that it puts an upper limit on the size of the force of interaction between two bodies which seem stuck to each other. The friction force must be less than or equal to $\mu N$ in magnitude during contact.

$$|F| \leq \mu N \tag{3.2}$$

All of the discussion above can be summarized with the following equations for the friction force

The friction force, the part of the force of interaction which is tangent to the surface.

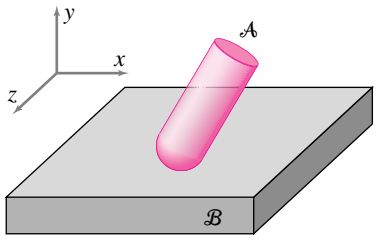
The relative slip velocity of the contacting points.

$$\vec{F}_{\text{on } \mathcal{A} \text{ from } \mathcal{B}} = -\mu \frac{\vec{v}_{\mathcal{A}/\mathcal{B}}}{|\vec{v}_{\mathcal{A}/\mathcal{B}}|} N \qquad \text{during slip}$$

$$|\vec{F}_{\text{on } \mathcal{A} \text{ from } \mathcal{B}}| \leq \mu N \qquad \text{during stationary contact}$$

The magnitude of the tangential part of the contact force

An upper bound on the tangential part of the contact force




$\mathcal{A}$ does not slide relative to $\mathcal{B}$

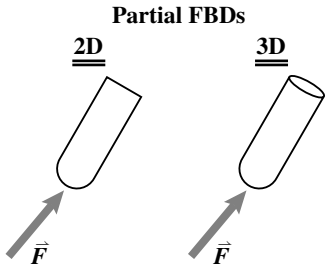



Figure 3.27: Object $\mathcal{A}$ does not slide relative to the plane $\mathcal{B}$.

Filename:tfigure2-noslip

For two-dimensional problems where slip can only be in one direction (or the opposite) this pair of functions describes the dark line in the friction graph of figure 3.26 in which $\dot{\delta}$ is the speed of relative slip.

The simplest friction law, the one we use in this book, uses a single constant coefficient of friction $\mu$. Almost always $.05 \leq \mu \leq 1.2$ and more commonly $.2 \leq \mu \leq 1$. We do not distinguish the static coefficient $\mu_s$ from the dynamic coefficient $\mu_d$ or $\mu_k$. That is $\mu = \mu_s = \mu_k = \mu_d$ for our purposes. We promote the use of this simplest law for a few reasons.

- All friction laws used are quite approximate, no matter how complex. Unless the distinction between static and dynamic coefficients of friction is essential to the engineering calculation, using $\mu_s \neq \mu_k$ doesn't add to the calculation's usefulness.
- The concept of a static coefficient of friction that is larger than a dynamic coefficient is, it turns out, not well defined if bodies have more than one point of contact, which they often do have. (See box **??** on page **??**.)
- Students learning mechanics are often confused about friction. Because the more complex friction laws are of questionable accuracy and usefulness anyway, it seems time is better spent understanding the simplest friction laws.

See box **??** on page **??** for more discussion of the pros and cons of the Coulomb-friction approximation.

In summary, the simple model of friction we use is:

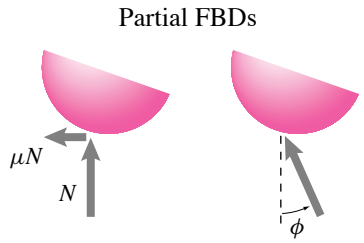



Figure 3.28: Two ways of characterizing friction: the friction coefficient $\mu$ and friction angle $\phi$.

Filename:tfigure2-friction-angle

Friction resists relative slipping motion. *During slip the friction force opposes relative motion and has magnitude* $F = \mu N$. When there is no slip the magnitude of the friction force $F$ cannot be determined from the friction law but it cannot exceed $\mu N$, that is $F \leq \mu N$.

## Friction angle

Sometimes people describe the friction coefficient with a *friction angle* $\phi$ rather than the coefficient of friction (see fig. 3.28). The friction angle is the angle between the net interaction force (normal force plus friction force) and the normal to the sliding surface when slip is occurring. The relation between the friction coefficient $\mu$ and the friction angle $\phi$ is

$$\tan\phi \equiv \mu.$$

The use of $\phi$ or $\mu$ to describe friction are equivalent. Which you use is a matter of taste and convenience. Sometimes analytic formulas in problems come out simpler looking with one or the other of $\mu$ and $\phi$ used to describe the friction.

## Rolling contact

An idealization for the non-skidding contact of balls, wheels, and the like is *pure rolling*.

> *Objects $\mathcal{A}$ and $\mathcal{B}$ are in pure rolling contact when their (relatively convex) contacting points have equal velocity. They are not slipping, separating, or interpenetrating.*

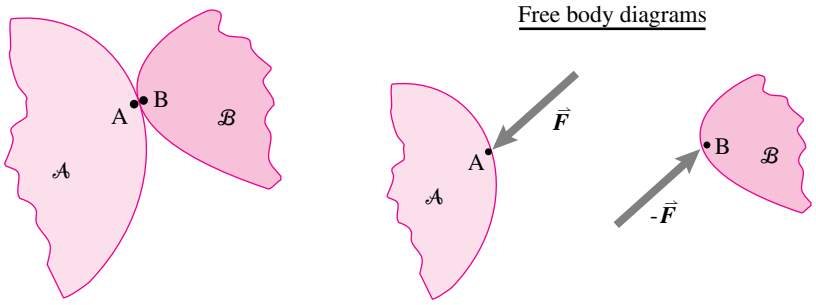



Figure 3.29: Rolling contact: Points of contact on adjoining bodies have the same velocity, $\vec{v}_A = \vec{v}_B$. But, $\vec{\omega}_{\mathcal{A}}$ is not necessarily equal to $\vec{\omega}_{\mathcal{B}}$.

Filename:tfigure2-rolling-contact

Most often, we are interested in cases where the contacting bodies have some non-zero relative angular velocity — a ball sitting *still* on level ground may be technically in rolling contact, but not interestingly so.

The simplest common example is the rolling of a round wheel on a flat surface in two dimensions. See figure 3.30.

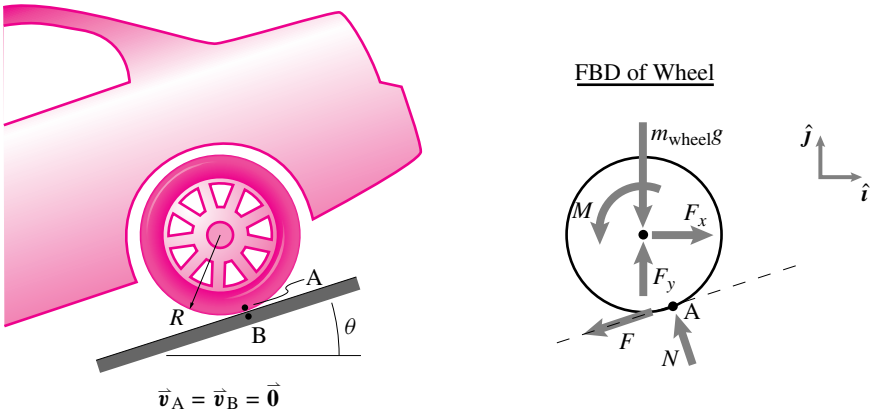



Figure 3.30: Pure rolling of a round wheel on a flat slope in two dimensions.

Filename:tfigure2-pure-rolling-wheel

In practice, there is often confusion about the direction and magnitude of the force $F$ shown in the free body diagram in figure 3.30. Here is a recipe:

1.) Draw F as shown in any direction which is tangent to the surface.

2.) Solve the statics or dynamics problem and find the scalar $F$. (It may turn out to be a negative, which is fine.)

3.) Check that rolling is really possible; that is, that slip would not occur. If the force is greater than the frictional strength, $|F| > \mu N$, the assumption of rolling contact is not appropriate. In this case, you must assume that $F = \mu N$ or $F = -\mu N$ and that slip occurs; then, re-solve the problem.

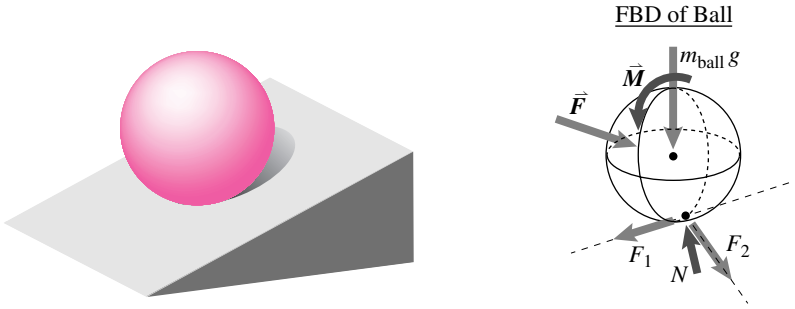



Figure 3.31: **Rolling ball in 3-D.** The force $\vec{F}$ and moment $\vec{M}$ are applied loads from, say, wind, gravity, and any attachments. $N$ is the normal reaction and $F_1$ and $F_2$ are the in plane components of the frictional reaction. One must check the no-slip condition, $\mu^2 N^2 \geq F_1^2 + F_2^2$.

Filename:tfigure2-3D-rolling

In three-dimensional rolling contact, we have a free body diagram that again looks like a free-body diagram for non-slipping frictional contact. Consider, for example, the ball shown in figure 3.31. For the friction force to be less than the friction coefficient times the normal force, we have the *no slip condition*

$$\sqrt{F_1^2 + F_2^2} \;\leq\; \mu N \qquad \text{or} \qquad F_1^2 + F_2^2 \;\leq\; \mu^2 N^2$$

*Note that the tangential forces in Fig. 3.32 and Fig. 3.2 are not rolling resistance.

Rolling is just a special case of frictional contact. It is the case where bodies contact at a single point (or on a line, as with cylinders) and have relative rotation yet have no relative velocity at their contacting points.

## Rolling resistance

Non-ideal rolling contact includes provision for *rolling resistance*. This resistance is simply represented by either moving the location of the point of contact force or by a contact couple. Rolling resistance leads to subtle questions which we skip here*.

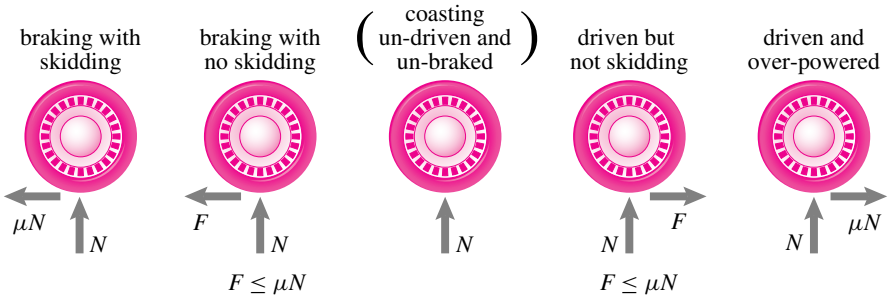



Figure 3.32: Partial free body diagrams of wheel in a braking or accelerating car that is pointed and moving to the right. The force of the ground on the tire is shown. But, for simplicity, the forces of the axle, gravity, and brakes on the wheel are not shown (that's why its a partial FBD). An ideal point-contact wheel is assumed. There is no 'rolling resistance' here.

Filename:tfigure3-2-car-breaking

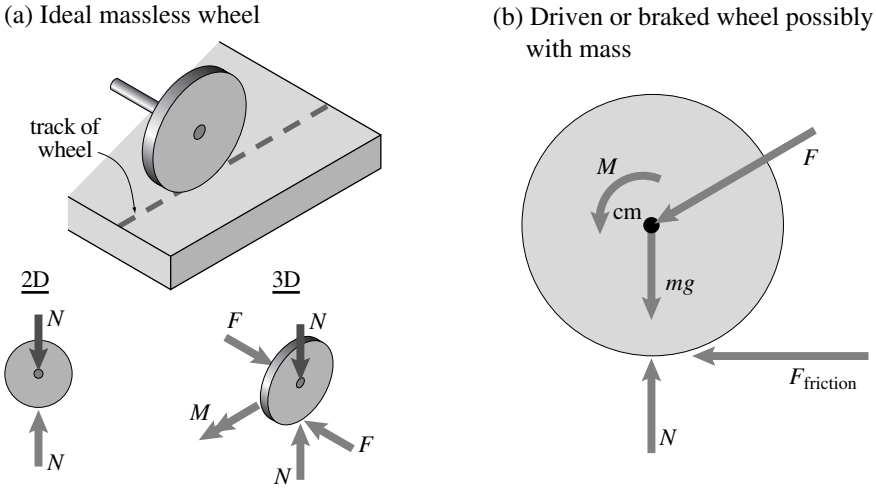



Figure 3.33: An ideal wheel is massless, rigid, undriven, round and rolls on flat rigid ground with no rolling resistance. Free body diagrams of ideal undriven wheels are shown in two and three dimensions. The force $F$ shown in the three-dimensional picture is perpendicular to the path of the wheel. (b) 2D free body diagram of a wheel with mass, possibly driven or braked. If the wheel has mass but is not driven or braked the figure is unchanged but for the moment $M$ being zero.

Filename:tfigure2-idealwheel

## Ideal wheels

An ideal wheel is an approximation of a real wheel. It is a sensible approximation if the mass of the wheel is negligible, bearing friction is negligible, and rolling resistance is negligible. Free body diagrams of undriven ideal wheels in two and three dimensions are shown in figure 3.33. This idealization is rationalized in chapter 4 in box 4.3 on page 187. Note that if the wheel is not massless, the 2-D free body diagram looks more like the one in figure 3.33b with $F_{\text{friction}} \leq \mu N$.

## Extended contact

When things touch each other over an extended region, like the block on the plane of fig. 3.34a, it is not clear what forces to put where on the free body diagram. On the one hand one imagines reality to be somewhat reflected by millions of small forces as in fig. 3.34b which may or may not be divided into normal ($n_i$) and frictional ($f_i$) components. But one generally is not interested in such detail, and even if interested one cannot find it easily (see box **??** on page **??**).

A simple approach is to replace the detailed force distribution with a single equivalent force, as shown in fig. 3.34c broken into components. The location of this force is not relevant for some problems. *

If one wants to make clear that the contact forces serve to keep the block from rotating, one may replace the contact force distribution with a pair of contacts at the corners as in fig. 3.34d.

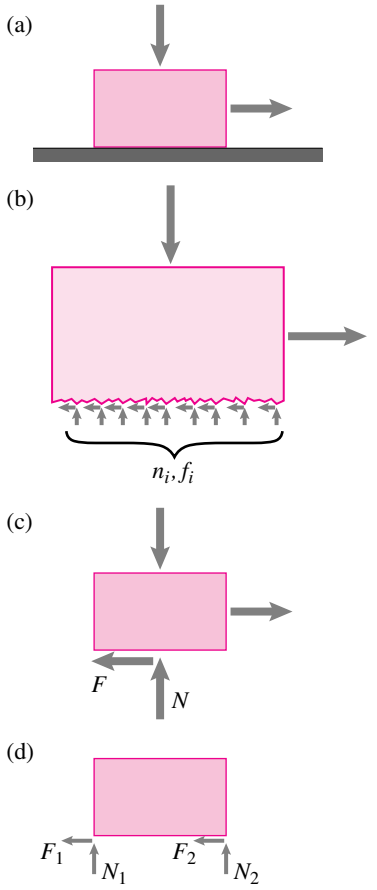



Figure 3.34: The contact forces of a block on a plane can be sensibly modeled in various ways.

Filename:tfigure-conformalblock

* In 3D, contact force distributions cannot always be replaced with an equivalent force at an appropriate location (see section 2.5). A couple may be required. Nonetheless, many people often make the approximation that a contact force distribution can be replaced by a force at an appropriate location. For example, this is the "center of pressure" approach used to describe the location of an imagined-equivalent ground force on a robot's foot. This approximation neglects any frictional resistance to twisting about the normal to the contact plane.

## 3.3 A short critique of Coulomb friction

This aside is not needed for homework. It is here for those who are interested in the place of Coulomb friction amongst more general friction laws.

In short, Coulomb's law of friction is good because

- Coulomb's law of friction is simple.
- Coulomb's law of friction usefully predicts many phenomena.
- It has the right trends in many regards, in that
  - sliding friction *is* roughly independent of slip rate, and
  - the friction force *is* roughly proportional to the normal force.
- Other candidate laws (generally) cost more in complexity than they gain in accuracy or usefulness.

On the other hand,

- The friction coefficient is not stable, it may vary from day to day or between samples of nearly identical materials.
- Coulomb's law, without a separate static coefficient of friction or an explicit dependence on rate of slip, cannot be used to explain frictional phenomena such as
  - the squeaking of doors,
  - the excitement of a violin string by a bow, and
  - earthquakes from sliding rocks.
- For some materials the dependence of friction on normal force is noticeably different from linear. Rubber on road, for example, has more friction force per unit normal force when the normal force is low. In other words the friction force for a given normal force is greater when the area of contact is greater. This dependence of friction on normal stress is presumably why racing cars have fat tires.

We expand on some of these points below.

**The friction coefficient is not a stable property** Jaeger, a famous rock mechanician, is said to have presented the following empirical friction law:

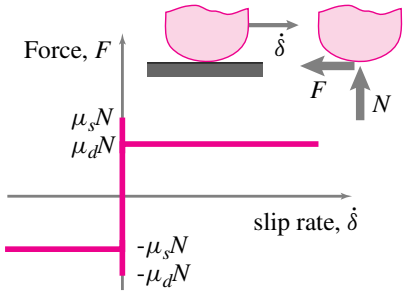
*A friction experiment will make a monkey out of you.*

For any pair of objects and any given experiment to measure the friction coefficient, the measured value will likely vary from day to day. This observation seems to violate our common notions of determinacy. Why does this apparent indeterminacy happen? Probably because friction involves the interaction of surfaces. The chemistry of a surface can be dramatically changed by very small quantities of material (a surface is a very small volume!). So any change in humidity, or perhaps a random finger touch, or a slight spray from here or there can dramatically change the surface chemistry and hence the friction.

This problem of the non-constancy of friction from day to day or sample to sample cannot be overcome by a better friction law. So unless one understands one's materials and their chemical environment extremely well, all friction laws, however sophisticated are doomed to large inaccuracy.

**Coulomb's friction law neglects the drop in the friction force at the start of sliding** Most simple treatments of friction immediately introduce two coefficients of friction. The sliding coefficient is also sometimes called the dynamic coefficient $\mu_d$ or the kinetic coefficient $\mu_k$. The other coefficient of friction is the 'static' coefficient of friction $\mu_s$.

According to standard lore, each pair of bodies has friction which is described by the static and dynamic coefficients of friction $\mu_s$ and $\mu_d$ with the understanding that the static coefficient of friction is greater than the dynamic coefficient of friction, $\mu_s > \mu_d$.




**Static-Dynamic Friction.** The relation between friction velocity and friction force is such that at all times the pair of values is found on the dark line shown.

According to this description, the relation between friction velocity and friction force is as given in the figure above. This description is useful for roughly characterizing the following phenomenon:

*It is harder to start something sliding than it is to keep it sliding.*

If the dynamics problem you are working on depends on this phenomenon the static-dynamic friction law is one way of treating it. But you should be forewarned that, though this law is great for qualitatively explaining how a bow excites a violin string, or why anti-lock brakes work better than all out skidding, it is not very accurate.

If one does careful experiments to try to understand in more detail how the friction force drops from a higher value to a lower one as slip starts, one discovers a world of phenomena that are not well captured with two simple coefficients of friction. Further, using two coefficients of friction leads to various paradoxes and indeterminacies when one studies slightly more complex problems. (See box **??** on page **??**.)

**Friction is not always proportional to normal force** The Coulomb friction equation, applicable during slip or at impending slip,

$$F = \mu N$$

is most directly translated into English as: *the friction force is proportional to the normal force*. This proportionality is, as far as we know, not fundamental, but rather an often reasonable approximation to many experiments. Why the interaction of so many solids obeys this proportionality so well is not known, though there are a few explanations that make this experimental result theoretically plausible.

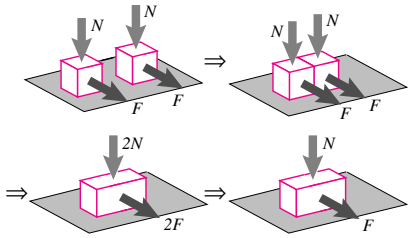
In some books you will see an additional law of friction stated as:

*The friction force is independent of the area of contact.*

By *'area of contact'* is meant the area you would measure macroscopically. For a 4 in × 8 in brick sliding on a pavement the area of contact is 32 $\text{in}^2$

[Another concept of area of contact is the actual area of contact at all the little asperities. This definition of area of contact is useful

for tribologists (people who study friction) but is of little concern to people interested in the mechanics of macroscopic things.]




Considering two blocks side by side as one block shows how friction being proportional to normal force means friction is independent of area of contact. The two blocks have twice the area as one block, but a given normal force causes the same friction force.

The independence of force with area is actually equivalent to the proportionality of friction force with normal force. Let's explain, or at least let's give the gist of the argument. Imagine two identical blocks side by side on a plane as in figure **??**. The force pushing down on each is $N$ and the friction force to cause slip is $F = \mu N$. The act of glueing the two together side-by-side should have no effect. Now we have one bigger block with twice the normal force, twice the friction force and twice the area of contact. If we assume that friction force is proportional to normal force, we know that if we now cut the normal force in half then the friction force will be cut in half. But now we have a new block with twice the area of contact as each of the original blocks and it carries the same normal force and the same friction force. Thus the friction force is unchanged by doubling the area of contact.

But in fact, some materials have friction force which does depend on the normal force, or for a given normal force, does depend on the area of contact. The most prominent example is the friction between rubber and pavement. For a given weight car, a larger friction force can be generated with a fat tire than a narrow one. That is, the ratio of the friction force to normal force decreases as the normal force increases.

**All things considered, Coulomb's law is alright** In this book we generally assume that $\mu_d = \mu_s = \mu$; there is just one coefficient of friction. Most often it is reasonable to assume that static friction is close enough to dynamic friction that it is not worth the trouble to distinguish them. Of course there are situations which one may want to understand where the transition from static to dynamic friction is essential. For these cases a static-dynamic friction model might provide some insight, but it may also cause basic modeling problems. Coulomb's law with one coefficient of friction is the simplest dry friction constitutive law. It is the appropriate description for most purposes. It is reasonably accurate, considering that it is one out of a mediocre crowd, and more elaborate laws are not particularly more accurate.

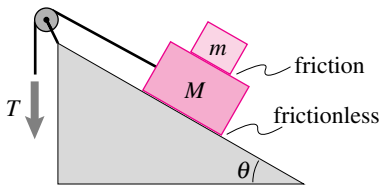



Figure 3.35: Two blocks held in place on an inclined surface

Filename:sfig2-1-4a

**SAMPLE 3.7 Stacked blocks at rest on an inclined plane.** Blocks $A$ and $B$ with masses $m$ and $M$, respectively, rest on a frictionless inclined surface with the help of force $T$ as shown in Fig. 3.35. There is friction between the two blocks. Draw free body diagrams of each of the the two blocks separately and a free body diagram of the two blocks as one system.

**Solution** The three free body diagrams are shown in Fig. 3.36 (a) and (b). Note the action and reaction pairs between the two blocks; the normal force $N_A$ and the friction force $F_f$ between the two bodies A and B. If we consider the two blocks together as a system, then the forces $N_A$ and $F_f$ do not show on the free body diagram of the system (See Fig. 3.36(b)), because now they are internal to the system.

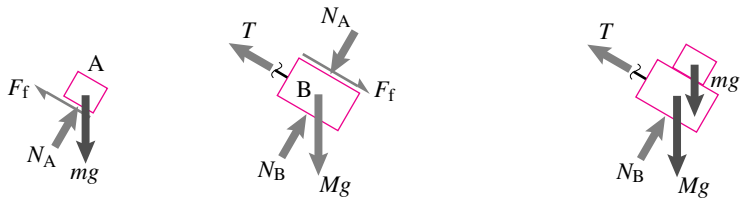



Figure 3.36: Free body diagrams of (a) block $A$ and block $B$ separately and (b) blocks $A$ and $B$ together.

Filename:sfig2-1-4b

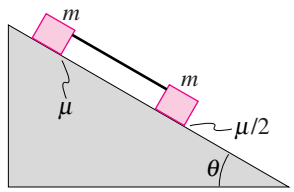



Figure 3.37: Two blocks slide down a frictional inclined plane. The blocks are connected by a light rigid rod.

Filename:sfig2-1-15

**SAMPLE 3.8 Two blocks slide down a frictional inclined plane.** Two blocks of identical mass but different material properties are connected by a massless rigid rod. The system slides down an inclined plane which provides different friction to the two blocks. Draw free body diagrams of the two blocks separately and of the system (two blocks with the rod).

**Solution** The Free body diagrams are shown in Fig. 3.38. Note that the friction forces on the two blocks are different because the coefficients of friction are different for the two blocks. The normal reaction of the plane, however, is the same for each block (why?).

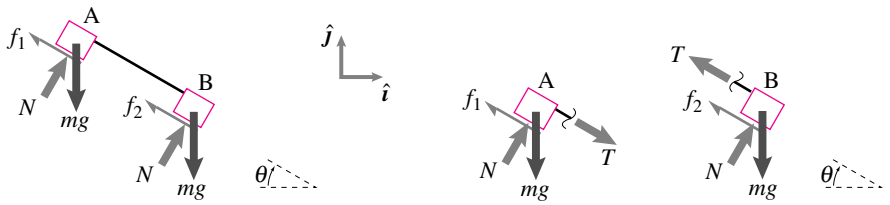



Figure 3.38: Free body diagrams of (a) the two blocks and the rod as a system and (b) the two blocks separately.

Filename:sfig2-1-15a

**SAMPLE 3.9 Massless pulleys.** A force $F$ is applied to the pulley arrangement connected to the cart of mass $m$ shown in Fig. 3.39. All the pulleys are massless and frictionless. The wheels of the cart are also massless but there is friction between the wheels and the horizontal surface. Draw a free body diagram of the cart, its wheels, and the two pulleys attached to the cart, all as one system.

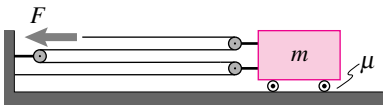



Figure 3.39: A cart with pulleys

Filename:sfig2-1-7a

**Solution** The free body diagram of the cart system is shown in Fig. 3.40. The force in each part of the string is the same because it is the same string that passes over all the pulleys.

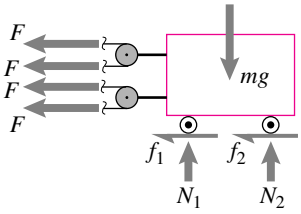



Figure 3.40: Free-body diagram of the cart.

Filename:sfig2-1-7b

**SAMPLE 3.10 A unicyclist in action.** A unicyclist weighing 160 lbs exerts a force on the front pedal with a vertical component of 30 lbf at the instant shown in figure 3.41. The rear pedal barely touches the other foot. Assume the wheel and the frame are massless. Draw free body diagrams of the cyclist and the cycle. Make other reasonable assumptions if required.

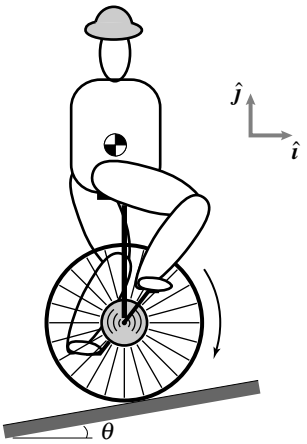



Figure 3.41: The unicyclist

Filename:sfig2-1-8

**Solution** Let us assume, there is friction between the seat and the cyclist and between the pedal and the cyclist's foot. Let's also assume a 2-D analysis. The free body diagrams of the cyclist and the cycle are shown in Fig. 3.42. We assume no couple interaction at the seat.

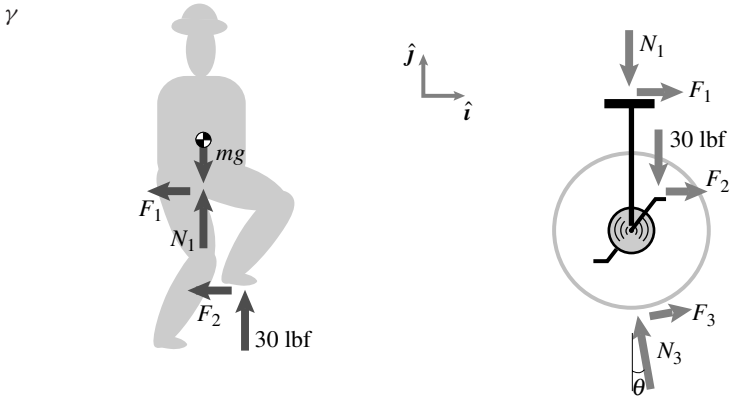



Figure 3.42: Free-body diagram of the cyclist and the cycle.

Filename:sfig2-1-8a

# Problems for Chapter 3

Free body diagrams

## 3.1 Interactions, Partial FBDs

**3.1** How does one know what forces and moments to use in

a) the statics force balance and moment balance equations?

b) the dynamics linear momentum balance and angular momentum balance equations?

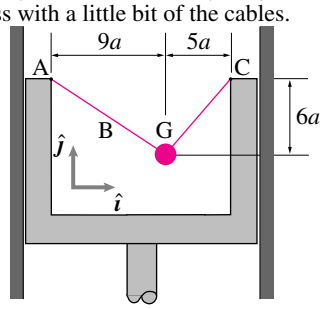
**3.2** In a free body diagram of a whole man standing with his right hand extended how do you show the force of his right arm on his body?

**3.3** Reproduce the first column of the table in Fig. **??** on page **??** for the force acting on your right foot from the ground as you step on a stair.

**3.4** Reproduce the second column of the table in Fig. **??** on page **??** for a force in the direction of $3\hat{\imath} + 4\hat{\jmath}$ but with unknown magnitude.

**3.5** Reproduce the third column of the table in Fig. **??** on page **??** for a 50 N force in the direction of the vector $3\hat{\imath} + 4\hat{\jmath}$.

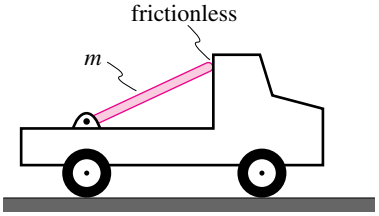
**3.6** A point mass $m$ at G is attached to a piston by two inextensible cables. There is gravity. Draw a free body diagram of the mass with a little bit of the cables.




**problem 3.6:**

Filename:pfigure2-1-suspended-mass

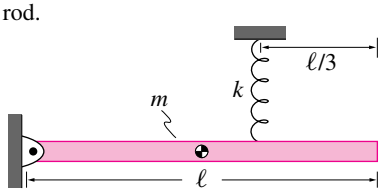
**3.7** A uniform rod of mass $m$ rests in the back of a flatbed truck as shown in the figure. Draw a free-body diagram of the rod.




**problem 3.7:**

Filename:pfig2-2-rp5

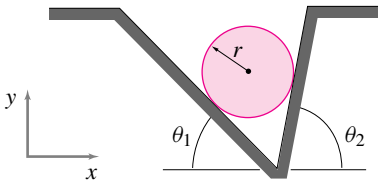
**3.8** The uniform rigid rod shown in the figure hangs in the vertical plane with the support of the spring shown. In this position the spring is stretched $\Delta\ell$ from its rest length. Draw a free body diagram of the spring. Draw a free body diagram of the rod.




**problem 3.8:**

Filename:pfig2-1-rp1

**3.9** A disc of mass $m$ sits in a wedge shaped groove. There is gravity and negligible friction. Draw a free body diagram of the disk.



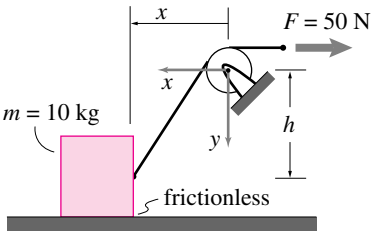

**problem 3.9:**

Filename:ch2-5

**3.10 FBD of a block.** The block of mass 10 kg is pulled by an inextensible cable over the pulley.

a) Assuming the block remains on the floor, draw a free diagram of the block. (There are various correct answers depending how you model the interaction of the bottom of the block with the ground. See Fig. **??** on page **??**)

b) Draw a free body diagram of the pulley with a little bit of the cable extending to both sides.




**problem 3.10:**

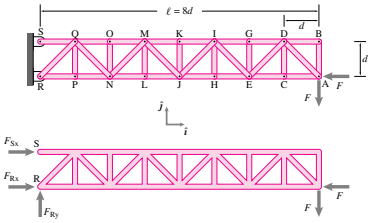
Filename:pfigure2-1-block-pulley

**3.11 Cantilevered truss** A truss is shown as well as a free body diagram of the whole truss.

a) Draw a free body diagram of that portion of the truss to the right of bar GE.

b) Draw a free body diagram of bar IE.

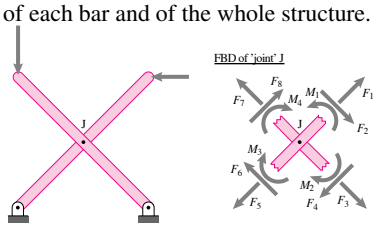
c) Draw a free body diagram of the joint at I with a small length of the bars protruding from I.



**problem 3.11:**

Filename:pfigure-derrick

**3.12 An X structure** A free body diagram of the joint J with a little bit of the bars near J is shown. Draw free body diagrams of each bar and of the whole structure.




**problem 3.12:**

Filename:pfigure-xstructure

**3.13 Pulleys** Draw free body diagrams of

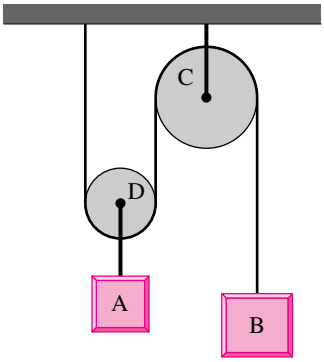
a) mass A with a little bit of rope

b) mass B with a little bit of rope

c) Pulley B with three bits of rope

d) Pulley C with three bits of rope

e) The system consisting of everything below the ceiling

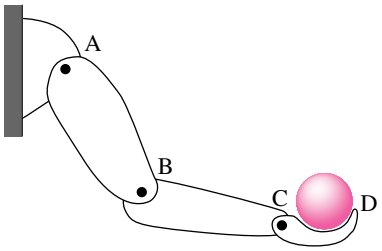



**problem 3.13:**

Filename:pfigure3-3-DH1

**3.14 FBD of an arm throwing a ball.** An arm throws a ball up. A crude model of an arm is that it is made of four rigid bodies (shoulder, upper arm, forearm and a hand) that are connected with hinges. At each hinge there are muscles that apply torques between the links. Draw a FBD of

a) the system consisting of the whole arm (three parts, but not the shoulder) and the ball.
b) the ball,
c) the hand, and
d) the fore-arm,
e) the upper arm,

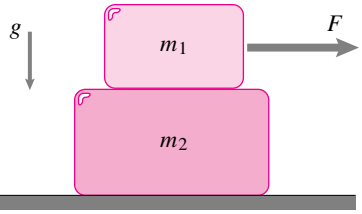



**problem 3.14:**

Filename:efig2-1-23

**3.15** Two frictionless blocks sit stacked on a frictionless surface. A force $F$ is applied to the top block. There is gravity.

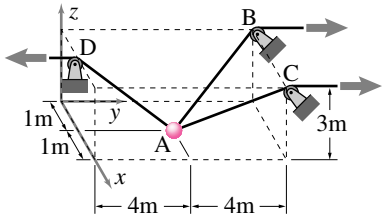
a) Draw a free body diagram of the two blocks together and a free body diagram of each block separately.




**problem 3.15:**

Filename:ch2-3

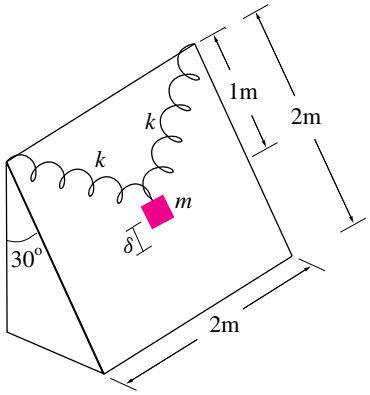
**3.16** The strings hold up the mass $m = 3\,\text{kg}$. There is gravity. Draw a free body diagram of the mass.




**problem 3.16:**

Filename:pfigure2-1-3D-pulley-fbd

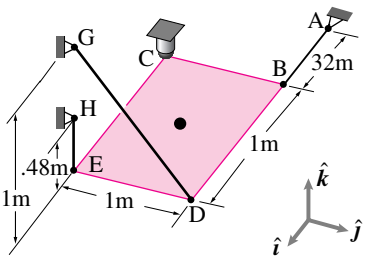
**3.17 Mass on inclined plane.** A block of mass $m$ rests on a frictionless inclined plane. It is supported by two stretched springs. The mass is pulled down the plane by an amount $\delta$ and released. Draw a FBD of the mass just after it is released.




**problem 3.17:**

Filename:efig2-1-24

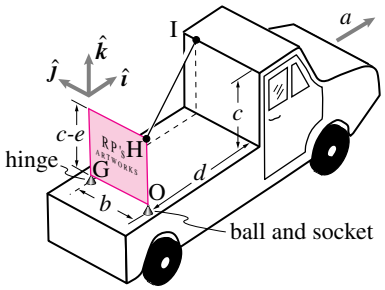
**3.18 Hanging a shelf.** A shelf with negligible mass supports a 0.5 kgmass at its center. The shelf is supported at one corner with a ball and socket joint and the other three corners with strings. Draw a FBD of the shelf.




**problem 3.18:**

Filename:ch3-14

**3.19 Sign** Draw a free body diagram of the sign shown.

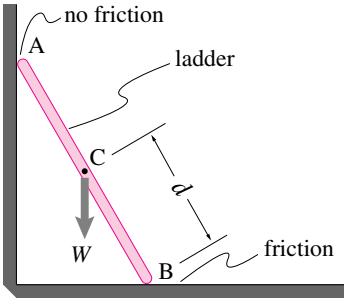



**problem 3.19:**

Filename:pfigure3-5-2

## 3.2 Contact and friction

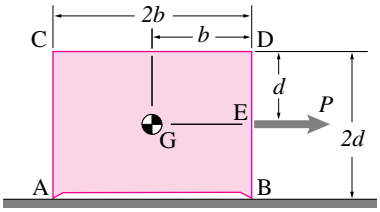
**3.20** A thin rod of mass $m$ rests against a frictionless wall and a floor with more than enough friction to prevent slip. There is gravity. Draw a free body diagram of the rod.




**problem 3.20:**

Filename:pfigure4-single-ladder

**3.21** A block of mass $m$ sits on a surface supported at points $A$ and $B$. A horizontal force $P$ acts at point E. There is gravity. The block is sliding to the right. The coefficient of friction between the block and the ground is $\mu$. Draw a free body diagram of the block.



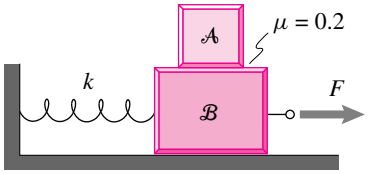

**problem 3.21:**

Filename:ch2-1

**3.22** For the system shown in the figure draw free-body diagrams of each mass separately and of the system of two blocks.

a) Assume there is friction with coefficient $\mu$. At the time of interest block $\mathcal{B}$ is sliding to the right and block $\mathcal{A}$ is sliding to the left relative to $\mathcal{B}$.

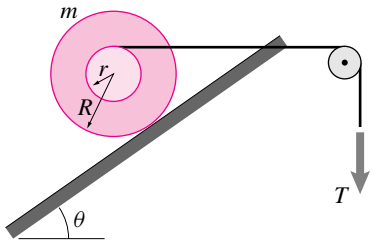
b) Assume there is so much friction that neither block slides.




**problem 3.22:**

Filename:pfig2-1-rp7

**3.23 Spool** Draw a free body diagram of the spool shown, including a bit of the rope. Assume the spool does not slid on the ramp.




**problem 3.23:**

Filename:pfigure4-single-spool

# Part II: Statics

CHAPTER 4

# Statics of one object

*Equilibrium of one object is defined by the balance of forces and moments. Force balance tells all for a particle. For an extended body moment balance is also used. There are special shortcuts for bodies with exactly two or exactly three forces acting. If friction forces are relevant the possibility of motion needs to be taken into account. Many real-world problems are not statically determinate and thus only yield partial solutions, or full solutions with extra assumptions.*

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4.2 Equilibrium of one object . . . . . **172**
4.3 Equilibrium with frictional contact . . . . . **184**
4.4 Internal forces . . . . . **198**
4.5 3D statics of one part . . . . . **204**
Problems for Chapter 4 . . . . . **212**

The goal here is to find unknown aspects of the forces acting on one part of a machine or structure. Such a part is also called an 'object' or 'body'. By 'unkown' we mean 'unknown at the outset' or 'you-need-to-know-mechanics-to-find'. Most often this will mean finding tensions in ropes or rods, contact forces where one part presses and rubs against another, and the force on an object at a point of connection to another object. We will also find the forces and moments that one part of an object applies to another part of the same object. Finally we might also find the direction or point of application of a force with *a priori* known magnitude.

## Needed skills

Throughout this and all later chapters you need vector and free body diagram skills and concepts from chapters 2 and 3.

## Statics is a subset of dynamics

Statics is the mechanics of things that don't move. But everything does move, at least a little. So strictly speaking dynamics is always the applicable subject. For many practical problems, however, statics is a good approximation of dynamics, very good. With little loss of accuracy, sometimes very little loss, and a great saving of effort, usually a great saving, statics can be used instead of dynamics. Statics is a useful model. Even for a fast moving system, say an accelerating car, statics calculations are appropriate for many of the parts. Although statics is a subset of dynamics (See box 4.1 on page 165) typical engineers do more statics calculations than dynamics calculations. Statics is the core of structural and strength analysis. Statics is the central tool used to predict when a structure or part will or will not break. Finally, Statics is good preparation for Dynamics *.

Here, and for all of statics, we neglect the role of inertia in small motions. We assume static equilibrium.

## Two dimensional and three dimensional mechanics

The world we live in is three dimensional and the theory of mechanics is a three dimensional theory. But three dimensions are harder to understand than two. So most learning and engineering analysis is done in two dimensions. You can't critically judge the degree of simplification this involves until you understand 3D mechanics. But you aren't ready to learn 3D mechanics until you understand 2D mechanics. We escape this catch-22 by being casual about the precise meaning of the 2D world view. For now we think of a

* Ironically perhaps, for some people a main gain from learning dynamics is the side effect of better learning the generally-more-useful statics.

cylinders and spheres as circles, of boxes as rectangles, and of cars as things with two wheels (one in front, one in back).

## Static equilibrium in a nutshell

The basic idea for all of statics is this: if the forces on a system (i.e., the forces showing on a free body diagram of the system) satisfy eqs. (Ic) and (IIc) (inside cover) the system is said to be in *static equilibrium* or just *in equilibrium*[*].

> A system is in static equilibrium if the applied forces and moments add to zero.

Another way to say this is that

> A system in static equilibrium satisfies the linear and angular momentum balance neglecting the inertial ($m\vec{a}$) terms.

A final alternative description of statics is:

> The full collection of forces on a system in static equilibrium are equivalent to (see Section 2.5 on page 83) a zero force and a zero couple.

The statics story is in-principle complete. You have the tools (vectors and free body diagrams) and you know the basic facts (the definition of statics, above). These are enough. But we'll guide you through some of the subtleties, warn you away from common misconceptions, and teach you some of the tricks of the trade. You will see that the simply-stated laws of statics allow you to accurately calculate things that most people who have not studied statics only vaguely understand.

# 4.1 Static equilibrium of a particle

### What is a particle?

The word particle usually means something small. In mechanics a *particle* is an object for which we don't worry about rotation, or the tendency of forces to cause rotation. A particle may or may not be small. Besides, smallness is in the eyes of the beholder. For some purposes a galaxy is well-modeled as a particle and for others a molecule is too big to be thought of as a particle. Big or small, the particle model of a system is defined by the lack of attention paid to the moment balance equations [*]. Either moment balance is trivially

[*] To be precise, static equilibrium requires that the system and all subsystems, all billion gazillion of them (all the different ways you could cut a piece out of your system), satisfy the equilibrium conditions. At this point we don't concern ourselves much with subsystems, just a single whole object.

[*] When trying to understand the motion of a galaxy in a cluster of galaxies, for example, the overall displacement (translation) through space is well-described by modeling the galaxy as a particle. Although the galaxy may rotate and distort in interesting ways, one can ignore the equations that describe that rotation and distortion when looking at the galaxy's overall average translation. Similarly when looking at the motion of an accelerating car, or a block on sliding on ramp or a part sliding on a rod, one may learn enough about the forces using a particle model without worrying about how various forces do or do not cause or prevent rotation or distortion.

satisfied or you can find what you need without worrying about how it is satisfied *.

* Equations are satisfied when the right side is equal to the left.

For statics of a particle force-balance tells all: $\sum_{\substack{\text{All forces}\\ \text{on the FBD}}} \vec{F}_i = \vec{0}$ $(Ic)$

In two dimensions this equilibrium equation makes up 2 independent scalar equations (2 components of the net force vector). In 3 dimensions we get 3 independent scalar equations. So we expect to be able to solve for 2 unknown quantities in 2D particle mechanics, and 3 in 3D.

## The statics-of-a-particle recipe

For particle statics we work with a simplified form of the general recipe from the inside back cover.

1) **Draw a free body diagram (FBD) of the part of interest.**
   Use knowledge of the contact conditions (see Chapter 3) to draw known and unknown aspects of the forces appropriately (see Fig. 3.4 on page 126);
2) **Set the sum of the forces on the FBD to zero: $\sum \vec{F}_i = \vec{0}$.**
   ('Equilibrium', 'force balance', or 'linear momentum balance in statics');
3) **Solve the equations for unkowns.**
   Use vector manipulation skills (Chapter 2) to solve the force balance equation for unknowns of interest.

## Scalar mechanics

In scalar, as opposed to vector, mechanics people sometimes like to take the dot product of Eqn. (Ic) with unit vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ and write the three scalar component equations *.

* If people who think in terms of scalar mechanics actually took these dot products they would be doing vector mechanics. Rather, people who do scalar mechanics think of the component equations as fundamental. This is fine.

$$\sum F_x = 0, \qquad \sum F_y = 0, \qquad \text{additionally, in 3D} \qquad \sum F_z = 0.$$




Figure 4.1: Equilibrium of a particle in 1D.

Filename:tfigure-particle-1Dstatics




Figure 4.2: Particle held by a string.

Filename:tfigure-particle-1Dstaticsa

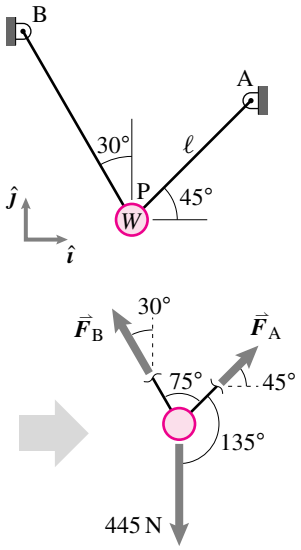



Figure 4.3: Weight hung from two lines with associated free body diagram.

Filename:tfigure-2lines

* **How to solve? Method 1)** Pull out $x$ and $y$ components of the vectors to get 2 equations in 2 unkowns; **Method 2)** Equivalently, dot both sides of the equation with $\hat{\imath}$ and $\hat{\jmath}$ to get 2 eauations in 2 unknowns; **Method 3)** dot both sides with a vector orthogonal to $r_{AP}$ to get one equation in $F_B$, similarly dot with a vector orthogonal to $\vec{r}_{BP}$ to get one equation in $F_A$, **Method 4)** cross both sides with $\vec{r}_{AP}$ to get one equation in $F_B$, similarly cross with $r_{BP}$ to get $F_A$. See section 2.4 starting on page 64 for more discussion about how to solve vector equations.

## 1D statics of a particle

Let's call the one dimension of interest the $x$ direction. The key governing equation is

$$\sum F_x = 0.$$

You could call the special direction $y$, $z$, $x'$ or $s$ if you like and then use, say $\sum F_z = 0$. The next two simple examples pretty much cover 1D particle statics.

Example: **Balance of two forces**
For the particle in Fig. 4.1, force balance gives

$$\sum \vec{F}_i = \vec{0} \quad \Rightarrow \quad 10\,\text{N}\hat{\imath} - F\hat{\imath} = \vec{0}.$$

Either by equating $x$ components of both sides, or equivalently dotting both sides with $\hat{\imath}$, we get $F = 10\,\text{N}$. Or, we could have just done scalar mechanics,

$$\sum F_x = 0 \quad \Rightarrow \quad 10\,\text{N} - F = 0 \quad \Rightarrow \quad F = 10\,\text{N}.$$

Most often we have to contend with forces which don't show up until we draw a free body diagram.

**Example.** Force pulling on a string. For the particle in Fig. 4.2 the quantity of interest, the tension in the cable, doesn't show in the sketch. We need to draw a free body diagram of the particle which means cutting the string. This FBD is shown in Fig. 4.1, where $F = T_{AB}$ represents the tension in cable AB. So force balance gives $T_{AB} = F = 10\,\text{N}$.

## 2D statics of a particle

The situation is less trivial when we go to 2D.

**Example.** A 100 pound weight hangs from 2 lines in Fig. 4.3. We cut the strings, draw a free body diagram and add the forces to get

$$\sum \vec{F}_i = \vec{0} \quad \Rightarrow \quad 445\,\text{N}(-\hat{\jmath}) + \underbrace{F_A \frac{(\hat{\imath}+\hat{\jmath})}{\sqrt{2}}}_{\vec{F}_A} + \underbrace{F_B(-\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath})}_{\vec{F}_B} = \vec{0}.$$

This can be solved various ways * to get $F_A = 230.3\,\text{N}$ and $F_B = 325.8\,\text{N}$.

Although moment balance is technically superfluous in particle mechanics, when the forces are concurrent moment balance can be used as a shortcut.

Example: **Weight hanging from 2 strings**
Consider again Fig. 4.3. Moment balance about point A gives

$$\sum \vec{M}_A = \vec{0} \quad \Rightarrow \quad \vec{r}_{P/A} \times \big(445\,\text{N}(-\hat{\jmath})\big) + \vec{r}_{P/A} \times \left( F_B(-\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath}) \right) + \vec{0} = \vec{0}.$$

Evaluating the cross products one way or another one again gives $F_B = 325.8\,\text{N}$. Similarly moment balance about B gives $F_A = 230.3\,\text{N}$.

Whether force or moment balance is used, for concurrent force systems we only have two independent scalar equilibrium equations in 2D, and three in 3D .

## Frictionless contact

As discussed in Chapter 3.1, engineered parts which slide often have bearings or lubrication which minimize the sliding resistance. To simplify analyses, that remaining resistance is often neglected and we model the contact as 'frictionless' ($\mu = 0$). This means the interaction force is normal to the contacting surfaces.

Example: **Pull a wagon uphill**
See Fig. 4.4. From the free body diagram we have

$$\sum \vec{F}_i = \vec{0} \quad \Rightarrow \quad -1000\,\text{N}\hat{\jmath} + N\hat{e}_2 + T_{\text{AB}}\hat{e}_1 = \vec{0}. \tag{4.1}$$

where $\hat{e}_1 = \cos(30^\circ)\hat{\imath} + \sin(30^\circ)\hat{\jmath}$ and $\hat{e}_2 = -\sin(30^\circ)\hat{\imath} + \cos(30^\circ)\hat{\jmath}$. $N$ and $T_{\text{AB}}$ are unkown forces. Here are two ways to solve for the unknowns.
**Method I.** Substitute the expressions for $\hat{e}_1$ and $\hat{e}_2$ above into eqn. (4.1), extract $x$ and $y$ components to get 2 equations in two unknowns which you can solve to get $T_{\text{AB}} = 500\,\text{N}$ and $N = 500\sqrt{3}\,\text{N}$ (note the font confusion that the force quantity $N$ and unit N have different meanings).
**Method II.** Using judiciously chosen dot products can simplify the algebra. Take the dot products of both sides of eqn. (4.1) with $\hat{e}_1$ and then with $\hat{e}_2$, to get two scalar equations. Dotting eqn. (4.1) with $\hat{e}_1$ eliminates terms orthogonal to $\hat{e}_1$, namely $N\hat{e}_2$. And dotting eqn. (4.1) with $\hat{e}_2$ 'kills' the $T_{\text{AB}}\hat{e}_1$ term. So the two equations each have only one unknown. See page 69 for more discussion of this method.

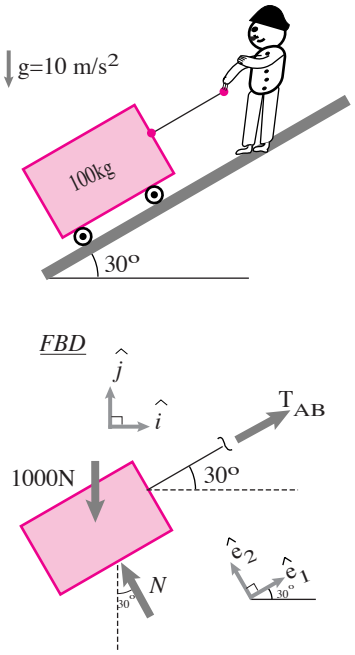



Figure 4.4: A wagon pulled uphill and associated free body diagram. Because we are doing particle mechanics we combine the two forces on the wheels into the single resultant $N$. Further, for particle mechanics we need not worry about where that $N$ is applied.
Filename:tfigure-particle-wagon

## Three dimensional particle mechanics

The basic idea is the same in 3D as in 2D.

Example: **One unknown force.**
Assume 3 known forces and one unknown force $\vec{F}$ are acting on a particle (Fig. 4.5). Then from force balance

$$\begin{aligned} \vec{0} &= \sum \vec{F}_i \\ \Rightarrow \quad \vec{0} &= (36\,\text{lbf}\hat{\imath} - 16\,\text{lbf}\hat{\jmath}) + (-52\,\text{lbf}\hat{k} + 5\,\text{lbf}\hat{\imath}) \\ &\quad +(-42\,\text{lbf}\hat{k} + 20\,\text{lbf}\hat{\imath} - 16\,\text{lbf}\hat{\jmath}) + \vec{F} \\ \Rightarrow \quad \vec{F} &= \left(-61\hat{\imath} + 32\hat{\jmath} + 94\hat{\jmath}\right)\,\text{lbf}. \end{aligned}$$

.

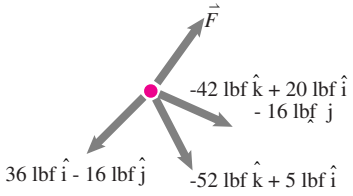



Figure 4.5: One unknown force $\vec{F}$.
Filename:tfigure-particle-trivial

The new difficulties in 3D particle mechanics are

- Visualization in 3D. (So practice making and reading 3D drawings.); and
- The vector force-balance equation is 3D and thus equivalent to 3 scalar equations. Solving these is at the upper boundary of what most people can do reliably by hand or even with a non-programmable calculator. So tricks to reduce the complexity of the solution are useful as is the ability to set up the resulting equations on a computer or programmable calculator.

Hint: If the direction of a force is given (possibly implicitly) express the force as a scalar times a unit vector: $\vec{F} = F\hat{\boldsymbol{\lambda}}$. (See top row, middle column of Fig. 3.4 on page 126.)

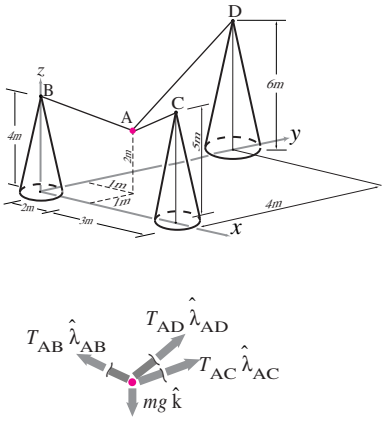



Figure 4.6: Mass A suspended by 3 ropes and a free body diagram of the mass.

Filename:tfigure-particle-3towers

Example: **Particle held by 3 ropes.**
Say $m = 100\,\text{kg}$ and $g = 10\,\text{N/kg}$ in Fig. 4.6. Force balance gives

$$\sum \vec{F}_i = \vec{0} \quad \Rightarrow \quad T_{AB}\hat{\lambda}_{AB} + T_{AC}\hat{\lambda}_{AC} + T_{AD}\hat{\lambda}_{AD} - mg\hat{k} = \vec{0} \tag{4.2}$$

which is a 3D vector equation in 3 unknowns (3=3, good). The $\hat{\lambda}$'s in eqn. (4.6) are known because, for example,

$$\hat{\lambda}_{AB} = \frac{\vec{r}_{AB}}{|\vec{r}_{AB}|} = \frac{\vec{r}_B - \vec{r}_A}{|\vec{r}_B - \vec{r}_A|}$$

and the position vectors are given in the picture. To get to a numerical answer for the the tensions you can use many methods such as (see Sample 4.4 on page 170)

1. Brute force by hand.
2. Systematically set up matrix equations for solution by some means.
3. Set up and solve equations on a computer.
4. Use a tricky dot product to extract one equation in one unknown.
5. Use moment about an axis to extract one equation in one unknown.

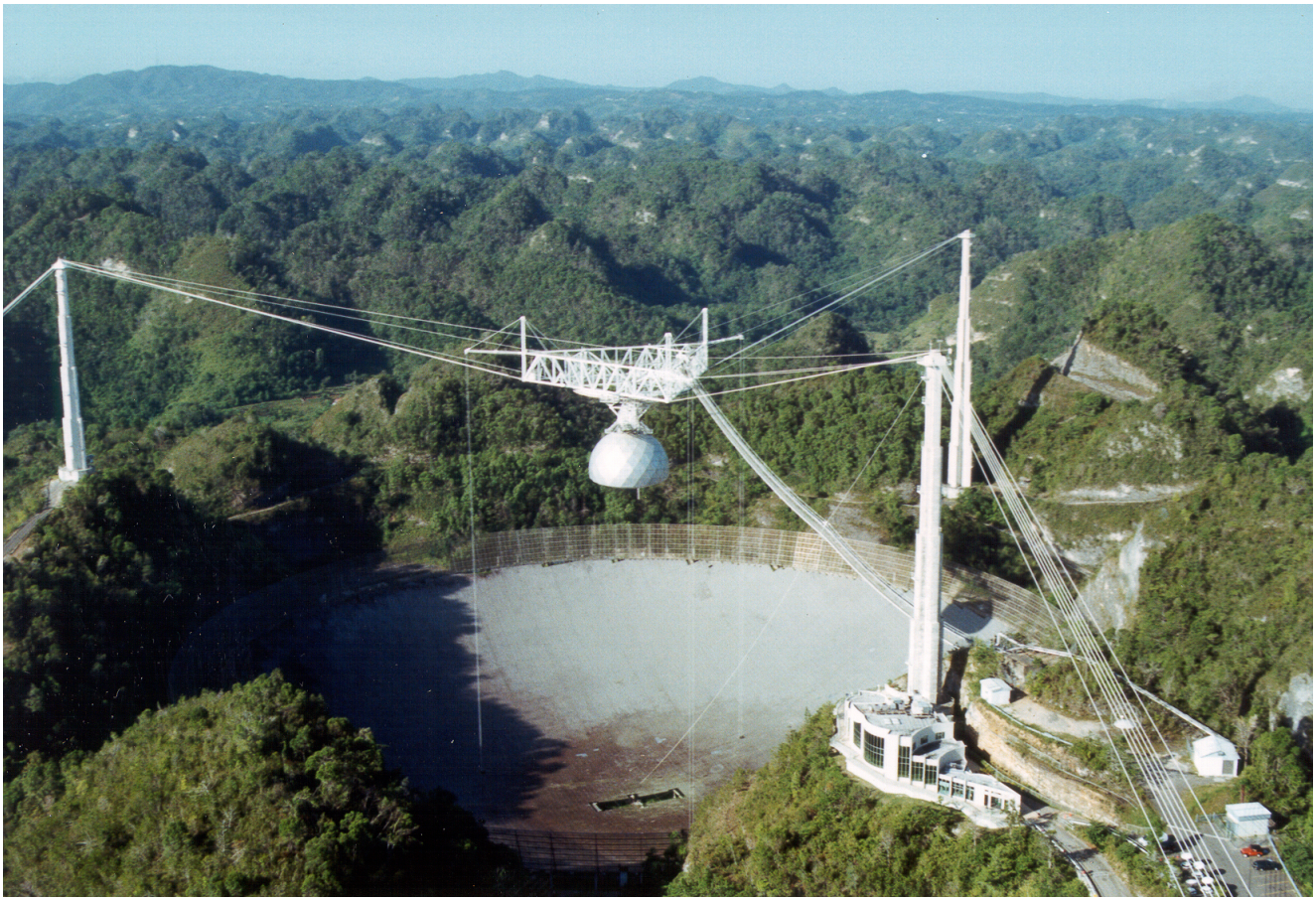


Figure 4.7: "Suspended 450 feet above the reflector is the 900 ton platform. Similar in design to a bridge, it hangs in midair on eighteen cables, which are strung from three reinforced concrete towers. One is 365 feet high, and the other two are 265 feet high." *Courtesy of the NAIC-Arecibo Observatory, a facility of the NSF.*

Filename:tfigure-arecibo

*With no string pulling it down*
*the wind would not hold a kite up*

## 4.1 *THEORY*
### *The simplification of dynamics to statics*

The bit of theory here is for people interested in the appropriateness of the statics model for systems that move. It will not help you with learning statics skills or doing statics homework problems.

The mechanics equations in the front inside cover are accurate enough for everything that 99.99% of engineers will ever encounter. The statics subset covers special case that apply less exactly to many things. But exactly enough for, say, 90% of the engineering mechanics calculations in the world. In statics we set the right hand sides of equations I and II to zero. We think that these terms are small enough, compared to other included terms, that they can be counted as zero. The neglected terms involve mass times acceleration and are called the *inertial terms*. Thus we replace the linear and angular momentum balance equations with their simplified statics forms

$$\sum_{\substack{\text{All external}\\ \text{forces}}} \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}} \qquad \text{and} \qquad \sum_{\substack{\text{All external}\\ \text{torques}}} \vec{\boldsymbol{M}}_{\mathrm{C}} = \vec{\boldsymbol{0}} \qquad \text{(Ic,IIc).}$$

which are sometimes called the *force balance* and *moment balance* equations and together are called the *equilibrium* equations. The forces to be summed (added) are the ones you see on a free body diagram. The torques that are summed are those due to the same forces (by means of $\vec{\boldsymbol{r}}_{\mathrm{i/C}} \times \vec{\boldsymbol{F}}_{\mathrm{i}}$) plus applied couples (force systems with zero resultant that have been replaced on the FBD with equivalent couples). The approximating assumption ('the model', see page 11) of an object being in static equilibrium is that the forces mediated by an object are much larger than the forces needed to accelerate it.

Detailed estimation of the errors from neglecting dynamics terms is a dynamics problem, so we can't fully address it here. But you can do a rough check by making sure that the mass times acceleration is a small fraction of typical forces you find from your statics analysis. This inertial term comes from the total of all the forces. So the approximation in statics is that the total of the forces is much less than any of the individual forces. If

$$\sum_{\substack{\text{All forces}\\ \text{on the part}}} \vec{\boldsymbol{F}}_i \ll F_{\text{typical}}$$

then statics is probably a good model; the forces are more canceling each other out, balancing each other, than causing acceleration. You can then figure out how these forces cancel each other, that is, you can do statics.

The statics equations are often accurate-enough for engineering purposes for

- Things that a normal person would call "still" such as a building or bridge on a calm day, and a sleeping person;
- Things that move with little acceleration, such as a tractor plowing a field and most of the parts in a smooth-flying airplane; and
- Parts that mediate the forces needed to accelerate more massive parts, such as gears in a transmission, the rear wheel of an accelerating bicycle, the strut in the landing gear of an airplane, and the individual structural members of a building swaying in an earthquake.

If your statics calculations make a bad prediction one of the possible errors is your neglect of dynamic terms. If the machine or structure seems relatively still it is more likely, however, that inaccuracies in your statics calculation come from in-accurate estimates of material properties (friction coefficient, failure strength, etc) or from mis-estimation of geometric features (a dimension, clearance, angle, etc).

## 4.2 *THEORY*
### *Existence and uniqueness*

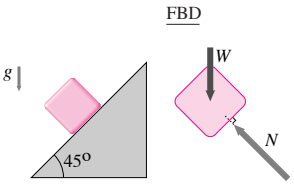
This is a mathematical aside for those interested in fine points.

Sometimes equations have no solutions and solutions are said to *not exist*. For example there *doesn't exist* a solution to the equations $x + 2y = 7, 2x + 4y = 15$ (subtracting twice the first equation from the second shows the contradiction that $0 = 1$). Sometimes equations have more than one solution and the solutions are said to be *non-unique*. For example the equation $x + y = 1$ has many solutions including $(x, y) = (1, 0)$ and $(x, y) = (0, 1)$ and $(x, y) = (10, -9)$ *etc.*

Although the words *existence* and *uniqueness* have a mathematically abstract irrelevant-to-the-real-world ring to them, they are relevant to real-world mechanics.

**Sometimes 'statics' problems have no solution** You could, conceivably, be given (in a class or in engineering practice) an *il-posed* problem.

Example: **A block on a frictionless sloped ramp.**
Using statics find the normal force for the block on the slippery ramp below.



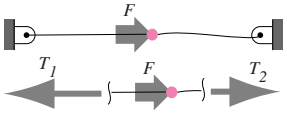

At a glance you can see that this is not a statics problem so there is no way to use statics to get a solution. Lacking this intuition we could look to the equations: force balance shows that there is no value of $N$ that can make the force vectors add to zero.

Of course when you run into such contradictions the setup will be more subtle. In the chapter on trusses there are a few examples where there is no solution which are beyond *a priori* expectation even with expert intuition.

Issues of existence, like for the example above, are not exceptional in engineering practice, but they are not common.

**Sometimes statics problems have more than one solution** In contrast to the relatively rare 'existence' issues above, issues of uniqueness are extremely common in the practice of statics. Perhaps annoyingly common.

Example: **Particle held by two strings.**
Find the tension in the strings to the sides of the point of application of a given load $F$.




Force balance along the strings gives us one equation for the two unknown tensions.

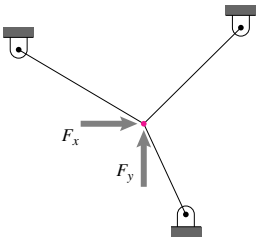
$$\left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{\imath} \quad \Rightarrow \quad -T_1 + F + T_2 = 0$$

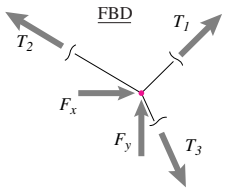
No other force balance or moment balance equation gives more information. For any given $F$ this equation has many solutions. The pair $(T_1, T_2)$ could be $(F, 0)$ or $(0, -F)$ $(2F, F)$ $(F + 7\,\text{N}, 7\,\text{N})$, etc.

Of course if you tie strings together like this and apply a force there is some actual tension in each string; reality, at any instant in time, is unique (as far as we know). For example, if you had tied the strings loosely together the right string gets slack and has $T_2 = 0$ and thus $T_1 = F$. But it takes an extra assumption of this nature to get a unique solution.

And just because you can make an assumption that leads you to a unique solution doesn't mean that assumption corresponds well to reality. You might assume your friend had tied the strings together loosely (and thus calculate $T_2 = 0$ and $T_1 = F$) when really she had really tied them together tightly (and so really, say, $T_2 = 30\,\text{N}$ and $T_1 = F + 30\,\text{N}$) Here is the same idea in 2D.

Example: **Particle held by three strings.**




Assume that $F_x$ and $F_y$ are given. What are the three tensions. Planar force balance gives two equations for the 3 unknown tensions. As in the previous example these equations have many solutions.

If you assume a) that one string goes slack and b) that no string can carry compression, the problem above has a unique answer. But, if this is a model of a real situation, you would have to know that the strings were not tied tightly at the start.

One could also make an example with 4 strings holding a particle in 3D. All of these examples allow a 'one parameter family of solutions'; specifying one number (say that the tension in cable 2 is zero) determines the other tensions. We could have more non-uniquenes than that by holding a particle with 3 strings in 2D, 4 strings in 2D, or 5 strings in 3D. And more non-uniqueness than even that is possible with even more strings. Sometimes the problem can lead to a unique solution with a simple reasonable physical assumption, and sometimes not (in which case you have to know details about the deformation properties of the objects to find a unique and accurate answer).

**Counting equations and unknowns** All of the examples above could be picked out as problematic by counting equations and unknowns. For the block on the ramp we had two equations for the one unknown $N$. Where-as for the string problems we had more unknowns than equations.

If you have more equations than unknowns existence is likely to be an issue; you probably can't find any solutions. If you have more unknowns than equations than uniqueness *is* and issue; any solution you find will be non-unique. But there are 'exceptional' cases for which equation counting does not tell all about existence and uniqueness, as discussed in detail in the advanced truss section 5.5 (see the lower right corner of the $2 \times 2$ tables there).

**SAMPLE 4.1 Equilibrium of a pin.** Two rods, AB and BC, are pinned together at point B and to the ground as shown in the figure. A force $F = 100$ N is applied at point B. Given that $\theta = 45^\circ$, find the tension in the two rods.

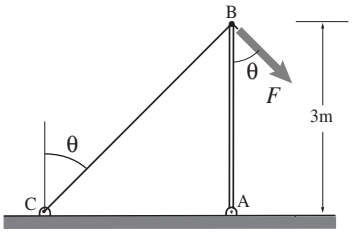



Figure 4.8:
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**Solution** The free-body diagram of the pin at B is shown in Fig. 4.9 where $T_1$ and $T_2$ are the tensions in rod AB and BC respectively. The static equilibrium of the pin at B requires that

$$\vec{F} + \vec{T}_1 + \vec{T}_2 = \vec{0}.$$

Dotting both sides of this equation with $\hat{\imath}$ and $\hat{\jmath}$ separately, we get the scalar equilibrium equations in the $x$ and $y$ directions:

$$\begin{aligned} F\sin\theta - T_2\sin\theta &= 0 \\ -F\cos\theta - T_1 - T_2\cos\theta &= 0. \end{aligned}$$

Solving these two equations simultaneously, we get

$$\begin{aligned} T_2 &= F \\ T_1 &= -(F + T_2)\cos\theta \\ &= -2F\cos\theta. \end{aligned}$$

Substituting the values of $\theta = 45^\circ$ and $F = 100\,\text{N}$, we get

$$\begin{aligned} T_1 &= -173.2\,\text{N} \\ T_2 &= 100\,\text{N}. \end{aligned}$$

$$\boxed{T_1 = -173.2\,\text{N}, \qquad T_2 = 100\,\text{N}}$$

Note that the tension in rod AB is negative, that is, rod AB is in *compression*. This is expected since the other two forces at B, $F$ and $T_2$, are pushing on rod AB. The equality of $F$ and $T_2$ is also expected from symmetry.

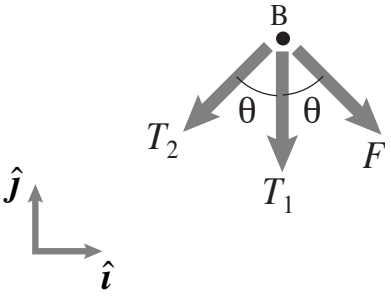



Figure 4.9: The free body diagram of the pin at B. Although we can see that rod AB is in compression, we use tension as positive and let the solution take care of the proper sign.
Filename:sfig4-1-new0a

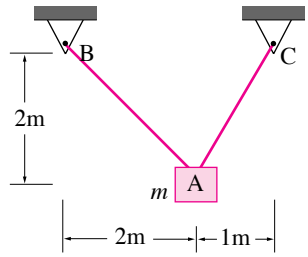



Figure 4.10:
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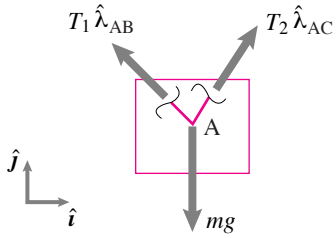



Figure 4.11:
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**SAMPLE 4.2** A 10 kg block $m$ hangs from strings $AB$ and $AC$ in the vertical plane as shown in the figure. Find the tension in the strings.

**Solution** The free body diagram of the block is shown in figure 4.11. The equation of force balance, $\sum \vec{F} = \vec{0}$, gives

$$\text{or} \qquad T_1\hat{\lambda}_{\text{AB}} + T_2\hat{\lambda}_{\text{AC}} - mg\,\hat{j} = \vec{0}, \tag{4.3}$$

where $\hat{\lambda}_{\text{AB}}$ and $\hat{\lambda}_{BC}$ are unit vectors in the AB and AC directions:

$$\begin{aligned}\hat{\lambda}_{AB} &= \frac{\vec{r}_{AB}}{|\vec{r}_{AB}|} = \frac{-2\,\text{m}\hat{i} + 2\,\text{m}\hat{j}}{2\sqrt{2}\,\text{m}} = \frac{1}{\sqrt{2}}(-\hat{i} + \hat{j})\\ \hat{\lambda}_{\text{AC}} &= \frac{\vec{r}_{AC}}{|\vec{r}_{AC}|} = \frac{1\,\text{m}\hat{i} + 2\,\text{m}\hat{j}}{\sqrt{5}\,\text{m}} = \frac{1}{\sqrt{5}}(\hat{i} + 2\hat{j}).\end{aligned}$$

Substituting in eqn. (4.3) and rearranging terms, we have

$$\left(-\frac{T_1}{\sqrt{2}} + \frac{T_2}{\sqrt{5}}\right)\hat{i} + \left(\frac{T_1}{\sqrt{2}} + \frac{2T_2}{\sqrt{5}} - mg\right)\hat{j} = \vec{0}.$$

Separating $x$ and $y$ components of this equation, we get the scalar equations

$$\begin{aligned}-\frac{T_1}{\sqrt{2}} + \frac{T_2}{\sqrt{5}} &= 0 \qquad (\textstyle\sum F_x = 0)\\ \frac{T_1}{\sqrt{2}} + \frac{2T_2}{\sqrt{5}} - mg &= 0 \qquad (\textstyle\sum F_y = 0).\end{aligned}$$

Solving these two equations simultaneously we get,

$$T_1 = \frac{\sqrt{2}}{3}mg = 46.24\,\text{N} \quad \text{and} \quad T_2 = \frac{\sqrt{5}}{3}mg = 73.12\,\text{N}.$$

$$\boxed{T_1 = 46.24\,\text{N},\ T_2 = 73.12\,\text{N}}$$

**Note:**

**Scalar equations** Separating the scalar equations in the $x$ and $y$ directions is equivalent to dotting eqn. (4.3) with $\hat{i}$ and $\hat{j}$ respectively, which gives

$$\begin{aligned}T_1(\underbrace{\hat{\lambda}_{\text{AB}}\cdot\hat{i}}_{-1/\sqrt{2}}) + T_2(\underbrace{\hat{\lambda}_{\text{AC}}\cdot\hat{i}}_{1/\sqrt{5}}) &= 0\\ \underbrace{\sqrt{\frac{2}{5}}\,T_2}_{T_1}(\underbrace{\hat{\lambda}_{\text{AB}}\cdot\hat{j}}_{1/\sqrt{2}}) + T_2(\underbrace{\hat{\lambda}_{\text{AC}}\cdot\hat{j}}_{2/\sqrt{5}}) - mg &= 0.\end{aligned}$$

**Matrix equation** The two scalar equations obtained from eqn. (4.3) can be written in the matrix form as

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} 0 \\ mg \end{pmatrix}.$$

Using $mg = (10\,\text{kg}) \cdot (9.81\,\text{m/s}^2) = 98.1\,\text{N}$, and solving the above matrix equation (see Sample 2.29 on page 79 and Sample 2.31 on page 81), we get

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} 46.24 \\ 73.12 \end{pmatrix} \text{N}$$

which is, of course, the same result as we got above.

**SAMPLE 4.3** A block of mass $m$ rests on a frictionless inclined plane with the help of a string that connects the mass to a fixed support at A. Find the force in the string.

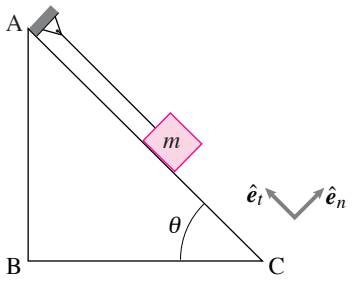



Figure 4.12: A mass-particle on an inclined plane.
Filename:sfig2-1-11

**Solution** The free-body diagram of the mass is shown in Fig. 4.13. The string force $F_s$ and the normal reaction of the plane $N$ are unknown forces. The force balance equation, $\sum \vec{F} = \vec{0}$, is

$$\vec{F}_s + \vec{N} + m\vec{g} = \vec{0}.$$

We can express the forces in terms of their components in various ways and then dot the vector equation with appropriate unit vectors to get two independent scalar equations. For example, we write the force balance equation using mixed basis vectors $\hat{e}_t$ and $\hat{e}_n$, and $\hat{\imath}$ and $\hat{\jmath}$:

$$F_s\hat{e}_t + N\hat{e}_n - mg\hat{\jmath} = \vec{0}. \tag{4.4}$$

We can now find $F_s$ directly by taking the dot product of the above equation with $\hat{e}_t$ since the other unknown $N$ is in the $\hat{e}_n$ direction and $\hat{e}_n \cdot \hat{e}_t = 0$:

$$\{\text{eqn. (4.4)}\} \cdot \hat{e}_t \quad \Rightarrow \quad F_s - mg\overbrace{(\hat{\jmath} \cdot \hat{e}_t)}^{\sin\theta} = 0 \quad \Rightarrow \quad F_s = mg\sin\theta.$$

$$\boxed{F_s = mg\sin\theta}$$

• • •

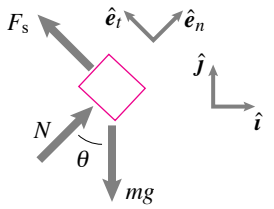



Figure 4.13: Free-body diagram of the mass and the geometry of force vectors.
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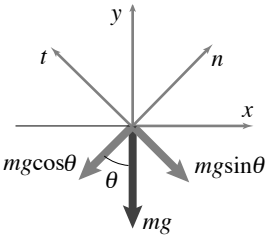
**Note:** We can find $N$ from a single equation by taking the dot product of eqn. (4.4) with $\hat{n}$:

$$\{\text{eqn. (4.4)}\} \cdot \hat{e}_n \quad \Rightarrow \quad N - mg\overbrace{(\hat{\jmath} \cdot \hat{e}_n)}^{\cos\theta} = 0 \quad \Rightarrow \quad N = mg\cos\theta.$$

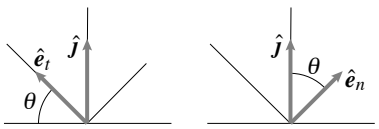
**Scalar approach:** We resolve all forces into their $\hat{e}_t$ and $\hat{e}_n$ components and then sum the forces. Here, $F_s$ is along the plane and therefore, has no component perpendicular to the plane. Force $N$ is perpendicular to the plane and therefore, has no component along the plane. We resolve the weight $mg$ into two components: (1) $mg\cos\theta$ perpendicular to the plane (along $\hat{e}_n$ ) and (2) $mg\sin\theta$ along the plane (along $\hat{e}_t$). Now we can sum the forces:

$$\sum F_t = 0 \quad \Rightarrow \quad F_s - mg\sin\theta = 0;$$
$$\text{and} \quad \sum F_n = 0 \quad \Rightarrow \quad N - mg\cos\theta = 0$$

which, of course, is essentially the same as the equations obtained above.




(a)




(b)

Figure 4.14: (a) Components of $mg$ along $t$ and $n$ directions. (b) The mixed basis dot products: $\hat{\jmath} \cdot \hat{e}_t = \sin\theta$ and $\hat{\jmath} \cdot \hat{e}_n = \cos\theta$
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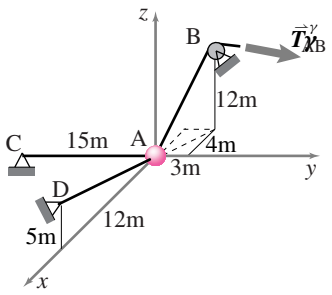



Figure 4.15:

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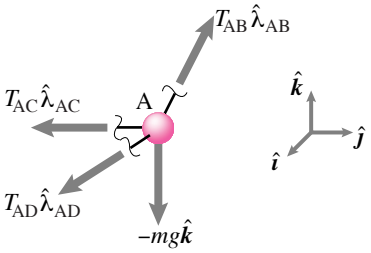



Figure 4.16:

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**SAMPLE 4.4 A particle in 3D.** A particle of mass 1 kg is attached to two strings tied at points C and D shown in the figure. Another string, AB, attached to the particle, passes over a pulley and is used to hold the particle in equilibrium under gravity such that it loses contact with the ground at point A. Find the tension in string AB.

**Solution** The free-body diagram of the particle is shown in Fig. 4.16. Assuming the tensions in strings AB, AC, and AD to be $T_{AB}$, $T_{AC}$, and $T_{AD}$ respectively, we can represent the string forces acting on the particle as $T_{AB}\hat{\boldsymbol{\lambda}}_{AB}$, $T_{AC}\hat{\boldsymbol{\lambda}}_{AC}$, and $T_{AD}\hat{\boldsymbol{\lambda}}_{AD}$, where the $\hat{\boldsymbol{\lambda}}$'s are the unit vectors along the strings.

The force balance on the particle gives us

$$T_{AB}\hat{\boldsymbol{\lambda}}_{AB} + T_{AC}\hat{\boldsymbol{\lambda}}_{AC} + T_{AD}\hat{\boldsymbol{\lambda}}_{AD} - mg\hat{\boldsymbol{k}} = 0. \tag{4.5}$$

This is the equation we need to solve to find $T_{AB}$. We show various methods below that you can use to get $T_{AB}$.

1. **Brute force (by hand).**

   From the given figure, the unit vectors are:

$$\begin{aligned}
\hat{\boldsymbol{\lambda}}_{AB} &= \frac{-4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 12\hat{\boldsymbol{k}}}{\sqrt{4^2+3^2+12^2}} = -\frac{4}{13}\hat{\boldsymbol{i}} + \frac{3}{13}\hat{\boldsymbol{j}} + \frac{12}{13}\hat{\boldsymbol{k}} \\
\hat{\boldsymbol{\lambda}}_{AC} &= -\hat{\boldsymbol{j}} \\
\hat{\boldsymbol{\lambda}}_{AD} &= \frac{12\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{k}}}{\sqrt{12^2+5^2}} = \frac{12}{13}\hat{\boldsymbol{i}} + \frac{5}{13}\hat{\boldsymbol{k}}.
\end{aligned}$$

   Substituting these vectors in eqn. (4.5) and equating the $x$, $y$ and $z$ components of the equation to zero separately, we get

$$\begin{aligned}
-\frac{4}{13}T_{AB} + \frac{12}{13}T_{AD} &= 0 \\
\frac{3}{13}T_{AB} - T_{AC} &= 0 \\
\frac{12}{13}T_{AB} + \frac{5}{13}T_{AD} &= mg.
\end{aligned} \tag{4.6}$$

   We can solve the three equations simultaneously to get

$$T_{AB} = \frac{39}{41}mg, \quad T_{AC} = \frac{9}{41}mg, \quad \text{and} \quad T_{AD} = \frac{13}{41}mg.$$

   Substituting $m = 1\,\text{kg}$ and $g = 9.81\,\text{m/s}^2$, we get the required values.

$$\boxed{T_{AB} = 9.33\,\text{N}, \quad T_{AC} = 2.15\,\text{N}, \quad T_{AD} = 3.11\,\text{N}}$$

2. **Systematically set up matrix equations.** Eqn. 4.5 can be written in matrix form as

$$\underbrace{\begin{bmatrix} \left[\hat{\boldsymbol{\lambda}}_{AB}\right]'_{xyz} & \left[\hat{\boldsymbol{\lambda}}_{AC}\right]'_{xyz} & \left[\hat{\boldsymbol{\lambda}}_{AD}\right]'_{xyz} \end{bmatrix}}_{3\times 3\text{ matrix}} \begin{bmatrix} T_{AB} \\ T_{AB} \\ T_{AB} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix}$$

   where $\left[\hat{\boldsymbol{\lambda}}_{AB}\right]'_{xyz}$ is a column of 3 numbers, namely the $x$, $y$, and $z$ components of $\hat{\boldsymbol{\lambda}}_{AB}$; similarly for the other two columns of the $3 \times 3$ matrix. This matrix equation is

then ready to hand to a calculator or computer for a matrix solution. Thus, eqn. (4.6) can be written as,

$$\begin{bmatrix} -4/13 & 0 & 12/13 \\ 3/13 & -1 & 0 \\ 12/13 & 0 & 5/13 \end{bmatrix} \begin{pmatrix} T_{\rm AB} \\ T_{\rm AC} \\ T_{\rm AD} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ mg \end{pmatrix}.$$

Using the pseudo code shown on the side * we solve the equations on a computer and get,

```
T = [9.33   2.15   3.11]
```

which is the solution that we obtained above by hand calculation.

3. **Computer solution.** All the math can be handed to a computer by a sequence of commands like this, working from the knowns to the unknowns (see page xiii), all in consistent units:

```
% Get all the knowns into the computer
  rA = [0 0 0]' ;   rB = [-4 3 12]'
  rC = [0 -15 0]' ;   rD = [12 0 5]'
  m = 1 ;  g = 9.81 ;
% Make relative position vectors
  rAB = rB- rA;   rAC = rC - rA;  rAD = rD - rA
% Make unit vectors
  lamdaAB = rAB/magnitude(rAB);
  lamdaAC = rAC/magnitude(rAC);
  lamdaAD = rAD/magnitude(rAD);
% Set up and solve the matrix equation
  M = [lamdaAB  lamdaAC  lamdaAD]  %3x3 matrix
  F = [0 0 mg]'   %column vector of known force
  solve {M T = F} for T
```

The column of numbers `T` will be the tensions in the 3 cables. Using this pseudo-code on a computer, we get `T = [9.33  2.15  3.11]` again.

4. **Be tricky to get one equation in one unknown.** Since we are interested only in $T_{\rm AB}$, we can get rid of the terms we don't know or care about. * The vector $\vec{\boldsymbol{r}}_{\rm AC} \times \vec{\boldsymbol{r}}_{\rm AD}$ is orthogonal to both $\vec{\boldsymbol{r}}_{\rm AC}$ and $\vec{\boldsymbol{r}}_{\rm AD}$, so it is orthogonal to $\hat{\boldsymbol{\lambda}}_{\rm AC}$ and $\hat{\boldsymbol{\lambda}}_{\rm AD}$. So taking the dot product of both sides of eqn. (4.5) with $\vec{\boldsymbol{r}}_{\rm AC} \times \vec{\boldsymbol{r}}_{\rm AD}$, we get

$$\begin{aligned} \left(\vec{\boldsymbol{r}}_{\rm AC} \times \vec{\boldsymbol{r}}_{\rm AD}\right) \cdot \left\{ T_{\rm AB}\hat{\boldsymbol{\lambda}}_{\rm AB} + T_{\rm AC}\hat{\boldsymbol{\lambda}}_{\rm AC} + T_{\rm AD}\hat{\boldsymbol{\lambda}}_{\rm AD} - mg\hat{\boldsymbol{k}} \right\} &= \left(\vec{\boldsymbol{r}}_{\rm AC} \times \vec{\boldsymbol{r}}_{\rm AD}\right) \cdot \vec{\boldsymbol{0}} \\ \left( \left(\vec{\boldsymbol{r}}_{\rm AC} \times \vec{\boldsymbol{r}}_{\rm AD}\right) \cdot \hat{\boldsymbol{\lambda}}_{\rm AB} \right) T_{\rm AB} &= mg \left( \left(\vec{\boldsymbol{r}}_{\rm AC} \times \vec{\boldsymbol{r}}_{\rm AD}\right) \cdot \hat{\boldsymbol{k}} \right) \\ T_{\rm AB} &= \frac{mg\left(\vec{\boldsymbol{r}}_{\rm AC} \times \vec{\boldsymbol{r}}_{\rm AD}\right) \cdot \hat{\boldsymbol{k}}}{\left(\vec{\boldsymbol{r}}_{\rm AC} \times \vec{\boldsymbol{r}}_{\rm AD}\right) \cdot \hat{\boldsymbol{\lambda}}_{\rm AB}}. \end{aligned}$$

Since, $\vec{\boldsymbol{r}}_{\rm AC} \times \vec{\boldsymbol{r}}_{\rm AD} = (-15\hat{\boldsymbol{\jmath}})\,\text{m} \times (12\hat{\boldsymbol{\imath}} + 5\hat{\boldsymbol{k}})\,\text{m} = (180\hat{\boldsymbol{k}} - 75\hat{\boldsymbol{\imath}})\,\text{m}^2$, substituting this cross product and other known quantities, we get

$$T_{\rm AB} = \frac{mg \cdot 180}{180 \cdot 12/13 + 75 \cdot 4/13} = 0.95\,mg = 9.33\,\text{N}.$$

$$\boxed{T_{\rm AB} = 9.33\,\text{N}}$$

* Pseudo-code:

```
Let m=1, 9=9.81
A = [ -4/13 0 12/13
       3/13 -1 0
       12/13 0 5/13 ]
b = [ 0 0 m*g]'
solve A*T = b for T
```

* Another way of doing this is by taking **Moment about an axis.** This approach is similar in spirit to the previous approach. Instead of the equilibrium eqn. (4.5) we could have used moments about axis CD to 'kill off' the tensions in ropes AC and AD (they have no moment about that axis), like this,

$$\sum M_{\text{axis CD}} = 0$$

$$\vec{\boldsymbol{r}}_{\rm CD} \cdot \left\{ \vec{\boldsymbol{r}}_{\rm DA} \times \left\{ T_{\rm AB}\hat{\boldsymbol{\lambda}}_{\rm AB} - mg\hat{\boldsymbol{k}} \right\} \right\} = 0$$

$$T_{\rm AB} = \frac{mg\,\vec{\boldsymbol{r}}_{\rm CD} \cdot \left(\vec{\boldsymbol{r}}_{\rm DA} \times \hat{\boldsymbol{k}}\right)}{\vec{\boldsymbol{r}}_{\rm CD} \cdot \left(\vec{\boldsymbol{r}}_{\rm DA} \times \hat{\boldsymbol{\lambda}}_{\rm AB}\right)}.$$

Again we have found one equation for one unknown, $T_{\rm AB}$. All the quantities on the right can be evaluated give $T_{\rm AB}$.

## 4.2 Equilibrium of one object

For particle statics we used that the forces acting on an object in equilibrium have no net push or pull; the forces add to zero. Now we will use that the forces have no tendency to cause rotation; the moments add to zero. These are not two in a long list of facts about equilibrium, but the whole story. As stated on the inside front cover and this chapter's introduction (page 160)

An object is in static equilibrium if and only if the force balance and moment balance equations hold.

$$\underbrace{\sum_{\substack{\text{All external}\\ \text{forces}}} \vec{F}_i = \vec{0}}_{\text{force balance}} \quad \text{and} \quad \underbrace{\sum_{\substack{\text{All external}\\ \text{torques}}} \vec{M}_{i/C} = \vec{0}}_{\text{moment balance}} \qquad \text{(Ic,IIc)}$$

The total force system acting on the object is then equivalent to a zero force and zero moment acting at C.

By supplementing the force-balance equation with moment balance we can determine more about the forces that act on an object.

### Rigid-body statics

To start with one often thinks of the object of interest is one piece, for example a whole car, a wheel, a person, a limb, a chair or a derrick. We often think of such an object as rigid, meaning that the object's shape and size only change negligibly due to the forces of interest. Thus the phrase *rigid-body mechanics*. Actually, however, the equilibrium equations, force and moment balance, apply just as well things to all things with little acceleration, whether or not they are stiff and solid. For a first pass at the subject, one thinks of applying the principles of statics to single rather-solid simply-defined objects. And such will be our main initial concern in this section. But really the delineation of an 'object' is up to you. And in later chapters we apply the same statics equations to clearly-non-rigid systems like water and rope. For statics the only concern is the delineation of the system at the instant of interest.

Once you know its shape, whether an object is rigid or not is irrelevant for statics.

## The reference point C in moment balance.

The moment balance equation is calculated by calculating the moments of forces relative to a point C using

$$\vec{M}_{\mathrm{i}} = \vec{r}_{\mathrm{i/C}} \times \vec{F}_i .$$

C is any convenient point, possibly the origin O of your coordinate system. C is not a special point. As discussed in Section 2.5 if a force system is equivalent to zero force and zero couple at C it is equivalent to a zero force and zero couple at any and every point D, E, Q, *etc*.

**Example.** As you sit still reading, gravity is pulling you down and forces from the floor on your feet, the chair on your seat, and the table on your elbows hold you up. All of these forces add to zero. The net moment of these forces about the front-left corner of your desk adds to zero. And the net moment of these forces about the mole near your left elbow is also zero.

The freedom to use any point you like for moment balance provides and oft-used shortcut.

## Number of equations and number of unknowns

In two dimensions the equilibrium equations make up 3 independent scalar equations. These could be:

- 2 components of force balance and the one non-trivial component of moment balance ; or
- moment balance about any two points and force balance in any direction (except in the direction orthogonal to the line connecting the two moment-balance points).
- moment balance about 3 points (any three points not on a straight line suffice, see box **??**); or

Note that moment balance necessarily is part of the equilibrium equations, but that force balance can be finessed. With one 2D free body diagram the equilibrium equations can be solved to find three unknown scalars, for example,

- The magnitudes of three forces whose directions are known *a priori*; or
- One unknown force vector (two components, or angle and magnitude) and one unknown magnitude; or
- Some other list of three scalars associated with the forces on the free body diagram. Besides force components and magnitudes these could include a force angle $\theta$, a friction coefficient $\mu$, or the location of force application.

Once you have three independent equations any additional equations you write, say moment about still another point, contains no new information*. In some problems the forces shown on a free body diagram automatically satisfy one or more of the equilibrium equations; in making the drawing you may have implicitly solved some equilibrium equations. The equilibrium

* A fourth equilibrium equation may superficially look different from an equation already written, but it could be derived from the other equations.

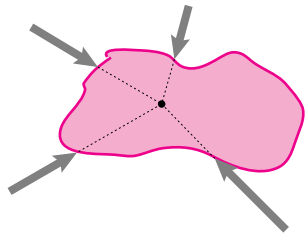


Figure 4.17: A set of forces acting *concurrently* on an object. All force lines of action intersect at one point.

Filename:tfigure-particleequilib

* If forces are not concurrent the particle model may still be useful, as demonstrated in the previous section.

equations then offer less new information, and sometimes none at all (see 2-force bodies below).

In 3 dimensions the equilibrium equations make up 6 independent scalar equations. Most directly these are 3 components of force and 3 components of moment. But there are many combinations of equilibrium equations that yield 6 independent scalar equations.

## Special cases: concurrent forces, two-force bodies, three-force bodies

We now discuss some special loading situations for which there are special insights or problem-solution tricks. In principle you don't need to know any of them because force balance and moment balance spell out the whole statics story. In practice it is best to know these special cases.

### Concurrent forces

In the special case when the lines of actions of all applied forces intersect at one point, moment balance is trivially satisfied (because none of the forces has a moment about the intersection point). Such a system of forces is called *concurrent* (Fig. 4.17) and the particle model is particularly appropriate*. In such a case the 2D equilibrium equations only provide two independent scalar equations and one can only use them to solve for two unknown scalars. In 3D one gets three independent scalar equations for a concurrent force system.

## One-force body

Lets first treat "one-force" bodies. Consider a finite body with only one force acting on it. Assume it is in equilibrium. Force balance says that the sum of forces must be zero. So that force must be zero.

> If only one force is acting on a body in equilibrium that force is zero.

That was too easy. But a count to 3 wouldn't feel complete if it didn't start at 1.

## Two-force body

When only two forces act on an object the situation is also simplified, though not so drastically as the case with one force. An object with only two forces acting on it is called a *two-force body* or *two-force member*.

> If a body in static equilibrium is acted on by two forces, then those forces are equal in magnitude, opposite in direction, and have a common line of action (the line connecting the two points of application).

This result is shown in Fig. 4.18 and explained in box **??**. If you recognize a two-force body you can draw it in a free body diagram as in fig. 4.18c and the equations of force and moment balance provide no new information. The two-force-body shortcut is especially useful for systems with several parts some of which are two-force members. Springs, dashpots, struts, and strings are generally idealized as two-force bodies.

Example: **Tower and strut**
Consider an accelerating cart (Fig. 4.21) holding up massive tower $AB$ which is pinned at $A$ and braced by the light strut $BC$. The rod $BC$ qualifies as a two-force member. The rod $AB$ does not because it has three forces and is also not in static equilibrium (non-negligible accelerating mass). Thus, the free body diagram of rod $BC$ shows the two equal and opposite colinear forces at each end parallel to the rod and the tower AB does not.

Example: **Logs as bearings**
Consider the ancient egyptian dragging a big stone Fig. 4.20. If the stone and ground are flat and rigid, and the log is round, rigid and much lighter than the stone we are led to the free body diagram of the log shown. *With these assumptions there can't be any resistance to rolling.* Note that this effectively frictionless rolling occurs no matter how big the friction coefficient between the contacting surfaces. That the egyptian got tired comes from logs not being perfectly round, the ground or stone not being perfectly flat, and, most importantly, the ground, log or stone not being perfectly rigid. In any case it takes effort to pick up the logs in the back and move them to the front.

Example: **One point of support**
If an object with weight is supported at just one point (Fig. 4.19), that point must be directly above or below the center-of-mass. Why? The gravity forces are equivalent to a single force at the center-of-mass. The body is then a two force body. Since the direction of the gravity force is down, the support point and center-of-mass must be above one another.

Similarly,

> If a body is suspended from one point, the center of gravity must be directly above or below that point.

## Three-force body

If a body in equilibrium has only three forces on it, the equilibrium equations again restrict the forces in a geometrically describable manner. The simplification is not as great as for two-force bodies but is remarkably useful for both calculation and intuition. In box 4.3 on page 178 moment balance about various axes is used to prove that

> If exactly three forces act on a body (2D and 3D) the body is in equilibrium only if
> 1. the three force vectors are coplanar,
> 2. and either

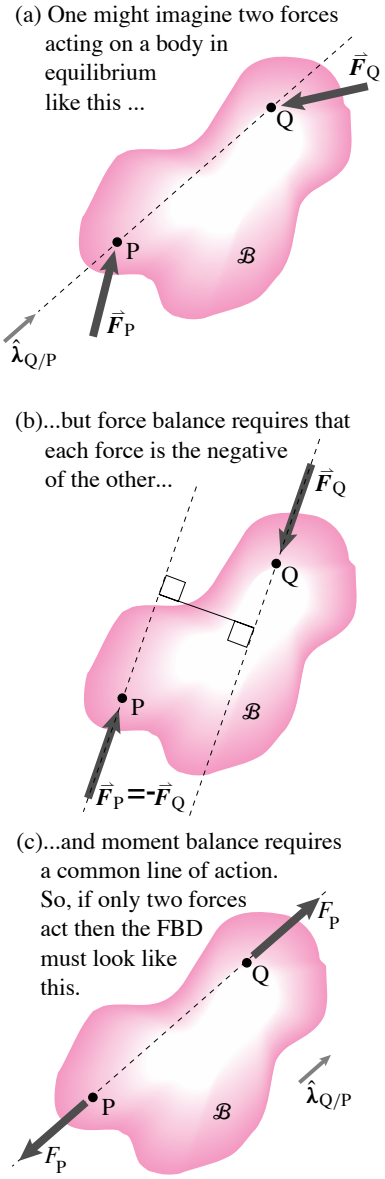



Figure 4.18: (a) Two forces acting on a body $\mathcal{B}$. (b) force balance implies that the forces are equal in magnitude and opposite in direction ($\vec{F}_P = -\vec{F}_Q$. (c) moment balance implies that the forces are colinear. Body $\mathcal{B}$ is a two-force member; the equilibrium equations imply that the two forces must be equal in magnitude, opposite in direction, and collinear. If the free body diagram shows equal-in-magnitude, opposite-in-direction and collinear forces then the equilibrium equations add no new information.

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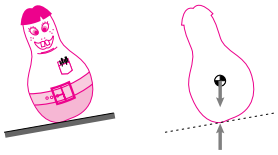


Figure 4.19: An object balancing on one point of support has to have its center-of-mass over the support point.

Filename:tfigure-groundbalance



Figure 4.20: Dragging a heavy stone on rolling round logs. If there is only point contact between the logs and the ground and stone, the logs are two-force bodies. If the logs are also round there is no resistance to forward motion.

Filename:tfigure-pyramid

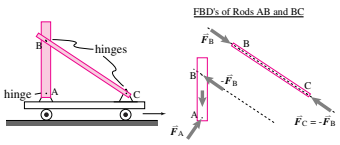



Figure 4.21: A cart is propelled by forces not showing. We are concerned about light strut BC and heavy tower AB. The strut is probably well-modeled as a two-force body.

Filename:tfigure2-cart-two-force

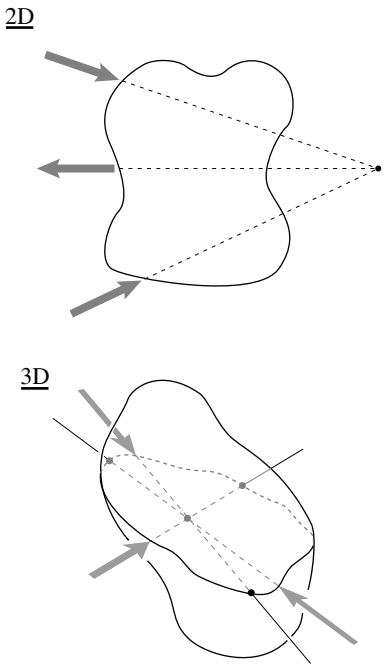



Figure 4.22: If exactly 3 forces act on a body the lines of action of the forces intersect at a single point and are coplanar. The point of intersection does not have to lie within the body. A special case is when that point is at infinity and the three forces are parallel.

Filename:tfigure2-three-force

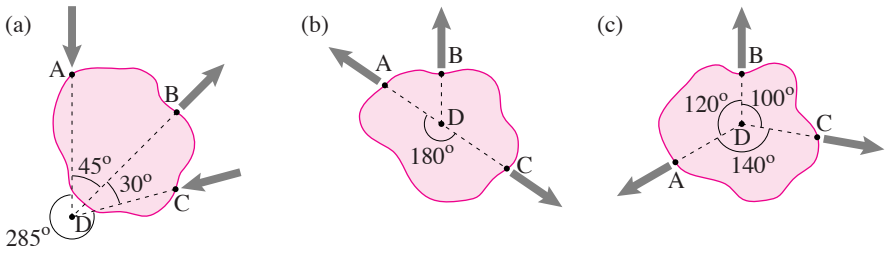
a) have lines of action which intersect at a single point (ie, they are concurrent), or
b) they are parallel.

One could imagine three random forces acting on a body. But, for equilibrium they must be coplanar and either concurrent or parallel. Unlike the case for 2-force bodies where the 2-force-body conditions imply the satisfaction of all equilibrium equations, for 3-force bodies planar concurrency still leaves two independent equilibrium equations possibly unsatisfied (for both 2D and 3D). That is, one still needs the equations of force balance in the plane (or, in the special case of three parallel forces, one scalar force balance and one moment balance equation ).

Example: **Hanging book box**
A box with a book inside is hung by two strings so that it is in equilibrium on when level. The lines of action of the two strings must intersect directly above the center-of-mass of the box/book system.

Example: **Which way do the forces go?**
The maximum angle between pairs of forces in a 3-force body can be (a) greater than, (b) equal to, or (c) less than $180^\circ$ (see figure below). In each case we can know something about the directions of the forces. Call the point of force concurrency D.

(a) Forces spread over more than $180^\circ$. Force balance perpendicular to the middle force implies that the outer two forces are both directed from D or both directed away from D. Force balance in the direction of the middle force shows that it has to have the opposite sense than the outer forces. If the others are pushing in then it is pulling away. If the outer forces are pulling away than it is pushing in.
(b) Forces spread exactly $180^\circ$. Force balance in the direction perpendicular to line ADC shows that the odd force must be zero. The other forces must obviously oppose each other.
(c) Forces spread over more than $180^\circ$. Force balance perpendicular to the force at C shows that the other two forces must both pull away towards D or both push in. Then force balance along C shows that all three forces must have the same sense. All three forces are pulling away from D or all three are pushing in.




## The idealized massless pulley

Both real machines and mechanical models are built of various building blocks. One of the standards is a pulley. We often draw pulleys schematically something like in figure 4.24a which shows that we believe that the tension in a string, line, cable, or rope that goes around an ideal pulley is the same on both sides, $T_1 = T_2 = T$. An ideal pulley is

(i.) Round,
(ii.) Has frictionless bearings,
(iii.) Has negligible inertia, and
(iv.) Is wrapped with a line which only carries forces along its length.

We now show that these assumptions lead to the result that $T_1 = T_2 = T$. First, look at a free body diagram of the pulley with a little bit of string at both ends (Fig. 4.24b). Since we assume the bearing has no friction, the interaction between the pulley bearing shaft and the pulley has no component tangent to the bearing.

To find the relation between tensions, we apply angular momentum balance (equation II) about point O

$$\left\{\sum \vec{M}_O = \dot{\vec{H}}_O\right\} \cdot \hat{k}. \tag{4.7}$$

Evaluating the left hand side of eqn. 4.7

$$\begin{aligned}\sum \vec{M}_O \cdot \hat{k} &= R_2T_2 - R_1T_1 + \underbrace{\text{bearing friction}}_{0} \\ &= R(T_2 - T_1), \text{ since } R_1 = R_2 = R.\end{aligned}$$

Because there is no friction, the bearing forces acting perpendicular to the round bearing shaft have no moment about point O (see also the short example on page 84). Because the pulley is round, $R_1 = R_2 = R$.

When mass is negligible, dynamics reduces to statics.

Putting these assumptions and results together gives

$$\begin{aligned}&\left\{\sum \vec{M}_O = \vec{0}\right\} \cdot \hat{k} \\ &\Rightarrow R(T_2 - T_1) = 0 \\ &\Rightarrow T_1 = T_2\end{aligned}$$

Thus, the tensions on the two lines of an ideal massless pulley are equal.

Lopsided pulleys are not often encountered, so it is usually satisfactory to assume round pulleys. But, in engineering practice, the assumption of frictionless bearings is often suspect. In dynamics, you may not want to neglect pulley mass.

## Lack of equilibrium as a sign of dynamics

Surprisingly, statics calculations often give useful information about dynamics. If, in a given problem, you find that forces or moments cannot be balanced* this is a sign that the related physical system will accelerate in the direction of imbalance (See the example 'block on ramp' on page 185).

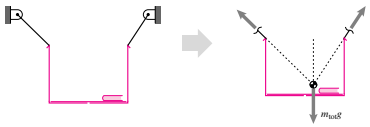



Figure 4.23: A book is in a box which hangs from two ropes.

Filename:tfigure-bookinbox

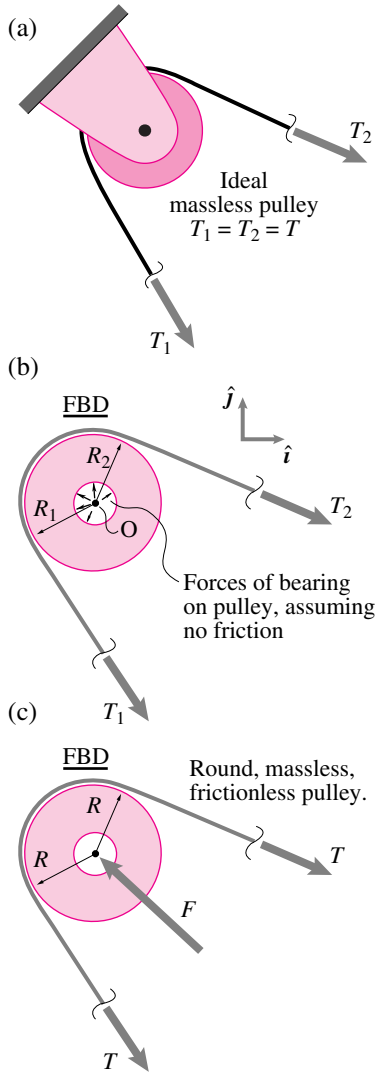



Figure 4.24: (a) An ideal massless pulley, (b) FBD of idealized massless pulley, detailing the frictionless bearing forces and showing forces at the cut strings, (c) final FBD after analysis.

Filename:tfigure3-pulleytheory1

* For more about non-existance of a statics solution, see box 4.2 on page 166.

## Linearity and superposition

For a given geometry the equilibrium equations are *linear*. That is: If for a given object you know a set of forces that is in equilibrium and you also know a second set of forces that is in equilibrium, then the sum of the two sets is also in equilibrium.

Example: **A bicycle wheel**

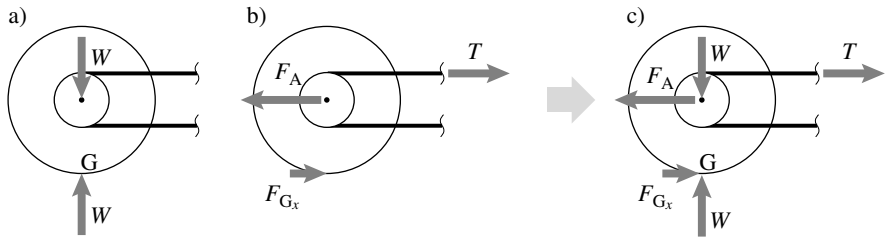



Figure 4.25: A bicycle wheel.

Filename:tfigure-wheelsuperposition

The free body diagram of an ideal massless bicycle wheel with a vertical load is shown in (a) above. The same wheel driven by a chain tension but with no weight is shown in equilibrium in (b) above. The sum of these two load sets (c) is therefor in equilibrium.

The idea that you can add two solutions to a set of equations is called the *principle of superposition*, sometimes called the principle of superimposition*. The principle of superposition provides a useful shortcut for some mechanics problems.

* Here's a bad pun to help you remember the idea. When talkative Sam comes over you get bored. When hungry Sally comes over you reluctantly go get a snack for her. When Sam and Sally come over together you get bored *and* reluctantly go get a snack. Each one of them is imposing. By the principle of superimposition their effects add when they are together. They are super imposing.

### 4.3 *THEORY*
### *Three-force bodies*

Here is a brief derivation of the result for three force bodies. The derivation is not needed for problem solving. However understanding the derivation may help build intuition.

Consider a body in static equilibrium with just three forces on it; $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$ acting at $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$. Taking moment balance about the axis through points at $\vec{r}_2$ and $\vec{r}_3$ implies that the line of action of $\vec{F}_1$ must pass through that axis. Similarly, for equilibrium to hold, the line of action of $\vec{F}_2$ must intersect the axis through points at $\vec{r}_1$ and $\vec{r}_3$ and the line of action of $\vec{F}_3$ must intersect the axis through $\vec{r}_1$ and $\vec{r}_2$. So, the lines of action of all three forces are in the plane defined by the three points of action and the lines of action of $\vec{F}_2$ and $\vec{F}_3$ must intersect. Taking moment balance about this point of intersection implies that $\vec{F}_1$ has line of action passing through the same point. A special case is when $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$ are parallel and have a common plane of action (equivalent to the concurrency point being at infinity).

**SAMPLE 4.5** Find force $F$ for equilibrium of the angle shown in the figure. The dimensions of the angle are $d = 0.3\,\text{m}$ and $a = 0.2\,\text{m}$.

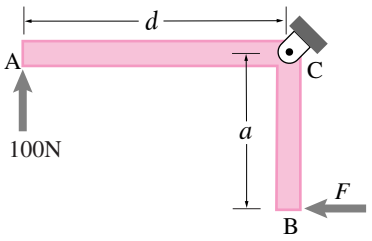



Figure 4.26:
Filename:sfig4-single-angle

**Solution** The free-body diagram of the angle is shown in Fig. Fig. 4.27. Since we are interested in force $F$, we can write the scalar moment balance equation (in $\hat{\boldsymbol{k}}$ direction) about point C (and thus get rid of the other unknown force $\vec{\boldsymbol{R}}$):

$$\begin{aligned} -Fa - (100\,\text{N})d &= 0 \\ \Rightarrow \quad F &= -(100\,\text{N})\frac{d}{a} \\ &= -100\,\text{N} \cdot \frac{0.3\,\text{m}}{0.2\,\text{m}} \\ &= -150\,\text{N}. \end{aligned}$$

$$\boxed{\vec{\boldsymbol{F}} = -(150\,\text{N})\hat{\boldsymbol{\imath}}}$$

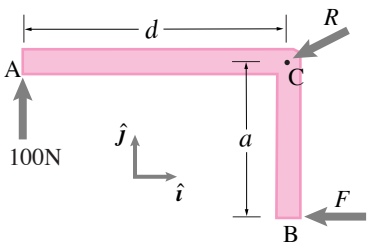



Figure 4.27:
Filename:sfig4-single-angle-a

**SAMPLE 4.6** Consider the angle shown in the figure with the applied forces. Can the angle be in equilibrium for some value of $F$? Explain.

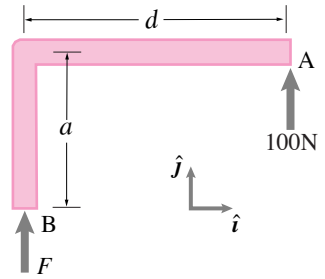



Figure 4.28:
Filename:sfig4-single-angle2

**Solution** Let us assume that the angle is in equilibrium. Then the forces acting on the angle must satisfy the force and moment balance equations. Now the force balance in the $\hat{\boldsymbol{\jmath}}$ direction gives

$$\begin{aligned} F + (100\,\text{N}) &= 0 \\ \Rightarrow \quad F &= -100\,\text{N}. \end{aligned}$$

The moment balance about point A gives

$$\begin{aligned} Fd &= 0 \\ \Rightarrow \quad F &= 0 \end{aligned}$$

Thus,

$$-100\,\text{N} = 0$$

which is a contradiction. Thus the angle cannot be in equilibrium with the applied forces.

It is easy to see that no matter which way $F$ acts (up or down), it cannot simultaneously balance the applied force at A and its moment. If $F = 100\,\text{N}$ acts upwards at B, the angle will accelerate up because it has a net force in the $\hat{\boldsymbol{\jmath}}$ direction. If $F = 100\,\text{N}$ acts downwards at B, the two equal and opposite forces at A and B produce a net moment on the angle and therefore the angle will start spinning about the $\hat{\boldsymbol{k}}$ direction. In fact, no matter what the value or direction of $F$ is, as long as it acts at point B, the angle cannot be in equilibrium. This is because the angle, as given, is a two force body, and for equilibrium, the two applied forces must be equal, opposite and colinear.

Equilibrium not possible.

## 4.4 *THEORY*
### *Two-force bodies*

Here we derive the ubiquitously-used result that if only two forces act on a body the two forces must be equal in magnitude, opposite in direction, and on a common line of action. You can (and will) use this result even if you do not master the reasoning in this box. But learning this reasoning may help your intuition.

Consider the free body diagram of a body $\mathcal{B}$ in figure 4.18a. Forces $\vec{F}_P$ and $\vec{F}_Q$ are acting on $\mathcal{B}$ at points $P$ and $Q$. Let's apply the equilibrium equations. First, we have that the sum of all forces on the body are zero,

$$\begin{aligned} \sum_{\substack{\text{All external}\\ \text{forces}}} \vec{F} &= \vec{0} \\ \vec{F}_P + \vec{F}_Q &= \vec{0} \qquad \Rightarrow \qquad \vec{F}_P = -\vec{F}_Q. \end{aligned}$$

Thus, the two forces must be equal in magnitude and opposite in direction. So, thus far, we can conclude that the forces must be parallel as shown in figure 4.18b. But the forces still seem to have a net turning effect, thus still violating the concept of static equilibrium. The sum of all external torques on the body about any point are zero. So, summing moments about point $P$, we get,

$$\begin{aligned} \sum_{\substack{\text{All external}\\ \text{torques}}} \vec{M}_{/P} &= \vec{0} \\ \vec{r}_{Q/P} \times \vec{F}_Q &= \vec{0} \qquad (\vec{F}_P \text{ produces no torque about } P.) \\ \left|\vec{r}_{Q/P}\right| \left(\hat{\lambda}_{Q/P} \times \vec{F}_Q\right) &= \vec{0} \qquad (\hat{\lambda}_{Q/P} = \frac{\vec{r}_{Q/P}}{|\vec{r}_{Q/P}|} = -\frac{\vec{r}_{P/Q}}{|\vec{r}_{P/Q}|}) \end{aligned}$$

So $\vec{F}_{\mathrm{Q}}$ has to be parallel to the line connecting P and Q. Similarly, taking the sum of moments about point $Q$, we get

$$-\hat{\lambda}_{Q/P} \times \vec{F}_Q = \vec{0}$$

and $\vec{F}_P$ also must be parallel to the line connecting P and Q. So, not only are $\vec{F}_P$ and $\vec{F}_Q$ equal and opposite, they are collinear as well since they are parallel to the axis passing through their points of action (see fig. 4.18c).

## 4.5 *THEORY*
### *Moment balance about 3 points is sufficient in 2D*

This is a theoretical aside showing that moment balance can totally replace force balance.

In 2D one can solve any statically determinate problem using moment balance about any 3 non-colinear points. Force balance adds no information.

Here we show the math behind this useful trick. The derivation here is only for logical completeness, it does not help with problem solving.

Consider two points A and B. Moment balance about these two points gives

$$\sum \vec{r}_{\mathrm{i/A}} \times \vec{F}_i = \vec{0} \text{ and } \sum \vec{r}_{\mathrm{i/B}} \times \vec{F}_i = \vec{0}.$$

Subtracting one of these equations from the other gives:

$$\begin{aligned} \sum \vec{r}_{\mathrm{i/A}} \times \vec{F}_i - \sum \vec{r}_{\mathrm{i/B}} \times \vec{F}_i &= \vec{0} \\ \sum \left(\vec{r}_{\mathrm{i/A}} - \vec{r}_{\mathrm{i/B}}\right) \times \vec{F}_i &= \vec{0} \\ \sum \left((\vec{r}_{\mathrm{i}} - \vec{r}_{\mathrm{A}}) - (\vec{r}_{\mathrm{i}} - \vec{r}_{\mathrm{B}})\right) \times \vec{F}_i &= \vec{0} \\ \sum \left(\vec{r}_{\mathrm{B}} - \vec{r}_{\mathrm{A}}\right) \times \vec{F}_i &= \vec{0} \\ \left(\vec{r}_{\mathrm{B}} - \vec{r}_{\mathrm{A}}\right) \times \sum \vec{F}_i &= \vec{0} \end{aligned}$$

Dotting both sides with a vector $\hat{k}$ normal to the plane we get (recalling the mixed triple product identity from page 52 in Section 2.3 that $(\vec{A} \times \vec{B}) \cdot \vec{C} = (\vec{C} \times \vec{A}) \cdot \vec{B}$) we can re-arrange terms to get

$$\begin{aligned} \left((\vec{r}_{\mathrm{B}} - \vec{r}_{\mathrm{A}}) \times \sum \vec{F}_i\right) \cdot \hat{k} &= \vec{0} \cdot \hat{k} \\ \left(\hat{k} \times (\vec{r}_{\mathrm{B}} - \vec{r}_{\mathrm{A}})\right) \cdot \sum \vec{F}_i &= 0. \end{aligned}$$

Thus moment balance about the points A and B implies force balance in the direction $\hat{k} \times (\vec{r}_{\mathrm{B}} - \vec{r}_{\mathrm{A}})$. This is force balance in the direction normal to the line AB (and in the plane).

Now consider a third point C. By the same reasoning moment balance about B and C implies force balance in the direction orthogonal to BC. So long as BC is not parallel to AB then we have force balance in two independent directions. So

$$\sum \vec{F}_i = \vec{0}.$$

The result only goes sour if the two directions are parallel, which occurs when two of the points A, B, and C are on a line. If A, B, and C are not on a line, moment balance about them implies force balance. So use of moment balance replaces the force balance equilibrium equations.

Moment balance about convenient points A, B, and C can simplify the equilibrium equations if the points are picked so that, by inspection, some forces have no moment.

**SAMPLE 4.7 A bar as a 2-force body:** A 4 ft long horizontal bar AC supports a load of 60 lbf at one end and is pinned to a wall at the other end. The bar is also supported by a string BC as shown in the figure. Find the forces applied by the pin and the string on the bar.

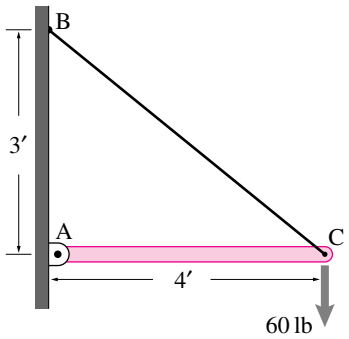



Figure 4.29:
Filename:sfig4-single-bar

**Solution** Let us do this problem two ways — using equilibrium equations without much thought, and using those equations with some insight.

The free-body diagram of the bar is shown in Fig. 4.30. The moment balance about point A, $\sum \vec{M}_{\rm A} = 0$, gives

$$\begin{aligned} \vec{r}_{\rm C/A} \times T\hat{\boldsymbol{\lambda}} + \vec{r}_{\rm C/A} \times (-P\hat{\boldsymbol{\jmath}}) &= \vec{\mathbf{0}} \\ \underbrace{\ell\hat{\boldsymbol{\imath}} \times T(-\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}})}_{\ell T \sin\theta\hat{\boldsymbol{k}}} + \underbrace{\ell\hat{\boldsymbol{\imath}} \times (-P\hat{\boldsymbol{\jmath}})}_{-\ell P\hat{\boldsymbol{k}}} &= \vec{\mathbf{0}} \\ (T\ell\sin\theta - P\ell)\hat{\boldsymbol{k}} &= \vec{\mathbf{0}} \qquad (4.8) \\ \Rightarrow \qquad T = \frac{P}{\sin\theta} = \frac{60\,\text{lbf}}{3/5} &= 100\,\text{lbf}. \end{aligned}$$

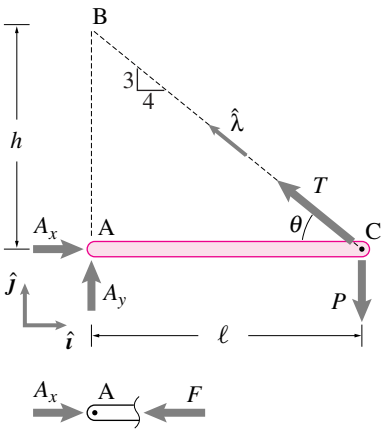



Figure 4.30:
Filename:sfig4-single-bar-a

The force equilibrium, $\sum \vec{F} = 0$, gives

$$(A_x - T\cos\theta)\hat{\boldsymbol{\imath}} + (A_y + T\sin\theta - P)\hat{\boldsymbol{\jmath}} = \vec{\mathbf{0}} \qquad (4.9)$$

Separating out $x$ and $y$ components of this equation, we get

$$\begin{aligned} A_x = T\cos\theta = (100\,\text{lbf}) \cdot \frac{4}{5} &= 80\,\text{lbf} \\ A_y = P - T\sin\theta &= 0 \end{aligned}$$

where the last equation, $A_y = P - T\sin\theta = 0$ follows from eqn. (4.8). Thus, the force in the rod is $\vec{A} = (80\,\text{lbf})\hat{\boldsymbol{\imath}}$, *i.e.*, a purely compressive force, and the tension in the string is 100 lbf.

$$\boxed{\vec{A} = (80\,\text{lbf})\hat{\boldsymbol{\imath}}, \quad T = 100\,\text{lbf}}$$

**Alternate Solution:** From the free-body diagram of the rod (see Fig. 4.31), we realize that the rod is a *two-force body*, since the forces act at only two points of the body, A and C. The reaction force at A is a single force $\vec{A}$, and the forces at end C, the tension $\vec{T}$ and the load $\vec{P}$, sum up to a single net force, say $\vec{F}$. So, now using the fact that the rod is a two-force body, the equilibrium equation requires that $\vec{F}$ and $\vec{A}$ be equal, opposite, and colinear (along the longitudinal axis of the bar). Thus,

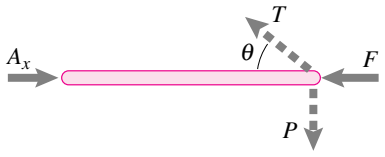



Figure 4.31:
Filename:sfig4-single-bar-b

$$\vec{A} = -\vec{F} = -F\hat{\boldsymbol{\imath}}.$$

Now,

$$\begin{aligned} \vec{F} &= \vec{P} + \vec{T} \\ -F\hat{\boldsymbol{\imath}} &= -P\hat{\boldsymbol{\jmath}} + T\sin\theta\hat{\boldsymbol{\jmath}} - T\cos\theta\hat{\boldsymbol{\imath}} \qquad (4.10) \end{aligned}$$

Separating out $x$ and $y$ components of this equation, we get

$$\begin{aligned} -F + T\cos\theta &= 0 \qquad (4.11) \\ P - T\sin\theta &= 0. \qquad (4.12) \end{aligned}$$

Solving these two equations simultaneously, we get $T = P/\sin\theta = 100\,\text{lbf}$ and $F = T\cos\theta = 80\,\text{lbf}$. The answers, of course, are the same.

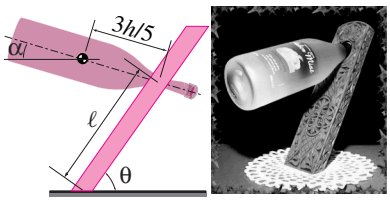



Figure 4.32:

Filename:sfig4-2-winestand

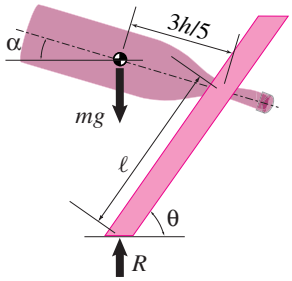



Figure 4.33:

Filename:sfig4-2-winestanda



Figure 4.34: A bottle holder with a wider base.

Filename:sfig4-2-wineholder2

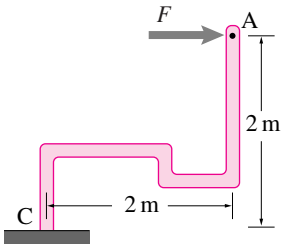



Figure 4.35:

Filename:sfig4-3-new4

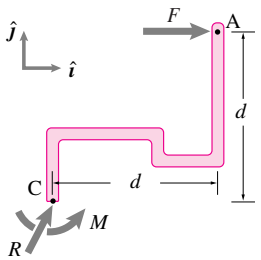



Figure 4.36:

Filename:sfig4-3-new4a

**SAMPLE 4.8 A bottle holder:** A clever design of a bottle holder (a plank with a hole) is shown in the figure. Note that the holder is not fixed to the support; it stands freely, but only when the bottle is in. Assume that the mass of the bottle is 1 kg and that the center-of-mass of the bottle is at 3/5th of its length ($h = 35\,\text{cm}$) from the neck support point. The bottle in its rest position is slightly tipped down ($\alpha = 15^\circ$). Assuming the mass of the stand to be negligible and $\ell = 30\,\text{cm}$, find the angle $\theta$ of the stand so that the bottle and the stand can stand together as shown.

**Solution** Let us draw the free-body diagram of the bottle and the stand together as one system. The forces acting are shown in Fig. **??**. Since the only forces acting on the system are $R$ and $mg$, they must be equal, opposite and colinear. Thus the line of action of the weight, $mg$, must pass through the center of the stand's footprint. From the given geometry, then, we must have,

$$\begin{aligned} \ell\cos\theta &= \frac{3h}{5}\cos\alpha \\ \Rightarrow\quad \theta &= \cos^{-1}\left(\frac{3h}{5\ell}\cos\alpha\right) \\ &= \cos^{-1}\left(\frac{3\cdot 35\,\text{cm}}{5\cdot 30\,\text{cm}}\cos 15^\circ\right) \\ &= 47.5^\circ. \end{aligned}$$

$$\boxed{\theta = 47.5^\circ}$$

**Note:** The latitude in design of the angle $\theta$ depends on the width of the base of the stand. The two forces acting on the system must be colinear and must pass through the base. Therefore, a wider base (perhaps at the expense of elegance) provides more freedom for the forces to move sideways, giving a range of $\theta$ and $\alpha$ for design. (see Fig. 4.34.)

---

**SAMPLE 4.9 Reactions at fixed ends.** For the bent bar shown in the figure, find the reaction forces at the fixed end for $F = 10\,\text{kN}$.

**Solution** The free-body diagram of the rod is shown in Fig. 4.36. Note that in addition to the reaction force $\vec{\boldsymbol{R}}$, there is a reaction moment $\vec{\boldsymbol{M}} = M\hat{\boldsymbol{k}}$ acting on the rod because of the fixed support.

The force balance equation, $\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$, gives us

$$\begin{aligned} F\hat{\boldsymbol{\imath}} + \vec{\boldsymbol{R}} &= \vec{\boldsymbol{0}} \\ \Rightarrow\quad \vec{\boldsymbol{R}} &= -F\hat{\boldsymbol{\imath}} = -(10\,\text{kN})\hat{\boldsymbol{\imath}}. \end{aligned}$$

Now, we can write the moment balance equation about point C, $\sum \vec{\boldsymbol{M}}_\text{C} = \vec{\boldsymbol{0}}$, to give

$$\begin{aligned} M\hat{\boldsymbol{k}} - Fd\hat{\boldsymbol{k}} &= \vec{\boldsymbol{0}} \\ \Rightarrow\quad M &= Fd = (20\,\text{kN}\cdot\text{m}). \end{aligned}$$

$$\boxed{\vec{\boldsymbol{R}} - (2\,\text{kN})\hat{\boldsymbol{\imath}},\ \vec{\boldsymbol{M}} = (20\,\text{kN}\cdot\text{m})\hat{\boldsymbol{k}}}$$

**SAMPLE 4.10** Consider the structure (a rocker arm) shown in the figure. Assume that bar CD can only take axial load (tension or compression). If a horizontal force, $F = 2\,\text{kN}$ is applied at point A, what is the tension in rod CD?

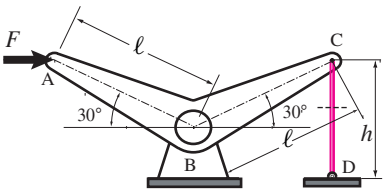



Figure 4.37:

Filename:sfig4-3-new2

**Solution** Let $T$ be the tension in the rod. Then, the free-body diagram of the rocker arm ABC is as shown in Fig. 4.38. We need to find $T$.

The easiest way to solve this problem is to apply moment balance, $\sum \vec{M}_{\text{B}} = \vec{0}$, about point B. Taking moments about this point gets rid of another unknown reaction force $R_B$ and relates $T$ to $F$ directly:

$$\vec{r}_{\text{A/B}} \times \vec{F} + \vec{r}_{\text{C/B}} \times \vec{T} = \vec{0}$$

We can evaluate the cross products vectorially or use the scalar form of the moment calculation (force times the lever arm) to give

$$\begin{aligned} \vec{r}_{\text{A/B}} \times \vec{F} &= -F\ell \sin\theta \hat{k} \\ \vec{r}_{\text{C/B}} \times \vec{T} &= -T\ell \cos\theta \hat{k}. \end{aligned}$$

So, the scalar moment balance equation in the $\hat{k}$ direction is

$$\begin{aligned} -F\ell \sin\theta - T\ell \cos\theta &= 0 \\ \Rightarrow \quad T &= -F \tan\theta. \end{aligned}$$

Now substituting the given values, $F = 2\,\text{kN}$, and $\theta = 30^\circ$, we get

$$T = -(2\,\text{kN}) \cdot (\tan 30^\circ) = -1.15\,\text{kN}.$$

Thus the rod is under compression, not tension. It is also clear from the picture that if we push at A, ABC will try to rotate clockwise about B, thus pushing down on the rod at C.

$$\boxed{T = -1.15\,\text{kN}}$$

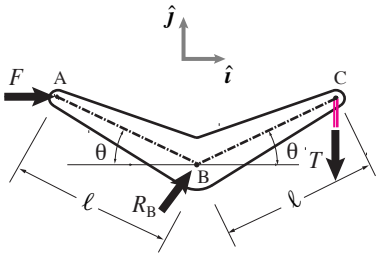



Figure 4.38: Free body diagram of the structure with rod CD cut just below C.

Filename:sfig4-3-new2a

**Comments:**

In this problem, we can find the tension in the rod also by using the force balance equation, $\sum \vec{F} = \vec{0}$. However, force balance will involve two unknown forces $T$ and $R$. Thus to solve for $T$, we will have to solve two scalar equations (force balance in $x$ and $y$-directions) simultaneously. Moment balance equation, on the other hand, gives just one scalar equation involving $T$ and $F$.

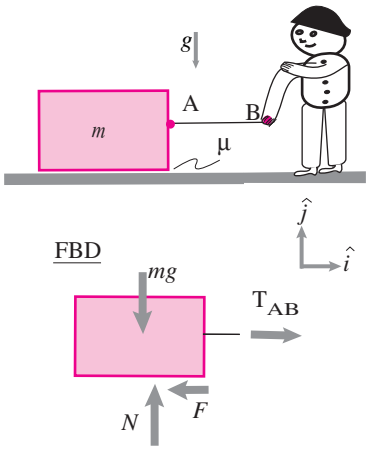



Figure 4.39: Pulling a block on a frictional ground, with related FBD. The complicated distribution of normal forces and friction forces on the block from the ramp have been replaced with the equivalent pair $N$ and $F$. For particle mechanics we don't worry about where $N$ and $F$ are applied.

Filename:tfigure-particle-blocklevel

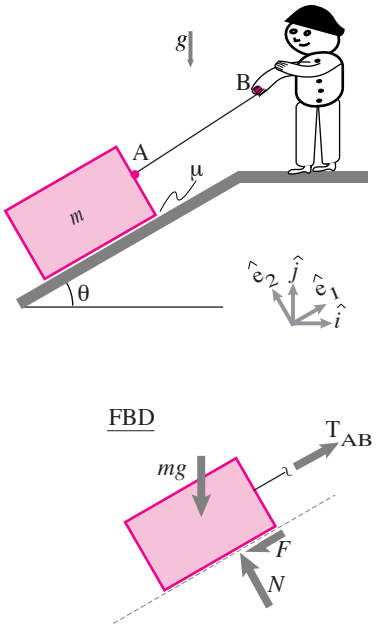



Figure 4.40: Pulling a block up a frictional slope with related FBD. As in the previous example, the pair $N$ and $F$ represent the net normal and frictional force.

Filename:tfigure-particle-blockslope

# 4.3 Equilibrium with frictional contact

Contacting objects are prevented from passing through each other by pressing against each other. Generally there is also some frictional resistance to relative slip. We have neglected friction so far for simplicity and because the neglect of friction is a reasonable approximation for some lubricated contact problems. On the other extreme, in some situations we have assumed that friction so well resists slip that we assumed 'no slip' and that frictional contact acts like a hinge or weld. Either way, with friction negligibly small, or reliably large, we have not worried about it.

However, for some purposes friction forces are not reasonably neglected during slip. Or, when there is no slip, sometimes we have to worry about whether the frictional bond is strong enough to prevent slip.

Although slip means motion and motion sounds like dynamics (contradicting the premise of statics), there are many situations where there is enough motion for friction to be important but not so much acceleration that inertial terms ($m\vec{a}$) are important.

How friction forces are represented on free body diagrams was discussed in Section 3.1 which you should review before proceeding further here. We will now consider friction forces in equilibrium conditions.

For simplicity, and because of the relatively high accuracy to complexity ratio, we consider only Coulomb friction with a single coefficient of friction $\mu$.

Example: **Drag a block with friction.**
Consider the block with friction on a slope (Fig. 4.39). You want to pull it slowly to the right with rod AB. Say $m = 100\,\text{kg}$, $g = 10\,\text{m/s}^2$, and $\mu = 0.3$.
Force balance, using the forces on the free body diagram gives:

$$\sum \vec{F}_i = \vec{0} \quad \Rightarrow \quad -mg\hat{\jmath} + T_{\text{AB}}\hat{\imath} + N\hat{\jmath} - F\hat{\imath} = \vec{0} \tag{4.13}$$

This, with the friction relation $F = \mu N$, is 3 scalar equations in $T_{\text{AB}}$, $F$ and $N$ with solution $N = mg = 1000\,\text{N}$, $F = \mu mg = 300\,\text{N}$, and $T_{\text{AB}} = \mu mg = 300\,\text{N}$.

Example: **Drag a block on a ramp with friction.**
Consider the block with friction on a slope (Fig. 4.40). You want to hold it with rod AB. Maybe you want to (i) slide it up slowly, or (ii) down slowly or (iii) hold it still. Say $m = 100\,\text{kg}$, $g = 10\,\text{m/s}^2$, $\theta = 45^\circ$, and $\mu = 0.3$.
Force balance, using the forces on the free body diagram, gives:

$$\sum \vec{F}_i = \vec{0} \quad \Rightarrow \quad -mg\hat{\jmath} + T_{\text{AB}}\hat{\boldsymbol{e}}_1 + N\hat{\boldsymbol{e}}_2 - F\hat{\boldsymbol{e}}_1 = \vec{0} \tag{4.14}$$

This, with the friction relation, is 3 scalar equations in $T_{\text{AB}}$, $F$ and $N$.

Summing forces in the rope direction and normal to the plane we get:

$$\begin{aligned} \{(\text{Eqn. } 4.15)\} \cdot \hat{\boldsymbol{e}}_1 \quad &\Rightarrow \quad -mg\sin\theta + T_{\text{AB}} - F = 0 \\ \{(\text{Eqn. } 4.15)\} \cdot \hat{\boldsymbol{e}}_2 \quad &\Rightarrow \quad mg\cos\theta + N = 0 \end{aligned} \tag{4.15}$$

or, for the quantities given $N = (100\,\text{kg})(10\,\text{m/s}^2)(\cos 45^\circ) = 1414\,\text{N}$. We assume that $F$ and $N$ are related by friction described with the standard Coulomb's friction model*:

i) $F = \mu N$ if the block is sliding up;
ii) $F = -\mu N$ if the block is sliding down; or
iii) $-\mu N \le F \le \mu N$ if the block is not sliding.

Solving eqn. (4.15) with the friction relations gives*.

i) $T_{AB} = mg\,(\mu\cos\theta + \sin\theta)$ if the block is sliding up;
ii) $T_{AB} = mg\,(-\mu\cos\theta + \sin\theta)$ if the block is sliding down;
Note that if $\tan\theta < \mu$ then $T_{AB} < 0$ and it then takes a push to slide down; or
iii) $mg\,(-\mu\cos\theta + \sin\theta) \le T_{AB} \le mg\,(\mu\cos\theta + \sin\theta)$ if the block is not sliding.
If $\tan\theta < \mu$ then $T_{AB} = 0$ is amongst the solutions for, so no sliding and the block can sit still on the slope with no pull on the rope.

Note that the tension $T_{AB}$ scales with $mg$. So doubling $m$ or $g$ doubles all the forces in all of these answers, as you might guess from dimensional considerations. The mathematically-abstract-sounding issues of existence and uniqueness often show up in friction problems. For example, sometimes there is no statics solution (non-existence).

Example: **Block on ramp.**
A statics problem without a solution. A block with coefficient of friction $\mu = .5$ is in static equilibrium sliding steadily down a $45^\circ$ ramp (Fig. 4.41). *Not!* If there is constant velocity motion then statics would apply. But the forces in the free body diagram cannot add to zero (since the resulant of the friction and normal force is tipped up and to the left and thus cannnot be parallel to the vertical gravity force). The assumptions are not consistent with statics (actually this is a dynamics problem, the block accelerates down the ramp). If you saw a block just sitting there on a ramp, then you can be sure that the slope and friction coefficient are not those given above.

Friction problems might be studied with a particle model, as above, or also with moment balance.

Example: **Dragged block as an extended body.**
This is a repeat of the first example on page 184. One might wonder if the dragging causes an uneven distribution of force up on the block. Does the block dragging back, for example, cause a bigger pressure on the back? As a simple model assume all the ground force is at the front and back edge of the block. Force balance gives basically the same information as for the particle model, namely that:

$$N_C + N_D = W \text{ and } \underbrace{F_C}_{\mu N_C} + \underbrace{F_D}_{\mu N_D} = T_{AB} \quad \Rightarrow \quad T_{AB} = \mu W.$$

One can find more with moment balance about any point you like, say C, with force balance gives

$$\sum M_{/C} = 0 \quad \Rightarrow \quad N_D = \frac{W}{2} + \frac{\mu h W}{2\ell} \text{ and } N_C = \frac{W}{2} - \frac{\mu h W}{2\ell}$$

So there is more pressure on the front than back. This difference goes away if the either the friction or the height of the string attachment vanish.

* **Caution:** A common mistake amongst beginners is to assume the equation $F = \mu N$ applies when there is friction. Rather, if the friction is preventing slip $F$ could be anything so long as $|F| \le \mu N$. And if the slip is opposite in direction from that implicitly assumed in the free body diagram then $F = -\mu N$ (see case (ii) in the example above).

* We could be tricky and get a single equation for the scalar $T_{AB}$ by dotting both sides of eqn. (4.15) with a vector orthogonal to the resultant of $N\hat{\boldsymbol{e}}_2 - F\hat{\boldsymbol{e}}_1$. For the case of uphill sliding such a vector would be $\hat{\boldsymbol{e}}_1 + \mu\hat{\boldsymbol{e}}_2$.

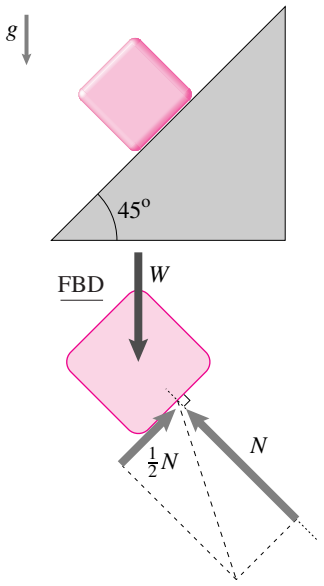



Figure 4.41: Block on steep ramp and related FBD.

Filename:tfigure-blockcontradiction

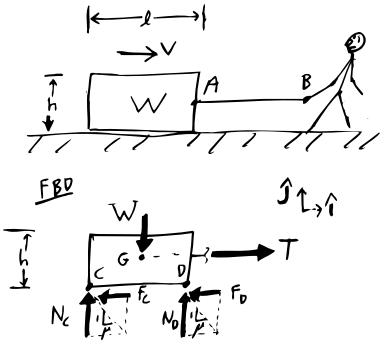



Figure 4.42: Dragging a block, taking account (in a simple way) the distribution of contact forces from the ground. Assume slip to the right is occuring.

Filename:tfigure-blockslide2pt

## Conditional contact, consistency, and contradictions

There is a natural hope that a subject will reduce to the solution of some well defined equations. For better and worse, things are not always this simple. For better because it means that the recipes are still not so well defined that computers can easily steal the subject of mechanics from people. For worse because it means you have to think hard to do some mechanics problems.

One source of these difficulties is the conditional nature of the equations that govern contact. For example:

- The ground pushes up on something to prevent interpenetration *if* the pushing is positive, *otherwise* the ground does not push up.
- The force of friction opposes motion and has magnitude $\mu N$ *if* there is slip, *otherwise* the force of friction is something less than $\mu N$ in magnitude.
- The distance between two points is kept from increasing by the tension in the string between them *if* the tension is positive, *otherwise* the tension is zero.

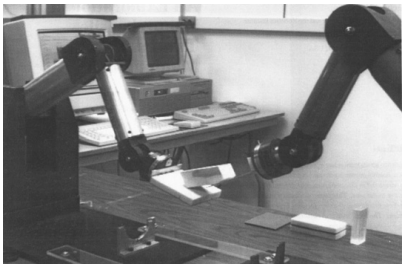
These conditions are, implicitly or explicitly, in the equations that govern these interactions. One does not always know which of the alternative contact conditions, if either, apply when one starts a problem. Sometimes multiple possibilities need to be checked*.

On a FBD at every point of frictional contact

- *If* the direction of slip or impending slip is known, *either*
  - Draw a normal force $N$ and a friction force $F = \mu N$ opposing the relative slip, *or*
  - Draw a single force $R$ at an angle $\phi$ from the normal of the contact in the direction which resists slip (with $\tan\phi = \mu$)
- *If* there is no slip, *either*
  - Draw a normal force $N$ and tangential force $F$ *or*
  - Draw a single force vector $\vec{\boldsymbol{R}}$ with unknown components
- *If* you don't know whether of not there is slip, *first*
  - Guess that there is no slip *then*
  - Solve the equilibrium equations, *then*
    - *If* $F \leq \mu N$: you guessed right and have found a solution to both the equilibrium and friction equations.
    - *If* $F > \mu N$: you guessed wrong and have to guess that there is slip in one direction (guess which), *then*
      - see if you can solve the equilibrium equations, if not *then*
      - assume slip in the opposite direction and try to solve the equilibrium equations, if you can't *then*
      - the problem has no solution

* All of the combinatorics in this recipe follows from the inability of our mathematics to deal easily with relations between variables with the step shape of Coulomb friction (Fig. 3.26 on page 144).

Example: **Robot hand**



Roboticist Michael Erdmann has designed a palm manipulator that manipulates objects without squeezing them. The flat robot palms just move around and the object consequently slides. Determining whether the object slides on one the other or possibly on both hands in a given movement is a matter of case study. The computer checks to see if the equilibrium equations can be solved with the assumption of sticking or slipping at one or the other contact.

Once you find a solution to a problem with friction there remains the possibility of multiple solutions, in this case for different reasons than the usual static indeterminacy. The following problem shows a case where a statics problem has multiple solutions due to friction effects.

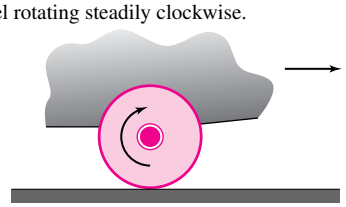
## 4.6 Wheels and two force bodies (part 1)

One often hears whimsical reverence for the "invention of the wheel." Now, using elementary mechanics, we can gain some appreciation for this revolutionary way of sliding things.

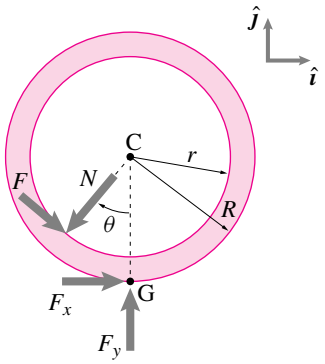
Without a wheel the force it takes to drag something is about $\mu W$. Since $\mu$ ranges between about .1 for teflon, to about .6 for stone on ground, to about 1 for rubber on pavement, you need to pull with a force that is on the order of a half of the full weight of the thing you are dragging.

You have seen how rolling on round logs cleverly take advantage of the properties of two-force bodies (page 175). But that good idea has the major deficiency of requiring that logs be repeatedly picked up from behind and placed in front again.

The simplest wheel design uses a dry "journal" bearing consisting of a non-rotating shaft protruding through a near close fitting hole in the wheel. Here is shown part of a cart rolling to the right with a wheel rotating steadily clockwise.



To figure out the forces involved we draw a free body diagram of the wheel. We neglect the wheels weight because it is generally much smaller than the forces it mediates. To make the situation clear the picture shows too-large a bearing hole $r$.




The force of the axle on the wheel has a normal component $N$ and a frictional component $F$. The force of the ground on the wheel has a part holding the cart up $F_y$ and a part along the ground $F_x$ which will surely turn out to be negative for a cart moving to the right. If we take the wheel dimensions to be known and also the vertical part of the ground reaction force $F_y$ we have as unknowns $N, F, \theta$ and $F_x$. To find these we could use the friction equation for the sliding bearing contact

$$F = \mu N;$$

force balance

$$F_x\hat{\imath} + F_y\hat{\jmath} + N(-\sin\theta\hat{\imath} - \cos\theta\hat{\jmath}) + F(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) = \vec{\mathbf{0}},$$

which could be reduced to 2 scalar equations by taking components or dot products; and moment balance which is easiest to see in terms of forces and perpendicular distances as

$$Fr + F_x R = 0.$$

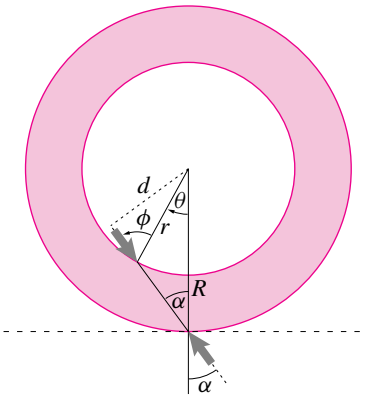
(continued on next page)

Example: **Rod pushed in a channel.**

## 4.7 Wheels and two force bodies (continued)

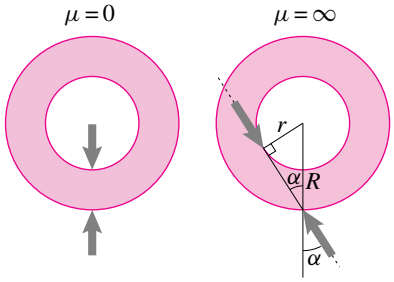
Of key interest is finding the force resisting motion $F_x$. With some mathematical manipulation we could solve the 4 scalar equations above for any of $F_x$, $N$, $F$, and $\theta$ in terms of $r$, $R$, $F_y$, and $\mu$. We follow a more intuitive approach instead.

As modeled, the wheel is a two-force body so the free body diagram shows equal and opposite colinear forces at the two contact points.




The friction angle $\phi$ describes the friction between the axle and wheel (with $\tan\phi = \mu$). The angle $\alpha$ describes the *effective* friction of the wheel. This is not the friction angle for sliding between the wheel and ground which is assumed to be larger (if not, the wheel would skid and not roll), probably much larger. The *specific resistance* or the *coefficient of rolling resistance* or the *specific cost of transport* is $\mu_{\text{eff}} = \tan\alpha$. (If there was no wheel, and the cart or whatever was just dragged, the specific resistance would be the friction between the cart and ground $\mu_{\text{eff}} = \mu$.)

Although we can solve for $\alpha$ in terms of $\mu$ or $\phi$ let's first consider two extreme cases: one is a frictionless bearing and the other is a bearing with infinite friction coefficient $\mu \to \infty$ and $\phi \to 90^\circ$.




In the case that the wheel bearing has no friction we satisfyingly see clearly that there is no ground resistance to motion. The case of infinite friction is perhaps surprising. Even with infinite friction we have that

$$\sin\alpha = \frac{r}{R}.$$

Thus if the axle has a diameter of 10 cm and the wheel of 1 m then $\sin\alpha$ is less than .1 no matter how bad the bearing material. For such small values we can make the approximation $\mu_{\text{eff}} = \tan\alpha \approx \sin\alpha$ so that the effective coefficient of friction is .1 or less no matter what the bearing friction.

> The genius of the wheel design is that it makes the effective friction less than $r/R$ no matter how bad the bearing friction.

Going back to the two-force body free body diagram we can see that

$$\begin{aligned} & d &=& d \\ \Rightarrow\quad & r\sin\phi &=& R\sin\alpha \\ \Rightarrow\quad & \sin\alpha &=& \frac{r}{R}\sin\phi. \qquad (*) \end{aligned}$$

From this formula we can extract the limiting cases discussed previously ($\phi = 0$ and $\phi = 90^\circ$). We can also plug in the small angle approximations ($\sin\alpha \approx \tan\alpha$ and $\sin\phi \approx \tan\phi$) if the friction coefficient is low to get

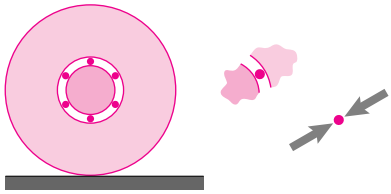
$$\mu_{\text{eff}} \approx \mu\frac{r}{R}.$$

The effective friction is the bearing friction attenuated by the radius ratio. Or, we can use the trig identity $\sin = \sqrt{1+\tan^{-2}}^{-1}$ to solve the exact equation (*) for

$$\mu_{\text{eff}} = \mu\frac{r}{R}\left(\frac{1}{\sqrt{1+\mu^2(1-r^2/R^2)}}\right),$$

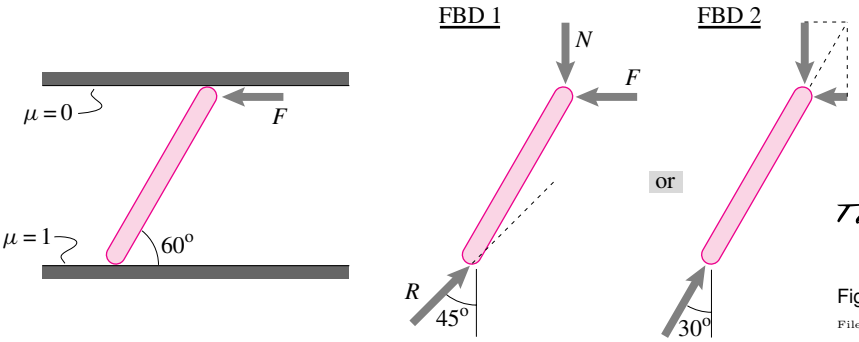
where the term in parenthesis is always less than one and close to one if the sliding coefficient in the bearing is low.

Finally we combine the genius of the wheel with the genius of the rolling log and invent a wheel with rolling logs inside, a ball bearing wheel.



Each ball is a two force body and thus only transmits radial loads. Its as if there were no friction on the bearing and we get a specific resistance of zero, $\mu_{\text{eff}} = 0$. Of course real ball bearings are not perfectly smooth or perfectly rigid, so its good to keep $r/R$ small as a back up plan even with ball bearings.

By this means some wheels have effective friction coefficients as low as about .003. The force it takes to drag something on wheels can be as little as one three hundredth the weight.




A light rod is just long enough to make a $60^\circ$ angle with the walls of a channel. One channel wall is frictionless and the other has $\mu = 1$. What is the force needed to keep it in equilibrium in the position shown? If we assume it is sliding we get the first free body diagram. The forces shown can be in equilibrium if all the forces are zero. Thus we have the solution that the rod slides in equilibrium with no force. If we assume that the block is not sliding the friction force on the lower wall can be at any angle between $\pm 45^\circ$. Thus we have equilibrium with the second FBD for arbitrary positive $F$. This is a second set of solutions. A rod like this is said to be *self locking* in that it can hold arbitrary force $F$ without slipping. That we have found freely slipping solutions with no force and jammed solutions with arbitrary force corresponds physically to one being able to easily slide a rod like this down a slot and then have it totally jamb. Some rock-climbing equipment depends on such self-locking and easy release.

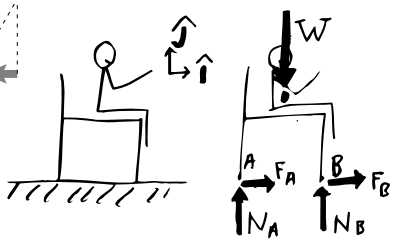



Figure 4.43: A chair with friction at the feet.
Filename:tfigure-2legchairfriction

## Statically indeterminate problems

When there are two or more points of frictional contact and there is no slip nor impending slip indeterminacy is likely.

Example: **Chair with friction**
If we assume Coulomb friction at the chair feet we know that

$$|F_A| \le \mu N_A \text{ and } |F_B| \le \mu N_B$$

The equilibrium equations tell us (assuming for simplicity that $W$ acts in the middle of the chair):

$$F_A + F_B = 0, N_A = N_B = W/2.$$

Putting these equations together we find that

$$-W/2 \le F_A \le W/2 \text{ and } F_B = -F_A$$

and no more. That is, all we can tell that both are within the friction limits and that the horizontal forces cancel each other.

If a free body diagram shows two forces with a common line of action, like the friction forces $F_A$ and $F_B$ on the chair above, the laws of statics might only find their sum, but otherwise can't untangle them.

Only if there is independent information, as would be the case if we knew the chair was sliding to the right (which it clearly isn't in this static example), could we find the friction forces.

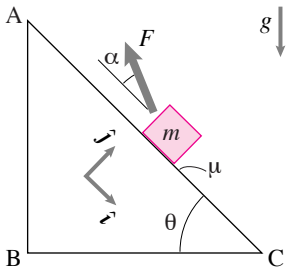



Figure 4.44:
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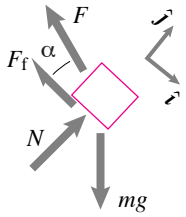



Figure 4.45: Free body diagram of the block. The friction force $\vec{F}_f$ may be known or unknown depending on whether the block is sliding or not. Under downhill sliding, $F_f = \mu N$, whereas for no sliding $|F_f| \le \mu N$.
Filename:sfig4-1-friction-a

**SAMPLE 4.11** Consider the block of mass $m = 10\,\text{kg}$ pushed up by the force $F$ on the ramp as shown in the figure. The coefficient of friction between the ramp and the block is $\mu = 0.7$.

1. Let $\theta = 60^\circ$ and $\alpha = 0^\circ$. Assuming that the block slides steadily downhill, find the tension in the string.
2. Let $\theta = 30^\circ$ and $\alpha = 30^\circ$. If the applied force $F = 20\,\text{N}$, find the force of friction on the block.
3. Let $\theta = 60^\circ$ and $\alpha = 30^\circ$. If the applied force $F = 10\,\text{N}$, find the force of friction on the block.

**Solution** The free-body diagram of the block is shown in Fig. 4.55. The force balance equation for the static equilibrium of the block ($\sum \vec{F} = \vec{0}$) gives

$$\underbrace{\vec{F}}_{F(-\cos\alpha\hat{\imath}+\sin\alpha\hat{\jmath})} + \overbrace{\vec{N}}^{N\hat{\jmath}} + \underbrace{\vec{F}_f}_{-F_f\hat{\imath}} + \overbrace{\vec{W}}^{mg(\sin\theta\hat{\imath}-\cos\theta\hat{\jmath})} = \vec{0}. \tag{4.16}$$

1. **Block sliding down:**
   If the block slides down steadily and slowly, we can use the static equilibrium equation written above with $\vec{F}_f = -\mu N\hat{\imath}$ (that is, the friction force is known). Substituting this value of $\vec{F}_f$ and separating out the $\hat{\imath}$ and $\hat{\jmath}$ components of eqn. (4.16) (by dotting the equation with $\hat{\imath}$ and $\hat{\jmath}$ separately), we get

$$\begin{aligned} -F\cos\alpha - \mu N + mg\sin\theta &= 0 \qquad (4.17)\\ F\sin\alpha + N - mg\cos\theta &= 0. \qquad (4.18)\end{aligned}$$

   Adding $\mu$ times eqn. (4.18) to eqn. (4.17) and rearranging terms, we get

$$\begin{aligned} F(\cos\alpha - \mu\sin\alpha) &= mg(\sin\theta - \mu\cos\theta)\\ \Rightarrow \quad F &= \frac{\sin\theta - \mu\cos\theta}{\cos\alpha - \mu\sin\alpha}\,mg. \qquad (4.19)\end{aligned}$$

   Substituting $\alpha = 0^\circ$, $\theta = 60^\circ$, and $\mu = 0.7$ in eqn. (4.19), we get

$$F = (\sin 60^\circ - 0.7\cdot\cos 60^\circ)\,mg = 0.52\,mg. = 51\,\text{N}.$$

$$\boxed{\vec{F} = -(51\,\text{N})\hat{\imath}}$$

2. **Block sliding or not sliding – not known:**
   Now, we are given that $F = 20\,\text{N}$, $\alpha = 30^\circ$, and $\theta = 30^\circ$. We do not know if the block is sliding or not. So, let us assume static equilibrium in the given configuration and solve for the friction force $F_f$. Then, we will check if it satisfies friction law for static equilibrium ($|F_f| \le \mu N$).

   Substituting $\vec{F}_f = -F_f\hat{\imath}$ in eqn. (4.16) and separating out the $\hat{\imath}$ and $\hat{\jmath}$ components of the equation, we get

$$\begin{aligned} -F\cos\alpha - F_f + mg\sin\theta &= 0\\ F\sin\alpha + N - mg\cos\theta &= 0\end{aligned}$$

   which are easily solved for $F$ and $N$ to give

$$\begin{aligned} F_f &= mg\sin\theta - F\cos\alpha\\ N &= mg\cos\theta - F\sin\alpha.\end{aligned}$$

Substituting the given values of $F$, $\theta$, and $\alpha$, we get

$$\begin{aligned} F_f &= 10\,\text{kg} \cdot 9.81\,\text{m/s}^2 \cdot \sin 30^\circ - 20\,\text{N} \cdot \cos 30^\circ = 31.73\,\text{N} \\ N &= 10\,\text{kg} \cdot 9.81\,\text{m/s}^2 \cdot \cos 30^\circ - 20\,\text{N} \cdot \sin 30^\circ = 74.96\,\text{N}. \end{aligned}$$

Now, the maximum possible value of friction force is $\mu N = 0.7 \cdot 74.96\,\text{N} = 52.47\,\text{N}$. Thus, $|F_f| < \mu N$, and therefore, our assumption of static equilibrium is valid. This equilibrium requires that $F_f = 31.73\,\text{N}$.

$$\boxed{\vec{F}_\text{f} = -(31.73\,\text{N})\hat{\imath}}$$

3. **Block sliding or not sliding – not known, again:**
In this case, $F = 10\,\text{N}$, $\alpha = 30^\circ$, and $\theta = 60^\circ$. Again, assuming static equilibrium, we do exactly the same calculations as above (in fact, use the same expressions) and substituting the given values, we get

$$\begin{aligned} F_f &= 10\,\text{kg} \cdot 9.81\,\text{m/s}^2 \cdot \sin 60^\circ - 10\,\text{N} \cdot \cos 30^\circ = 76.3\,\text{N} \\ N &= 10\,\text{kg} \cdot 9.81\,\text{m/s}^2 \cdot \cos 60^\circ - 10\,\text{N} \cdot \sin 30^\circ = 44\,\text{N}. \end{aligned}$$

Here, $|F_f| > \mu N$. Clearly, $F_f$ is not less than or equal to $\mu N$, and therefore, our assumption of static equilibrium is not valid. In fact, the given parameters of the problem will make the block accelerate downhill — a problem of dynamics that cannot be solved with statics equations.

$$\boxed{\vec{F}_\text{f} = \text{no solution}}$$

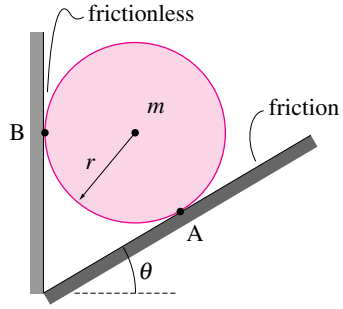



Figure 4.46:
Filename:sfig4-single-ball

**SAMPLE 4.12 How much friction does the ball need?** A ball of mass $m$ sits between an incline and a vertical wall as shown in the figure. There is no friction between the wall and the ball but there is friction between the incline and the ball. Take the coefficient of friction to be $\mu$ and the angle of incline with the horizontal to be $\theta$. Find the force of friction on the ball from the incline.

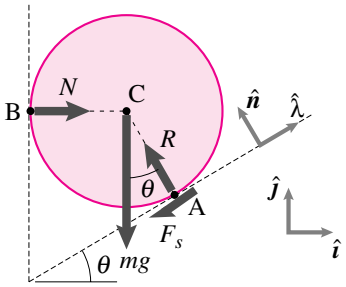



Figure 4.47:
Filename:sfig4-single-ball-a

**Solution** The free body diagram of the ball is shown in Fig. 4.47. Note that the normal reaction of the vertical wall, $N$, the force of gravity, $mg$, and the normal reaction of the incline, $R$, all pass through the center C of the ball. Therefore, the moment balance about point C, $\sum \vec{M}_{\mathrm{C}} = \vec{0}$, gives

$$\underbrace{\vec{r}_{\mathrm{A/C}} \times F_s \hat{\boldsymbol{\lambda}}}_{r F_s} = \vec{0}$$
$$\Rightarrow \quad F_s = 0.$$

Thus the force of friction on the ball is zero! Note that $F_s$ is independent of $\theta$, the angle of incline. Thus, irrespective of what the angle of incline is, in the static equilibrium condition, there is no force of friction on the ball.

$$\boxed{F_s = 0}$$

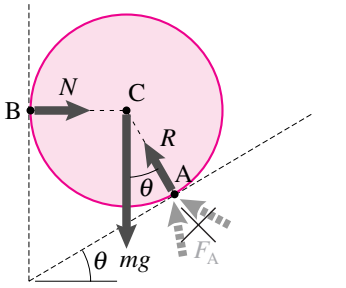



Figure 4.48:
Filename:sfig4-single-ball-b

**Note:** The ball here is a three force body since there are three forces acting on it — two contact forces (at A and B) and one gravity force. Therefore, for equilibrium, all the three forces must intersect at a single point. Now, lines of action of the gravity force and the normal reaction at B intersect at the center C of the ball. Therefore, the line of action of the contact force at A also must pass through the center. This is clearly not possible if the contact force is not normal to the incline (see the candidate contact forces marked by the dashed gray arrows in Fig. 4.48. If there is any non-zero friction force at A, the contact force (the resultant of the normal reaction and the friction force) at A will be tipped away from the normal, thus making its line of action miss the center of the ball and, therefore, violate equilibrium condition.

**SAMPLE 4.13 Will the ladder slip?** A ladder of length $\ell = 4\,\text{m}$ rests against a wall at $\theta = 60°$. Assume that there is no friction between the ladder and the vertical wall but there is friction between the ground and the ladder with $\mu = 0.5$. A person weighing 700 N starts to climb up the ladder.

1. Can the person make it to the top safely (without the ladder slipping)? If not, then find the distance $d$ along the ladder that the person can climb safely. Ignore the weight of the ladder in comparison to the weight of the person.
2. Does the "no slip" distance $d$ depend on $\theta$? If yes, then find the angle $\theta$ which makes it safe for the person to reach the top.

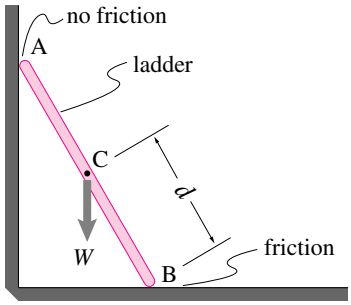



Figure 4.49:
Filename:sfig4-single-ladder

**Solution**

1. The free-body diagram of the ladder is shown in Fig. 4.50. There is only a normal reaction $\vec{\boldsymbol{R}} = R\hat{\boldsymbol{\imath}}$ at A since there is no friction between the wall and the ladder. The force of friction at B is $\vec{\boldsymbol{F}}_s = -F_s\hat{\boldsymbol{\imath}}$ where $F_s \le \mu N$. To determine how far the person can climb the ladder without the ladder slipping, we take the critical case of impending slip. In this case, $F_s = \mu N$. Let the person be at point C, a distance $d$ along the ladder from point B.

   From moment balance about point B, $\sum \vec{\boldsymbol{M}}_B = \vec{\boldsymbol{0}}$, we find

$$\begin{aligned} \vec{\boldsymbol{r}}_{A/B} \times \vec{\boldsymbol{R}} + \vec{\boldsymbol{r}}_{C/B} \times \vec{\boldsymbol{W}} &= \vec{\boldsymbol{0}} \\ -R\ell\sin\theta\hat{\boldsymbol{k}} + Wd\cos\theta\hat{\boldsymbol{k}} &= \vec{\boldsymbol{0}} \\ \Rightarrow \quad R &= W\frac{d\cos\theta}{\ell\sin\theta}. \end{aligned}$$

   From force equilibrium, we get

$$(R - \mu N)\hat{\boldsymbol{\imath}} + (N - W)\hat{\boldsymbol{\jmath}} = \vec{\boldsymbol{0}}. \tag{4.20}$$

   Dotting eqn. (4.20) with $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{\imath}}$, respectively, we get

$$\begin{aligned} N &= W \\ R &= \mu N = \mu W. \end{aligned}$$

   Substituting this value of $R$ in eqn. (4.20) we get

$$\begin{aligned} \mu W &= W\frac{d\cos\theta}{\ell\sin\theta} \\ \Rightarrow \quad d &= \mu\ell\tan\theta \qquad (4.21) \\ &= 0.5 \cdot (4\,\text{m}) \cdot \tan 60° = 3.46\,\text{m}. \end{aligned}$$

   Thus, the person cannot make it to the top safely.

$$\boxed{d = 3.46\,\text{m}}$$

2. The "no slip" distance $d$ depends on the angle $\theta$ via the relationship in eqn. (4.21). The person can climb the ladder safely up to the top if

$$\tan\theta = \frac{1}{\mu} \quad \Rightarrow \quad \theta = \tan^{-1}(\mu^{-1}) = 63.43°.$$

   Thus, any reasonable angle $\theta \ge 64°$ will allow the person to climb up to the top safely.

$$\boxed{\theta \ge 64°}$$

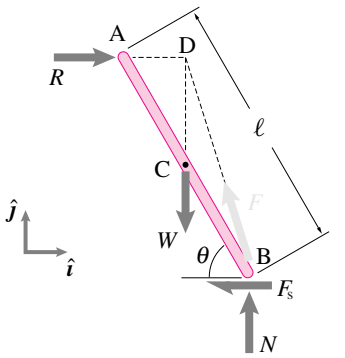



Figure 4.50: The free body diagram of the ladder indicates that it is a three force body. Since the direction of the forces acting at points B and C are known (the normal, horizontal reaction at B and the vertical gravity force at C), it is easy to find the direction of the net ground reaction at A — it must pass through point D. The ground reaction $F$ at A (shown by light grey arrow) can be decomposed into a normal reaction $N$ and a horizontal reaction (the force of friction) $F_s$ at A.
Filename:sfig4-single-ladder-a

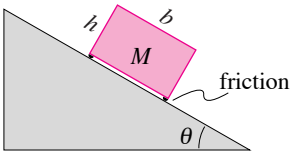



Figure 4.51:
Filename:sfig4-1-friction1

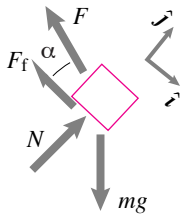



Figure 4.52:
Filename:sfig4-1-friction-a

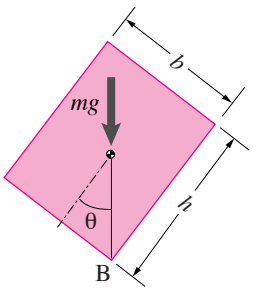



Figure 4.53: The impending tipping condition $\tan\theta = \gamma \equiv b/h$, provided $\tan\theta \le \mu$.
Filename:sfig4-1-friction1b

**SAMPLE 4.14 Will it tip or will it slide?** Whether or not a box of a given width and height will slide or tip over on an inclined plane depends on the slope of the plane and the coefficient of friction. For a given slope $\theta$, find the relationship between the coefficient of friction $\mu$ and the aspect ratio of the box, $\gamma = b/h$ for impending tipping.

**Solution** Let us imagine that we put the box on a flat surface and then slowly start tilting the surface up with respect to the horizontal. At some slope, the box will either tip over or slide. Just before the instant the box starts to tip over or slide, it is in static equilibrium. The magnitude of the friction force at the contact points is $|F| \le \mu N$ where $N$ is the magnitude of the normal force at the contact, and the equality holds only in the case of impending slip. That is, if the box is about to slip, then $F = \mu N$ at each contact point.

The free body diagram of the box is shown in Fig. 4.55. Let us first write the equations of static equilibrium assuming there is no impending slip.

The force balance in the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ directions (see Fig. 4.55) gives

$$F_A + F_B = mg\sin\theta \tag{4.22}$$
$$N_A + N_B = mg\cos\theta. \tag{4.23}$$

The moment equilibrium about the center-of-mass, $\sum \vec{\boldsymbol{M}}_C = \vec{\boldsymbol{0}}$, in the $\hat{\boldsymbol{k}}$ direction gives

$$N_B \cdot \frac{b}{2} - N_A \frac{b}{2} - (F_A + F_B)\frac{h}{2} = 0. \tag{4.24}$$

Substituting $F_A + F_B = mg\sin\theta$ from eqn. (4.22) in eqn. (4.24), and solving eqns. (4.23) and (4.24) simultaneously, we get

$$N_A = \frac{1}{2}mg\left(\cos\theta - \frac{h}{b}\sin\theta\right), \text{ and } \quad N_B = \frac{1}{2}mg\left(\cos\theta + \frac{h}{b}\sin\theta\right).$$

If the box were to tip over (about point B), the support forces at A will go to zero (because of loss of contact). Thus, for impending tipping,

$$N_A = 0 \quad \Rightarrow \quad \cos\theta - \frac{h}{b}\sin\theta = 0 \quad \Rightarrow \quad \tan\theta = \frac{b}{h} = \gamma.$$

Thus, the condition for impending tipping is

$$\tan\theta = \gamma. \tag{4.25}$$

This condition, however, does not guarantee that the box *will* tip over. In fact, it may start sliding before it tips over. We need to check if sliding condition is met before eqn. (4.25) is satisfied. In other words, we need to check the value of friction forces and make sure that $|F_A + F_B| \le \mu(N_A + N_B)$. Thus, for no slipping,

$$F_A + F_B \le \mu(N_A + N_B) \quad \Rightarrow \quad mg\sin\theta \le \mu mg\cos\theta \quad \Rightarrow \quad \tan\theta \le \mu.$$

Using this condition (with equality) in eqn. (4.25), we get the *critical* condition for tipping:

$$\gamma = \mu.$$

$$\boxed{\gamma = \tan\theta \le \mu}$$

You may know this condition geometrically as the line of action of the weight of the box must pass through B and beyond for tipping over (see Fig. 4.53).

**SAMPLE 4.15 How big does the friction force get?** Consider the box on the inclined plane of Sample 4.14 again. The box has aspect ratio $\gamma = b/h$. The coefficient of friction is $\mu$. Imagine that the angle $\theta$ of the inclined plane can be varied. How does the force of friction on the box vary with $\theta$? How does the maximum value of this force depend on $\mu$?

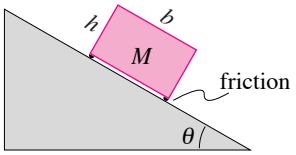



Figure 4.54:
Filename:sfig4-1-friction2

**Solution** If we imagine the inclined plane to be not inclined ($\theta = 0$) but horizontal and the box to be just sitting there, the force of friction on the box has to be zero. As we tilt the plane up ($\theta > 0$), the friction force starts increasing. It increases up to the point of impending slip unless the box tips over before that. Assuming that the aspect ratio of the box prevents it from tipping (see Sample 4.14), we can determine the maximum value up to which the friction force rises before the box starts slipping.

From Sample 4.14, we know that the total friction force $F_s = F_A + F_B = mg\sin\theta$. Thus the normalized friction force (as a fraction of the weight of the block), $F_s/mg$ is

$$\frac{F_s}{mg} = \sin\theta.$$

Thus the total friction force varies as sine of the ramp angle. Howver, this variation is valid only upto the maximum value of the friction force ($\mu N$) when the block starts sliding. The critical angle at which this maximum is attained is $\theta_{\text{slip}} = \tan^{-1}\mu \equiv \phi$ (friction angle). Thus,

$$\left.\frac{F_s}{mg}\right|_{\max} = \sin\phi.$$

Figure 4.56 shows how the maximum normalized friction force varies with $\mu$. Note that for lower values of $\mu$ (which covers most practical values of $\mu$), the relationship is almost linear. Thus, $|F_s/mg| \approx \mu$ for $\mu \leq 0.5$.

$$\boxed{\frac{F_s}{mg} = \sin\theta, \quad \left.\frac{F_s}{mg}\right|_{\max} = \sin\phi}$$

What happens to the friction force after it attains the maximum value $F_s = mg\sin\phi$? For a given ramp angle, the friction force remains constant and the box slides.

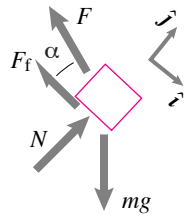



Figure 4.55:
Filename:sfig4-1-friction-a

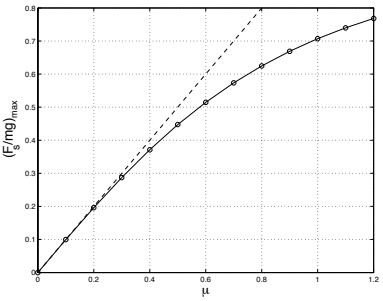



Figure 4.56: The maximum net friction force on the box as a function of the friction coefficient $\mu$. Note that the relationship is almost linear for most practical values of $mu$.
Filename:sfig4-1-friction2-Fvsmu

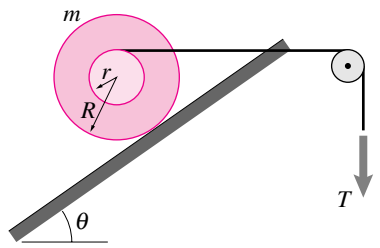



Figure 4.57:
Filename:sfig4-single-spool

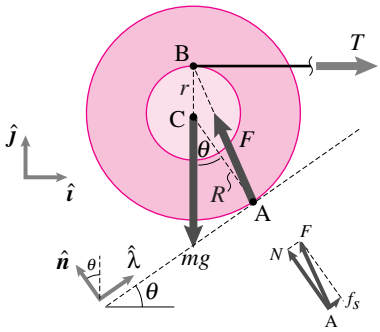



Figure 4.58:
Filename:sfig4-single-spool-a

**SAMPLE 4.16** A spool of mass $m = 2\,\text{kg}$ rests on an incline as shown in the figure. The inner radius of the spool is $r = 200\,\text{mm}$ and the outer radius is $R = 500\,\text{mm}$. The coefficient of friction between the spool and the incline is $\mu = 0.4$, and the angle of incline $\theta = 60°$.

1. Which way does the force of friction act, up or down the incline?
2. What is the required horizontal pull $T$ to balance the spool on the incline?
3. Is the spool about to slip?

**Solution**

1. The free-body diagram of the spool is shown in Fig. 4.58. Note that the spool is a 3-force body. Therefore, in static equilibrium all the three forces — the force of gravity $mg$, the horizontal pull $T$, and the incline reaction $F$ — must intersect at a point. Since $T$ and $mg$ intersect at the top of the inner drum (point B), the reaction force $\vec{F}$ of the incline must be along the direction AB. Now the incline reaction $\vec{F}$ is the vector sum of two forces — the normal (to the incline) reaction $N$ and the friction force $F_s$ (along the incline). The normal reaction force $N$ passes though the center C of the spool. Therefore, the force of friction $F_s$ must point up along the incline to make the resultant $\vec{F}$ point along AB.

   Up the incline

2. From the moment equilibrium about point A, $\sum \vec{M}_\text{A} = \vec{0}$, we get

$$\vec{r}_{\text{C/A}} \times (-mg\hat{j}) + \vec{r}_{\text{B/A}} \times (T\hat{i}) = \vec{0}.$$

   These cross products can be easily evaluated by using the scalar form of the moment of a force as the product of force and the lever arm. Thus the moment of $mg$ is $mg \cdot R\sin\theta$ and the moment of $T$ is $-T \cdot (r + R\cos\theta)$ about point A in the $\hat{k}$ direction. The cross product can also be evaluated using vectors with mixed basis.* Thus the scalar form of the moment balance equation gives

$$\begin{aligned} mgR\sin\theta &= T(R\cos\theta + r) \\ \Rightarrow \quad T &= mg\,\frac{\sin\theta}{\cos\theta + r/R} \\ &= 2\,\text{kg} \cdot 9.81\,\text{m/s}^2 \cdot \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2} + \frac{.2\,\text{m}}{.5\,\text{m}}} \\ &= 18.88\,\text{N}. \end{aligned}$$

   $T = 18.88\,\text{N}$

3. To find if the spool is about to slip, we need to find the force of friction $|F_s|$ and see if $F_s = \mu N$. The force balance on the spool, $\sum \vec{F} = \vec{0}$ gives

$$T\hat{i} - mg\hat{j} + F_s\hat{\lambda} + N\hat{n} = \vec{0} \tag{4.26}$$

   where $\hat{\lambda}$ and $\hat{n}$ are unit vectors along the incline and normal to the incline, respectively. Dotting eqn. (4.26) with $\hat{\lambda}$ we get

$$\begin{aligned} F_s &= -T(\underbrace{\hat{i}\cdot\hat{\lambda}}_{\cos\theta}) + mg(\underbrace{\hat{j}\cdot\hat{\lambda}}_{\sin\theta}) \\ &= -T\cos\theta + mg\sin\theta \\ &= -18.88\,\text{N}(1/2) + 19.62\,\text{N}(\sqrt{3}/2) \\ &= 7.55\,\text{N}. \end{aligned}$$

* Note that

$$\vec{r}_{\text{C/A}} = R\hat{e}_n \text{ and } \vec{r}_{\text{B/A}} = R\hat{e}_n + r\hat{j}.$$

Therefore,

$$\begin{aligned} \vec{r}_{\text{C/A}} \times (-mg\hat{j}) &= -mgR(\hat{e}_n \times \hat{j}) \\ \vec{r}_{\text{B/A}} \times T\hat{i} &= T[R(\hat{e}_n \times \hat{i}) + r(\hat{j} \times \hat{i})]. \end{aligned}$$

Now, from the geometry of the basis vectors, we have, $\hat{e}_n \times \hat{j} = -\sin\theta\hat{k}$ and $\hat{e}_n \times \hat{i} = -\cos\theta\hat{k}$. Therefore,

$$\begin{aligned} \vec{r}_{\text{C/A}} \times (-mg\hat{j}) &= mgR\sin\theta\hat{k}, \\ \vec{r}_{\text{B/A}} \times T\hat{i} &= -TR\cos\theta\hat{k} - r\hat{k}. \end{aligned}$$

Similarly, we compute the normal force $N$ by dotting eqn. (4.26) with $\hat{\boldsymbol{n}}$:

$$\begin{aligned} N &= -T(\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{n}}) + mg(\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{n}}) \\ &= T\sin\theta + mg\cos\theta \\ &= 18.88\,\text{N}(\sqrt{3}/2) + 19.62\,\text{N}(1/2) \\ &= 26.16\,\text{N}. \end{aligned}$$

Now we find that $\mu N = 0.4(26.16\,\text{N}) = 10.46\,\text{N}$ which is greater than $F_s = 7.55\,\text{N}$. Thus $F_s < \mu N$, and therefore, the spool is not about to slip.

Not about to slip

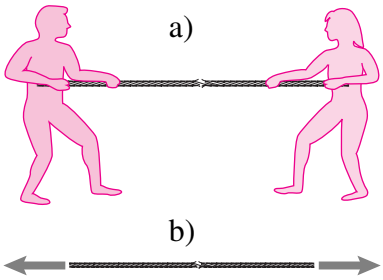



Figure 4.59: a) Two people pulling on a rope that is likely to break in the middle, b) A free body diagram of the rope.

Filename:tfigure-ropeexternal

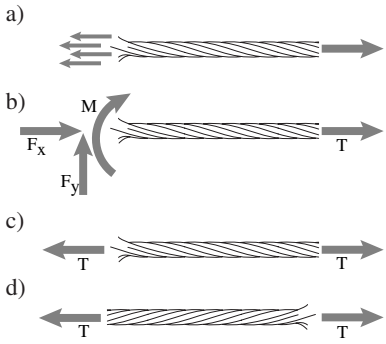



Figure 4.60: a) free body diagram of the right part of the rope, b)the same free body diagram, with the force distribution at the cut replaced with an equivalent force couple system, c) further simplified by using the laws of mechanics, and d) a free body diagram of the left portion of the rope.

Filename:tfigure-ropeinternal

# 4.4 Internal forces

The vague concept of 'forces inside' a structure is in superficial conflict with the subject of mechanics. Mechanics equations only concern the forces *on* an object shown in a free body diagram;'internal forces' have no place on a free body diagram and thus no place in mechanics.

Example: **Pulling on the ends of a rope; nothing internal**

Consider two people pulling apart the frayed rope of fig. 4.59a. A free body diagram of the rope is shown in fig. 4.59b. The laws of mechanics use the *external* forces on an isolated system. These are the forces that show on a free body diagram. For the rope these are the forces at the ends. The free body diagram does not include internal forces. Thus nothing about the 'internal forces' at the fraying part of the rope shows up in the mechanics equations describing the rope.

Mechanics has nothing to say about so called 'internal forces' and thus nothing to say about the rope breaking in the middle. 'Internal forces' are meaningless in mechanics. The section title describes a non-existent subject.

Something's wrong. The problem is somewhat one of language: 'internal forces' are not really internal and they are not really forces!

## 'Internal forces' represent external forces on a smaller body

On page 3 we advertised mechanics as being useful for predicting when things will break. And our intuitions strongly tell us that there is something about the forces *in* the rope that make it break. Yet mechanics equations are based on the forces that show on free body diagrams. And free body diagrams only show external forces. How can we use mechanics based on external forces to describe the 'forces' inside a body? We use an idea whose simplicity hides its incredible utility:

> You cut the body, and what was inside is now on the outside of a smaller body.

In the case of the rope, we cut it in the middle. Then we fool the rope into thinking it wasn't cut using forces (remember, 'forces are *the* measure of mechanical interaction'), one force, say, at each fiber that is cut. Then we get the free body diagram of fig. 4.60a. We can simplify this to the free body diagram of fig. 4.60b because we know that every force system is equivalent to a force and couple at any point, in this case the middle of the rope. If we apply the equilibrium conditions to this cut rope we see that

$$\begin{array}{rcrcr} \text{Sum of vertical forces is zero} & \Rightarrow & F_y & = & 0 \\ \text{Sum of horizontal forces is zero} & \Rightarrow & F_x & = & -T \\ \text{Sum of moments about the cut is zero} & \Rightarrow & M & = & 0. \end{array}$$

Thus we get the simpler free body diagram of fig. 4.60c as you probably already guessed without using the equilibrium equations explicitly.

## Tension

We have just derived the concept of 'tension in a rope' also sometimes called the 'axial force'. The tension is the pulling force on a free body diagram of the cut rope. If we had used the same cut for a free body diagram of the left half of the rope we would see the free body diagram of fig. 4.60d. Either by the principle of action and reaction, or by the equilibrium equations for the left half of the rope, you see also a tension $T$. The force vector is the opposite of the force vector on the right half of the rope. So it doesn't make sense to talk about the tension force vector *in* the rope since different (opposite) force vectors manifest themselves on the two sides of the cut ($-T\hat{\boldsymbol{\imath}}$ on the left end of the right half and $T\hat{\boldsymbol{\imath}}$ on the right end of the left half). Instead we talk about the scalar tension $T$ which expresses the force vector at the cut as

$$\vec{\boldsymbol{F}} = T\hat{\boldsymbol{\lambda}}$$

where $\hat{\boldsymbol{\lambda}}$ is a unit vector pointing out from the free body diagram cut. Because $\hat{\boldsymbol{\lambda}}$ switches direction depending on which half rope you are looking at, the same scalar $T$ works for both pieces.

The *tension* in a rope, cable, or bar is the amount of force pulling out on a free body diagram of the cut rope, cable, or bar. Tension is a scalar.

## Internal 'forces' are not force vectors

Note our abuse of language: force is a vector, tension is an 'internal force' and tension is a scalar. What we call 'internal forces' are not really forces. We can't talk about the internal force vector at a point in the string because there are two different vectors for each cut, one for left half of string and one for the right. An 'internal force' isn't a force vector. Rather it is a quantity from which we can find a force vector once we have made a cut and picked which side of the cut we care about. We use this confusing language because of its firm place in the engineering workplace.

The common phrase *internal force* means 'a scalar with dimensions of force from which you can find the force on one side of a free body diagram cut'.

* Summarizing:

- Internal forces are not internal. Rather they describe the forces on the boundary of a smaller system that has a free body diagram cut that is inside the system of previous interest.

* Calling tension a scalar is a deception for pedagogical purposes. The best representation of 'internal forces' is with tensors which are too mathematically advanced for this book. But it is fun to notice that the concept of a tensor, something prominent in Einstein's theory of general relativity for example, has its origin in tension, our object of study here. Note the non-coincidental similarity of the words *tensor* and *tension*. What is a tensor? Loosely, a tensor is a quantity that helps you find a vector (the force at a cut) once you are told another vector (the unit vector pointing outwards from the cut). [Aside for hyper-experts: The relation between the tension tensor and tension scalar can be expressed by the dyadic representation $\underline{\underline{\boldsymbol{T}}} = T\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_1$.]

- Internal forces are not force vectors. Rather they are scalars from which you can find the force vector acting on one side of a free body diagram cut.

## What is the strength of a structural piece?

Getting back to the question of whether or not the rope will break, we can now characterize the rope by the tension it can carry. A $10kN$ cable can carry a tension of $10,000\,\text{N}$ all along its length. This means a free body diagram of the rope, cut anywhere along its length, could show forces up to but not bigger than $10,000\,\text{N}$. If the rope is frayed it may break at, say, a tension of $2,000\,\text{N}$, meaning a free body diagram with a cut at the fray can only show forces up to $2,000\,\text{N}$.

Note that tension is not always positive. A negative tension (negative pulling out from the ends) is also called a positive compression (positive pushing in at the ends). For ropes we don't see much negative tension, the rope bends with just a hint of compression. But for metal and wood bars, and bones, compression is as important as tension.

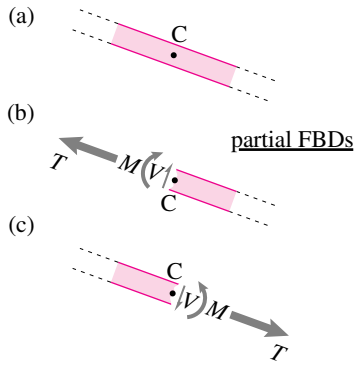



Figure 4.61: a) A piece of a structure, loads not shown; b) a partial free body diagram of the right part of the bar; c) a partial free body diagram of the left part of the bar.

Filename:tfigure-signs

## Shear force and bending moment

To characterize the strength of more than just 2-force bodies we need to generalize the concept of tension. The main idea, which was emphasized in Chapter 3, is this:

> You can make a free body diagram cut anywhere on any body no matter how it is loaded.

As for tension, we define internal forces in terms of the forces (and moments) that show up on a free body diagram cut. Again we consider things (bars) that are rather longer than they are wide or thick because

- Long narrow pieces are commonly used in construction of buildings, machines, plants and animals.
- Internal forces in long narrow things are easier to understand than in bulkier objects.

For now we limit ourselves to 2D statics. At an arbitrary cut we can find the force and moment on the remaining piece in the same manner as in Section 4.2. And we could look at the $x$ and $y$ components of the force. Fine. The problem is that the force and moment we find do not just depend on the cut, but on which body we look at. One one side of the cut a force and moment act, on the other body on the other side of the cut, the opposite force and moment act. Another problem with $xy$ components is that they don't necessarily line up with the natural directions for the structural part. So, for the purposes of thinking about internal forces we break the force into two com-

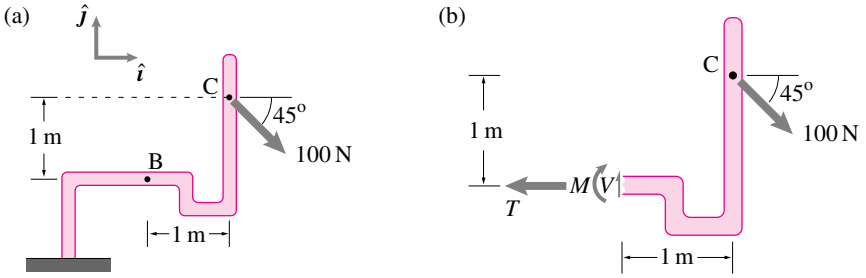
ponents (see fig. 4.61) lined up with the part. And we measure the internal forces with scalars that are the same for both sides of the cut:

- The *tension* $T$ is the scalar part of the force directed along the bar assumed positive when pulling away from the free body diagram cut.
- The *shear force* $V$ is the force perpendicular to the bar (tangent to the free body diagram cut). Our sign convention is that shear is positive if it tends to rotate the cut object clockwise. An equivalent statement of the sign convention is that shear is positive if down on cuts at the right of a bar and positive if up on a cut on the left of bar (and to the right on top and to the left on the bottom).

Since we are just doing 2D problems now, the moment is always in the out-of-plane (typically $\hat{\boldsymbol{k}}$) direction.

- *The bending moment* $M$ is the scalar part of the bending moment. The sign convention is that for a smiling beam (Fig. 4.62): A clockwise ($-\hat{\boldsymbol{k}}$) couple is positive on a left cut and a counterclockwise ($\hat{\boldsymbol{k}}$) couple is positive on a right cut*.

The tension $T$, shear $V$, and bending moment $M$ on fig. 4.61 follows these sign conventions.




Example: **Internal forces in a bent rod**
The internal forces at B can be found by making a free body diagram of a portion of the structure with a cut at B.

| | | | | |
|---|---|---|---|---|
| Sum of vertical forces is zero | $\Rightarrow$ | $V$ | $=$ | $(100/\sqrt{2})$ N |
| Sum of horizontal forces is zero | $\Rightarrow$ | $T$ | $=$ | $(100/\sqrt{2})$ N |
| Sum of moments about the cut at B is zero | $\Rightarrow$ | $M$ | $=$ | $-100\sqrt{2}$ N m. |

You may have noticed that we did get ahead of ourselves and use the concept of tension in a rope or rod as a source of loading with known direction on a particle and rigid body. We will use the concept of tension extensively in our analysis of trusses. Calculating how internal forces vary from point to point in a structure is picked up in Section 8 on page 359.

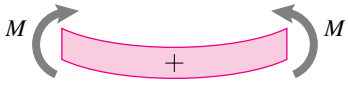



Figure 4.62: The smiling beam sign convention for bending moment. For a horizontal beam, moments which tend to make the beam smile (curve up) are called positive.

Filename:tfigure-smilingbeam

* Note that neither $V$ nor $T$ changes if you rotate your paper until the picture is upside down. However, the definition for the sign convention for $M$ has the disadvantage that the bending moment does changes sign if you turn your paper upside down. (This ambiguity can only be avoided if one picks a favored side or end of the beam).

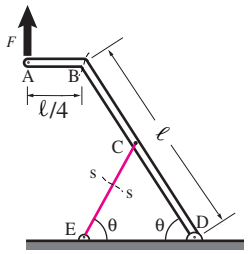



Figure 4.63:
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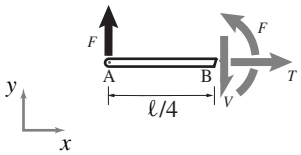



Figure 4.64:
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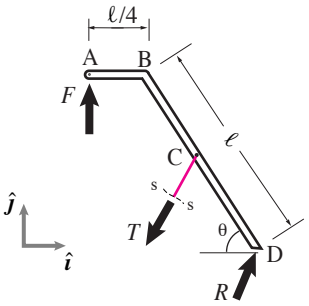



Figure 4.65:
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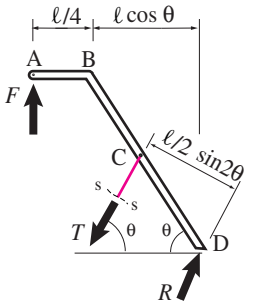



Figure 4.66: Moments of $F$ and $T$ about point D can be evaluated by multiplying the forces with their respective lever arms $\ell/4 + \ell\cos\theta$ and $\ell/2\sin 2\theta$, respectively.
Filename:sfig4-3-new3b

**SAMPLE 4.17** A structure is made up of two bars – a thick bent bar ABC and a thin bar CE. Point C is halfway between B and D, $\ell = 0.8\,\text{m}$ and $\theta = 60^\circ$. Bar ABC is pulled up by a force $F = 500\,\text{N}$ at point A.

1. Find the internal forces in the bar ABC just to the right of point B.
2. Find the force in bar CE at the section s-s shown in the figure.

**Solution** We cut the bar ABC at point B. The free-body diagram of the left part AB is shown in Fig. 4.64. The internal forces acting at the cut section are tension $T$, shear force $V$ and the bending moment $M$. From force balance of part AB in $x$ and $y$ directions, we have

$$T = 0, \quad \text{and} \quad V = F = 500\,\text{N}.$$

From the moment balance about point B, we have

$$M - F\ell/4 = 0 \quad \Rightarrow \quad M = F\ell/4 = 100\,\text{N·m}.$$

$$\boxed{T = 0, \quad V = 500\,\text{N}, \quad M = 100\,\text{N·m}}$$

For finding the tension in rod CE at the given section, we cut the rod at s-s and draw the free-body diagram of the structure along with the upper part of the rod attached at point C. The tension in bar CE is $T$ and the reaction of the support at pin D is $R$. We need to find $T$.

We can write the moment balance equation about point D, $\sum \vec{M}_\text{D} = \vec{0}$, so that the unknown force $R$ (that we are not interested in) disappears from the equation:

$$\vec{r}_{\text{A/D}} \times \vec{F} + \vec{r}_{\text{C/D}} \times \vec{T} \quad = \quad \vec{0}.$$

The moments of $F$ and $T$ about point D can be easily evaluated using the scalar formula 'force times the lever arm' (see Fig. 4.66). Thus, the moment balance equation in $\hat{k}$ direction is:

$$-F\ell(1/4 + \cos\theta) + T\frac{\ell}{2}\sin 2\theta \quad = \quad 0$$

$$\Rightarrow \quad T = \frac{(1/4 + \cos\theta)}{1/2 \cdot \sin 2\theta}.$$

Substituting the given values, $F = 500\,\text{N}$ and $\theta = 60^\circ$, we get

$$T = 866\,\text{N}.$$

$$\boxed{T = 866\,\text{N}}$$

**Note:** Evaluation of the moment equation about point D using vectors and cross products is as follows. Since $\vec{r}_{\text{A/D}} = \vec{r}_{\text{A/B}} + \vec{r}_{\text{B/D}} = -\frac{\ell}{4}\hat{i} + \ell(-\cos\theta\hat{i} + \sin\theta\hat{j})$, $\vec{r}_{\text{C/D}} = \frac{\ell}{2}(-\cos\theta\hat{i} + \sin\theta\hat{j})$, $\vec{F} = F\hat{j}$, and $\vec{T} = T(-\cos\theta\hat{i} - \sin\theta\hat{j})$,

$$\vec{r}_{\text{A/D}} \times \vec{F} = -F\left(\frac{\ell}{4} + \ell\cos\theta\right)\hat{k}, \text{ and } \vec{r}_{\text{C/D}} \times \vec{T} = T\ell\cos\theta\sin\theta\hat{k}.$$

Therefore, the moment balance equation is

$$-F\ell(1/4 + \cos\theta)\hat{k} + T\frac{\ell}{2}\sin 2\theta\hat{k} = \vec{0}.$$

**SAMPLE 4.18** A ladder of length $2d = 4\,\text{m}$ rests against a wall as shown. A person of weight $W = 700\,\text{N}$ stands at C. Assume that the ladder does not slip. Neglecting the weight of the ladder, find the internal forces in the ladder at sections $a$-$a$ and $b$-$b$, at mid points of AC and AB, respectively. (See Sample 4.13.)

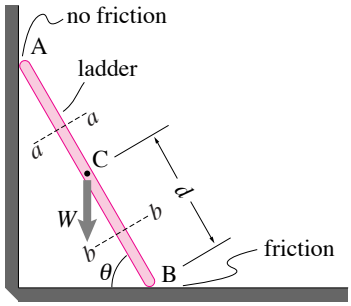



Figure 4.67:
Filename:sfig4-ladder-internalf

**Solution** To find the internal forces at the indicated sections, we need to cut the ladder at those sections, one at a time, draw the free body diagram of each part and carry out the force and moment balance equations. A little anticipation shows that we will need the support reactions at A and B in our calculations. So, let us first determine the support reactions. The free-body diagram of the ladder is shown in Fig. 4.68. The moment balance about point B in $\hat{\boldsymbol{k}}$ direction gives

$$-R(2d\sin\theta) + W(d\cos\theta) = 0 \quad \Rightarrow \quad R = \frac{W}{2}\frac{\cos\theta}{\sin\theta}.$$

The force balance, $\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$, gives

$$R\hat{\boldsymbol{\imath}} - W\hat{\boldsymbol{\jmath}} + \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}} \quad \Rightarrow \quad \vec{\boldsymbol{F}} = -R\hat{\boldsymbol{\imath}} + W\hat{\boldsymbol{\jmath}}.$$

Substituting the given values of $\theta(60^\circ)$ and $W(700\,\text{N})$, we get,

$$\vec{\boldsymbol{R}} = (202\,\text{N})\hat{\boldsymbol{\imath}}, \quad \text{and} \quad \vec{\boldsymbol{F}} = (-202\hat{\boldsymbol{\imath}} + 700\hat{\boldsymbol{\jmath}})\,\text{N}.$$

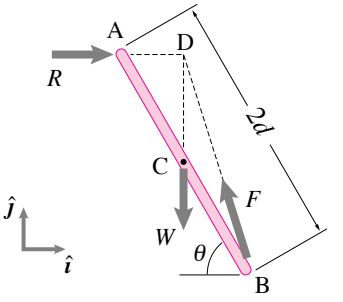



Figure 4.68:
Filename:sfig4-ladder-internalf-a

**Section** $a$-$a$**:** Now, we cut the ladder at $a$-$a$ and draw the free-body diagram of the upper part of the ladder as shown in Fig. 4.69. The force balance for this part gives

$$\begin{aligned} T\hat{\boldsymbol{\lambda}} - V\hat{\boldsymbol{n}} + R\hat{\boldsymbol{\imath}} &= \vec{\boldsymbol{0}} \\ \Rightarrow \quad T &= -R(\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{\lambda}}) = -R\cos\theta \\ \text{and} \quad V &= R(\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{n}}) = R\sin\theta. \end{aligned}$$

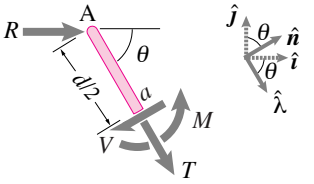



Figure 4.69: Free body diagram of the upper part of the ladder cut at section $a$-$a$. Note that $\hat{\boldsymbol{\lambda}} = \cos\theta\hat{\boldsymbol{\imath}} - \sin\theta\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{n}} = \sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}$.
Filename:sfig4-ladder-internalf-b

Substituting the numerical values of $R$ and $\theta$, we get $T = -101\,\text{N}$ and $V = 175\,\text{N}$. Now, the moment balance equation about $a$ (the cut) gives

$$M - R(d/2)\sin\theta = 0 \quad \Rightarrow \quad M = (1/2)Rd\sin\theta$$

which, with numerical values, gives $M = 175\,\text{N·m}$.

$$\boxed{T = -101\,\text{N}, \quad V = 175\,\text{N}, \quad M = 175\,\text{N·m}}$$

**Section** $b$-$b$**:** Now we consider the internal forces at section $b$-$b$. We cut the ladder at the given section. We can consider the free-body diagram of the upper part or the lower part of the ladder to find the internal forces. Considering the upper part, (see Fig. 4.70) we get, from force balance,

$$T\hat{\boldsymbol{\lambda}} - V\hat{\boldsymbol{n}} + R\hat{\boldsymbol{\imath}} - W\hat{\boldsymbol{\jmath}} = \vec{\boldsymbol{0}}$$

which, as the analysis above, gives

$$\begin{aligned} T &= -R(\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{\lambda}}) + W(\hat{\boldsymbol{\jmath}}\cdot\hat{\boldsymbol{\lambda}}) = -R\cos\theta + W(-\sin\theta) = -707\,\text{N} \\ V &= R(\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{n}}) - W(\hat{\boldsymbol{\jmath}}\cdot\hat{\boldsymbol{n}}) = R\sin\theta - W\cos\theta = -175\,\text{N}. \end{aligned}$$

Similarly, the scalar moment balance equation about point $b$ gives

$$M - R\frac{3d}{2}\sin\theta + W\frac{d}{2}\cos\theta = 0 \quad \Rightarrow \quad M = 175\,\text{N·m}.$$

$$\boxed{T = -707\,\text{N}, \quad V = -175\,\text{N}, \quad M = 175\,\text{N·m}}$$

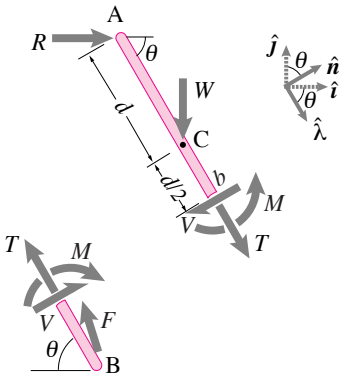



Figure 4.70:
Filename:sfig4-ladder-internalf-c

## 4.5 3D statics of one part

The world we live in, along with the structures and machines we study, is often adequately modeled as 2D, and but sometimes 2D analysis is too crude. Here we use 3D statics to find various unknown aspects of forces acting on one part. Sometimes the 3D analysis results in an answer we could have found accurately enough using a 2D model, and sometimes a 2D model is clearly inadequate. By learning the 3D approach you can get a better sense of when to use a 2D model (which is most of the time for most engineers).

The 3D topic is conceptually the same as in 2D: we use a free body diagram and use the force and moment balance equations. However, the geometry can be more of a challenge, the moment balance equation becomes a full vector equation (instead of just having one non-zero component)*, and the number of scalar equations from one free body diagram increases from 3 to 6. And issues related to static-determinacy arise more often and more subtly.

* For 2D problems we used the phrase 'moment about a point' to be short for 'moment about an axis in the $z$ direction that passes through the point. In 3D moment about a point is a 3-component vector.

* Most elementary text-book problems are statically determinate. Unfortunately most real-world problems, when you first model them, are not statically determinate.

### The statics-of-a-3D-object recipe

Our recipe here:

1) **Draw a free body diagram (FBD)** of the part of interest.
   Use knowledge of the contact conditions (see Chapter 3) to draw known and unknown aspects of the forces appropriately (seeFig. 3.4 on page 126) [hint: use of the form $F\hat{\boldsymbol{\lambda}}$ is often appropriate];
2) **Write equilibrium equations** in terms of the forces (and couples) shown on the FBD;
3) **Solve the equilibrium equations** for unkowns.

### The brute-force approach to statically determinate problems

A problem is *statically determinate* when all as-yet-unknown forces can be found using the equilibrium equations. In 3D statics this generally means that the two vector equilibrium equations

$$\sum \vec{\boldsymbol{F}}_i = \vec{\boldsymbol{0}} \text{ and } \sum \vec{\boldsymbol{M}}_{\text{i/C}} = \vec{\boldsymbol{0}}$$

(where C is any one point that you chose) make up 6 independent scalar equations which you can solve for 6 unknown aspects of the applied forces (say the magnitudes of 6 forces whose directions are known *a priori*)*.

### Alternative equation sets

In 2D single-part statics we noted various alternative to using vector force balance and moment about one point (see page 173). Similarly, here there

are also an infinite number of true equilibrium equations, for example

- $\left(\sum \vec{\boldsymbol{F}}_{\mathrm{i}}\right) \cdot \hat{\boldsymbol{\lambda}} = 0$ where $\hat{\boldsymbol{\lambda}}$ is a vector in any direction you please. ; and
- $\left(\sum \vec{\boldsymbol{M}}_{/\mathrm{C}}\right) \cdot \hat{\boldsymbol{\lambda}} = 0$. This is moment balance about an axis through C in the $\hat{\boldsymbol{\lambda}}$ direction.

From these there are various ways to extract 6 independent scalar equations, including:

- Cartesian components of force balance and moment balance about any point C: $\sum F_x = 0$, $\sum F_x = 0$, $\sum F_x = 0$, $\sum M_{Cx} = 0$, $\sum M_{Cy} = 0$, and $\sum M_{Cz} = 0$. This always works, although it does not necessarily minimize algebra.
- Force balance in any 3 non-coplanar directions and moment balance about point C resolved in any three non-coplanar directions.
- Moment balance about 6 independent axes. There seems to be no simple description of indendent axes but for that they give independent equilibrium equations *. Practically speaking, six moment-about-an-axes equations are likely to be independent if not too many axes are parallel with each other, not too many are coplanar, and not too many intersect at one point.

In any case force balance contributes at most 3 independent equations and moment balance can contribute up to 6 (thus rendering force balance a non-essential tool).

Solving 6 equations in 6 unknowns, or even setting up such for computer solution, is relatively time consuming and error prone. Thus one looks for shortcuts when one can, namely:

Useful shortcuts:

- Use moment balance about an axis that intersects, or is parallel to, as many unknown force lines-of-action as possible (thus those forces do not show up in that equilibrium equation);
- Use force balance in a direction orthogonal to as many of the unknown forces as possible (so those forces don't show up in that equation).

* One can test the sufficiency of the equations by this check: if a force at the origin and a couple are the only forces applied to a system, do the equations demand that they must both be zero.

## Special loadings

### Two- and three-force bodies

The concepts of two-force (page 174) and three-force (page **??**) bodies are identical in 3D.

- If there are only two forces applied to a body in equilibrium they must be equal and opposite and acting along the line connecting the points of application. The full set of six equations tell you no more.
- If there are only three force applied to a body they must all be in the plane of the points of application and the three forces must have lines of action that intersect at one point. The three equations of force balance are an additional restriction on these three forces.

There are other special loadings where the equilibrium equations offer less than 6 independent equations:

- **2D**. If all of the forces have **lines of action in one plane** then there are only three independent scalar equations and thus one can solve for 3 unknowns. For example, if all the forces lie in the $xy$ plane then automatically $\sum F_{iz} = 0$, $\sum M_{ix/C} = 0$, and $\sum M_{iy/C} = 0$.
- **Concurrent forces**. If all the lines of action intersect in one point, say D, then $\sum \vec{M}_{/\mathrm{D}} = \vec{\mathbf{0}}$ is automatically satisfied and only the 3 equations of force balance are independent.
- If all the forces are **parallel** in, say the $\hat{\boldsymbol{k}}$ direction then force balance in the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ directions as well as moment balance about any axis in the $\hat{\boldsymbol{k}}$ direction are automatically satisfied and there are only three independent equilibrium equataions (say $\sum F_z = 0$, $\sum M_x = 0$ and $\sum M_y = 0$.

### What does it mean for a problem to be '2D'?

The world we live in is three dimensional, all the objects to which we wish to study mechanically are three dimensional, and if they are in equilibrium they satisfy the three-dimensional equilibrium equations. How then can an engineer justify doing 2D mechanics? There are a variety of overlapping justifications.

- The 2D equilibrium equations are a subset of the 3D equations. In both 2D and 3D, $\sum F_x = 0$, $\sum F_y = 0$, and $\sum \vec{M}_{/0} \cdot \hat{\boldsymbol{k}} = 0$. So, if when doing 2D mechanics, one just neglects the $z$ component of any applied forces and the $x$ and $y$ components of any applied couples, one is doing correct 3D mechanics, just not all of 3D mechanics. If the forces or conditions of interest to you are contained in the 2D equilibrium equations then 2D mechanics is really 3D mechanics, ignoring equations you don't need.
- If the $xy$ plane is a plane of symmetry for the object and any applied loading, then the three dimensional equilibrium equations not covered by the two dimensional equations, are automatically satisfied. For a car, say, the assumption of symmetry implies that the forces in the $z$ direction will automatically add to zero, and the moments about the $x$ and $y$ axis will automatically be zero.
- If the object is thin and there are constraint forces holding it near the $xy$ plane, and these constraint forces are not of interest, then 2D statics is

also appropriate. This last case is caricatured by all the poor mechanical objects you have drawn so. They are conceptually constrained to lie in your flat paper by invisible slippery glass in front of and behind the paper.

## "Internal forces" in 3D

At a free body diagram cut on a long narrow structural piece in 2D there showed two force components, tension and shear, and one scalar moment. In 3D such a cut shows a force $\vec{\boldsymbol{F}}$ and a moment $\vec{\boldsymbol{M}}$ each with three components. If one picks a coordinate system with the $x$ axis aligned with the bar at the cut, the concept of tension remains the same. Tension is the force component along the bar.

$$T = F_x = \vec{\boldsymbol{F}} \cdot \hat{\boldsymbol{\imath}}.$$

The two other force components, $F_x$ and $F_y$, are two components of shear. The net shear force is a vector in the plane orthogonal to $\hat{\boldsymbol{\imath}}$.

The new concept, often called *torsion* is the component of $\vec{\boldsymbol{M}}$ along the axis:

$$\text{torsion} = M_x = \vec{\boldsymbol{M}} \cdot \hat{\boldsymbol{\imath}}$$

Torsion is the part of the moment that twists the shaft.

The remaining part of the $\vec{\boldsymbol{M}}$, in the $yz$ plane, is the bending moment. It has two components $M_x$ and $M_y$.

## The preponderance of statically indeterminate problems

Unfortunately the real world does not often present problems which are at first blush statically determinate. The statics equations are relevant and provide useful information, they are just not sufficient for finding all unknowns of interest. Finding the forces depends on knowing the deformation properties of the structures as well as details of their initial state.

Example: **Four-leg furniture**
Take the table, chair or bed you are now interacting with. It probably has 4 legs. To keep it simple imagine the legs of the, say, table are on a slippery (negligible-friction) floor and the table is symmetric (left-right and front-back). What are for forces of the floor on the legs? The most we can get from the statics equations is that

$$R_1 = R_3, \quad R_2 = R_4, \text{ and } R_1 + R_2 = W/2.$$

If we insist that there is no glue between the floor and table then $R_1 \geq 0$, $R_2 \geq 0$, $R_3 \geq 0$, $R_4 \geq 0$. But we still can't find the reactions. Here is the variety of solutions:

$$\begin{array}{lllll} & R_1 = R_3 = W/2 & \text{and} & R_2 = R_4 = 0 \\ \text{or} & R_1 = R_3 = 0 & \text{and} & R_2 = R_4 = W/2 \\ \text{or} & R_1 = R_3 = W/4 & \text{and} & R_2 = R_4 = W/4 \\ \text{or} & R_1 = R_3 = C & \text{and} & R_2 = R_4 = W/2 - C \\ & & & \text{(with } C \text{ anything in the interval } \quad 0 \leq C \leq W/2). \end{array}$$

It takes more than just statics to find the forces. One has to know the exact initial shape of the table and floor and how the table and floor 'give' in response to loads.

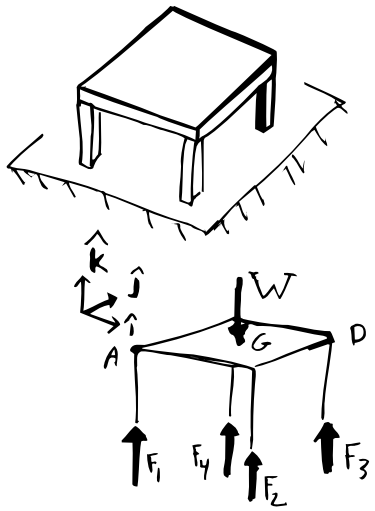



Figure 4.71: A symmetric 4 leg table on a flat floor. Statics is insufficient to find the 4 forces of the ground on the table. The full weight could be carried by either diagonal pair of legs.

Filename:tfigure-4legtable

The lack of static determinacy of a table is not merely an academic curiosity. If you measured the forces of the floor on your table legs they could well differ noticeably from $W/4$ each. Once friction is taken into account the situation is near hopeless.

Example: **Statically determinate stool**
Is it even possible to make a stool in 3 dimensions that is statically determinate? Here's one way. Give it three legs. One leg can have a point frictional contact (3 reaction components), one leg can have a wheel (2 reaction components) and one can be frictionless (like with a castored wheel, 1 reaction component). $3 + 2 + 1 = 6$.

In general it is hard to hold an object in place in three dimensions in a statically determinate manner. Here are some other ways (besides the unusual stool above):

- with six rods that have ball-and-socket joints at both the object-end and at the ground-end. The rods need to have a variety of orientations and attachment points (this ideas is used in a 'Stewart Platform').
- With one ball-and-socket joint and three rods.
- A 3 leg stool with three wheels (at the contact points one can draw a line in the direction normal to rolling, the three such lines must not intersect at a point).
- With one hinge and one two-force-member rod.
- With one axially sliding hinge and two rods.
- With a single welded connection.

Given that many things are held in place in a manner that seems statically indeterminate what what can one do in practice? A common approach is to remove reaction components that you think are relatively unimportant. Some examples:

- A door held by two hinges. That's 10 reaction components. Usually one replaces, in the analysis, the hinges with ball-and-socket joints. That makes 6 unknown reaction components but is still statically indeterminate no matter what the loading (the force along the line connecting the joints cannot be decomposed into parts acting at each joint). So one joint is allowed to slide along the nominal hinge axis.
- 4 leg furniture. Counting friction there are 12 reaction components. If side loads are not an issue than we can assume-away friction. Thus we have only 4 reaction components for 3 equations (see table example above). We can get a unique solution by assuming the forces share the symmetry of the table (thus $F_1 = F_2$).

Given this sad state of affairs in 3D it is easy to see why engineers often resort to the more-easily-made determinate 2D world for their models and analyses.

**SAMPLE 4.19 3-D moment at the support:** A 'T' shaped cantilever beam is loaded as shown in the figure. Find all the support reactions at A.

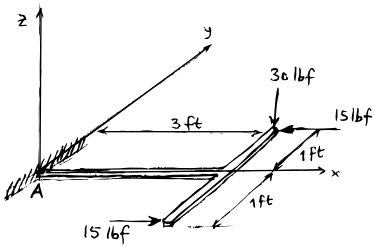



Figure 4.73:
Filename:sfig4-intern-cant3D

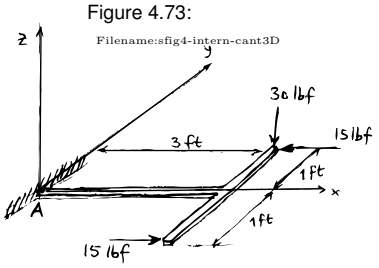



Figure 4.72:
Filename:sfig4-intern-cant3D

**Solution** The free-body diagram of the beam is shown in Fig. **??**. Note that the forces acting on the beam can produce in-plane as well as out of plane moments. Therefore, we show the unknown reactions $\vec{\boldsymbol{R}}$ and $\vec{\boldsymbol{M}}_{\mathrm{A}}$ as general 3-D vectors at A. The moment equilibrium about point A, $\sum \vec{\boldsymbol{M}}_{\mathrm{A}} = \vec{\mathbf{0}}$, gives

$$\vec{\boldsymbol{M}}_{\mathrm{A}} + \vec{\boldsymbol{r}}_{\mathrm{C/A}} \times (\vec{\boldsymbol{F}}_1 + \vec{\boldsymbol{F}}_2) + \vec{\boldsymbol{r}}_{\mathrm{D/A}} \times \vec{\boldsymbol{F}}_3 = \vec{\mathbf{0}}.$$

$$\begin{aligned} \Rightarrow \quad \vec{\boldsymbol{M}}_{\mathrm{A}} &= (\vec{\boldsymbol{r}}_{\mathrm{B/A}} + \vec{\boldsymbol{r}}_{\mathrm{C/B}}) \times (\vec{\boldsymbol{F}}_1 + \vec{\boldsymbol{F}}_2) + (\vec{\boldsymbol{r}}_{\mathrm{B/A}} + \vec{\boldsymbol{r}}_{\mathrm{D/B}}) \times \vec{\boldsymbol{F}}_3 \\ &= (\ell\hat{\boldsymbol{\imath}} + a\hat{\boldsymbol{\jmath}}) \times (-F_1\hat{\boldsymbol{k}} - F_2\hat{\boldsymbol{\imath}}) + (\ell\hat{\boldsymbol{\imath}} - a\hat{\boldsymbol{\jmath}}) \times F_3\hat{\boldsymbol{\imath}}. \end{aligned}$$

But $F_3 = -F_2 = F$ (say). Therefore,

$$\begin{aligned} &= (\ell\hat{\boldsymbol{\imath}} + a\hat{\boldsymbol{\jmath}}) \times (-F_1\hat{\boldsymbol{k}} - F\hat{\boldsymbol{k}}) + (\ell\hat{\boldsymbol{\imath}} - a\hat{\boldsymbol{\jmath}}) \times F\hat{\boldsymbol{\imath}} \\ &= F_1\ell\hat{\boldsymbol{\jmath}} - F_1a\hat{\boldsymbol{\imath}} - 2Fa\hat{\boldsymbol{k}} \\ &= 30\,\mathrm{lbf} \cdot 3\,\mathrm{ft}\,\hat{\boldsymbol{\jmath}} - 30\,\mathrm{lbf} \cdot 1\,\mathrm{ft}\,\hat{\boldsymbol{\imath}} - 2(30\,\mathrm{lbf} \cdot 1\,\mathrm{ft})\hat{\boldsymbol{k}} \\ &= (-30\hat{\boldsymbol{\imath}} + 90\hat{\boldsymbol{\jmath}} - 60\hat{\boldsymbol{k}})\,\mathrm{lb{\cdot}ft}. \end{aligned}$$

The force equilibrium, $\sum \vec{\boldsymbol{F}} = \vec{\mathbf{0}}$, gives

$$\begin{aligned} \vec{\boldsymbol{R}} &= -\vec{\boldsymbol{F}}_1 - \vec{\boldsymbol{F}}_2 - \vec{\boldsymbol{F}}_3 \\ &= -\vec{\boldsymbol{F}}_1 - \vec{\boldsymbol{F}} + \vec{\boldsymbol{F}} \\ &= -(-F_1\hat{\boldsymbol{k}}) = F_1\hat{\boldsymbol{k}} \\ &= 30\,\mathrm{lbf}\hat{\boldsymbol{k}}. \end{aligned}$$

$$\boxed{\vec{\boldsymbol{A}} = 30\,\mathrm{lbf}\hat{\boldsymbol{k}},\ \text{and}\ \vec{\boldsymbol{M}}_{\mathrm{A}} = (-30\hat{\boldsymbol{\imath}} + 90\hat{\boldsymbol{\jmath}} - 60\hat{\boldsymbol{k}})\,\mathrm{lb{\cdot}ft}}$$

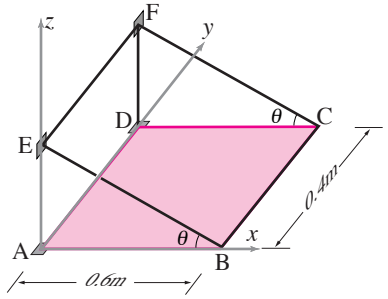



Figure 4.74:
Filename:sfig4-3d-plate

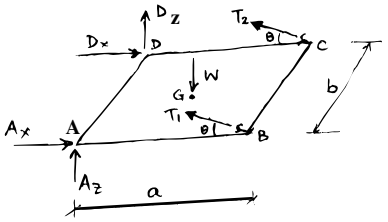



Figure 4.75:
Filename:sfig4-3d-plate-a

**SAMPLE 4.20 An unsolvable problem?** A 0.6 m × 0.4 m uniform rectangular plate of mass $m = 4$ kg is held horizontal by two strings BE and CF and linear hinges at A and D as shown in the figure. The plate is loaded uniformly with books of total mass 6 kg. If the maximum tension the strings can take is 100 N, how much more load can the plate take?

**Solution** The free-body diagram of the plate is shown in Fig. 4.75. Note that we model the hinges at A and D with no resistance in the $y$-direction. Since the plate has uniformly distributed load (including its own weight), we replace the distributed load with an equivalent concentrated load $\vec{W}$ acting vertically through point G.

The various forces acting on the plate are

$$\vec{W} = -W\hat{k},\ \vec{T}_1 = T_1\hat{\lambda}_{\rm BE},\ \vec{T}_2 = T_2\hat{\lambda}_{\rm CF},\ \vec{A} = A_x\hat{\imath} + A_z\hat{k},\ \vec{D} = D_x\hat{\imath} + D_z\hat{k}.$$

Here, $\hat{\lambda}_{\rm BE} = \hat{\lambda}_{\rm CF} = -\cos\theta\hat{\imath} + \sin\theta\hat{k} = \hat{\lambda}$(let). Now, we apply moment equilibrium about point A, *i.e.*, $\sum \vec{M}_{\rm A} = \vec{0}$.

$$\vec{r}_{\rm B} \times \vec{T}_1 + \vec{r}_{\rm C} \times \vec{T}_2 + \vec{r}_{\rm G} \times \vec{W} + \vec{r}_{\rm D} \times \vec{D} = \vec{0} \tag{4.27}$$

where,

$$\begin{aligned}
\vec{r}_{\rm B} \times \vec{T}_1 &= a\hat{\imath} \times T_1\hat{\lambda} = -aT_1\sin\theta\hat{\jmath} \\
\vec{r}_{\rm C} \times \vec{T}_2 &= (a\hat{\imath} + b\hat{\jmath}) \times T_2\hat{\lambda} = T_2 b\sin\theta\hat{\imath} - T_2 a\sin\theta\hat{\jmath} + T_2 b\cos\theta\hat{k} \\
\vec{r}_{\rm G} \times \vec{W} &= \frac{1}{2}(a\hat{\imath} + b\hat{\jmath}) \times (-W\hat{k}) = -\frac{Wa}{2}\hat{\imath} + \frac{Wa}{2}\hat{\jmath} \\
\vec{r}_{\rm D} \times \vec{D} &= b\hat{\jmath} \times (D_x\hat{\imath} + D_z\hat{k}) = D_z b\hat{\imath} - D_x b\hat{k}.
\end{aligned}$$

Substituting these products in eqn. (4.27) and dotting with $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$, we get

$$T_2\sin\theta + D_z = \frac{W}{2} \tag{4.28}$$

$$T_2\cos\theta - D_x = 0 \tag{4.29}$$

$$(T_1 + T_2)\sin\theta = \frac{W}{2}. \tag{4.30}$$

The force equilibrium, $\sum \vec{F} = \vec{0}$, gives

$$\vec{A} + \vec{D} + \vec{T}_1 + \vec{T}_2 + \vec{W} = \vec{0}.$$

Again, substituting the forces in their component form and dotting with $\hat{\imath}$ and $\hat{k}$ (there are no $\hat{\jmath}$ components), we get

$$\begin{aligned}
A_x + D_x - (T_1 + T_2)\cos\theta &= 0 \\
\Rightarrow \quad A_x - T_1\cos\theta &= 0 \qquad (4.31) \\
A_z + D_z + (T_1 + T_2)\sin\theta &= 0 \\
\Rightarrow \quad A_z + T_1\sin\theta &= \frac{W}{2}. \qquad (4.32)
\end{aligned}$$

These are all the equations that we can get. Now, note that we have five independent equations (eqns. (4.28) to (4.32)) but six unknowns. Thus we cannot solve for the unknowns uniquely. This is an indeterminate structure! No matter which point we use for our moment equilibrium equation, we will always have one more unknown than the number of independent equations. We can, however, solve the problem with an extra assumption (see comments below) — the

structure is symmetric about the axis passing through G and parallel to $x$-axis. From this symmetry we conclude that $T_1 = T_2$. Then, from eqn. (4.31) we have

$$2T \sin\theta = \frac{W}{2} \quad \Rightarrow \quad T = \frac{W}{4 \sin\theta}.$$

We can now find the maximum load that the plate can take subject to the maximum allowable tension in the strings.

$$\begin{aligned} W &= 4T \sin\theta \\ \Rightarrow \quad W_{\max} &= 4T_{\max} \sin\theta \\ &= 4(100\,\text{N}) \cdot \frac{1}{2} = 200\,\text{N}. \end{aligned}$$

The total load as given is $(6+4)\,\text{kg} \cdot 9.81\,\text{m/s}^2 = 98.1\,\text{N} \approx 100\,\text{N}$. Thus we can double the load before the strings reach their break-points. Now the reactions at D and A follow from eqns. (4.28), (4.29), (4.31), and (4.32).

$$\begin{aligned} D_z = A_z = \frac{W}{2} - T \sin\theta &= \frac{W}{2} \\ D_x = A_x = T \cos\theta &= \frac{W}{4} \cot\theta. \end{aligned}$$

$$\boxed{W_{\max} = 200\,\text{N}}$$

**Comments:**

1. We got only five independent equations (instead of the usual 6) because the force equilibrium in the $y$-direction gives a zero identity (0 = 0). There are no forces in the $y$-direction. The structure seems to be unstable in the $y$-direction — if you push a little, it will move. Remember, however, that it is so because we chose to model the hinges at A and D that way keeping in mind the only vertical loading. The actual hinges used on a bookshelf will not allow movement in the $y$-direction either. If we model the hinges as ball and socket joints, we introduce two more unknowns, one at each joint, and get just one more scalar equation. Thus we are back to square one. There is no way to determine $A_y$ and $D_y$ from equilibrium equations alone.
2. The assumption of symmetry and the consequent assumption of equality of the two string tensions is, mathematically, an extra independent equation based on deformations (strength of materials). At this point, you may not know any strength of material calculations or deformation theory, but your intuition is likely to lead you to make the same assumption. Note, however, that this assumption is sensitive to accuracy in fabrication of the structure. If the strings were slightly different in length, the angles were slightly off, or the wall was not perfectly vertical, the symmetry argument would not hold and the two tensions would not be the same.

Most real problems are like this — indeterminate. Our modelling, which requires insight, makes them determinate and solvable.

# Problems for Chapter 4

Statics of one object

## 4.1 Static equilibrium of a particle

### Preparatory Problems

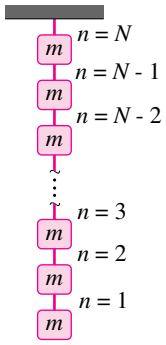
**4.1** What is a particle?

**4.2** What are the equations of equilibrium for a particle (also called "*equilibrium conditions*", "*force balance*", or "*linear momentum balance for statics*"?

**4.3** A string connects a particle A at (1m, 2m) to a support B at (3m, 5m). The tension in the string is 10N. There are other strings also holding the particle in place. What is the force of string AB on the particle?

**4.4** A frictionless ramp connects A at (3m, 5m) to B at (12m, 17m). The ramp pushes a block with a force of-50N. Express the force from the ramp as a vector $\vec{F}$ (ignore the other forces that also act on the block holding it in place).

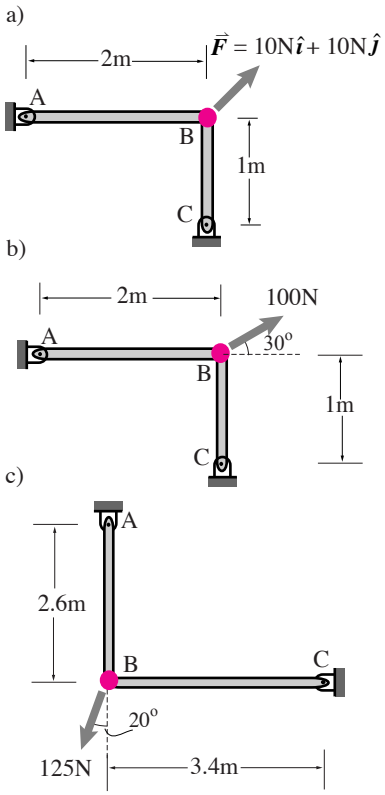
**4.5** $N$ small blocks each of mass $m$ hang vertically as shown, connected by $N$ inextensible strings. Find the tension $T_n$ in string $n$.




**problem 4.5:**

Filename:pfigure2-hanging-masses

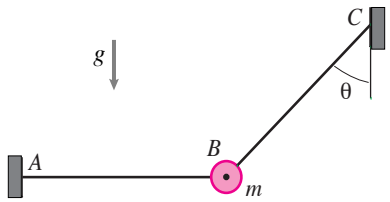
**4.6** For each situation below, find the tensions in the two rods.




**problem 4.6:**

Filename:pfigure-particle-1

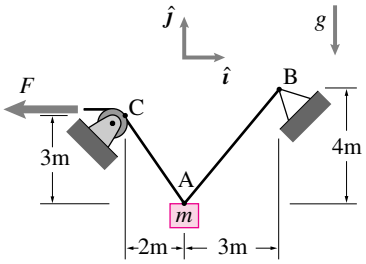
**4.7** A particle of mass $m = 2\,\text{kg}$ hangs from strings AB and AC as shown. AB is horizontal and $\theta = 45^\circ$. Find the tension in the two strings.




**problem 4.7:**

Filename:pfigure4-1-rp1

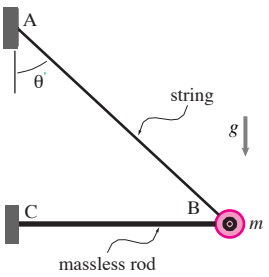
**4.8** What force should be applied to the end of the string over the pulley at $C$ so that the mass at $A$ is at rest?




**problem 4.8:**

Filename:f92h1p1-a

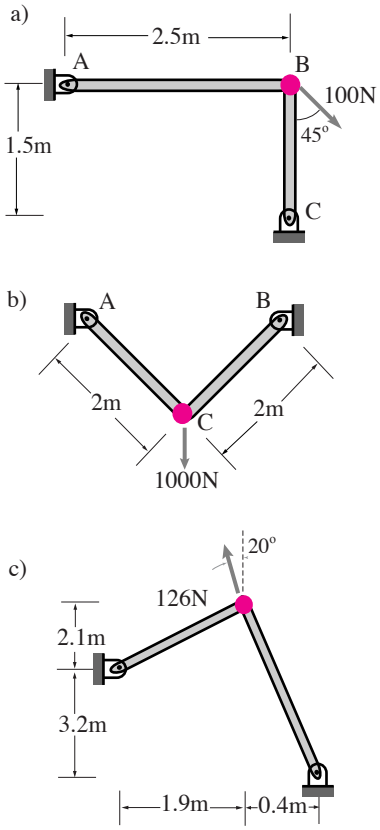
**4.9** A particle of mass $m = 5\,\text{kg}$ at the end of a horizontal massless rod CB of length 1.2 m is held in place with the help of a string AB that makes an angle $\theta = 45^\circ$ with the vertical in the equilibrium position. Find the tension in the bar CB (it is ok to have negative tension).




**problem 4.9:**

Filename:pfigure4-1-rp2

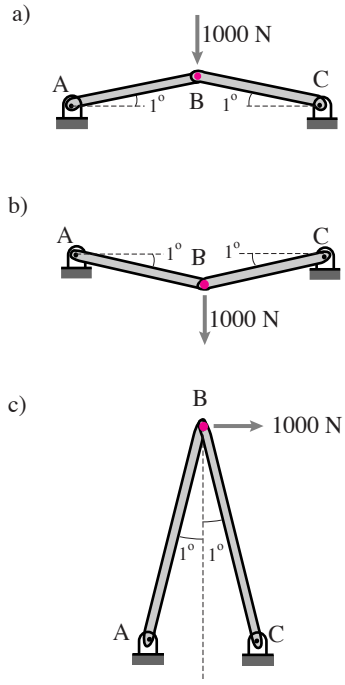
**4.10** For each structure shown below, find the tension in each rod. (Note the tension can be less than zero.)




**problem 4.10:**

Filename:pfigure-particle-2a

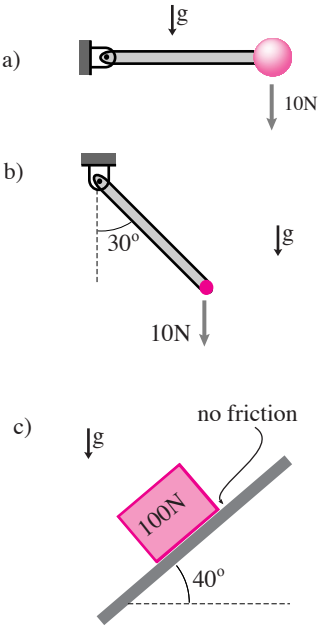
**4.11** In the following structures, a pin connects two thin bars that are very nearly either horizontal or vertical. Find the tensions in each rod under the applied loads. (Note the tension is less than zero for some of the rods.)




**problem 4.11:**

Filename:pfigure-particle-2b

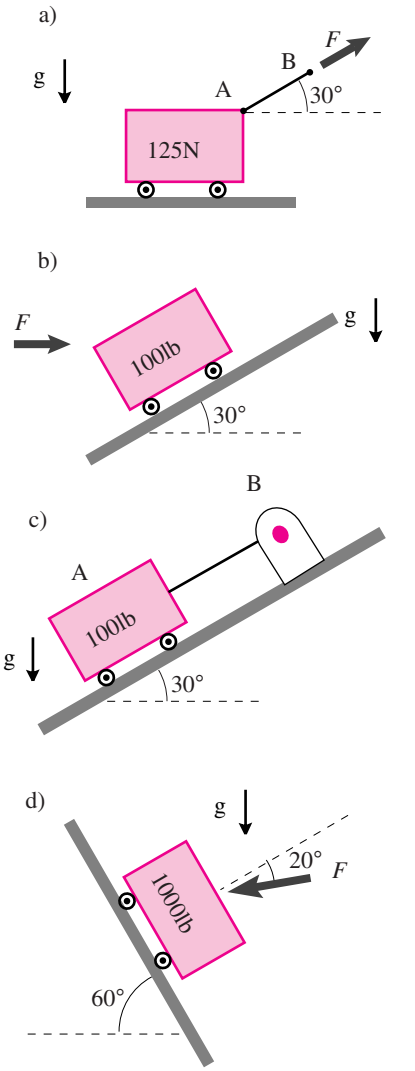
**4.12** For each situation shown below, equilibrium is not possible. Write the vector equation for force balance and show that it has no solutions (*i.e.*, leads to an equation like $7 = 0$).




**problem 4.12:**

Filename:pfigure-particle-3

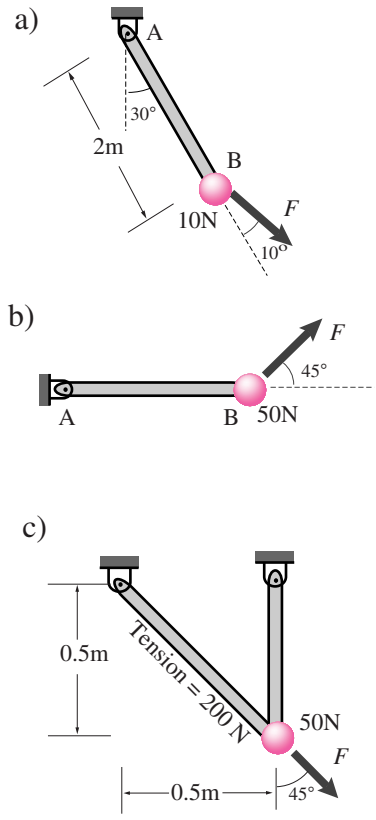
**4.13** Assume no sliding friction ($\mu = 0$). Assume equilibrium. Find all reactions, tensions, and forces.




**problem 4.13:**

Filename:pfigure-particle-4

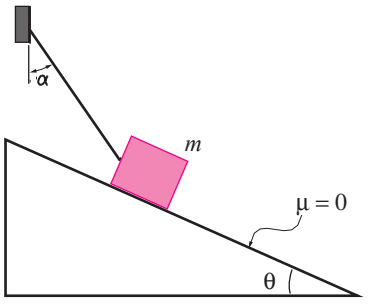
**4.14** Find the unknown forces and tensions in each structure shown below.




**problem 4.14:**

Filename:pfigure-particle-5

**4.15** A block of mass $m = 5\,\text{kg}$ rests on a frictionless inclined plane as shown in the figure. Let $\theta = \alpha = 30^\circ$. Find the tension in the string.

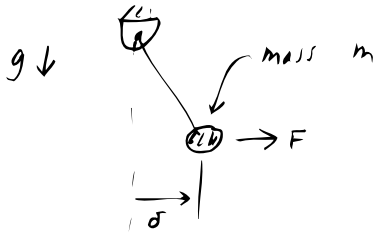



**problem 4.15:**

Filename:pfigure4-1-rp3

## More-Involved Problems

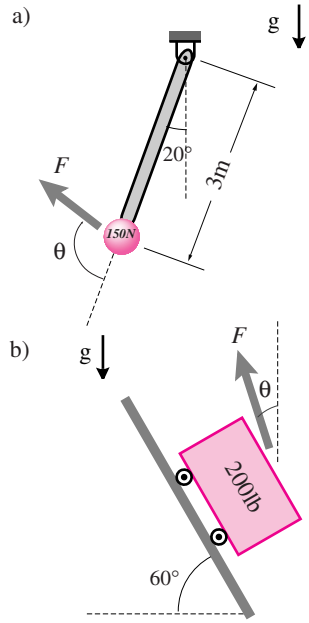
**4.16** For small $\delta$ what is the relation between $F$ and $\delta$ (and $g$ and $\ell$) for a static pendulum?




**problem 4.16:** A pendulum as a spring.

Filename:pfigure-spring3

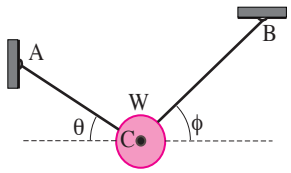
**4.17** In the situations shown in the figures, find the value of $\theta$ that minimizes $F$. What is the corresponding value of $F$ in each case?




**problem 4.17:**

Filename:pfigure-particle-6

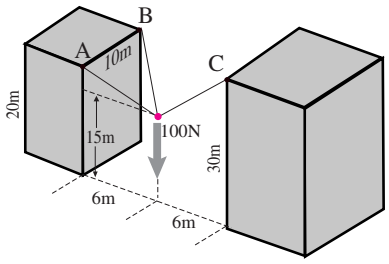
**4.18** An object of weight $W = 10\,\text{N}$ is held in equilibrium in the vertical plane by two strings AC and BC. Let $\theta = 30^\circ$ and $0 \le \phi \le 90^\circ$. Find and plot the tension in the two strings against $\phi$ and comment on the variation of the tension.




**problem 4.18:**

Filename:pfigure4-1-rp5

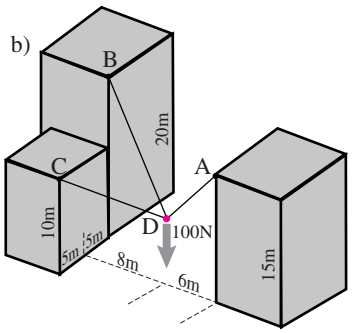
**4.19** Find the tensions in the three strings shown in the figure.




**problem 4.19:**

Filename:pfigure-particle-7

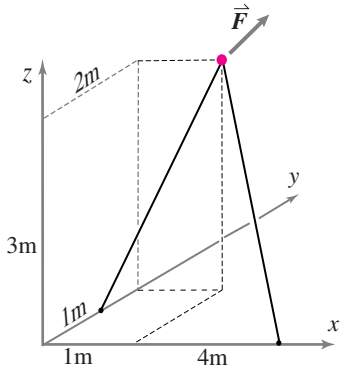
**4.20** Find the tensions in the three strings shown in the figure (string CD is horizontal).




**problem 4.20:**

Filename:pfigure-particle-8

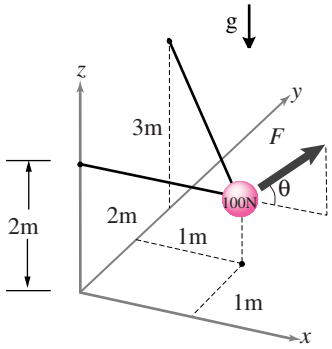
**4.21** Show that the particle acted upon by the given force $\vec{F} = (3\hat{\imath} + 4\hat{\jmath} + 5\hat{k})$ N, and held by the two bars as shown in the figure cannot be in equilibrium.




**problem 4.21:**

Filename:pfigure-particle-3D3

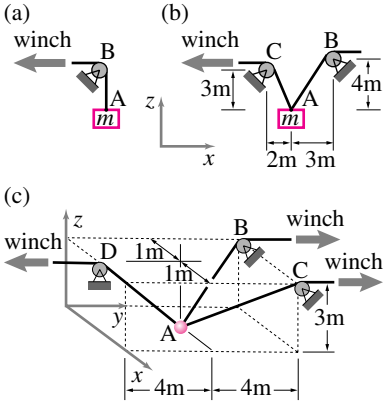
**4.22** In the figure shown, the force $\vec{F}$ acts on the particle (weighing 100 N) in the $x$-$z$ plane. Find $F$ as a function of $\theta$ for equilibrium of the particle. For what value of $\theta$, the required force is minimum?




**problem 4.22:**

Filename:pfigure-particle-3D4

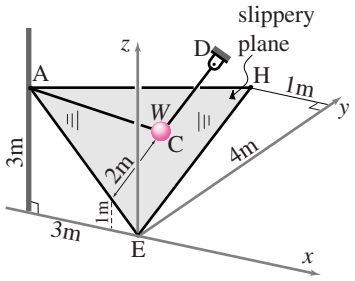
**4.23** For the three cases (a), (b), and (c), below, find the tension in the string AB. In all cases the strings hold up the mass $m = 3\,\text{kg}$. You may assume the local gravitational constant is $g = 10\,\text{m/s}^2$. In all cases the winches are pulling in the string so that the velocity of the mass is a constant 4 m/s upwards (in the $\hat{k}$ direction). [ Note that in problems (b) and (c), in order to pull the mass up at constant rate the winches must pull in the strings at an unsteady speed.]




**problem 4.23:**

Filename:f92h1p1

**4.24** A block of weight W , held by two strings AC and DC, rests on a slippery plane AEH. String CD is parallel to EH. Find the tensions in the two strings and the reaction of the plane. You may approximate AC to lie in the plane AEH.




**problem 4.24:**

Filename:pfigure-particle-9

## 4.2 Static equilibrium of one body

## Preparatory Problems

**4.25** For problems below, assume a 2D free-body diagram has been drawn where forces $\vec{F}_1, \vec{F}_2, \ldots, \vec{F}_5$ are applied at positions $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_5$ relative to the origin. Use this information in the answers below.

a) What is the force balance equation?
b) What is the moment balance equation about the origin?
c) What are equilibrium conditions?
d) Write equilibrium conditions as many different ways as you can.
e) How many independent scalar equations can one write using various force and moment balance equations?
f) If force $\vec{F}_4$ is moved to a new position along the its direction, which equilibrium equations are changed and which are not?
g) If force $\vec{F}_4$ is displaced sideways relative to its direction, which equilibrium equations are changed and which are not?

**4.26** What is the meaning of *the line of action of a force*?

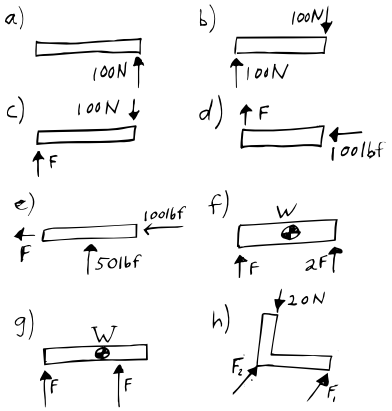
**4.27** If only two forces, $\vec{F}_1$ and $\vec{F}_2$, act on a body at $\vec{r}_1$ and $\vec{r}_2$, what do the equilibrium conditions tell you about the two forces?

**4.28** If only three forces, $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$, act on a body at $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$, what do the equilibrium conditions tell you about the three forces?

**4.29** Which of the bars below cannot possibly be in equilibrium and which ones can? (Where the center of mass is indicated, assume non-zero weight acting vertically downwards. Assume dimensions as needed.)

i) Explain in words.
ii) Explain using equations.

Note that scalars (e.g., $F$, $F_1$, etc.) can be positive or negative.



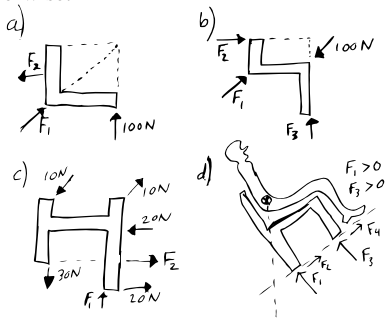

**problem 4.29:**

Filename:pfigure-4-2-manybars1

**4.30** Which of the objects below cannot possibly be in equilibrium and which ones can? (Where the center of mass is indicated, assume non zero weight acting vertically downwards. Assume dimensions as needed.)

a) Explain in words.
b) Explain using equations.

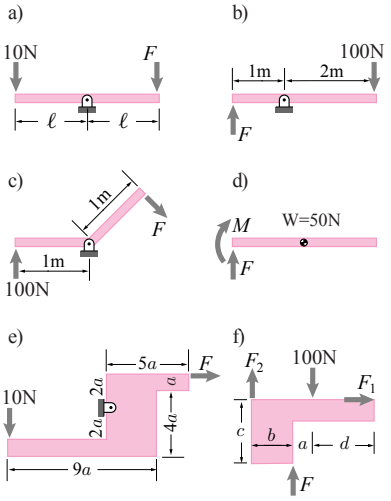
Note that scalars (e.g., $F$, $F_1$, etc.) can be positive or negative unless mentioned otherwise.




**problem 4.30:**

Filename:pfigure-4-2-manybars2

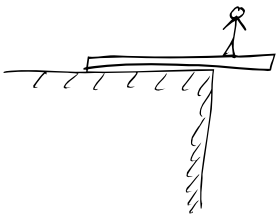
**4.31** In the problems shown below, find $F$ for equilibrium.




**problem 4.31:**

Filename:pfigure-4-2-basics

**4.32** A straight uniform 2000 N beam is 6 m long. It rests on a flat roof with a 2 m overhang. How far out the overhang can an 800 N person walk without the beam tipping over?

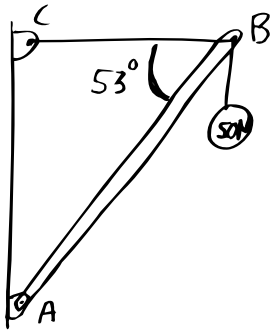


**problem 4.32:**

Filename:pfigureSoodak4-50

**4.33** The uniform bar AB is 5 m long and weighs 100 N. It is pinned at A and supported by the horizontal cord BC attached at end B. A 50 N weight hangs from end B.

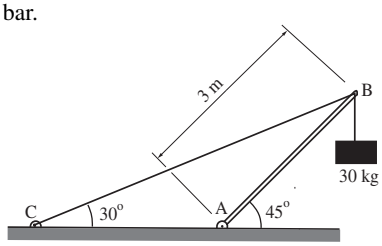
a) Find the tension in cord C.
b) Find the magnitude and direction of the force exerted on the pin at A by the bar.




**problem 4.33:**

Filename:pfigureSoodak4-11

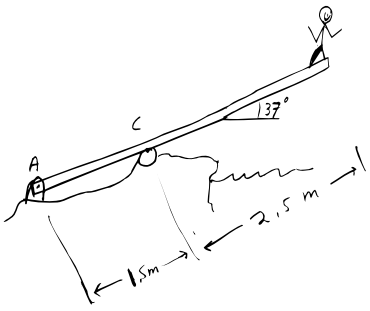
**4.34** For static equilibrium of the system and the configuration shown in the figure, find the support reaction at end A of the bar.




**problem 4.34:**

Filename:pfigure4-1-rp10

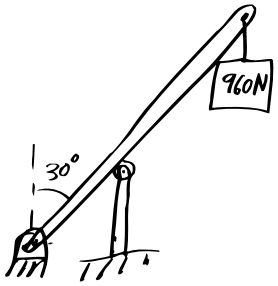
**4.35** A 400 N child stands on the end of a uniform 800 N diving plank which is pinned on one end and which also rests on a log (idealized as frictionless). Find the force of the log on the plank and of the pin on the plank.




**problem 4.35:**

Filename:pfigureSoodak4-46

**4.36** A negligible weight 6 m rod is pinned at one end and leans over a frictionless wall a third of the way up from the bottom. Find the forces of the wall and the pin on the rod.



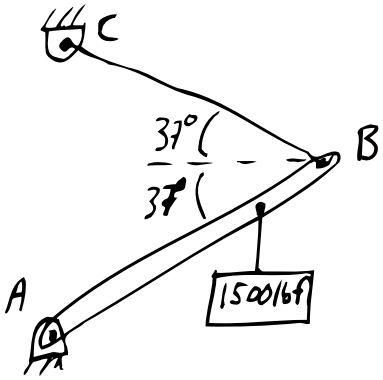

**problem 4.36:**

Filename:pfigureSoodak4-47

## More-Involved Problems

**4.37** The uniform boom AB is 20 ft long and weighs 150 lbf. A 1500 lbf weight is suspended from a point 5 ft from end B. The boom is pinned at A and supported by the cable BC attached at end B.

a) Find the tension in the cable.
b) Find the force exerted on the boom by the pin at A.

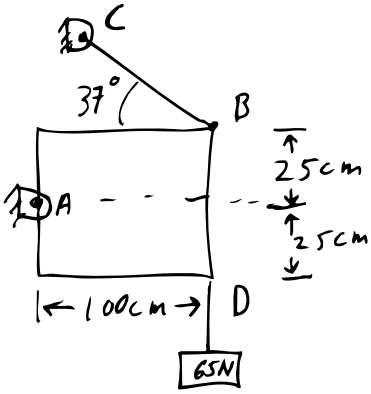



**problem 4.37:**

Filename:pfigureSoodak4-12

**4.38** The 30 N uniform rectangular plate is supported by a pin at A and cable BC attached at corner B. A 65 N weight hangs from corner D.

a) Find the tension in the cable.
b) Find the force exerted on the plate by the pin at A.



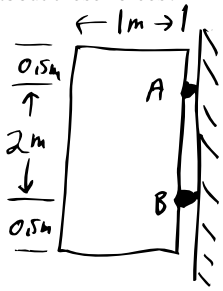

**problem 4.38:**

Filename:pfigureSoodak4-13

**4.39** A uniform door of width 1 m and weight 200 N is supported by two hinges a distance 2 m apart.

a) Find the horizontal component of the force by the door on the upper hinge.
b) Find the horizontal component of the force by the door on the lower hinge.

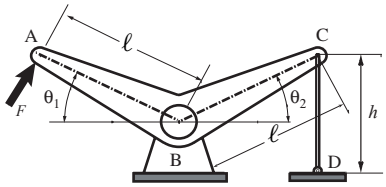
c) Can you find the vertical force of the door on the upper or lower hinge? If not, what do you know about these forces?




**problem 4.39:**

Filename:pfigureSoodak4-15

**4.40** In the mechanism shown, find the maximum force $F$ that can be applied at A normal to the link AB such that the magnitude of the force in rod CD does not exceed 10 kN.

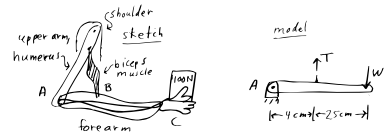



**problem 4.40:**

Filename:pfigure4-1-rp13

**4.41** For biomechanics purposes muscles are commonly modeled as massless cables and joints (elbow, shoulder, hip, ankle, etc) as frictionless hinges connecting rigid bones. You will find that the muscle tension and joint reaction forces are large compared to the loads being carried. This is a general feature in biomechanics because muscles usually have short lever-arms relative to the bone lengths.

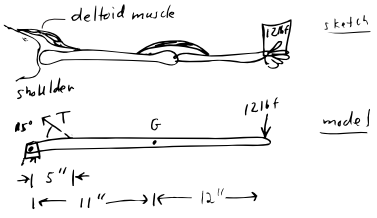
A human forearm weights 14 N and supports a 100 N weight. Find the muscle tension and the force of the upper arm on the forearm at the elbow.




**problem 4.41:**

Filename:pfigureSoodak4-23

**4.42** See Problem 4.41. An arm weighs 7 pounds and supports a 12 pound weight. Find the tension in the deltoid muscle and the force of the body on the arm at the shoulder joint.



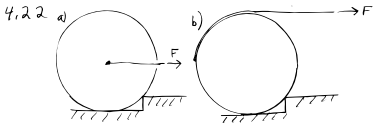

**problem 4.42:**

Filename:pfigureSoodak4-24

**4.43** A 240 N roller is 1 m in diameter. It is being pulled over a 0.1 m curb with a horizontal rope. The roller does not slide on the curb.

a) What is the force required to lift the roller over the curb with the rope attached at the middle?

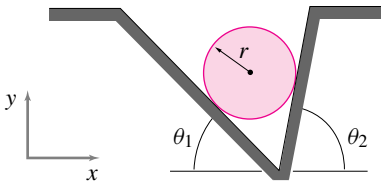
b) What is the force required if instead the rope is instead wrapped around the roller as shown?




**problem 4.43:**

Filename:pfigureSoodak4-22

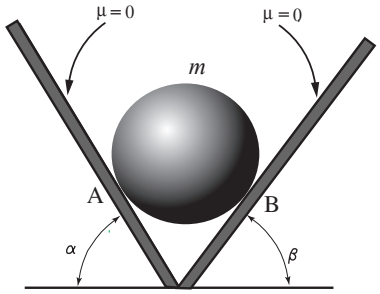
**4.44** What are the forces on the disk due to the groove? Define any variables you need.




**problem 4.44:**

Filename:ch2-5-b

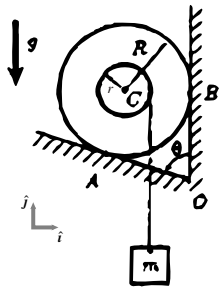
**4.45** A solid sphere of mass $m = 5$ kg and radius $R = 250$ mm rests between two frictionless inclined planes. Let $\alpha = 60^\circ$. Find the magnitudes of normal reactions of the plane as functions of $\beta$ and plot normalized reactions ($N_1/mg$ and $N_2/mg$ for $0 < \beta \leq 90^\circ$). Comment on the plot.




**problem 4.45:**

Filename:pfigure4-1-rp6

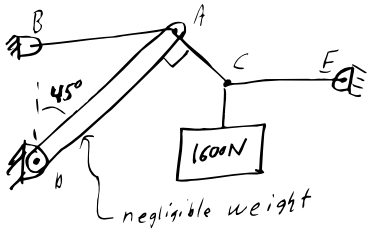
**4.46** Assuming the spool is massless and that there is no friction at point A, find the force on the spool at point B in order to maintain equilibrium. Answer in terms of some or all of $r$, $R$, $g$, $\theta$, and $m$.




**problem 4.46:**

Filename:pfigure-spoolinwedge

**4.47** Find the tension in cord AB.




**problem 4.47:**

Filename:pfigureSoodak4-48

## 4.3 Friction and equilibrium

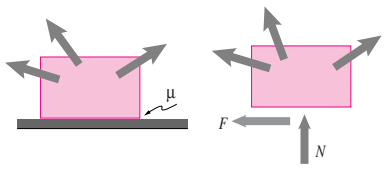
## Preparatory Problems

**4.48** For the block shown in the figure, what do you know about $F$ if

a) the block is sliding to the right

b) the block is sliding to the left

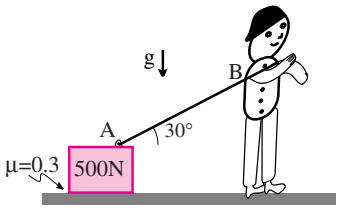
c) the block is not sliding.




**problem 4.48:**

Filename:pfigure-particle-block1

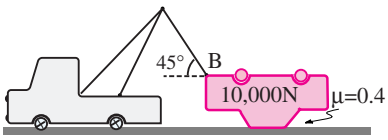
**4.49** A block weighing 500 N is dragged slowly on the ground as shown in the figure. Find the tension in the string?




**problem 4.49:**

Filename:pfigure-particle-10a

**4.50** Find the tension in the two rods on the tow truck as well as the tension in the string assuming the car is dragged at constant speed.




**problem 4.50:**

Filename:pfigure-particle-10b

**4.51** Consider the tow truck dragging the car in Problem 4.50 again. In order to ensure safety, you would like to minimize the tension in the rope attached to the car. Assume that the angle shown at point B is $\theta$.

a) What value of $\theta$ minimizes the tension in the rope?
b) What is the corresponding value of $T$?
c) What is the force of the ground on the car?

.

## More-Involved Problems

**4.52** A 30, 000 N stone cube one meter on a side was dragged up a 20° ramp by 100 of a Pharaoh's slaves by a rope parallel to the slope. The coefficient of friction was $\mu = 0.2$. Assume all the ground contact is at the front and back edges of the cube.

a) Find the dragging force.
b) Find the force on the front and back edges of the cube.

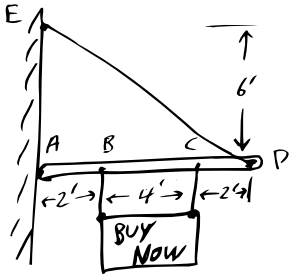



**problem 4.52:**

Filename:pfigureSoodak4-52

**4.53** The 20 lbf uniform rectangular sign is suspended form the strut ABCD by two wires. The strut is supported by cable DE and a pin at A.

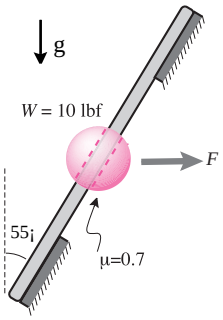
a) Find tension DE.
b) Suppose the workers who hung the sign forgot to pin the strut to the wall at point A. What is the least value of $\mu$ between the strut and wall for the system to maintain equilibrium.




**problem 4.53:**

Filename:pfigureSoodak4-14

**4.54** A horizontal force $F$ is applied to slide the bead on the rod shown in the figure. Find the value of $F$ that is required to initiate sliding. Why is $F$ so big or small?

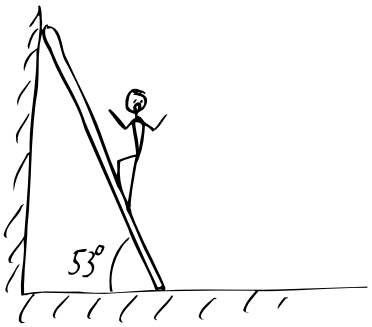



**problem 4.54:**

Filename:pfigure-particle-12a

**4.55** A 130 pound person climbs a 120 pound ladder that is 30 ft long. The ladder leans against a frictionless wall and makes an angle of 53° with the ground.

a) Find the force of the ground on the ladder when the person is one third of the way up the ladder.
b) When the person gets two thirds of the way up the bottom of the ladder starts to slip. What is $\mu$ between the ladder and ground?

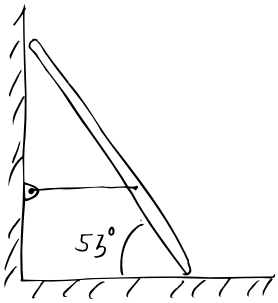



**problem 4.55:**

Filename:pfigureSoodak4-16

**4.56** A uniform 200 N, 10 m ladder leans between a frictionless ground and wall. It is kept from sliding away from the wall by a horizontal cable 2 m above the ground. Find

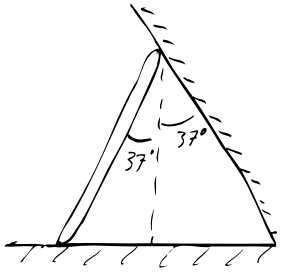
a) The tension in the cable.
b) The force of the ground on the ladder.
c) The force of the wall on the ladder.




**problem 4.56:**

Filename:pfigureSoodak4-17

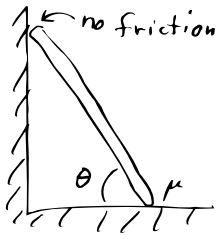
**4.57** A uniform ladder of length $\ell$ and weight $W$ rests against a frictionless slanted wall. What is the minimum $\mu$ between ladder and ground that is needed to hold the ladder in position?




**problem 4.57:**

Filename:pfigureSoodak4-19

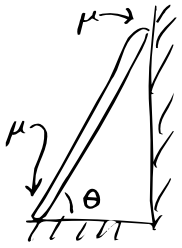
**4.58** A uniform ladder with weight $W$ and length $\ell$ leans against a frictionless vertical wall and makes an angle $\theta$ with the ground. In terms of the given quantities, find the values of $\mu$ at the ground for which the ladder will not slip.




**problem 4.58:**

Filename:pfigureSoodak4-20

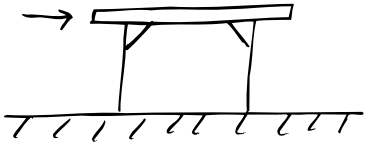
**4.59** A uniform ladder with weight $W$ and length $\ell$ leans against a frictional vertical wall and is supported by the frictional ground. The same coefficient of friction $\mu$ applies to the wall and to the ground. In terms of the given quantities, find the values of $\theta$ between the ladder and ground for which the ladder can be in equilibrium without slipping.




**problem 4.59:**

Filename:pfigureSoodak4-21

**4.60** A 2 m square 500 N 4-leg table is pushed across a floor by a horizontal force at its top surface and normal to one edge. Assume the table is 0.8 m high, that its center of mass is 0.6 m high and that all four legs slide on the floor with friction coefficient $\mu = 0.3$. Which legs carry the most load and what is the magnitude of the force from the ground on one of those legs?



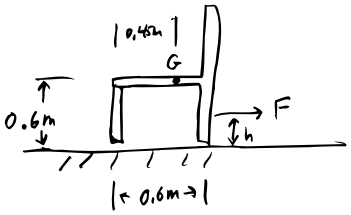
**problem 4.60:**

Filename:pfigureSoodak4-53

**4.61** An 80 N chair is pulled steadily to the right by a rope. The coefficient of friction between the ground and floor is $\mu = 0.25$.

a) What is the force needed to pull the chair?

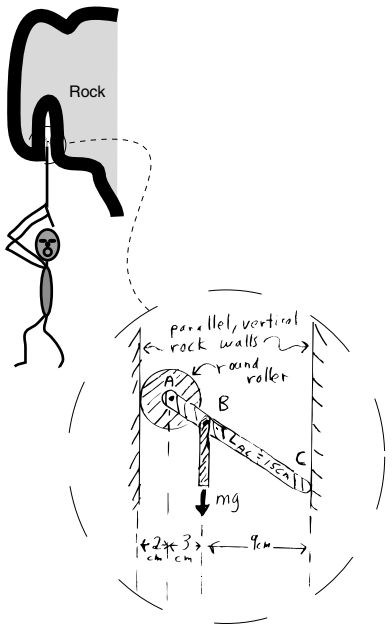
b) What is the highest point on the chair that the rope can be tied without the chair tipping over?




**problem 4.61:**

Filename:pfigureSoodak4-54

**4.62** A candidate rock-climbing device consists of a roller (radius 2 cm) frictionlessly pinned at A to diagonal-member AC. The length of AC from point A to the wall-contact point at C is $L_{AC} = 15cm$. The climber ($m = 60$ kg) hangs from a rope connected to AC by a pin at B. B is on the line AC and located as shown in the figure. If needed, assume $g = 10$N/kg. What is the minimum coefficient of friction $\mu$ at C that is needed to hold up the climber?




**problem 4.62:**

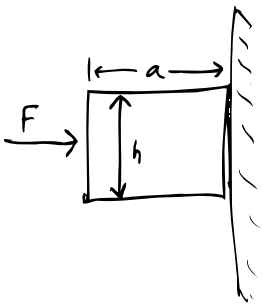
Filename:F02final1-Climber

**4.63** A uniform $W = 50$ N block with width $a$ and height $h$ is held against a wall with a horizontal force of $F$ acting on the left side half way up the block. The block is prevented from sliding down the wall by friction. There is no glue (no tension between wall and block).

a) Assuming friction is high enough to prevent slip, what is the minimum $F$ to keep the block from tipping away from the wall?

b) For twice that $F$ what is the minimum friction to keep the block from sliding down the wall?

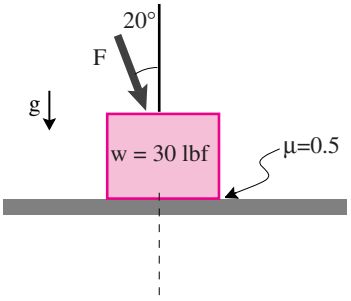
c) For $a = h$ and $F = 3W$ the resultant of all the wall normal and contact forces is a single force that acts on the right side of the block at what position $y$ above the bottom of the block?




**problem 4.63:**

Filename:pfigureSoodak4-51

**4.64** In the figure shown, what is force $F$ required to push the block along the floor? This problem has no solution. Explain why (using free-body diagrams and mechanics equations).




**problem 4.64:**

Filename:pfigure-particle-13

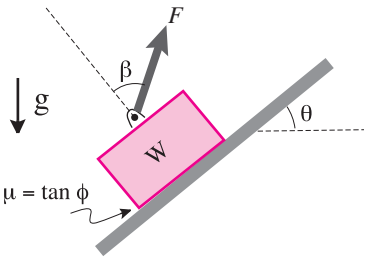
**4.65** Consider the situation shown shown in the figure. Give your answers to the following questions in terms of some or all of $W, \theta, \beta, g$, and $\phi$ or $\mu$. Assume all values of $\beta$ and $0 \le \theta \le \pi/2$, $0 \le \mu$, $g > 0$, $W > 0$.

a) Assume the block slides seadily uphill. Find F. For what values of $\theta$, $\beta$, and $\mu$ does no such $F$ exist (allow $F < 0$)?

b) Assume the block slides downhill. What is F? For what values of $\theta$, $\beta$, and $\mu$ does no such solution exist?

c) assume the block is not sliding. What are the possible values of F? For what values of $\theta$, $\beta$, and $\mu$ does such a solution exist?

d) For what values of $\theta$, $\beta$, and $\mu$ can you have the block slide up, slide down, or lock (that is, no incipient slip) depending on the value of $F$?



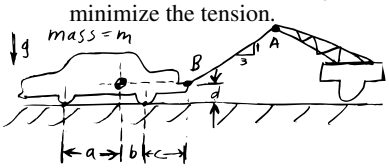

**problem 4.65:**

Filename:pfigure-particle-12b

**4.66** A car is being towed. Unfortunately all the wheels are locked and skidding with friction coefficient $\mu$.

a) In terms of some or all of $a, b, c, d, m, g$& $\mu$, find the tension in the tow cable AB.

b) Instead of an angle with slope 1/3, what should the cable angle be to minimize the tension.




**problem 4.66:**

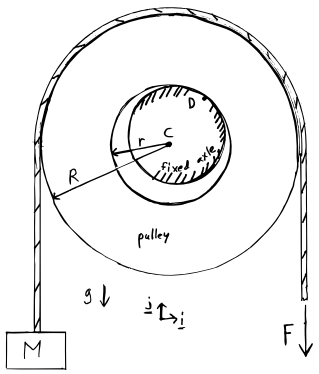
Filename:F01p1-2-carfriction

**4.67** A weight $M$ is steadily raised by pulling with a force $F$ on a rope going over a negligible-mass pulley on an unlubricated journal bearing (no ball bearings). For an ideal frictionless pulley $F = Mg$. Here we have a friction coefficient between the bearing and its axle which is is $\mu = \tan\phi$. [Hint: Finding the location of the contact point D is probably part of your solution.]

a) Find $F$ in terms of $M$, $g$, $R$, $r$ and $\mu$ (or $\phi$ or $\sin\phi$ or $\cos\phi$ — whichever is most convenient. For example $\cos(\tan^{-1}(\mu))$ is more simply expressed as $\cos\phi$), *and*

b) Evaluate $F$ in the special case that $M = 100\,\text{kg}$, $g = 10\,\text{m/s}^2$, $r = 1\,\text{cm}$, $R = 2\,\text{cm}$, and $\mu = \sqrt{3}/3$ (so $\phi = \pi/6$, $\sin\phi = 1/2$, $\cos\phi = \sqrt{3}/2$).

c) Referring back to the general case, for fixed $r$, $R$, $M$, and $g$ what happens to $F$ as $\mu \to \infty$ (does it go to $\infty$)?




**problem 4.67:**

Filename:pfigure-s01-p2-3

**4.68** A reel of mass $M$ and outer radius $R$ is connected by a horizontal string from point $P$ across a pulley to a hanging object of mass $m$. The inner cylinder of the reel has radius $r = \frac{1}{2}R$. The slope has angle $\theta$. There is no slip between the reel and the slope. There is gravity.

a) Find the ratio of the masses so that the system is at rest.

b) Find the corresponding tension in the string, in terms of $M$, $g$, $R$, and $\theta$.

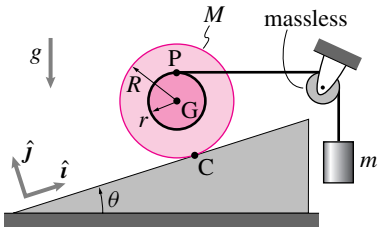
c) Find the corresponding force on the reel at its point of contact with the slope, point $C$, in terms of $M$, $g$, $R$, and $\theta$.

Harder Draw a careful sketch and find a point where the lines of action of the gravity force and string tension intersect. For the reel to be in static equilibrium, the line of action of the reaction force at $C$ must pass through this point. Using this information, what must the tangent of the angle $\phi$ of the reaction force at $C$ be, measured with respect to the normal to the slope? Does this answer agree with that you would obtain from your answer in part(c)?

d) What is the relationship between the angle $\phi$ of the reaction at $C$, measured with respect to the normal to the ground, and the mass ratio required for static equilibrium of the reel?

e) What is the minimum coefficient of friction $\mu$ at C needed to prevent slip.

Check that for $\theta = 0$, your solution gives $\frac{m}{M} = 0$ and $\vec{F}_C = Mg\hat{\jmath}$ and for $\theta = \frac{\pi}{2}$, it gives $\frac{m}{M} = 2$ and $\vec{F}_C = Mg(\hat{\imath} + 2\hat{\jmath})$.



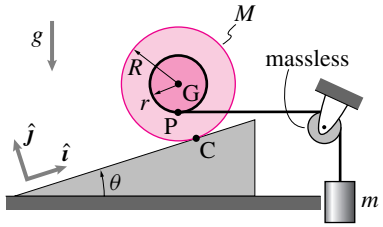

**problem 4.68:**

Filename:pfigure2-blue-47-3-a

**4.69** This problem is identical to problem 4.68 except for the location of the connection point of the string to the reel, point $P$. A reel of mass $M$ and outer radius $R$ is connected by an inextensible string from point $P$ across a pulley to a hanging object of mass $m$. The inner cylinder of the reel has radius $r = \frac{1}{2}R$. The slope has angle $\theta$. There is no slip between the reel and the slope. There is gravity. In terms of $M$, $m$, $R$, and $\theta$, find:

a) the ratio of the masses so that the system is at rest,
b) the corresponding tension in the string, and
c) the corresponding force on the reel at its point of contact with the slope, point $C$.
d) What is the minimum coefficient of friction $\mu$ at C needed to prevent slip.

Check that for $\theta = 0$, your solution gives $\frac{m}{M} = 0$ and $\vec{F}_C = Mg\hat{\jmath}$ and for $\theta = \frac{\pi}{2}$, it gives $\frac{m}{M} = -2$ and $\vec{F}_C = Mg(\hat{\imath} - 2\hat{\jmath})$.The negative mass ratio is impossible since mass cannot be negative and the negative normal force is impossible unless the wall or the reel or both can 'suck' or they can 'stick' to each other (that is, provide some sort of suction, adhesion, or magnetic attraction).

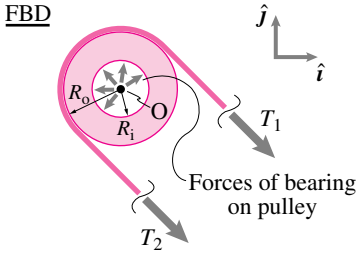



**problem 4.69:**

Filename:pfigure2-blue-47-3-b

**4.70** Assume a massless pulley is round and has outer radius $R_2$. It slides on a shaft that has radius $R_i$. Assume there is friction between the shaft and the pulley with coefficient of friction $\mu$, and friction angle $\phi$ defined by $\mu = \tan(\phi)$. Assume the two ends of the line that are wrapped around the pulley are parallel.

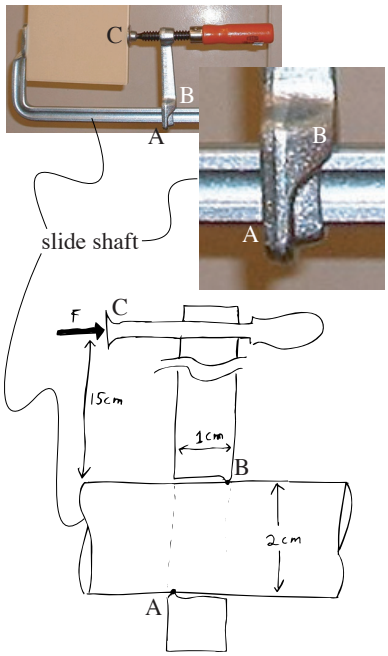
a) What is the relation between the two tensions when the pulley is turning? You may assume that the bearing shaft touches the hole in the pulley at only one point. .
b) Plug in some reasonable numbers for $R_i$, $R_o$ and $\mu$ (or $\phi$) to see one reason why wheels (say pulleys) are such a good idea even when the bearings are not all that well lubricated.




**problem 4.70:**

Filename:pfigure-blue-20-2

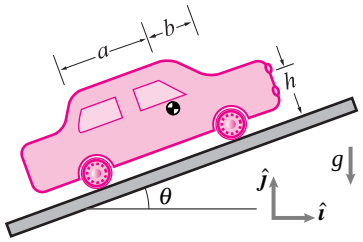
**4.71** The so-called *pipe-clamp* has a bracket ABC which loosely fits around the slide-shaft (the'pipe'). When not clamped there is no big force at C and the bracket freely slides on the shaft. However the bracket frictionally locks once the load $F$ at C gets large. Neglecting gravity, find the minimum coefficient of friction $\mu$ at A and B for which this clamp holds well (which it does).




**problem 4.71:**

Filename:pfigure-frictionclamp

**4.72** Find the minimum coefficient of friction $\mu$ needed for a front wheel drive car to go up hill. Answer in terms of some or all of $a$, $b$, $h$, $m$, $g$ and $\theta$.




**problem 4.72:**

Filename:pfigure-frontwheelup

**4.73** Solve Problem 4.72 for a rear wheel drive car.

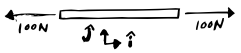
**4.74** Solve Problem 4.72 for a four wheel drive car.

## 4.4 Internal forces

## Preparatory Problems

**4.75** For the bar shown which ones of the following statements are true?

a) The two forces cancel so the tension is zero.
b) The two forces add so the tension is 200 N.
c) The tension is 100 N$\hat{\imath}$.
d) The tension is $-100$ N$\hat{\imath}$.
e) The tension is 100 N$\hat{\imath}$ on the right end and $-100$ N$\hat{\imath}$ on the left end.




**problem 4.75:**

Filename:pfig4-3-100N

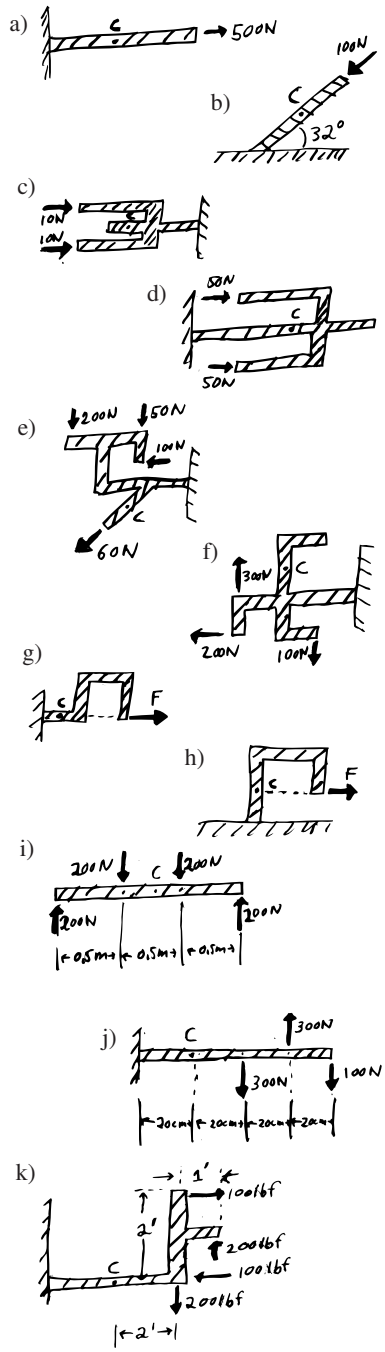
**4.76** What letters and case (upper or lower) are used in this book for tension, shear force, and bending moment?

**4.77** Mechanics depends on free body diagrams. And free body diagrams only show the external forces on an object. So how can mechanical sense be made of the concept of "internal" force?

**4.78** A string is conceptually cut in half by making free body diagrams of the left and right halves of the string. At the cut on the left half of the string acts the force 50 N$\hat{\imath}$. At the cut on the right half of the string acts the force $-50$ N$\hat{\imath}$. With two different forces acting on the two halves how can one define a single 'tension'?

**4.79** Define as precisely as you can:
a) Shear force
b) Bending moment

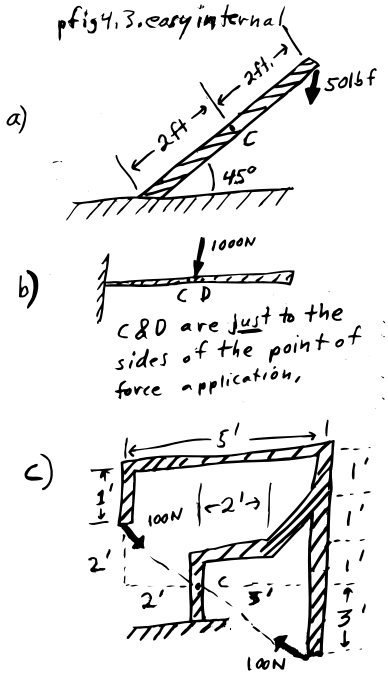
**4.80** Find the tension, shear force and bending moment at C for each of the structures below. Neglect gravity. Assume dimensions as needed.




**problem 4.80:**

Filename:pfig4-3-simpleinternal

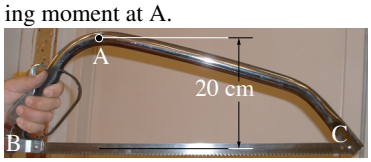
**4.81** Find the tension, shear and bending moment at section C for each of the structures below. And also at D, if marked. Assume reasonable dimensions as needed.




**problem 4.81:**

Filename:pfig4-3-easyinternal

**4.82** The tension in the bow-saw blade BC is 250 N. Find the tension, shear, and bending moment at A.

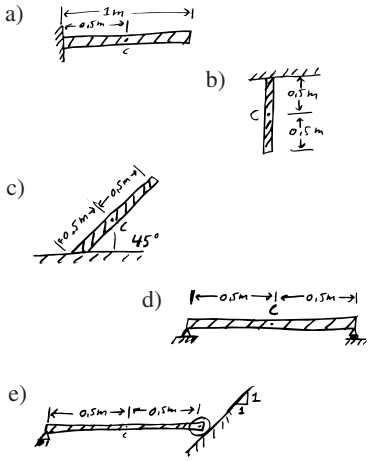



**problem 4.82:**

Filename:pfigure-bowsaw

## More-Involved Problems

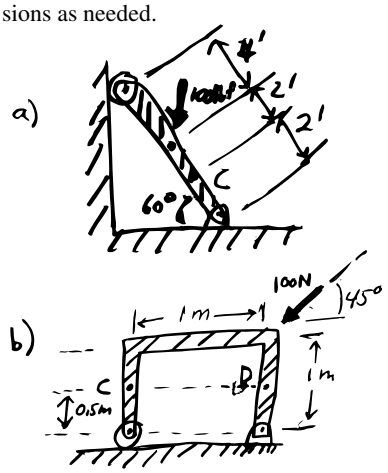
**4.83** Find the tension, shear and bending moment at section C for each of the structures below. And also at D, if marked. Include gravity, assume all bars are uniform with density of (100 N/m). Assume reasonable dimensions as needed.




**problem 4.83:**

Filename:pfig4-3-mediuminternal

**4.84** Find the tension, shear and bending moment at section C for each of the structures below. And also at D, if marked. Neglect gravity. Assume reasonable dimensions as needed.




**problem 4.84:**

Filename:pfig4-3-realinternal

## 4.5 Advanced statics

## Preparatory Problems

**4.85** In 2D, the force balance and moment balance equations for equilibrium of a body give three independent scalar equations that can be used to solve for three unknowns. How many independent scalar equations can you get from force and moment balance in 3D? Write down a set of such equations.

**4.86** In 3D, how many independent scalar equations can you write for equilibrium of a particle?

**4.87** How is moment balance equation about an axis different from moment balance about a point? Illustrate your answer with an example.

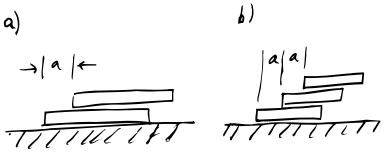
**4.88** How many independent scalar equations of equilibrium can you get by writing moment balance equations about different lines or axes in 3D?

## More-Involved Problems

**4.89** Assume identical uniform rigid blocks with weight $W = 1\,\text{N}$, height $h = 1\,\text{cm}$, and length $\ell = 10\,\text{cm}$ are put one on top of the other. Assume there is no glue so blocks can only push against each other.

a) For two blocks what is the biggest overhang $a$ so that the top block does not tip over?

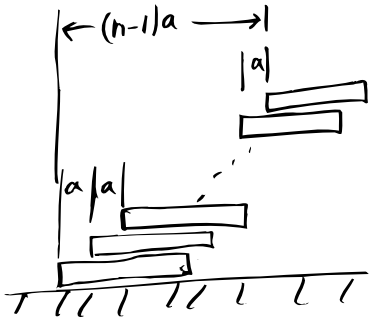
b) For three blocks what is the biggest total overhang $= 2a$ (the same overhang $a$ at each layer) so that the top block doesn't tip, nor does the middle block?




**problem 4.89:**

Filename:pfigureSoodak4-43

**4.90** See Problem 4.89. For $n$ stacked blocks what is the biggest possible overhang ($= na$) so that there is no tipping of any part of the pile relative to the rest? What is the maximum overhang in the limit $n \to \infty$?




**problem 4.90:**

Filename:pfigureSoodak4-44

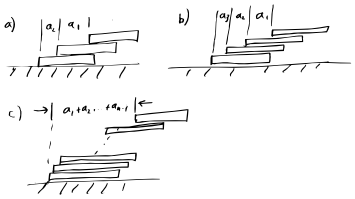
**4.91** See simpler problems 4.89 and 4.90. Stacking identical rigid blocks one on top of each other one wants to get the biggest overhang possible without the tower toppling. Each block has, say, $W = 1\,\text{N}$, height $h = 1\,\text{cm}$, and length $\ell = 10\,\text{cm}$.

a) For three blocks find the biggest $a_1$ and $a_2$ so there is no toppling. [First put the top block as far to the right as you can, $a_1$, for no toppling. Then put that pair as far to the right as possible for no toppling over the bottom block.] The total overhang is $a_1 + a_2$.

b) For 4 blocks find the largest possible overhang $a_1+a_2+a_3$ by placing the tower of three above as far to the right as possible relative to the bottom block. [Note that you place the center of mass of the top 3 blocks over the right edge of the fourth bottom block].

c) For $n$ blocks what is the biggest possible overhang ($= a_1+a_2+a_3+\cdots+a_{n-1}$)?

d) Using blocks with length $\ell = 10\,\text{cm}$ how many blocks $n$ are needed to get an overhang of 1 m? 2 m?




**problem 4.91:**

Filename:pfigureSoodak4-45

**4.92** Uniform plate ADEH with mass $m$ is connected to the ground with a ball and socket joint at A. It is also held by three massless bars (IE, CH and BH) that have ball and socket joints at each end, one end at the rigid ground (at I, C and B) and one end on the plate (at E and H).

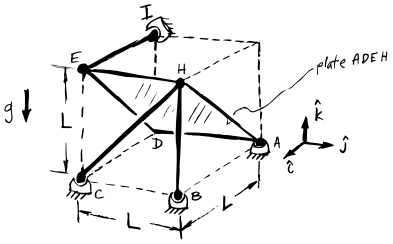
In terms of some or all of $m$, $g$, and $L$ find

a) the reaction at A (the force of the ground on the plate),

b) $T_{\text{IE}}$,

c) $T_{\text{CH}}$,

d) $T_{\text{BH}}$.




**problem 4.92:**

Filename:pfigure-01-f1

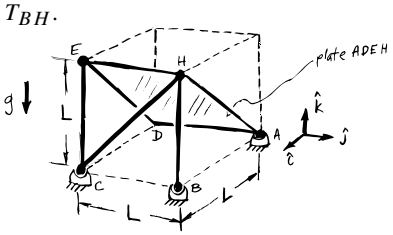
**4.93** An 80 kg square table has one quarter cut away. The remaining 60 kg are supported on 3 massless legs on a level floor. Use $g = 10\text{N/kg}$. What is the load carried by leg AB? (State your assumptions clearly.)



**problem 4.93:**

Filename:S03q4p7table

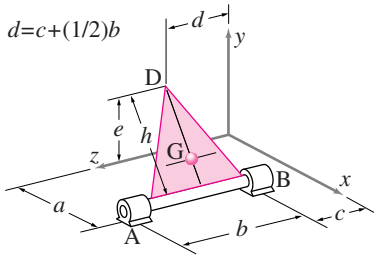
**4.94** Uniform plate ADEH with mass $m$ is connected to the ground with a ball and socket joint at A. It is also held by three massless bars (IE, CH and BH) that have ball and socket joints at each end, one end at the rigid ground (at I, C and B) and one end on the plate (at E and H). In terms of some or all of $m$, $g$, and $L$ find the reaction at A (the force of the ground on the plate) and the three bar tensions $T_{IE}$, $T_{CH}$ and $T_{BH}$.




**problem 4.94:**

Filename:F01final1-plate

**4.95** A massless triangular plate rests against a frictionless wall at point $D$ and is rigidly attached to a massless rod supported by two ideal bearings fixed to the floor. A ball of mass $m$ is fixed to the centroid of the plate. There is gravity and the system is at rest What is the reaction at point $D$ on the plate?



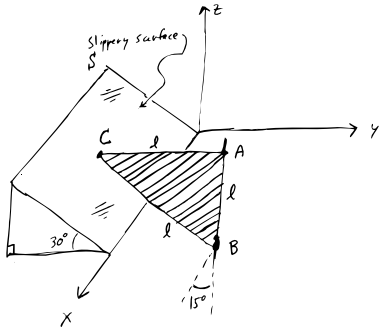

**problem 4.95:**

Filename:ch3-1b

**4.96** A uniform equilateral triangular plate with weight $W = 1000N$ and sides $\ell = 2\,\text{m}$ rests against a slippery plane S. Point C is 0.5 m above the $xy$ plane. The bottom edge of the triangle has ball-and-socket joints at A and B, with the line AB on the $xy$ plane making an angle of $15^\circ$ with the $x$ direction.

a) Find the reaction at C

b) Find all you can about the reactions at A and B.




**problem 4.96:**

Filename:pfigure-restingtriangle

**4.97** A uniform 5 kg shelf is supported at one corner with a ball and socket joint and the other three corners with strings. At the moment of interest the shelf is at rest. Gravity acts in the $-\hat{\boldsymbol{k}}$ direction. The shelf is in the $xy$ plane.

a) Draw a FBD of the shelf.

b) Challenge: without doing any calculations on paper can you find one of the reaction force components or the tension in any of the cables? Give yourself a few minutes of staring to try to find this force. If you can't, then come back to this question after you have done all the calculations.

c) Write down the equation of force equilibrium.

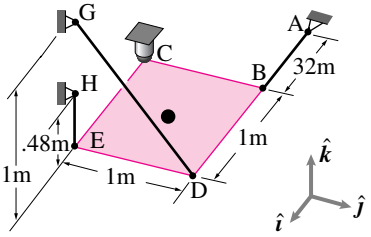
d) Write down the moment balance equation using the center of mass as a reference point.

e) By taking components, turn (b) and (c) into six scalar equations in six unknowns.

f) Solve these equations by hand or on the computer.

g) Instead of using a system of equations try to find a single equation which can be solved for $T_{EH}$. Solve it and compare to your result from before.

h) Challenge: For how many of the reactions can you find one equation which will tell you that particular reaction without knowing any of the other reactions? [Hint, try moment balance about an appropriate axis as well as force balance in an appropriate direction. It is possible to find five of the six unknown reaction components this way.] Must these solutions agree with (d)? Do they?




**problem 4.97:**

Filename:pfigure-s94h2p10-a

**4.98** The sign is held up by 6 rods. Find the tension in bars

a) BH

b) EB

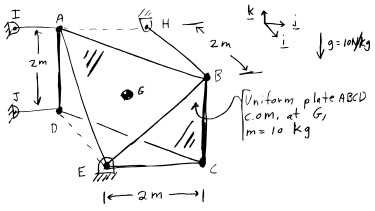
c) AE

d) IA

e) JD

f) EC

[One game you can play is to see how many of the tensions you can find *without* knowing any of the others. Another approach is to set up and solve 6 equations in 6 unknowns.]




**problem 4.98:**

Filename:pfigure-crookedsign

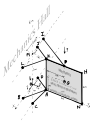
**4.99** The 100 kg, 2 m square, uniform sign KHNA is held up by 6 bars.

**Structure and geometry clarifications:** The sign is held vertically, 1m in front of, and orthogonal-to a vertical wall. Each bar holding the sign has a ball-and-socket joint both where it attaches to the sign and where it attaches to the wall. The points L, M, J, I, K, P and H lie in the same horizontal plane that includes the top edge of the sign. The points M, O, and C lie on a vertical line that is coplanar with the sign. Points B, O, D, A, and N lie in a horizontal plane shared with the bottom ege of the sign. The center of mass of the sign is at G. $g = 10\text{N/kg}$.

a) Find the "bar force" in bar AC. [hint: $\Sigma F_z = \left\{\Sigma \vec{\boldsymbol{F}}\right\} \cdot \hat{\boldsymbol{k}} = 0$].

b) Find the "bar force" in bar IP. [hint: $\Sigma M_{AK} = \left\{\Sigma \vec{\boldsymbol{M}}_{/A}\right\} \cdot \hat{\boldsymbol{k}} = 0$].

c) Find the "bar force" in bar KL.



**problem 4.99:**

Filename:F02p1p2sign

**4.100** Below is a highly schematic picture of a tricycle. The wheels are at C, B and A. The person-trike system has center of mass at G directly over the rear axle. The wheels at C and A are good free-turning, high friction wheels. The wheel at B is in a small ditch and can't move. Assume no slip and that $F$, $m$, $g$, $w$, $\ell$, and $h$ are given.

a) Of the 9 possible reaction components at A, B, and C, which do you know are zero *a priori*.

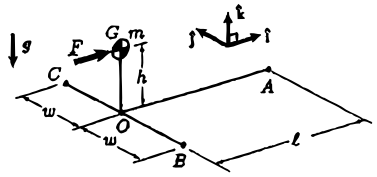
b) Find all the reaction components (the full reaction force) at A.

c) Find the vertical component of the reaction at C.

d) Find the $x$ and $z$ reaction components at B.

e) Find the sum of the $y$ components of the reactions at B and C.

f) Can you find the $y$ component of the reaction at C? Why or why not?

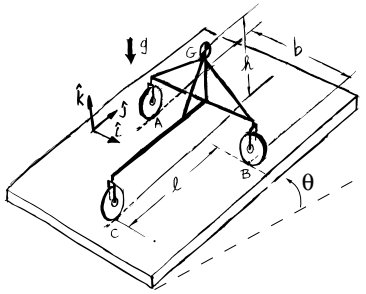



**problem 4.100:**

Filename:pfigure-trikerock

**4.101** A 3-wheeled robot with mass $m$ is parked on a hill with slope $\theta$. The ideal massless robot wheels are free to roll but not to slip sideways. The robot steering mechanism has turned the wheels so that wheels at A and C are free to roll in the $\hat{\boldsymbol{\jmath}}$ direction and the wheel at B is free to roll in the $\hat{\boldsymbol{\imath}}$ direction. The center of mass of the robot at G is $h$ above (normal to the slope) the trailer bed and symmetrically above the axle connecting wheels A and B. The wheels A and B are a distance $b$ apart. The length of the robot is $\ell$.

Find the force vector $\vec{\boldsymbol{F}}_A$ of the ground on the robot at A in terms of some or all of $m$, $g$, $\ell$, $\theta$, $b$, $h$, $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ .




**problem 4.101:**

Filename:pfigure-threewheelparked

CHAPTER 5

# Trusses and frames

*Here we consider collections of parts assembled so as to hold something up or hold something in place. Emphasis is on trusses, assemblies of bars connected by pins at their ends. Trusses are analyzed by drawing free body diagrams of the pins or of bigger parts of the truss (method of sections). Frameworks built with other than two-force bodies are also analyzed by drawing free body diagrams of parts. Structures can be rigid or not and redundant or not, as can be determined by the collection of equilibrium equations.*

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Many structures are built from two or more parts. If the parts are well-modeled as rigid and the connections between them are also well-modeled as rigid then the separateness of the parts is not visible to the laws of mechanics. The collection is then, effectively, a single object. And the best we can do with statics is to treat the group as one object. And this has been the approach of the previous chapter.

Either by accident or design, however, the connections between solid parts often are not well-modeled as rigid. Rather, the connections are sometimes reasonably approximating as freely allowing some relative motion. Such a connection does not transmit the force or moment associated with the free motion.

The standard non-rigid models for motion-allowing connections between parts are

- with pin joints. A pin joint allows relative rotation of the parts and does not transmit moments. Forces are transmitted in all directions. The pin connection is, by far, the most common model for connections in structures.
- a round pin in a slot. A pin in slot allows relative rotation of the two parts and relative motion in one direction. The only force transmitted is orthogonal to the slot.
- square pin in a slot (or shaft around a rod). This connection allows sliding in the slot but does not allow rotation. Force orthogonal to the slot is transmitted as is a moment.

This chapter concerns the analysis of arrays of parts connected by these means. In 3D the array of standard connections is more complex, as discussed in Chapter 3.

In the previous chapter we only considered one object, and thus one free body diagram, at a time. Here we need to consider, all at once, a collection of objects and the associated collection of free body diagrams. The new skills that are thus needed are

- Use of the principal of action and reaction in the representation of forces on the free body diagrams of pairs of interacting objects, and
- The solution of a larger number of simultaneous equilibrium equations.

We start with the analysis of bodies built out of straight bars connected to each other by pins at their ends.

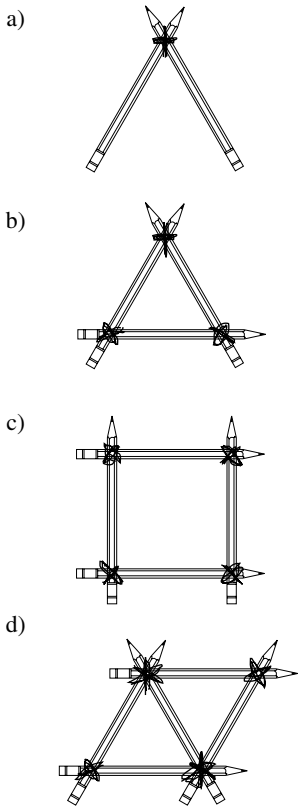



Figure 5.1: a) Two pencils strapped together with a rubber band are not sturdy. b) A triangle made of pencils feels sturdy. c) A square made of 4 pencils easily distorts into a parallelogram. d) A structure made of two triangles feels sturdy (assuming it is kept flat).

Filename:tfigure-pencil

# 5.1 Introduction to trusses and the method of joints

Trusses are good. They are useful in engineering practice, they are easy to analyze, and they provide a good example of more general structural concepts. You can quickly get a tactile sense of the truss concept. Get 9 short sticks and 9 rubber bands*. Put them together a few different ways and feel the resulting rigidity or lack thereof.

a) First join two sticks tightly together with a rubber band so that they cannot easily slide along the connection, as in fig. 5.1a. Despite the tight joint connection you can feel that the sticks rotate relative to each other relatively easily; it is easy to open and close the upside-down V.

b) Add a third stick to complete the triangle (fig. 5.1b). The relative rotation of the first two pencils is now almost totally prohibited. Even though each joint on its own made a relatively flexible V, together the 3 joints make a very stiff triangle.

c) Now tightly strap four sticks into a square as in fig. 5.1c, making 4 rubber band joints at the corners. Put the square down on a table. The sticks don't stretch or bend visibly, nor do they slide much along eachother's lengths, but the connections allow the sticks to rotate relative to each other so the square easily distorts into a parallelogram.

d) Now add two more sticks to your triangle to make two triangles (fig. 5.1d). So long as you keep this structure flat on the table, it is also sturdy.

Because a triangle is fully determined by the lengths of its sides and the V and quadrilateral are not, the structures made of triangles are *much* harder to distort. A triangle is sturdy even without rigid joints. And a V and a square (and a pentagon, etc) are not. You have just observed the essential inspiration of a truss:

> Triangles make sturdy structures.

## Swiss cheese

A different way to discover a truss is by means of subtraction. Imagine your first initial design for a bridge is to make it from one huge chunk of solid steel. This would be wildly heavy and expensive. So you could cut holes out of the chunk here and there, greatly diminishing the weight and amount of material used, but not much reducing the strength. Between these holes you would see other heavy regions of metal from which you might cut more holes leading to a more savings of weight at not much cost in strength. In fact, the reduced weight in the middle decreases the load on the outer parts of the structure

* Some useable supplies are surely on hand. For sticks you can use pencils, pens, paper tightly rolled and taped into tubes, chopsticks, popsicle sticks, big wooden matches, knitting needles, barbeque skewers, plastic drinking straws, straw from a broom, Tinkertoy rods, table knives, or strips cut from cardboard. Similarly rubberbands can be replaced with masking, scotch or duct tape, used chewing gum, lashings using thread or string, hotmelt glue, or (for some of the sticks above) with a paperclip punched through the sticks and bent to make a home-made rivet.

possibly making the whole structure stronger rather than weaker*. Eventually you would find yourself with very holy swiss cheese, a structure that looks something like a collection of bars attached from end to end in vaguely triangular patterns; like a microscopic picture of spongy bone (Fig. 5.2): As opposed to a solid block, a truss

- Uses less material;
- Puts less gravity load on other parts of the structure;
- Leaves space for other things of interest (*e.g.*, cars, cables, wires, people).

Real trusses are usually not made by removing material from a solid but by joining bars of steel, wood, or bamboo with welds, bolts, rivets, nails, screws, glue, or lashings. Once you are aware you will notice trusses in bridges, radio towers, and large-scale construction equipment. Early airplanes were flying trusses (Fig. 5.3). Bamboo trusses have been used as scaffoldings for millennia. Birds have had bones whose internal structure is truss-like since they were dinosaurs.

Trusses are practical sturdy light structures.

But trusses are also prominent near the front of elementary mechanics books because

- They are perhaps the easiest example of a complex mechanical system that a student can analyze;
- They illustrate a variety of more general structural mechanics issues;
- They help build intuition about structures that are not really trusses (The engineering mind can see an underlying conceptual truss where no physical truss exists.).

What is a truss?

A *truss* is a structure made from long narrow bars connected at their ends.

The sturdiness of most trusses comes from the inextensibility of the bars, not the resistance to rotation at the joints (as in the sticks and rubberband examples at the start of this section). To make the analysis simpler the (generally small) resistance to rotation in the joints is totally neglected in truss analysis. Thus the interaction of the bars with their neighbors is by forces, with no couples; each bar has one net force acting on each end. So:

An *ideal truss* is an assembly of two force members.

* Advanced aside: There are three ways that having more material in a structure can make it weaker: 1) the extra material adds to the gravitational load, as for the imagined bridge here, 2) The added material can be wedged in, causing the structure to fight itself (so called *locked-in* or *internal* stresses), and 3) material in the wrong place can cause *stress concentrations* and thus weak spots.

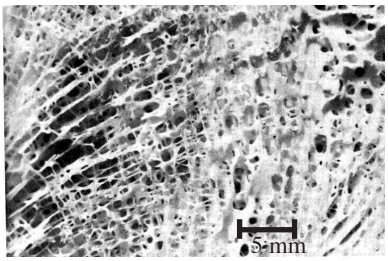



Figure 5.2: The truss inside you. The structure of spongy bone is vaguely truss-like. Shown here is human cancellous bone from the proximal femur, with the marrow removed. (courtesy Rod Lakes).

Filename:tfigure-BoneLakes



Figure 5.3: Chanute Glider, 1897. The Wright brothers first planes were near copies of the planes built a few years earlier by Octave Chanute, a retired bridge designer. Structurally these early biplanes were essentially flying bridges. The Wright bothers were preoccupied with trusses. Their first inspiration and main airplane patent was about how to control the distortion of a truss ('wing warping'). The Wrights even thought of the threads in their wing fabric as bars in a truss. Take away the outer skin from many small modern planes and you will also find trusses. (National Air and Space Museum Negative 1A-2036084-10697)

Filename:tfigureChanute33



Figure 5.4: Many bridges, especially old train bridges, are essentially trusses. Here's one that is partially obscured by the truss on a nearby car bridge.

Filename:tfigure-truss

Or, if you like, an ideal truss is a collection of bars connected at their ends with frictionless pins. Loads are only applied at the pins. In engineering analysis, the word 'truss' refers to an ideal truss even though the object of interest might have, say, welded joint connections. Had we assumed the presence of welding equipment in your study room, the opening paragraph of this section would have described the welding of metal bars instead of the attachment of pencils with rubber bands. Even with welded-together steel you would have found that the triangles would be much more rigid than the V or square.

## Bars, joints, loads, and supports

An ideal truss is a collection of *bars* connected at frictionless *joints* at which are applied *loads* as shown in fig. 5.5b (the load at a joint can be $\vec{\mathbf{0}}$ and thus not show on either the sketch of the truss or the free body diagram of the truss). A truss is held in place with *supports* which are idealized in 2D as either being fixed pins (as for joint E in fig. 5.5a) or as a pin on a roller (as for joint G in fig. 5.5a). *Reaction forces*, the forces on the truss at the supports, show on a FBD of the whole truss (Fig. 5.5b) and also on a FBD of any joint at a support. Each bar is a two-force body (Fig. 5.5c), with the same magnitude of tension pulling away from each end. A joint can be cut free with a conceptual chain saw, fooling each bar stub with the bar tension, as in the free body diagram of a joint in Fig. 5.5d.

The bar tensions can be negative. A bar with a tension of, say, $T = -5000\,\text{N}$ is said to be in *compression*. A tension of $-5000\,\text{N}$ is a compression of $5000\,\text{N}$.

## Elementary truss analysis

In elementary truss analysis you are given a truss design to which given loads are applied. Your goal is to 'solve the truss' which means you are to find the reaction forces and the tensions in the bars (sometimes called the 'bar forces'). As an engineer, this allows you to determine the needed strengths for the bars.

## The 'method of free body diagrams'

Trusses are always analyzed by the same basic method used in all of mechancis, the 'method of free body diagrams'.

- Free body diagrams are drawn of the whole truss and of various parts of the truss.
- The equilibrium equations are applied to each free body diagram, and
- The resulting equations are solved for the unknown bar forces and reactions.

The 'method of free body diagrams' is classically subdivided into two submethods.

- In the *method of joints* you draw free body diagrams of every joint and apply the force balance equations to each free body diagram.
- In the *method of sections* you draw a free body diagrams of one or more parts of the structure each of which includes 2 or more joints and apply force and moment balance to the part or parts.

These two methods can be used separately or in conjunction. In the rest of this chapter we cover the method of joints, the method of sections, computer solution using the method of joints, and miscellaneous advanced truss topics.

## Why aren't trusses everywhere?

Trusses can carry big loads with little use of material and can look nice (See fig. 5.11).They are used in many structures. Why don't engineers use trusses for all structural designs? Here are some reasons to consider not using a truss:

- Trusses are relatively difficult to build, involving many small parts* and thus requiring much time and effort to assemble.
- Trusses can be sensitive to damage when forces are not applied at the anticipated joints. They are especially sensitive to loads on the middle of the bars.
- Trusses inevitably depend on the tension strength in some bars. Some common building materials (*e.g.*, concrete, stone, and clay) crack easily when pulled.
- Trusses often have little or no redundancy, so failure in one part can lead to total structural failure.
- The triangulation that trusses require can use space that is needed for other purposes (*e.g.*, doorways, rooms).
- Trusses tend to be stiff, and sometimes more flexibility is desirable (*e.g.*, diving boards, car suspensions).
- In some places some people consider trusses unaesthetic. (e.g., the Washington Monument is not supposed to look like the Eiffel Tower).

None-the-less, for situations where you want a stiff, light structure that can carry known loads at pre-defined points, a truss is often a great design choice.

The elementary truss analysis you are about to learn is straightforward and fun. You will learn it without difficulty. However, the analysis of trusses at a more advanced level is mysteriously deep and has occupied great minds from the mid-nineteenth century (*e.g.*, Maxwell and Cauchy) to the present (see, e.g., box 5.2 on page 272).

## Method of joints

Let's start with an example.

Example: **Derrick arm.**
Consider this planar model to the arm of a construction derrick (see fig.5.7). Assume $F$ and $d$ are known. This truss has joints A-S (skipping 'F' to avoid confusion with the load). As is

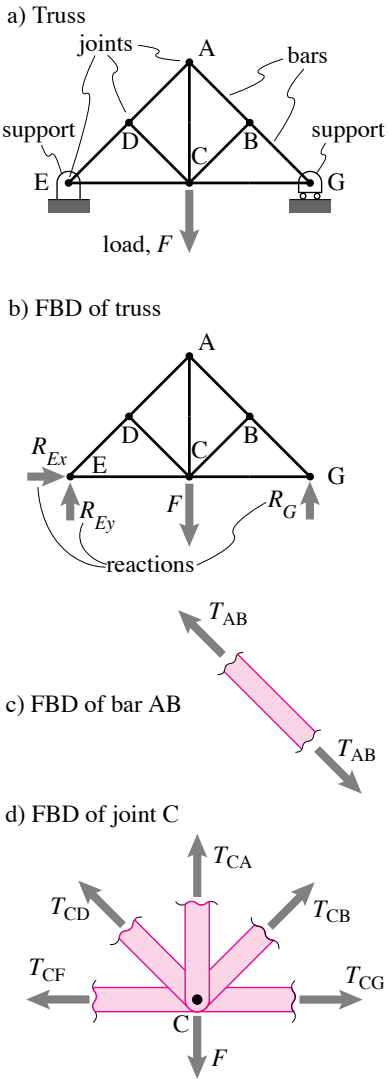



Figure 5.5: a) a truss, b) FBD of truss (It is OK to show $R_{Ex}$ even if it turns out to be zero), c) FBD of a bar, each bar is a two-force body, d) FBD of joint C, a joint is acted on by bar tensions and by applied loads. See Sample 5.1 on page 235 for an analysis of this truss.

Filename:tfigure-trussdef



Figure 5.6: The Howrah bridge. Trusses are such a good idea that they are used recursively. The bars in some large trusses, like this one or the Eiffel Tower, are themselves made of trusses (courtesy Bipin Chandra, www.bipinchandra.co.uk).

Filename:tfigureHowrah

* If an Indian says to you "Go and count the rivets in the Howrah bridge." she means go away and do something that will take a very long time. The bridge has many rivets (and bars and joints).

* To include the force of gravity on the truss elements replace the single gravity force at the center of each bar with a pair of equivalent forces at the ends. The gravity loads then all apply at the joints and the truss can still be analyzed as a collection of two-force members. Trusses are so efficient, however, that the load they carry is often much greater than their weight. So weights of the truss parts are often neglected when calculating bar forces.

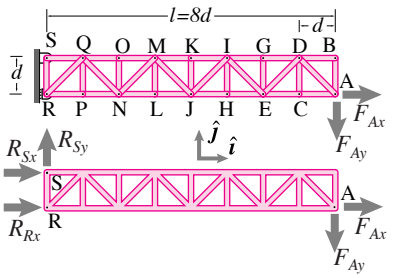



Figure 5.7: A truss with 33 bars, 18 joints and 3 unknown reaction forces. The reactions are sometimes written as, say, $R_{Sx}$ instead of $F_{Sx}$.

Filename:tfigure-derrick

common in truss analysis, we totally neglect the force of gravity on the truss elements *. The goal is to find the tensions in the bars (the so-called 'bar forces').

The method of joints is a subset of the more general method of free body diagrams. Free body diagrams are drawn of the joints. Here is the method-of-joints recipe:

- Draw a free body diagram of the whole structure and write 3 independent equilibrium equations (6 in 3D) and solve for unknown reactions if you can. This step is technically superfluous, but is so-often a time-saver that its best to just do it.
- Draw free body diagrams of all $n$ joints, 18 such in the example above.
- For each joint free body diagram you write the force balance equations, each of which can be broken down into 2 scalar equations (3 in 3D).
- Solve the $2n$ joint equations ($3n$ in 3D) for the unknown bar forces and reactions. In the example above this is $18 * 2 = 36$ equations for 33 unknown tensions and 3 unknown reactions (which you may have found from the FBD of the whole structure, but need not have).

Solving 36 simultaneous equations is generally only feasible with a computer, which is one way to go about things. However, for simple triangulated structures, like the one in fig. 5.7, you can find a sequence of joints for which hand solution is possible. If you solve the equilibrium equations as you go there are at most two unknown bar forces at each joint. By this means, hand-calculator solution of the joint force-balance equations is feasible for simple trusses.

Example: **Using the FBD of the whole structure**
From the free body diagram of the whole structure (Fig. 5.7) we find that

$$\begin{aligned}
\left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{j} \quad &\Rightarrow \quad F_{Sy} = F_{Ay} \\
\left\{\sum \vec{M}_{/S} = \vec{0}\right\} \cdot \hat{k} \quad &\Rightarrow \quad F_{Rx} = 8F_{Ay} - F_{Ax}. \\
\left\{\sum \vec{M}_{/R} = \vec{0}\right\} \cdot \hat{k} \quad &\Rightarrow \quad F_{Sx} = -8F_{Ay}.
\end{aligned}$$

Note, we picked a sign convention for the graphical representation of forces on the Free Body Diagram (see pages 23 and 126) and let the algebra possibly generate negative numbers: at S the support pushes on the arm with a force of $-8F_{Ay}$ which is pulling (if $F_{Ay} > 0$)

Note that for tension the order of subscripts is not meaningful. The tension $T_{BC}$ is the same scalar as the tension $T_{CB}$. $T_{BC} = T_{CB}$ is the amount of pulling on joint B and also the amount of pulling on joint C. That the two force vectors are negatives of each other is accounted for by the definition of tension as pulling. This unimportance of the order of subscripts is in contrast with the case of position vectors where $\vec{r}_{BC}$ is the position vector from B to C (also called $\vec{r}_{C/B}$). For position vectors $\vec{r}_{BC} = \vec{r}_{C/B} = -\vec{r}_{B/C} = -\vec{r}_{CB}$. Summarizing, the subscript order has meaning for $\vec{r}_{AB}$ but not for $T_{AB}$.

## FBDs of the joints

In the solve-by-hand method of joints we first find a joint with at most 2 bars connected. Then we work our way through the structure, one joint at a time, picking joints with at most 2 unknown bar tensions.

For the truss in Fig. 5.7

- Joint B has only two bars connected (see fig. 5.8). Force balance using FBD 5.8 tells us at a glance that

$$\sum F_x = 0 \Rightarrow T_{\mathrm{DB}} = 0 \text{ and } \sum F_y = 0 \Rightarrow T_{\mathrm{AB}} = 0$$

- Now you can draw a free body diagram of joint A where there are only two unknown tensions (since we just found $T_{\mathrm{AB}}$), namely $T_{\mathrm{AD}}$ and $T_{\mathrm{AC}}$. Force balance gives two scalar equations

$$\begin{aligned} \sum F_x = 0 \quad &\Rightarrow \quad F_{Ax} - T_{AC} - \sqrt{2}T_{AD}/2 = 0 \\ \sum F_y = 0 \quad &\Rightarrow \quad -F_{Ay} + \underbrace{T_{AB}}_{0} + \sqrt{2}T_{AD}/2 = 0 \end{aligned}$$

  which you can solve to find $T_{AD} = \sqrt{2}F_{Ay}$ and $T_{AC} = F_{Ax} - \sqrt{2}F_{Ay}$.
- Next is joint C. Force balance for joint C will tell you $T_{\mathrm{CD}}$ and $T_{\mathrm{CE}}$.
- Then you can work your way through the alphabet of joints. Using the bar tensions you have already found you can find, one at a time, joints with only two unknown tensions.

That's it for the method of joints for simple structures.

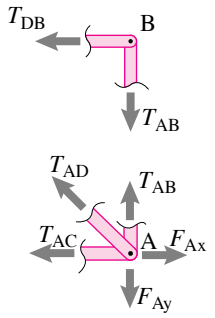



Figure 5.8: Free body diagram of joints B and A.

Filename:tfigure-derrickB

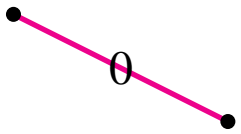



Figure 5.9: A zero force member is sometimes indicated by writing a zero on top of the bar.

Filename:tfigure-zeroforce

## Zero force members

Just by looking at joint B and thinking about the free body diagram you could probably pick out that bars DB and AB must be *zero force members*. Here we explain the unnecessary but useful trick of recognizing such zero-force members even before systematically using the method of joints. *Zero-force members* are bars with $T = 0$, like bars AB, BD and CD in the truss of Fig. 5.7. The basic idea is this:

> If there is any direction for which only one bar contributes a force on a joint, then that bar is a zero-force member.

In particular:

- At any joint where
  - there are no loads, and
  - where there are only two unknown non-parallel bar forces, and
  - where all known bar-tensions are zero,

  then the two new bar tensions are both zero (e.g., joint B in Fig. 5.8).
- At any joint where all bars but one are in the same direction as the applied load (if any), the one bar is a zero-force member (see joints C, G, H, K, L, O, and P in Fig. 5.7).

In the truss of fig. 5.7 bars AB, BD, CD, EG, IH, JK, ML, NO, and PQ are all zero force members. Sometimes it is useful to keep track of the zero force members by marking them with a zero (see fig. 5.9).

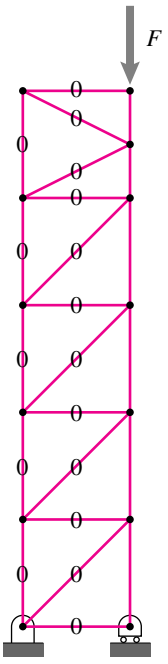



Figure 5.10: A tower with many zero force members. Although they carry no load they prevent structural collapse.

Filename:tfigure-zeroforcetower

## Zero-force members often have a non-zero purpose

Although with the given loading zero-force members have no tension, they are often needed because there are small loads not considered in the basic analysis. These could be from imperfections, or load induced asymmetries in a structure. This gives the 'zero-force' bars a small job to do, a job not noticed by the equilibrium equations in elementary truss analysis, but one that can prevent total structural collapse. Imagine, for example, the tower of fig. 5.10. In a real tower of that design the zero-force members might carry very small loads, say 100 or 1000 times smaller than the tensions (or compressions) calculated for the other bars. But if the zero-force members were removed the tower would collapse. Thus, in practice, you may observe large heavy structures with some very thin bars. Bars which in simple analyses carry no loads. But bars which prevent structural collapse*.

## Simple and not-simple trusses

Most elementary texts, like this one, start with structures that yield easily to the method of joints. These are structures where you can totally solve the equilibrium equations for the joints one at a time; each new joint only introduces two new unknown bar-tensions.

For more complex trusses this straightforward approach can fail a few ways:

- Some structures are not designed in a straightforward triangulated manner and cannot be solved 2 equations at a time. Although the method of joints may still yield a solution, it may require simultaneous solution of all of the equilibrium equations*.
- Many structures cannot be solved (the bar tensions can't be found) by using the laws of statics alone. Such are called 'statically indeterminate' structures.

For this first truss section we only consider structures that are statically determinate and easily solved.

* A common use of zero-force members is to brace long bars that are in compression and which would otherwise buckle (pop out to one side).

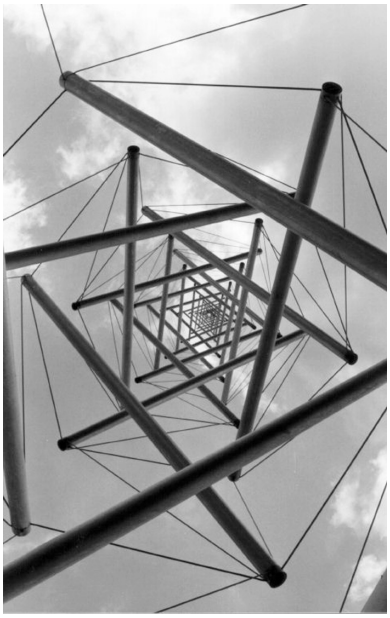


Figure 5.11: Sometimes trusses are used *only* because they look nice. The tensegrity structure 'Needle Tower' was designed by artist Kenneth Snelson and is on display in the Hirshhorn Museum in Washington, DC. It doesn't hold up anything but itself. Here you are looking straight up the middle. (courtesy Christopher Rywalt)

Filename:tfigure-Needle

* In the language of linear algebra: simple structures yield equilibrium equations that are naturally in upper triangular form, more complicated structures do not yield an upper-triangular form.

**SAMPLE 5.1** The truss shown in the figure carries a load $F = 10\,\text{kN}$ at joint D. The truss is designed with nine rods, six of which (the inclined ones) have the same length $d = 2\,\text{m}$. Rods BC, EC, DE and BD form a square.

1. Find the support reactions at joints A and F.
2. Find the tensions in rods BD and BC.

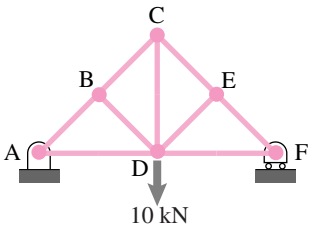



Figure 5.12:
Filename:sfig5-trussjoints-1

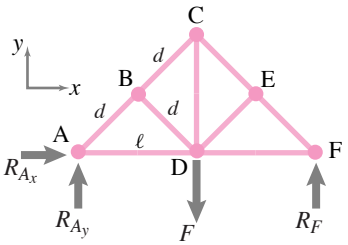



Figure 5.13:
Filename:sfig5-trussjoints-1a

### Solution

1. **Support reactions:** To find the support reactions at A and F, we draw the free-body diagram of the entire truss (see Fig. 5.13). We are given that $d = 2\,\text{m}$ and that $\angle\text{ABD} = \angle\text{DEF} = \pi/2$. Therefore, $\ell = \sqrt{2}d = 2\sqrt{2}\,\text{m}$.

   The scalar force balance equation in $x$-direction readily gives $R_{A_x} = 0$. The scalar moment balance equation about point A gives

   $$2\ell R_F - \ell F = 0 \quad \Rightarrow \quad R_F = \frac{F}{2} = 5\,\text{kN}.$$

   Now, from the scalar force balance in the $y$-direction, we have

   $$R_{A_y} + R_F - F = 0 \quad \Rightarrow \quad R_{A_y} = F - R_F = 5\,\text{kN}.$$

   $$\boxed{R_{A_x} = 0, \quad R_{A_y} = 5\,\text{kN}, \quad R_F = 5\,\text{kN}}$$

2. **Tensions in BD and BC:** We can find the tensions in rods BC and BD by analysing the equilibrium of joint B. As you can see, joint B has three unknown forces acting on it, namely the tensions of rods AB, BC and BD. Since the joint equilibrium equations (only two scalar equations) can only solve for two unknowns, we need to start at joint A, determine $T_{AB}$ first and then move on to joint B.

   The free-body diagrams of the joints A and B are shown in Fig. 5.14. Let us first consider the equilibrium of joint A. From the scalar force balance equations, we have

   $$\begin{aligned}
   \sum F_y = 0 \quad &\Rightarrow \quad R_{A_y} + T_{AB}\sin\theta = 0 \\
   &\Rightarrow \quad T_{AB} = -R_{A_y}/\sin\theta = -5\,\text{kN}/(1/\sqrt{2}) = -7\,\text{kN}. \\
   \sum F_x = 0 \quad &\Rightarrow \quad T_{AB}\cos\theta + T_{AD} = 0 \\
   &\Rightarrow \quad T_{AD} = -T_{AB}\cos\theta = -7\,\text{kN}(1/\sqrt{2}) = 5\,\text{kN}.
   \end{aligned}$$

   Now, we analyze joint B. From the geometry of forces, it is clear that writing scalar force balance equations in the $x'$ and $y'$ directions will be advantageous. For example, the force balance in the $x'$ direction immediately gives $T_{AD} = 0$. The force balance in the $y'$ direction gives

   $$-T_{AB} + T_{BC} = 0 \quad \Rightarrow \quad T_{BC} = T_{AB} = -7\,\text{kN}.$$

   $$\boxed{T_{BC} = -7\,\text{kN}, \quad T_{BD} = 0}$$

   Note that it is easy to spot bar BD as a zero force member since it is perpendicular to rods AB and BC.

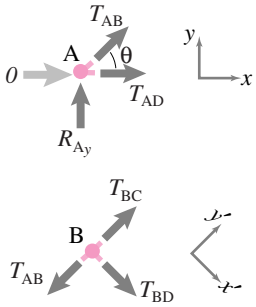



Figure 5.14: Free-body diagrams of joints A and B. From the given geometry, $\theta = pi/4$.
Filename:sfig5-trussjoints-1b

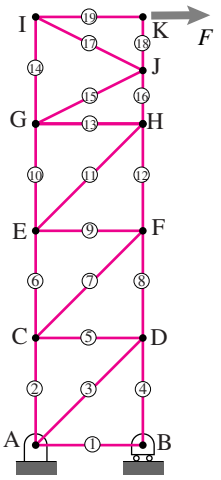



Figure 5.15:
Filename:sfig5-trussjoints-2

**SAMPLE 5.2** For the truss tower shown in the figure, assume all horizontal and vertical rods to be 1 mlong and rods numbered 16 and 18 to be 0.5 mlong. Given that the horizontal load on the truss $F = 500\,\text{N}$, find the tension in rod 15.

**Solution** To find the tension in rod 15, we can use the equilibrium of either joint G or joint K. In either case, the free-body diagram will have four unknown bar tensions (for four bars connected to each of these joints) at the joint. Therefore, we will not be able to solve for them. So, let us start at joint K and work through joint I to joint J. This sequence gets us only two unknown forces at each joint.

The free-body diagrams of the three joints are shown in Fig. 5.16. Let us first consider the equilibrium of joint K. A simple inspection (or force balance in the $y$-direction) shows that bar 18 is a zero force member. The force balance in the horizontal direction then immediately gives $T_{19} = F = 500\,\text{N}$. Thus,

$$T_{19} = 500\,\text{N} \quad \text{and} \quad T_{18} = 0.$$

Next, we consider the equilibrium of joint I. Since $T_{19}$ is already known, there are only two unknown forces, $T_{14}$ and $T_{17}$ at this joint. The force balance in the horizontal direction gives

$$\begin{aligned} T_{19} + T_{17}\cos\theta &= 0 \\ \Rightarrow \quad T_{17} &= -\frac{T_{19}}{\cos\theta} \\ &= -\frac{500\,\text{N}}{\cos(\tan^{-1}(0.5))} \\ &= -559\,\text{N}. \end{aligned}$$

Now we proceed to joint J. Note that we used only one scalar equation (force balance in the $x$-direction) at joint I, since we do not need $T_{14}$. Similarly, to find $T_15$, we only need the force balance in the horizontal direction at joint J:

$$\begin{aligned} -T_{17}\cos\theta - T_{15}\cos\theta &= 0 \\ \Rightarrow \quad T_{15} &= -T_{17} = 559\,\text{N}. \end{aligned}$$

$$\boxed{T_{15} = 559\,\text{N}}$$

**Note:** We did not have to find support reactions first in order to proceed to other joints as in the previous sample. As long as you can find a sequence of joints with just two unknown forces at each joint, up to the force that you need to determine, you can easily find the force with hand calculations.

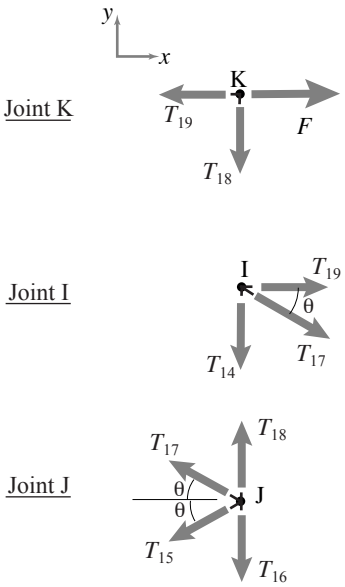



Figure 5.16: Free-body diagrams of joints K, I, and J. From the given geometry, $\theta = \tan^{-1}(\text{KJ/IK}) = 26.56^\circ$.
Filename:sfig5-trussjoints-2a

**SAMPLE 5.3** The truss shown in the figure is made up of five horizontal and six inclined rods. All inclined rods are 1 m long and at right angles to each other. The truss carries two vertical loads, $F_1 = 4\,\text{kN}$ and $F_2 = 1\,\text{kN}$ as shown. Find the tensions in rods CE, DE, and DF.

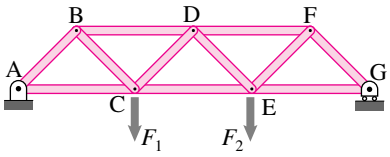



Figure 5.17:
Filename:sfig5-trussjoints-3

**Solution** To find tensions in rods CE, DE and DF, we can either use joints C and D, or joints E and F. However, for either set we need to start from other joints since there are more than two unknown forces at each joint. Let us start from joint G and work our way through joints F and E. To start at joint G, however, we first need to determine the support reaction G.

The free-body diagram of the entire truss is shown in Fig. 5.18 where we have numbered the rods for convenience. The scalar moment balance equation about point A in the $z$-direction gives

$$3\ell R_G - \ell F_1 - 2\ell F_2 = 0 \quad \Rightarrow \quad R_G = \frac{F_1 + 2F_2}{3} = 2\,\text{kN}.$$

The force balance equations give

$$\sum F_x = 0 \quad \Rightarrow \quad R_{A_x} = 0$$
$$\sum F_y = 0 \quad \Rightarrow \quad R_{A_y} = F_1 + F_2 - R_G = 3\,\text{kN}.$$

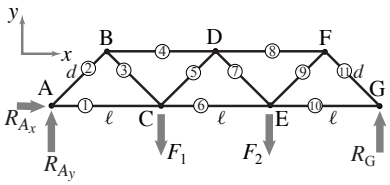



Figure 5.18:
Filename:sfig5-trussjoints-3a

Now, we are ready to proceed from joint G. The free-body diagrams of joints G, F, and E are shown in Fig. 5.19.

At joint G:

$$\sum F_y = 0 \quad \Rightarrow \quad T_{11}\sin\theta + R_G = 0$$
$$\Rightarrow \quad T_{11} = -\frac{R_G}{\sin\theta} = -\sqrt{2}R_G = -2.83\,\text{kN}.$$
$$\sum F_x = 0 \quad \Rightarrow \quad -T_{11}\cos\theta - T_{10} = 0$$
$$\Rightarrow \quad T_{10} = -T_{11}\cos\theta = 2\,\text{kN}.$$

At joint F:

$$\sum F_y = 0 \quad \Rightarrow \quad -T_{11}\sin\theta - T_9\sin\theta = 0$$
$$\Rightarrow \quad T_9 = -T_{11} = 2.83\,\text{kN}$$
$$\sum F_x = 0 \quad \Rightarrow \quad (T_{11} - T_9)\cos\theta - T_8 = 0$$
$$\Rightarrow \quad T_8 = (T_{11} - T_9)\cos\theta = -4\,\text{kN}.$$

At joint E:

$$\sum F_y = 0 \quad \Rightarrow \quad (T_7 + T_9)\sin\theta - F_2 = 0$$
$$\Rightarrow \quad T_7 = \frac{F_2}{\sin\theta} - T_9 = -1.41\,\text{kN}.$$
$$\sum F_x = 0 \quad \Rightarrow \quad (T_9 - T_7)\cos\theta + T_{10} - T_6 = 0$$
$$\Rightarrow \quad T_6 = (T_9 - T_7\cos\theta + T_{10} = 5\,\text{kN}.$$

$$\boxed{T_{\text{CE}} = 5\,\text{kN},\ T_{\text{DE}} = -1.41\,\text{kN},\ T_{\text{DF}} = -4\,\text{kN},}$$

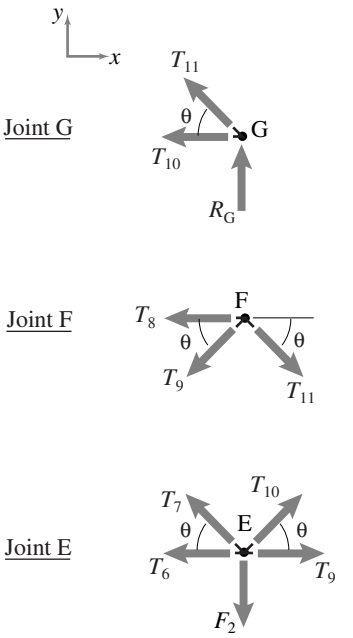



Figure 5.19: Free-body diagrams of joints G, F, and E. From the given geometry, $\theta = 45^\circ$.
Filename:sfig5-trussjoints-3b

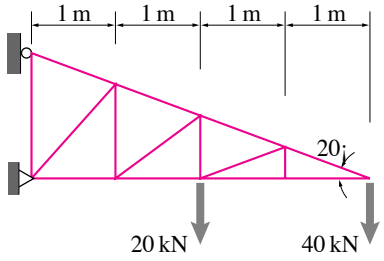



Figure 5.20:
Filename:sfig4-truss-comp

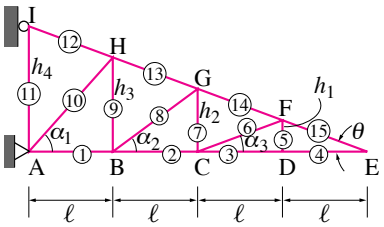



Figure 5.21:
Filename:sfig4-truss-comp-a

**SAMPLE 5.4** The truss shown in the figure has four horizontal bays, each of length 1 m. The top bars make 20° angle with the horizontal. The truss carries two loads of 40 kN and 20 kN as shown. Find the forces in each bar. In particular, find the bars that carry the maximum tensile and compressive forces.

**Solution** Since we need to find the forces in all the 15 bars, we need to find enough equations to solve for these 15 forces in addition to 3 unknown reactions $A_x$, $A_y$, and $I_x$. Thus we have a total of 18 unknowns. Note that there are 9 joints and therefore, we can generate 18 scalar equations by writing force equilibrium equations (one vector equation per joint) for each joint.

Number of unknowns $15 + 3 = 18$
Number of joints $= 9$
Number of equations $9 \times 2 = 18$.

So, we go joint by joint, draw the free-body diagram of each joint and write the equilibrium equations. After we get all the equations, we can solve them on a computer. All joint equations are just force equilibrium equations, *i.e.*, $\sum \vec{F} = \vec{0}$.

- Joint **A**:

$$(A_x + T_1 + T_{10}\cos\alpha_1)\hat{\imath} + (A_y + T_{11} + T_{10}\sin\alpha_1)\hat{\jmath} = \vec{0}. \tag{5.1}$$

- Joint **B**:

$$(-T_1 + T_2 + T_8\cos\alpha_2)\hat{\imath} + (T_9 + T_8\sin\alpha_2)\hat{\jmath} = \vec{0}. \tag{5.2}$$

- Joint **C**:

$$(-T_2 + T_3 + T_6\cos\alpha_3)\hat{\imath} + (T_7 + T_6\sin\alpha_3)\hat{\jmath} = P\hat{\jmath}. \tag{5.3}$$

- Joint **D**:

$$(T_4 - T_3)\hat{\imath} + T_5\hat{\jmath} = \vec{0}. \tag{5.4}$$

- Joint **E**:

$$(-T_4 - T_{15}\cos\theta)\hat{\imath} + T_{15}\sin\theta\,\hat{\jmath} = 2P\hat{\jmath}. \tag{5.5}$$

- Joint **F**:

$$(-T_6\cos\alpha_3 + (T_{15} - T_{14})\cos\theta)\hat{\imath} + (-T_6\sin\alpha_3 + (T_{14} - T_{15})\sin\theta - T_5)\hat{\jmath} = \vec{0}. \tag{5.6}$$

- Joint **G**:

$$(-T_8\cos\alpha_2 + (T_{14} - T_{13})\cos\theta)\hat{\imath} + ((T_{13} - T_{14})\sin\theta - T_8\sin\alpha_2 - T_7)\hat{\jmath} = \vec{0}. \tag{5.7}$$

- Joint **H**:

$$(-T_{10}\cos\alpha_1 + (T_{13} - T_{12})\cos\theta)\hat{\imath} + ((T_{12} - T_{13})\sin\theta - T_{10}\sin\alpha_1 - T_9)\hat{\jmath} = \vec{0}. \tag{5.8}$$

- Joint **I**:

$$(-I_x + T_{12}\cos\theta)\hat{\imath} + (-T_{11} - T_{12}\sin\theta)\hat{\jmath} = \vec{0}. \tag{5.9}$$

Dotting each equation from (5.1) to (5.9) with $\hat{\imath}$ and $\hat{\jmath}$, we get the required 18 equations. We need to define all the angles that appear in these equations ($\alpha_1, \alpha_2, \alpha_3$, and $\theta$) before we are ready to solve the equations on a computer.

Let $\ell$ be the length of each horizontal bar and let DF $= h_1$, CG $= h_2$, and BH $= h_3$. Then, $h_1/\ell = h_2/2\ell = h_3/3\ell = \tan\theta$. Therefore,

$$\tan\alpha_1 = \frac{h_3}{\ell} = \frac{3\ell\tan\theta}{\ell} \quad \Rightarrow \quad \alpha_1 = \tan^{-1}(3\tan\theta)$$

$$\tan\alpha_2 = \frac{h_2}{\ell} = 2\tan\theta \quad \Rightarrow \quad \alpha_2 = \tan^{-1}(2\tan\theta)$$

$$\tan\alpha_3 = \frac{h_1}{\ell} = \tan\theta \quad \Rightarrow \quad \alpha_3 = \tan^{-1}(\tan\theta) = \theta.$$

Now, we are ready for a computer solution. You can enter the 18 equations in matrix form or as your favorite software package requires and get the solution by solving for the unknowns. Here is a pseudocode to set up and solve the matrix equation. Let us order the unknown forces in the form

$$x = [T_1 \quad T_2 \quad \ldots \quad T_{15} \quad A_x \quad A_y \quad I_x]^T$$

so that $x_1$ to $x_{15} = T_1$ to $T_{15}$; $x_{16} = A_x$, $x_{17} = A_y$, and $x_{18} = I_x$.

**Entering and solving full matrix equation:**

```
theta = pi/9                          % specify theta in radians
alpha1 = atan(3*tan(theta)            % calculate alpha1
alpha2 = atan(2*tan(theta))           % calculate alpha2 from arctan
alpha3 = theta                        % calculate alpha3 from arctan
C = cos(theta), S = sin(theta)        % compute all sines and cosines
C1 = cos(alpha1), S1 = sin(alpha1)
C2 = .. ..
A = [1 0 0 0 0 0 0 0 0 C1 0 0 0 0 0 1 0 0     % enter matrix A row-wise
     0 0 0 0 0 0 0 0 0 S1 1 0 0 0 0 0 1 0
     .
     .
     0 0 0 0 0 0 0 0 0 0 -1 -S 0 0 0 0 0 0]
b = [0 0 0 0 0 20 0 0 0 40 0 0 0 0 0 0 0 0]'  % enter column vector b
solve A*x = b for x
```

The solution obtained from the computer is

$$T_1 = -128.22\,\text{kN}, \quad T_2 = T_3 = T_4 = -109.9\,\text{kN}, \quad T_5 = T_6 = 0,$$
$$T_7 = 20\,\text{kN}, \quad T_8 = -22.66\,\text{kN}, \quad T_9 = -T_{10} = 13.33\,\text{kN}, \quad T_{11} = -50\,\text{kN},$$
$$T_{12} = 146.19\,\text{kN}, \quad T_{13} = 136.44\,\text{kN}, \quad T_{14} = T_{15} = 116.95\,\text{kN},$$
$$A_x = 137.37\,\text{kN}, \quad A_y = 60\,\text{kN}, \quad I_x = -137.37\,\text{kN}.$$

## 5.2 The method of sections

The central concept for mechanics, and thus for truss analysis, is of a free body diagram. For truss analysis we have already found it fruitful to draw free body diagrams of the whole structure, of the bars (to see that they are two-force bodies), and of the individual joints. But you can draw a free body diagram of any part of a system you are studying. Assuming static equilibrium, force and moment balance apply to that subsystem.

In *the method of sections* you find bar tensions by drawing a free body diagram of a part of the truss that includes more than one joint and less than the whole structure.

The place where the truss is cut is called the *section*.

### What's wrong with the method of joints?

The method of joints can solve any solvable truss. So why learn a different method? There are two basic reasons.

1. Sometimes one only wants to know a little and the method of joints is cumbersome.

   Example: **Difficulty in finding just one bar tension.**
   Say you are interested only in $T_{\mathrm{KM}}$ in the truss of fig. 5.7 on page 232. With the method of joints we could find $T_{\mathrm{KM}}$ using the method of joints or by working through the joints one at time. To get to joint K we would have to draw free body diagrams of at least 8 other joints first. And for each we would have to solve two simultaneous equations.

2. Sometimes the method of joints doesn't best reveal basic structural ideas.

   Example: **Difficulty in understanding trends.**
   Again look at the truss of fig. 5.7. With the method of joints we would find, after all the algebra, that all the bars on the bottom (AC, CE, EH, HJ, JL, LN, NP, PR) have compression (negative tension) and that each bar has more compression than the one to its right. Similarly the top is all tension with the tension increasing with the bars more to the left. Are these trends just a consequence of lots of algebra?

The method of sections provides a shortcut, particularly for elementary textbook-like problems. And the method of sections can explain some structural trends.

### The basic method of sections recipe

Say you are just trying to find one bar tension, for example $T_{\mathrm{KM}}$ in the truss of Fig. 5.7. For simplicity we limit our attention to 2D structures.

- Find a way to cut the structure into two parts, using a *section cut* that
  - cuts the bar of interest and
  - cuts at most 3 bars in total and

  - where one of the two parts of the truss have all loads known because
    * all loads are given applied loads, or
    * the loads are reactions that have been found using a free body diagram of the whole structure.
- Write and solve the equations of moment balance for one side of the structure. This should be 3 equations in 3 unknowns.
  - Either use 3 random equations (say force balance and moment balance), or
  - Look for a shortcut. Try to find one equation that contains the unknown of interest and no other unknowns using
    * moment balance about the point of intersection of the lines of action of the two unknown forces that are not of interest, or
    * if the two uninteresting unknown forces are parallel, use force balance in a direction orthogonal to them.

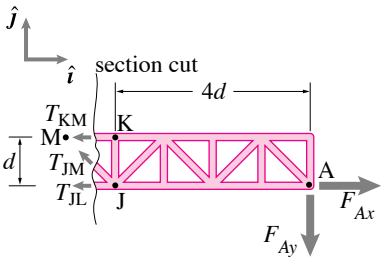



Figure 5.22: Free body diagram of a 'section' of the structure.

Filename:tfigure-derricksection

For a given truss and given bar tension of interest there is no guarantee that the recipe applies. You can always find a section cut through the bar of interest, but there may be too-many unknowns in the free body digrams of both of the resulting sub-structures.

Because 2D statics of finite bodies gives three scalar equations we can generally find all three unknown bar tensions from a section cut that goes through 3 bars.

Look at the free body diagram from a section cut in Fig. 5.22. Moment balance about point J (about an axis through J in the $z$ direction) gives:

$$\left\{\sum \vec{M}_J = \vec{0}\right\} \cdot \hat{k} \quad \Rightarrow \quad T_{KM} = 4F_{Ay}.$$

Using the FBD with this same section cut we can also find:

$$\left\{\sum \vec{M}_M = \vec{0}\right\} \cdot \hat{k} \quad \Rightarrow \quad T_{JL} = -4F_{Ay} + F_{Ax}, \text{ and}$$
$$\left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{j} \quad \Rightarrow \quad F_{JM} = \sqrt{2}F_{Ay}.$$

Note that in the free body diagram of fig. 5.22 moment balance about point J eliminates $T_{JM}$ and $T_{JL}$ and gives one equation for $T_{KM}$. And in the free body diagram of fig. 5.22 force balance in the $\hat{j}$ direction eliminates $T_{KM}$ and $T_{JL}$ and gives one equation for $T_{JM}$.

## Using sections to gain insight

In the method of joints, as you worked your way along the structure fig. 5.7 from right to left you would have found the tensions getting bigger and bigger on the top bars and the compressions (negative tensions) getting bigger and bigger on the bottom bars. With the method of sections you can see that this comes from the lever arm of the load $F$ being bigger and bigger for longer and longer sections of truss. The moment caused by the vertical load $F_{Ay}$ is carried by the tension in the top bars and compression in the bottom bars.

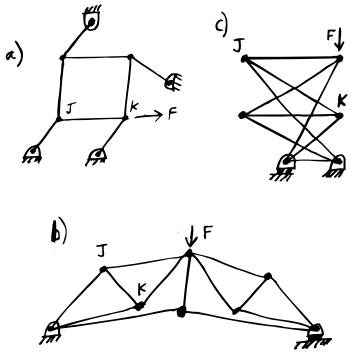



Figure 5.23: Some statically determinate trusses that do not yield easily to the method of sections. No section cut reveals a free body diagram from which you can find $T_{JK}$. In the truss (c) there are joints only where dots are marked (this mildly unusual truss has no closed triangles). To find $T_{JK}$ in these trusses one has to write the equilibrium equations for many joints and solve these simultaneously.

Filename:tfigure-Sectionsdontwork

## Final warning

Because of positive experiences with the method of sections for textbook-like problems and very simple structures, many people are left with the impression that the method of sections is more powerful than the method of joints. It isn't. The method of sections is of less general utility than the method of joints. And, unlike for the method of joints, there is no simple systematic way to find all of the bar tensions in all statically-determinate trusses (See Fig. 5.23).

**SAMPLE 5.5** The tower truss shown in the figure is fabricated with 19 rods. All the horizontal and vertical rods are one meter long. Joint J is halfway between joints K and H. The horizontal force applied at joint K is 1 kN. Find the tensions in

1. rod GJ, and
2. rod CE.

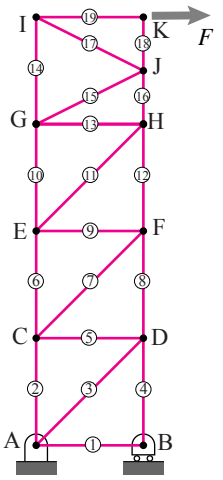



Figure 5.24:
Filename:sfig5-trusssections-1

**Solution** To find the tension in rod GJ, numbered 15, let us make a cut through the truss as shown in Fig. 5.25. The section taken here cuts rods 14, 15, and 16. The free-body diagram has only three unknown tensions acting on the part of the truss under consideration.

From the force balance in the $x$ direction, we see at once,

$$\begin{aligned} F - T_{15}\cos\theta &= 0 \\ \Rightarrow \quad T_{15} &= \frac{F}{\cos\theta} \\ &= \frac{1\,\text{kN}}{\cos 26.56^\circ} \\ &= 1.12\,\text{kN}. \end{aligned}$$

$$\boxed{T_{\text{GJ}} \equiv T_{15} = 1.12\,\text{kN}}$$

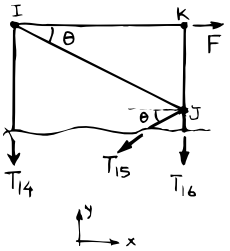



Figure 5.25:
Filename:sfig5-trusssections-1a

To determine the tension in rod CE, we consider a section that cuts rods CE, CF, and DF. The free-body diagram of the truss above this section is shown in Fig. 5.26. Once again, we have only three unknown forces on the body under consideration (note that we will have six unknown forces that include three support reactions if we considered the lower part of the truss, below the selected section).

To find $T_6$, we write the scalar moment balance equation in the $z$-direction about point F:

$$\begin{aligned} aT_6 - 2aF &= 0 \\ \Rightarrow \quad T_6 &= 2F \\ &= 2\,\text{kN}. \end{aligned}$$

$$\boxed{T_{\text{CE}} \equiv T_6 = 2\,\text{kN}}$$

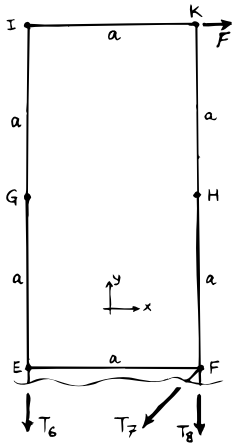



Figure 5.26:
Filename:sfig5-trusssections-1b

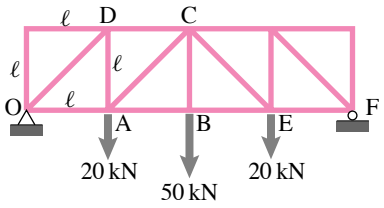



Figure 5.27:
Filename:sfig4-truss-simple

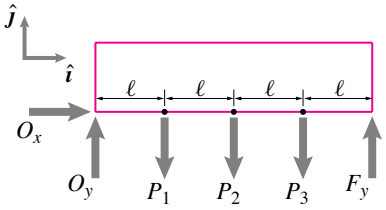
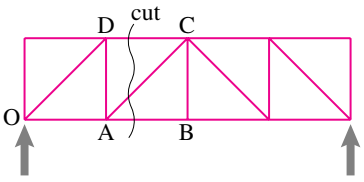



Figure 5.28:
Filename:sfig4-truss-simple-a



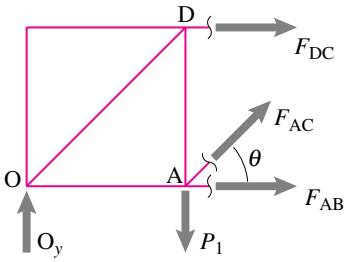




Figure 5.29:
Filename:sfig4-truss-simple-b

**SAMPLE 5.6 A 2-D truss:** The box truss shown in the figure is loaded by three vertical forces acting at joints A, B, and E. All horizontal and vertical bars in the truss are of length 2 m. Find the forces in members AB, AC, and DC.

**Solution** First, we need to find the support reactions at points O and F. We do this by drawing the free-body diagram of the whole truss and writing the equilibrium equations for it. Referring to Fig. 5.28, the force equilibrium, $\sum \vec{F} = \vec{0}$ implies,

$$O_x\hat{\imath} + (O_y + F_y - P_1 - P_2 - P_3)\hat{\jmath} = \vec{0}. \tag{5.10}$$

Dotting eqn. (5.10) with $\hat{\imath}$ and $\hat{\jmath}$, respectively, we get

$$\begin{aligned} O_x &= 0 \\ O_y + F_y &= P_1 + P_2 + P_3. \end{aligned} \tag{5.11}$$

The moment equilibrium about point O, $\vec{M}_{\rm O} = \vec{0}$, gives

$$(-P_1\ell - P_2\,2\ell - P_3\,3\ell + F_y\,4\ell)\hat{k} = \vec{0} \tag{5.12}$$

$$\text{or} \qquad F_y = \frac{1}{4}(P_1 + 2P_2 + 3P_3). \tag{5.13}$$

Solving eqns. (5.11) and (5.13), we get

$$F_y = 45\,\text{kN}, \text{ and } O_y = 45kN.$$

In fact, from the symmetry of the structure and the loads, we could have guessed that the two vertical reactions must be equal, *i.e.*, $O_y = F_y$. Then, from eqn. (5.11) it follows that $O_y = F_y = (P_1 + P_2 + P_3)/2 = 45\,\text{kN}$.

Now, we proceed to find the forces in the members AB, AC, and DC. For this purpose, we make a cut in the truss such that it cuts members AD, AC, and DC, just to the right of joints A and D. Next, we draw the free-body diagram of the left (or right) portion of the truss and use the equilibrium equations to find the required forces. Referring to Fig. 5.29, the force equilibrium requires that

$$(F_{AB} + F_{DC} + F_{AC}\cos\theta)\hat{\imath} + (O_y - P_1 + F_{AC}\sin\theta)\hat{\jmath} = \vec{0}. \tag{5.14}$$

Dotting eqn. (5.14) with $\hat{\imath}$ and $\hat{\jmath}$, respectively, we get

$$F_{AB} + F_{DC} + F_{AC}\cos\theta = 0 \tag{5.15}$$

$$O_y - P_1 + F_{AC}\sin\theta = 0. \tag{5.16}$$

So far, we have two equations in three unknowns ( $F_{AB}, F_{DC}, F_{AC}$ ). We need one more independent equation to be able to solve for the unknown forces. We now write moment equilibrium equation about point A, *i.e.*, $\sum \vec{M}_{\rm A} = \vec{0}$,

$$\begin{aligned} (-O_y\ell - F_{DC}\ell)\hat{k} &= \vec{0} \\ \Rightarrow \qquad O_y + F_{DC} &= 0. \end{aligned} \tag{5.17}$$

We can now solve eqns. (5.15–5.17) any way we like, *e.g.*, using elimination or a computer. The solution we get (see next page for details) is:

$$F_{AC} = -25\sqrt{2}\,\text{kN}, \quad F_{DC} = -45\,\text{kN}, \quad \text{and} \quad F_{AB} = 70\,\text{kN}.$$

$$\boxed{F_{AC} = -25\sqrt{2}\,\text{kN}, \quad F_{DC} = -45\,\text{kN}, \quad F_{AB} = 70\,\text{kN}.}$$

**Comments:**

- Note that the values of $F_{AC}$ and $F_{DC}$ are negative which means that bars AC and DC are in compression, not tension, as we initially assumed. Thus the solution takes care of our incorrect assumptions about the directionality of the forces.
- **Short-cuts:** In the solution above, we have not used any tricks or any special points for moment equilibrium. However, with just a little bit of mechanics intuition we can solve for the required forces in five short steps as shown below.
  (i) No external force in $\hat{\boldsymbol{\imath}}$ direction implies $O_x = 0$.
  (ii) Symmetry about the middle point B implies $O_y = F_y$. But,

$$O_y + F_y = \sum P_i = 90\,\text{kN} \quad \Rightarrow \quad O_y = F_y = 45\,\text{kN}.$$

  (iii) $(\sum \vec{\boldsymbol{M}}_\text{A} = \vec{\boldsymbol{0}}) \cdot \hat{\boldsymbol{k}}$ gives

$$O_y \ell + F_{DC} \ell = 0 \quad \Rightarrow \quad F_{DC} = -O_y = -45\,\text{kN}.$$

  (iv) $(\sum \vec{\boldsymbol{M}}_\text{C} = \vec{\boldsymbol{0}}) \cdot \hat{\boldsymbol{k}}$ gives

$$-O_y\, 2\ell + P_1 \ell + F_{AB} \ell = 0 \quad \Rightarrow \quad F_{AB} = 2O_y - P_1 = 70\,\text{kN}.$$

  (v) $(\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}) \cdot \hat{\boldsymbol{\jmath}}$ gives

$$O_y - P_1 + F_{AC} \sin\theta = 0 \quad \Rightarrow \quad F_{AC} = (P_1 - O_y)/\sin\theta = -25\sqrt{2}\,\text{kN}.$$

- **Solving equations:** On the previous page, we found $F_{AB}$, $F_{DC}$, and $F_{AC}$ by solving eqns. (5.14–5.16) simultaneously. Here, we show you two ways to solve those equations.
  1. *By elimination:* From eqn. (5.16), we have

$$F_{AC} = \frac{O_y - P_1}{\sin\theta} = \frac{20\,\text{kN} - 45\,\text{kN}}{1/\sqrt{2}} = -25\sqrt{2}\,\text{kN}.$$

  From eqn. (5.17), we get

$$F_{DC} = -O_y = -45\,\text{kN},$$

  and finally, substituting the values found in eqn. (5.14), we get

$$F_{AB} = -F_{DC} - F_{AC}\cos\theta = 45\,\text{kN} + 25\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 70\,\text{kN}.$$

  2. *On a computer:* We can write the three equations in the matrix form:

$$\underbrace{\begin{bmatrix} 1 & 1 & \cos\theta \\ 0 & 0 & \sin\theta \\ 0 & 1 & 0 \end{bmatrix}}_{\texttt{A}} \underbrace{\begin{Bmatrix} F_{AB} \\ F_{DC} \\ F_{AC} \end{Bmatrix}}_{\texttt{x}} = \begin{Bmatrix} 0 \\ P_1 - O_y \\ -O_y \end{Bmatrix} = \underbrace{\begin{Bmatrix} 0 \\ -25 \\ -45 \end{Bmatrix}}_{\texttt{b}} \text{kN}.$$

  We can now solve this matrix equation on a computer by keying in matrix `A` (with $\theta$ specified as $\pi/4$) and vector `b` as input and solving for `x`.*

* Pseudocode:

```
A = [1 1 cos(pi/4)
     0 0 sin(pi/4)
     0 1 0]
b = [0 -25 -45]
solve A*x = b for x
```

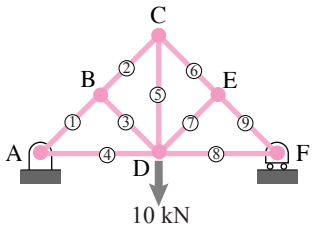



Figure 5.30:
Filename:sfig5-trusssections-2

**SAMPLE 5.7** Consider the truss shown in the figure. Let rods AB, BC, EC, EF, BD, and DE be each 2 m long. Find the tensions in rods DE and CD.

**Solution** We do not need any analysis to find the tension in rod DE. Since DE is normal to CF, DE has to be a zero force member for equilibrium of joint E.* However, let us find out the same result using the method of sections. Let us take a section just to the left of joint E that cuts through rods CE, DE and DF. The free-body diagram of the truss to the right of the section is shown in Fig. 5.31. The scalar moment balance equation, $\sum M_z = 0$, about point F gives at once,

$$aT_7 = 0 \quad \Rightarrow \quad T_7 = 0.$$

Thus rod CE is tension free. Now, we make another cut, taking the section shown in Fig. 5.32 to determine the tension in rod CD. Since $T_7 = 0$, we can write the scalar moment balance equation in the $z$-direction about point A to give

$$\begin{aligned} \ell T_5 - \ell F &= 0 \\ \Rightarrow \quad T_5 &= F = 10\,\text{kN}. \end{aligned}$$

$$\boxed{T_7 = 0, \quad T_5 = 10\,\text{kN}}$$

* In fact, that both rods BD and DE are zero force members. Since they carry no tension, rods AD and DF must also be zero force members for equilibrium of joint D

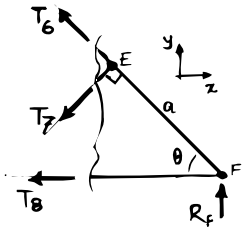



Figure 5.31:
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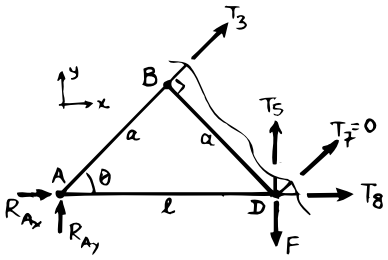



Figure 5.32:
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# 5.3 Solving trusses on a computer

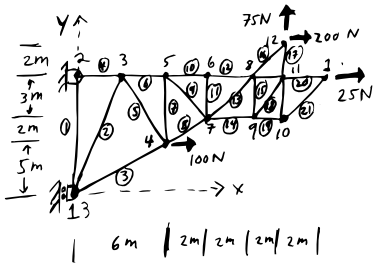



Figure 5.33: A truss with 13 bars (numbered with circles around them), 21 joints, 3 loads and 3 reaction components.

Filename:tfigure-trussnum

The method of joints is routine and is easily implemented on a computer.

- First, some software packages will accept a collection of algebraic equations, say the joint equilibrium equations, and solve them as a set for the unknowns.
- Second, on can take the set of algebraic equations as written by hand, and organize them into matrix form and solve that form on a computer as described in Section 2.4 (see page 75).
- Finally, one can treat the whole truss problem as one for which you want to do all the algebra and solution on the computer.

The first two approaches are general purpose, using the linearity of the equations and nothing special about trusses. They are as useful for trusses as for any other situation in which you have several simultaneous equations to solve.

Here we present a method for both setting up and solving the equations for a truss using no hand-calculations whatsoever. That is, we present a program which you can write in whatever your preferred computer package. The advantages of having a general purpose computer program available include:

- it is quicker to then solve any given truss
- you are less likely to make an error
- if you find an error in data entry, you can quickly correct it without having to redo all other data entry and calculation
- you can change the truss geometry easily to see the effect on the bar tensions and reactions
- you can just as easily solve non-simple trusses where neither the method of joints nor the method of sections allows solution of only 2 or 3 simultaneous equations at a time.

The rest of this section is a description of the recipe, a presentation of the final program (on page 254), and some samples using that program. This is all just a systematic use of the method of joints.

## The data that defines a truss problem

We first show how to define the truss, how it is supported, and the loads on it, with an organized collection of numbers rather than a picture. For definiteness, refer to the picture in Fig. **??** which we want to communicate to a computer. First pick an origin, coordinate directions, units to use for length and units to use for force. First a few numbers that say how many other numbers are needed.

$n_{joints}$ is the number of joints, often called $j$. In the example $n_{joints} = 13$.

$n_{bars}$ is the number of bars (rods), often called $b$. In the example $n_{bars} = 21$.

$n_{bcs}$ is the number of reaction components (or boundary conditions), often called $r$. $n_{bcs}$ is commonly 3: an $x$ and $y$ component at one joint and just an $x$ or $y$ component at another, as in the example.

The descriptions of the joints, the bars, the reactions and the loads are held in 4 matrices*.

[$J$] is a matrix defining the joints. Each joint is identified by a number (1 or 2 or ...) with each number from 1 to $n_{joints}$ associated with one joint. It doesn't matter which joint has which number. Each row of [$J$] is the information for one joint. The first entry of a row is the joint number, and the next two numbers are the coordinates of the joint. If joint 6 is at $x = 8$ m, $y = 10$ m then the row 6 of [$J$] would be [6 8 10]. [$J$] has $n_{joints}$ rows and 3 columns (Fig. 5.34).

[$B$] is a matrix defining the bars. It has one row for each bar ($n_{bars}$ of them) and three columns. The bars are identified by numbers 1, 2, ...(sometimes circled, to distinguish them from the joint numbering). It doesn't matter which bar has which number so long as every integer from 1 to $n_{bars}$ is associated with a bar (see Fig. 5.35).
The first row of [$B$] describes bar 1, the second describes bar 2, etc. The first element of each row is the bar number. This is also the number of the row, but it makes your data easier to read. The second two numbers are the numbers of the joints at the two ends of the bar. So if bar 11 connects base joint 7 with tip joint 6 the 11th row of [$B$] is [11 7 6]. It is equivalent and ok to have instead the 11th row of [$B$] be [11 6 7]; neither end of a bar is more special than the other. But once you have set the base and tip they are used to define angles in the calculations below.

[$R$] is a matrix of reactions. It has as many rows as there are unknown reaction components, typically 3. [$R$] has 4 columns. For easier reading, the first element of each row is the number of the row. The second element is the node at which the reaction applies. The next two numbers indicate the direction of the force acting on the truss ($x$ and $y$ components of a unit vector in the direction of the reaction):
- for a roller at a joint the last two numbers in the row are in the direction normal to the rollers. For normal support rollers they would be [0 1], for rollers against a vertical wall to the right of the structure they would be [−1 0]. For a roller on a 45° slope the two components could be [0.707 0.707]
- for a pin joint there are two rows in [$R$]: one for the $x$ direction and one for the $y$.

Often [$R$] will have exactly 3 rows. For the example matrix [$R$] would be

$$[R] = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \\ 3 & 13 & 1 & 0 \end{bmatrix}$$

[$F$] is a matrix of applied loads. It has a row for each joint at which there is a non-zero load. It has three columns. The first entry of each row is the joint to which the load is applied. The next two numbers are the $x$ and $y$ components of the load applied to that joint. Any units

*If you use and are comfortable with object-oriented programming some of the data structures below can be written in a more transparent form using suggestive naming rather than array locations for the various bits of data.

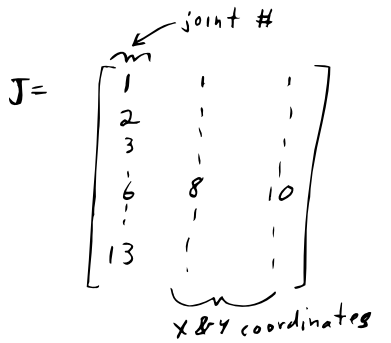



Figure 5.34: The matrix [$J$] gives the location of the joints.

Filename:tfigure-trussnumJ

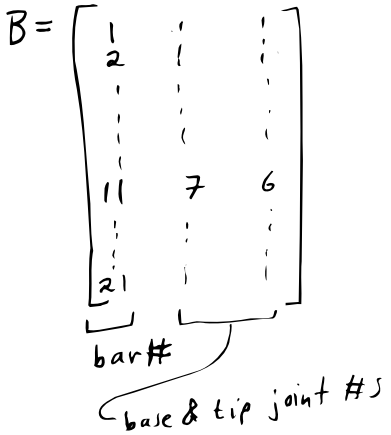



Figure 5.35: The matrix [$B$] gives the base and tip joint # of each bar.

Filename:tfigure-trussnumB

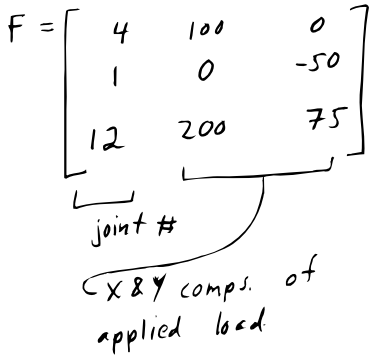



Figure 5.36: The matrix [$F$] defines the applied loads, one row for each joint at which a load is applied.

Filename:tfigure-trussnumF

can be used, they just have to be the same units for all loads. And the numerical answer for the tensions will be in these same units. If there is a rightwards load of 100 N at joint 4 one line of $[F]$ will read [4 100 0] (see Fig. 5.36).

All the information about a truss that we usually communicate with a sketch is in the 4 matrices $[J]$, $[B]$, $[R]$, and $[F]$.

These specify the locations of the joints, which joints the bars are connected to, the directions and locations of reaction forces and the applied loads. Given these matrices and nothing else one could draw the truss, supports, and loading.

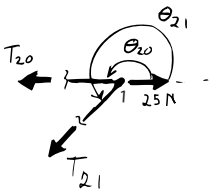



Figure 5.37: Free body diagram of joint 1.
Filename:tfigure-trussnumjoint

## The unknowns

Solving the truss, finding the tensions in the bars and reaction components, is just a matter of manipulating the numbers in the four data matrices. We will hold that answer in the list $[T]$:

$[T]$ is a column vector holding the unknowns. It has as many elements as there are unknowns ($n_u \equiv n_{bars} + n_{bcs}$). The first $n_{bars}$ elements are the unknown tensions, the last $n_{bcs}$ elements are the unknown reaction components.

## The problem

Our goal now is to use the data matrices $[J]$, $[B]$, $[R]$, and $[F]$ to find the unknowns $[T]$. We know it can be done by hand and, because the equations are linear, computer solutions should be straightforward.

## Setting up the joint equations in matrix form

We now apply the method of joints.

For each joint we draw a free body diagram (in our mind). And we apply force balance in the $x$ and $y$ directions. Thus we will have $2n_{joints}$ equations in terms of our $n_u \equiv n_{bars} + n_{bcs}$ unknowns. The strategy is to write all these equations long hand (in our mind) and then assemble those into matrix form.

If joint 1 has emanating from it bars 3 and 7 and also has a 25 N horizontal load to the right the first of these $2n_{joints}$ equations is (see Fig. 5.37):

$$\cos\theta_{20} T_{20} + \cos\theta_{21} T_{21} + 25\,\text{N} = 0,$$

where $\theta_{20}$ and $\theta_{21}$ are the angles of bars 20& 21, measured CCW from the plus $x$ direction. We can write this again as

$$0 * T_1 + 0 * T_2 + \cdots + A_{1,20} * T_{20} + A_{1,21} * T_{21} = -25\,\text{N}$$

where the cosines have been rewritten as elements of a matrix. If we assume that lots of these matrix elements are zero we can rewrite the first equation

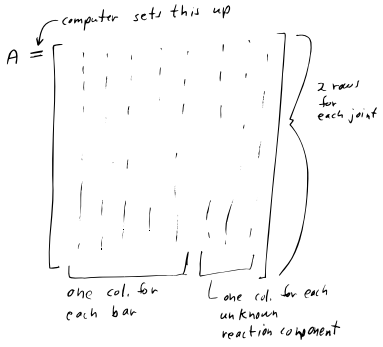



Figure 5.38: The matrix $[A]$ contains all the coefficients in the linear joint equations. It is made of sines and cosines of bar angles and the direction cosines of reaction-force directions.

Filename:tfigure-trussnumA

once again as

$$A_{1,1} * T_1 + A_{1,2} * T_2 + A_{1,3} * T_3 + \cdots + A_{1,n_u} * T_{n_u} = -F_{11}.$$

using $[A]$ as a matrix with lots of zeros, but sines and cosines of bar angles where appropriate. Recall that $n_u$ is the number of unknown bar tensions and reaction components and $F_{11} = 25\,\text{N}$ is the $x$ component of the load applied to joint 1.

For the second equation we similarly write the equation for force balance in the $y$ direction for joint 1.

$$\sin\theta_{20} T_{20} + \sin\theta_{21} T_{21} = 0,$$

which can also be written out with the terms of $[A]$ (see Fig. 5.38) as

$$A_{21} * T_1 + A_{22} * T_2 + A_{23} * T_3 + \cdots + A_{2n_u} * T_{n_u} = -F_{12}.$$

The next two equations describe joint 2, etc. Thus the assembly of $2n_{joints}$ equations looks like this

$$\begin{array}{cccc} A_{11} * T_1 & A_{12} * T_2 & \ldots & = & -F_{11} \\ A_{21} * T_1 & A_{22} * T_2 & \ldots & = & -F_{12} \\ \ldots & & & & \\ \ldots & & & & \\ A_{2n_{joints}\,2} T_1 & A_{2n_{joints}\,2} T_2 & \ldots & = & -F_{n_{joints}\,2} \end{array}$$

which we can write more compactly as

$$[A][T] = [L] \tag{5.18}$$

where $[A]$ is a matrix with cosines and sines of the bar angles and lots of zeros (because most bars don't touch a given joints) and $[L]$ is a list of negative of the loads applied in the $x$ and $y$ directions at the joints.

The point is, that all the information needed to calculate all the terms in $[A]$ and $[L]$ are in our four truss-definition matrices $[J]$, $[B]$, $[R]$ and $[F]$. And eqn. (5.18) for the unknown $[T]$ is exactly of the type that computers are great at solving.

## Some preliminary geometry

The matrix $[A]$ is made up of sines and cosines of bar angles and we have specified the truss by the $x$ and $y$ positions of the ends of the bars. We first tell the computer to do some simple trig to find the sines and cosines.

$[X]$ is a list of $x$ coordinates of each bar tip relative to its base. $[X]$ is a single column with $n_{bars}$ entries. To find the entries of $[X]$ subtract the base-joint $x$ coordinate from the tip-joint $x$ coordinate. For bar 13 this would be

```
X(13) = J( B(13,3), 2 ) - J( B(13,2), 2 )
```

because `B(13,3)` is the joint at the tip of bar 13 and `B(13,2)` is the joint at the base. Thus `J(B(13,3),2)` and and `J(B(13,2),2)` are the $x$ coordinates of the joints at the tip and base of bar 13. To find all of the elements of $[X]$ you may need to loop through all the bars or, depending on your package, you may be able to do the subtraction in one step.

$[Y]$ is a list of base-to-tip $y$ coordinates for the bars defined analogously to $[X]$ above. Thus

```
Y(13) = J( B(13,3), 3 ) - J( B(13,2), 3 )
```

$[D]$ is a list of bar lengths (distances), so

```
D(13) =  ( X(13)^2 + y(13)^2 )^.5
```

$[C]$ is a list of $n_{bars}$ cosines for the bars, one cosine for each bar. It is defined as the counter-clockwise angle of the base-to-tip bar relative to the positive $x$ axis. Thus

```
C(13) =  X(13)/D(13)    % cosine
```

$[S]$ is a similar list of sines so

```
S(13) =  Y(13)/D(13)    % sine
```

All we need from the above are the $[C]$ and $[S]$ column vectors*.

* The calculation of $[X]$, $[Y]$ and $[D]$ are just intermediate steps to simplify the presentation. If you can tolerate dense coding and use a package that deals well with matrices, $[C]$ and $[S]$ can be generated with as few as 2 dense lines of code.

## Building up $[A]$ from $[J]$, $[B]$ and $[R]$

The only difficult work in setting up a statically-determinate truss for computer solution is making up the matrix $[A]$. First lets set $[A]$ to be a matrix with $2n_{joints}$ rows and $n_u$ columns and with every entry zero.

```
A = [0]
```

We now need to put a bunch of cosines and sines into the right places.

**Cycling through the bars.** If we look at the whole $[A]$ matrix we see that the information about bar 7, say, only occurs in column 7 of $[A]$; column 7 of $[A]$ consists of the terms that multiply $T_7$. Furthermore, information about bar 7 only shows up in the rows corresponding to the $x$ and $y$ force balance for the the joints at its two ends; that's 4 places in total.

- Bar 7 pulls on its base joint `B(7, 2)` in the $x$ direction. Because we write 2 equations for each joint this equation corresponds to row `2*B(7, 2)-1`. Thus we can make the assignment
  ```
  A(  (2*B(7, 2)-1), 7  )  =   C(7)
  ```
- Bar 7 pulls on its base joint in the $y$ direction. This equation corresponds to the next row `2 *  B(7, 2)` Thus we can make the assignment
  ```
  A( (2*B(7, 2) + 1), 7 )  =   S(7)
  ```
- Bar 7 pulls in the opposite direction on its tip joint `B(7, 3)` so
  ```
  A( 2*B(7, 3),       7 )  =  -C(7)
  ```
- and
  ```
  A( (2*B(7, 3) + 1), 7 )  =  -S(7)
  ```

One needs to cycle through all the bars* and make these 4 assignments, 7 was just used as an example. In a package that deals well with matrices all four assignments associated with one bar could be in a single line of code.

**Cycling through the reactions to fill in the right-most columns of** $[A]$**.** The unknown reaction components have much the same role as do the bar tensions. But they act on only one joint. Thus each reaction component only affects 2 rows of $[A]$, the $x$ and $y$ components of that joint equation.

For reaction 3, say, the relevant joint is `R(3, 2)` and thus the relevant rows are `2*R(3, 2)-1` and `2*R(3, 2)` . The relevant column is $n_{bars} + 3$.

- for the x component of reaction 3
  `A( (2*R(3, 2)-1), (nbars+3) ) = R(3,3)`
- for the y component of reaction 3
  `A((2*R(3, 2) ), (nbars+3) ) = R(3,4)`

Most often, for trusses that are rigid even when floating, one only has three such reaction components to cycle through.

**The load vector** $[L]$ The load vector is just made up of the forces applied to the joints. For load 2, for example, applied at joint `F(2,1)` , the two relevant rows of $[L]$ are `2*F(2,1)` and `2*F(2,1)+1` at which act the $x$, and $y$ components of the force `F(6,2)` and `F(6,3)`, respectively. Thus, for load 6, we have

```
L( 2*F(2,1)   ) =  -F(2,2)
L( 2*F(2,1)+1 ) =  -F(2,3)
```

Recall that the minus sign follows from moving the applied load to the right side of the equation. This pair of commands needs to be applied to each line of the $[F]$ matrix.

## Solution

We have now constructed all the unknowns in eqn. (5.18)

$$[A][T] = [L] \tag{5.19}$$

and can thus hand the problem to the computer for solution

```
Solve {A T = L} for T
```

The resulting column vector $[T]$ is a list of bar tensions and reaction components.

*Naive approaches. One could imagine working one row at a time, corresponding to working one joint equation at a time rather than one bar at a time. For each joint we then would need to hunt through the list of bars and see which are connected to that joint. One could write a program to do this, its just more complex than the approach we present. Alternately, you might imagine that in our original data set we would have associated each joint with the bars that connect to it (rather than the other way around as we did). This is also legitimate. But, because the number of connected bars varies from joint to joint the data structure would be more complex. Finally, because the key information is the location of the bar ends, we could have used those coordinates in our data array for the bars. But this would have required our entering the coordinates of each joint over and over, once for each bar-end connected to that joint.

## The complete truss program

The complete truss program, in pseudo-code that you need to convert to your preferred computer language/package, is shown in Fig. 5.3 on page 254. Some of the loops can be 'vectorized' if your package supports such. The output $[T]$ is a column with the tensions followed by the reaction components.

## What can go wrong?

Besides the various careless errors you will discover the first 10 or so times you try to run your code, there are possible deeper problems.

Because we are not trying to write general purpose super-robust software we assume the simple check for determinacy (number of unknowns = number of equations):

$$n_{rods} + n_{bcs} = 2n_{joints} \quad \text{or} \quad b + r = 2j$$

has been satisfied. Thus $[A]$ will be square. If the truss is determinate the computer will give you a nice solution. If the truss is not determinate, with $[A]$ square or not, the result of the computer calculation will depend on the software package, ranging from an error message (e.g., "`Matrix singular!`" or "`Divide by zero!`") to the computer's making its best guess at what you want (even though the equations may have no solution, or may be many solutions to select from). Some computer packages don't tell you when they are guessing.

## How the pros solve trusses

The approach we show here is representative of how a systematic approach can be used to setup and solve a class of mechanics problems on a computer. In detail, however, the recipe presented here is simpler than that commonly employed in *finite-element* computer programs. These programs deal with statically-indeterminate problems, not as special pathological cases, but as the general case. A statically-indeterminate truss has tensions which can't be found from statics alone, but can be found if constitutive-laws for the bars are known. Thus finite-element programs must use more than statics, they use the properties of the materials.

A simple finite-element program for statically-indeterminate trusses would use the displacements of the joints as unknowns, rather than the tensions in the bars. Such a program is only a little longer than the one presented here, but requires introduction of the concept of a stiffness matrix*, a topic a shade too advanced to cover in detail here.

* Advanced aside. The stiffness matrix $[K]$ for a truss has $2n_{joints}$ rows and columns. It satisfies the equation $[\Delta] = [K][L]$ where $[L]$ is a list of $x$ and $y$ components of the loads applied to all the joints and $[\Delta]$ are the $x$ and $y$ displacements of the joints for those loads. The matrix $[K]$ can be *assembled* if the properties of all the bars are given.

```
%PSEUDO-CODE TO SOLVE ANY 2D STATICALLY DETERMINATE TRUSS
%%%
```

**SAMPLE 5.8** For the truss shown in the figure, the coordinates of the three joints are: A(0,0), B(2m,2m), and C(4m,0). Find all reactions and bar forces using computer analysis. Show the input data to the program used and the matrices [A] and [L] generated by the program.

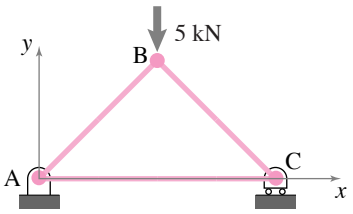



Figure 5.40:
Filename:sfig5-3-simpletruss

**Solution** The free-body diagram of the truss with the unknown reactions serially numbered is shown in Fig. 5.41. We have also numbered the bars and joints for preparing the input data file as described in the text. Here, we have three bars and three joints, three unknown reactions, and one externally applied load. Therefore, the input matrices [B] for bar data, [J] for joint data, [R] for support reaction data, and [F] for applied load data are as follows (see page 248 for row and column descriptions).

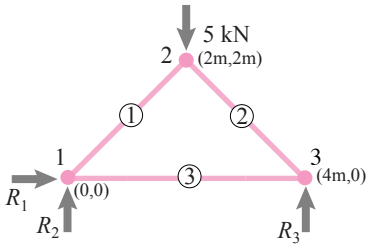



Figure 5.41:
Filename:sfig5-3-simpletruss-a

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 1 & 3 \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0.0 & 0.0 \\ 2 & 2.0 & 2.0 \\ 3 & 4.0 & 0.0 \end{bmatrix},$$

$$R = \begin{bmatrix} 1 & 1 & 1.0 & 0.0 \\ 2 & 1 & 0.0 & 1.0 \\ 3 & 3 & 0.0 & 1.0 \end{bmatrix}, \quad F = \begin{bmatrix} 2 & 0.0 & -5.0 \end{bmatrix}$$

The computer program based on the pseudocode described in the text generates the following matrices [A] and [L], before solving for the tensions and reactions:

```
A = [ 0.7071         0    1.0000    1.0000         0         0
      0.7071         0         0         0    1.0000         0
     -0.7071    0.7071         0         0         0         0
     -0.7071   -0.7071         0         0         0         0
           0   -0.7071   -1.0000         0         0         0
           0    0.7071         0         0         0    1.0000 ]
L = [ 0
      0
      0
      5
      0
      0 ]
```

The final step, `Solve {A T = F} for T,` gives the following output

```
T = [ -3.5355
      -3.5355
       2.5000
            0
       2.5000
       2.5000 ]
```

which means, $T_1 = T_2 = -3.5355\,\text{N}$, $T_3 = 2.5\,\text{N}$, $R_1 = 0$, and $R_2 = R_3 = 2.5\,\text{N}$.

$$\boxed{T_1 = T_2 = -3.5355\,\text{N},\ T_3 = 2.5\,\text{N},\ R_1 = 0,\ R_2 = R_3 = 2.5\,\text{N}}$$

**Note:** If you write a truss code, you can use this sample to check your code.

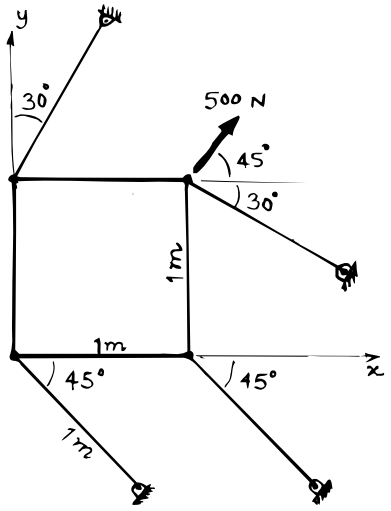



Figure 5.42:
Filename:sfig5-3-square

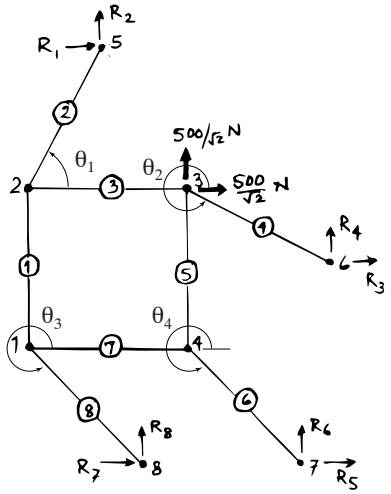



Figure 5.43: Free-body diagram of the truss along with bar and joint numbers. From the given geometry, $\theta_1 = \pi/3, \theta_2 = 11\pi/6 = -\pi/6$, and $\theta_3 = \theta_4 = \pi/4$. Assuming the bar length to be $\ell$, we can write the coordinates of all joints in terms of $\ell$ and $\theta$'s. For example, $x_5 = \ell\cos\theta_1$, $y_5 = \ell + \ell\sin\theta_1$.
Filename:sfig5-3-square-a

**SAMPLE 5.9** The truss shown in the figure has no triangles, yet it is rigid in the configuration shown as discussed in the text. It is also an example of a truss where you cannot find a sequence of joints that will let you solve for the bar forces 'locally', that is, without solving all joint equations simultaneously. Assume all bars to be 1 m long. Find all reactions and bar forces. Show the input data to the program used.

**Solution** The free-body diagram of the truss with the unknown reactions serially numbered is shown in Fig. 5.43. Note that support reactions have been taken as unknown $x$ and $y$ components of the reaction at each support point. We could have, alternatively, taken the reaction components to be along and normal to the bars at each support point.

The bars and joints are numbered as shown. Here, we have eight bars and eight joints, eight unknown reactions, and one externally applied load. Let the length of each bar be $\ell = 1$ m. The angle of outer bars with the $x$-axis are $\theta_1 = \pi/3, \theta_2 = -\pi/6, \theta_3 = \theta_4 = 5\pi/4$. Therefore, the input matrices [B] (bar data), [J] (joint data), [R] (support reaction data), and [F] (applied load data) are as follows (see page 248 for row and column descriptions).

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 5 \\ 3 & 2 & 3 \\ 4 & 3 & 6 \\ 5 & 3 & 4 \\ 6 & 4 & 7 \\ 7 & 1 & 4 \\ 8 & 1 & 8 \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0.0 & 0.0 \\ 2 & 0.0 & \ell \\ 3 & \ell & \ell \\ 4 & \ell & 0.0 \\ 5 & \ell\cos\theta_1 & \ell + \ell\sin\theta_1 \\ 6 & \ell + \ell\cos\theta_2 & \ell + \ell\sin\theta_2 \\ 7 & \ell + \ell\cos\theta_3 & \ell\sin\theta_3 \\ 8 & \ell\cos\theta_4 & \ell\sin\theta_4 \end{bmatrix},$$

$$R = \begin{bmatrix} 1 & 5 & 1.0 & 0.0 \\ 2 & 5 & 0.0 & 1.0 \\ 3 & 6 & 1.0 & 0.0 \\ 4 & 6 & 0.0 & 1.0 \\ 5 & 7 & 1.0 & 0.0 \\ 6 & 7 & 0.0 & 1.0 \\ 7 & 8 & 1.0 & 0.0 \\ 8 & 8 & 0.0 & 1.0 \end{bmatrix}, \quad F = \begin{bmatrix} 3 & 500/\sqrt{2} & 500/\sqrt{2} \end{bmatrix}$$

The computer program based on the pseudocode described in the text generates the following output, [T], for the tensions and reactions: The final step, `Solve {A T = F} for T,` gives the following result for bar tensions and reactions.

$$\begin{array}{rclrcl} T_1 &=& -418.26\,\text{N}, & R_1 &=& -241.48\,\text{N}, \\ T_2 &=& -482.96\,\text{N}, & R_2 &=& -418.26\,\text{N}, \\ T_3 &=& 241.48\,\text{N}, & R_3 &=& -112.07\,\text{N}, \\ T_4 &=& -129.41\,\text{N}, & R_4 &=& 64.70\,\text{N}, \\ T_5 &=& 418.26\,\text{N}, & R_5 &=& 418.26\,\text{N}, \\ T_6 &=& 591.51\,\text{N}, & R_6 &=& -418.26\,\text{N}, \\ T_7 &=& 418.26\,\text{N}, & R_7 &=& -418.26\,\text{N}, \\ T_8 &=& -591.51\,\text{N}, & R_8 &=& 418.26\,\text{N}. \end{array}$$

**Note:** It is easy to check that $R_1 + R_3 + R_5 + R_7 = -F\cos(45^\circ)$ and $R_2 + R_4 + R_6 + R_8 = -F\sin(45^\circ)$ where $F = 500$ N. That is, for the free-body diagram of the truss, $\sum F_x = 0$ and $\sum F_y = 0$.

# 5.4 Frames

A structure made of two or more pieces at least one of which is not a two-force body is called a *frame*. Although frames are sometimes considered a broader class that includes trusses as special cases. The vocabulary is not important. But the analysis of non-truss frameworks is a bit less formulaic than the method of joints for trusses.

Although trusses are good, they are not good enough for all purposes, nor necessarily good-enough models of very truss-looking things.

Example: **A-frame ladder.**
The two-diagonal parts of an A-frame ladder are not two-force bodies and thus truss analysis may not be most appropriate.

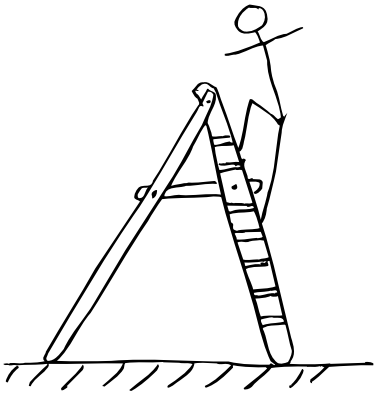


Figure 5.44: An A-frame ladder is not a truss.

Filename:tfigure-Aframe

The overall mechanics recipe applies to frames, of course: a) draw free body diagrams, b) apply the laws of mechanics to each free body diagram, and c) solve the mechanics equations for unknowns of interest. For trusses, the free body diagrams of each bar, with the 3 equilibrium equations (6 in 3D) just yield the "two-force" body result that the bar has equal tensions at the two ends. Because there was no more to learn from the bar free body diagrams we just let them go. And we used the bar tensions as forces on free body diagrams of the joints. Its as if the bars were just a means to mediate action-reaction pairs between joints.

For more general frameworks we we have to pay full respect to the free body diagrams of all of the parts, not just the pins. At least for all of the parts that are not two-force bodies. Here is the analysis of frameworks recipe:

Draw free body diagrams of

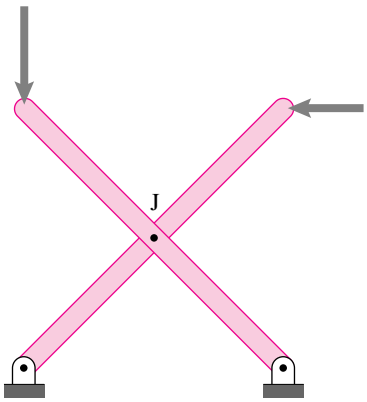
- the whole structure; and
- the separate parts of the structure; and
- collections of parts of the structure if such seems likely to be fruitful;
- Use the principal of action and reaction in the free body diagrams so that there is only one unknown force (2 components in 2D, 3 in 3D) at a point where two bodies contact;

for each free body diagram write equilibrium conditions. In 2D these should yield three independent scalar equations for each non-point part (6 in 3D)
solve some or all of the equilibrium equations for desired unknowns.

Because now the pieces are not all two-force bodies, we will not know the directions of the interaction forces *a priori* and the method of joints is next to useless *. Naturally one can be on the look out for tricks and shortcuts:

- for any two force bodies assign an equal valued tension to each end (thus eliminating any need or use for equilibrium equations for that

* Conversely, we could have analyzed trusses the way we are now going to analyze frames. This seldom used approach to trusses, the 'method of bars and pins' is discussed in box 5.4 on 260.




FBD of 'joint' J

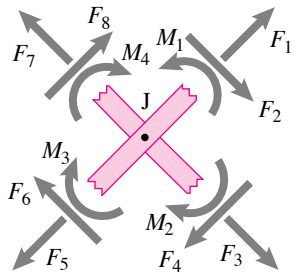



Figure 5.45: A simple frame and a free body diagram of a joint. Unlike the case for trusses, free body diagrams of joints (pins) are nearly useless in the study of more general frames. There are too many unknown forces and moments acting at the pin.

Filename:tfigure-xstructure

object)

- consider each pin as part of one of the bodies to which it is connected (ie, there is no need to draw a separate FBD of the pin).
- To minimize calculation, look for a subset of the equilibrium equations that
  - contains your unknowns of interest, and
  - has as many unknowns as scalar equations, and
  - contains as few equations as possible.

Our general goal here is to find the reaction forces, the interaction forces and the 'internal' forces in the components of a statically determinate structure.

Example: **An X structure**
Two bars are joined in an 'X' by a pin at J. Neither of the bars is a two-force body so a free body diagram of the 'joint' at J, made by cutting and leaving stubs as we did with trusses, has 12 unknown force and moment components.

Instead of drawing free body diagrams of the connections, our approach here is to draw free body diagrams of each of the structure or machine's parts. Sometimes, as was the case with trusses, it is also useful to draw a free body diagram of a whole structure or of some multi-piece part of the structure

## Static determinacy

A statically determinate structure has

- a solution for all possible applied loads, and
- only one solution, and
- this solution can be found by using equilibrium equations applied to each of the pieces.

Not all practical structures are statically determinate. Some structures are rigid but redundant, thus precluding finding all unknowns from statics. Some structures cannot carry all loads, but can carry the loads of interest (e.g., a vertical cable that can usefully carry a weight but cannot carry a side load). None-the-less, for starters here we emphasize determinate structures. The basic counting formula

$$\text{number of equations} = \text{number of unknowns}$$

is necessary for determinacy but does not guarantee determinacy. For frameworks in 2D there are three equilibrium equations for each (non-point) object. There are two unknown force components for every pin connection, whether to the ground or to another piece. And there is one unknown force component for every roller connection whether to the ground or between objects. Applied forces do not count in this determinacy check, even if they are unknown.

Example: **'X' structure counting**

In the ‘X’ structure above we can count as follows.

$$\begin{aligned} \text{number of equations} &\overset{?}{=} \text{number of unknowns} \\ \text{(3 eqs per bar)} \cdot \text{(2 bars)} &\overset{?}{=} \text{(2 unknown force comps per pin)} \cdot \text{(3 pins)} \\ \text{6 eqs} &\overset{\surd}{=} \text{6 unknown force components} \end{aligned}$$

So the ‘X’ structure passes the counting test for static determinacy.

## Redundant structures

A redundant structure can carry whatever loads it can carry in more than one way. If not also indeterminate, a redundant structure has fewer equilibrium equations than unknown reaction or interaction force components. Finding all the reaction components is only possible if one models the deformation, a topic for more advanced structural mechanics. Example: **Overbraced ‘X’**

The structure is evidently redundant because it has a bar added to a structure which was already statically determinate. By counting we get

$$\begin{aligned} \text{number of equations} &\overset{?}{=} \text{number of unknowns} \\ 3 \cdot \underbrace{\text{(number of bars)}}_{3} &\overset{?}{=} 2 \cdot \underbrace{\text{(number of joints)}}_{5} \\ \text{9 eqs} &< \text{10 unknown force components} \end{aligned}$$

thus demonstrating redundancy.

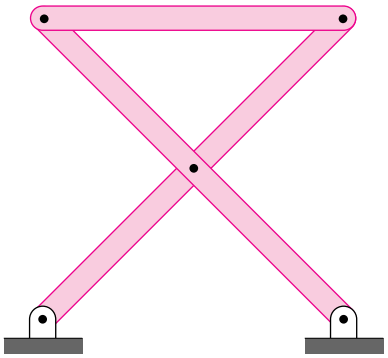


Figure 5.46: Overbraced X. A rigid frame that is not statically determinate.

Filename:tfigure-redundantX

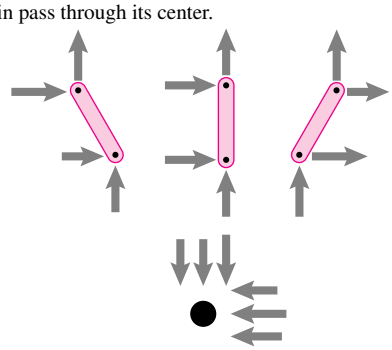
## 5.1 The 'method of bars and pins' for trusses

This is an aside for those who wonder why truss analysis seems so different than frame analysis.

Trusses are simple frameworks. So the methods used for more general frameworks should work for trusses. They do and the resulting method, which is essentially never used in such detail, we will call 'the method of bars and pins'.

In the method of bars and pins you treat a truss like any other structure. You draw a free body diagram of each part. One approach is to draw free body diagrams of each pin also. You use the principle of action and reaction to relate the forces on the different bars and pins. Then you solve the equilibrium equations.

Assuming a frictionless round pin at the hinge, all the bar forces on the pin pass through its center.

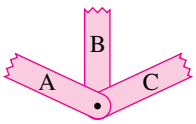


Thus, in 2D, you get two equilibrium equations for each pin and three for each bar. If you apply the three bar equations to a given bar you find that it obeys the two-force body relations. Namely, the reactions on the two bar ends are equal and opposite and along the connecting points. Now application of the pin equilibrium equations is identical to the joint equations we had previously. Thus, the 'method of bars and pins' reduces to the method of joints in the end.

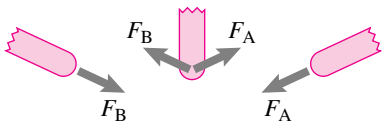
Another approach is to associate each pin with one of the bars to which it is attached. Then just think of a truss as bars that are connected with forces and no moments. Draw free body diagrams of each piece, use the principle of action and reaction, and write the equilibrium equations for each bar. This is the approach that is used in this section for other structures.

If three bars A, B, and C are connected to a pin, consider the pin as part of, say, A. Then consider action-reaction pairs between A and B, and between A and C, but not between B and C. Similarly if there are four or more bars, consider interactions between each bar and the one-bar that has the pin.

Partial Structure




Partial FBD's



**SAMPLE 5.10** The braced X-frame shown in the figure carries two vertical loads $F_1 = 2\,\text{kN}$ and $F_2 = 3\,\text{kN}$. Points G and H are directly above points A and B respectively. If $d = h = 2\,\text{m}$, find the tension in the brace CD.

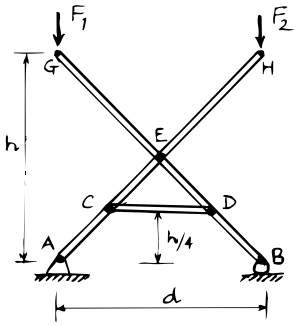



Figure 5.47:
Filename:sfig5-4frames-1

**Solution** The brace CD is pinned to the X-frame at C and D. The only loads acting on the brace are at its ends C and D. Therefore, it is a two-force body. Let us assume that the tension in brace is $C_x$. We need to find $C_x$ under the given loads.

The free-body diagram of the whole frame is shown in Fig. 5.48. Since the frame is supported by a hinge at A and a roller at B, there are three scalar support reactions acting on the frame. We can now determine all the three reactions from the static analysis of the frame:

$$\begin{aligned} \sum F_x = 0 \quad &\Rightarrow \quad A_x = 0 \\ \sum M_A = 0 \quad &\Rightarrow \quad B_y d - F_2 d = 0 \\ &\Rightarrow \quad B_y = F_2 \\ \sum F_y = 0 \quad &\Rightarrow \quad A_y = F_1 + F_2 - B_y = F_1. \end{aligned}$$

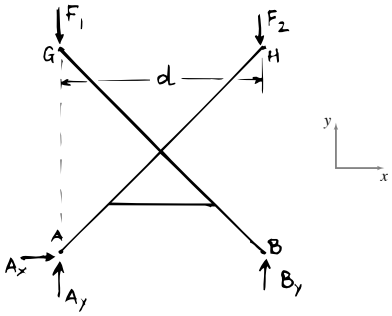



Figure 5.48:
Filename:sfig5-4frames-1a

Thus all the reactions are known. Now we can analyze either bar AH or bar BG (the analysis is identical) to determine the tension $C_x$ in the brace. The free-body diagram of bar AH is shown in Fig. 5.49. Since we are only interested in $C_x$, we can carry out moment balance about point E ($\sum M_E = 0$) to give

$$\begin{aligned} C_x \frac{d}{4} - F_2 \frac{d}{2} - A_y \frac{d}{2} &= 0 \\ \Rightarrow \quad C_x &= 2(F_2 + A_y) \\ &= 2(F_2 + F_1) \\ &= 2(3k\,\text{N} + 2\,\text{kN}) \\ &= 10\,\text{kN}. \end{aligned}$$

Thus the tension in the brace is twice the total load on the structure.

Tension in brace CD = 10 kN

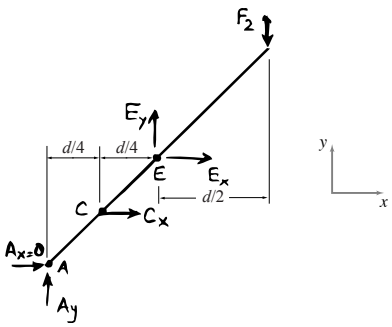



Figure 5.49:
Filename:sfig5-4frames-1b

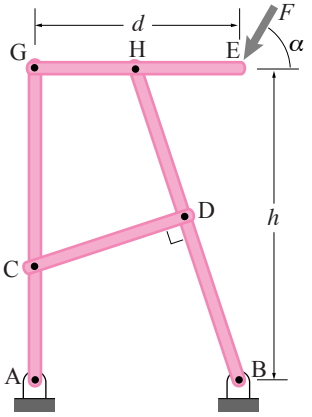



Figure 5.50:
Filename:sfig5-4frames-2

**SAMPLE 5.11** The frame shown in the figure is supported by hinges at both A and B. Bar GE is as long as the base AB and bar BH is pinned to GE at the mid point H. Brace CD is pinned at D, the mid-point of bar BH, and is orthogonal to bar BH. The load on the structure, $F = 1\,\text{kN}$, is applied at E, at an angle $\alpha = 60°$. Given that $d = 2\,\text{m}, h = 3\,\text{m}$, find the forces on the inclined bar BH and the support reactions at A and B.

[Note: Usually, determinate framed structures are made up of overhangs and extensions on a rigid triangle. This structure is an example of a frame that does not contain any rigid traingle.]

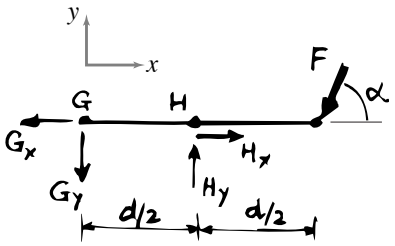



Figure 5.51:
Filename:sfig5-4frames-2a

**Solution** The given structure has hinges at both A and B. Therefore, there are four scalar support reactions, two each at A and B. So, from the free-body diagram of the whole structure, we cannot determine all support reactions. In fact, the free-body diagram of each rod will have more than three unknown forces (you can check this mentally). Thus, we are not likely to find all unknown forces on a bar without analyzing other bars. Since bar CD is a two-force member bar, it only contributes one scalar force, the tension in this rod. Now, there are two unknown scalar forces at each pin joint, A, B, G, and H, and one force at C and D (the same force). Thus we have nine unknown scalar forces. We have three bars AG, GE, and BH, each with three independent scalar equations of static equilibrium. Thus we have nine independent equations in nine unknowns. Therefore, we can solve for all the unknown forces.

Consider the free-body diagram of bar GE. The static equilibrium of this bar requires

$$\begin{aligned}
\sum M_H = 0 \quad &\Rightarrow \quad G_y(d/2) - F\sin\alpha(d/2) = 0 \\
&\Rightarrow \quad G_y = F\sin\alpha \\
\sum F_y = 0 \quad &\Rightarrow \quad -G_y + H_y - F\sin\alpha = 0 \\
&\Rightarrow \quad H_y = F\sin\alpha + G_y = 2F\sin\alpha \\
\sum F_x = 0 \quad &\Rightarrow \quad H_x - G_x - F\cos\alpha = 0. \qquad (5.20)
\end{aligned}$$

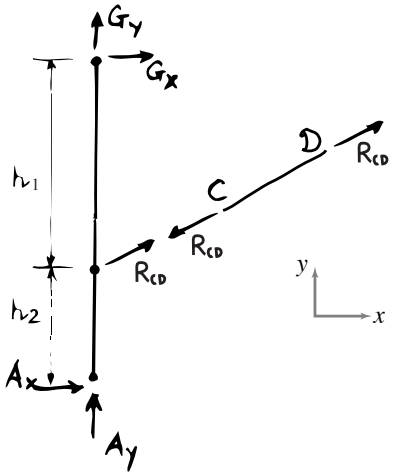



Figure 5.52:
Filename:sfig5-4frames-2b

Thus we have found $G_y$ and $H_y$ but only a relationship between $G_x$ and $H_x$. Since $G_x$ and $H_x$ are colinear, we cannot solve for them from the static analysis of bar GE alone. Now, let us consider bar AG (or bar BH; does not make a difference). The equilibrium analysis of this bar gives

$$\begin{aligned}
\sum F_y = 0 \quad &\Rightarrow \quad A_y + G_y + R_{CD}\sin\theta = 0 \qquad (5.21) \\
\sum M_C = 0 \quad &\Rightarrow \quad A_x h_1 - G_x h_2 = 0 \qquad (5.22) \\
\sum F_x = 0 \quad &\Rightarrow \quad A_x + G_x + R_{CD}\cos\theta = 0. \qquad (5.23)
\end{aligned}$$

Since none of these equations contains only one unknown, we cannot solve for these forces from the equilibrium equations of bar AG alone. Note that we have written these equations in terms of $h_1, h_2$, and $\theta$, thus far, undetermined geometric variables. However, we can easily find them from the given geometry. Now let us analyze bar BH.

$$\begin{aligned}
\sum F_y = 0 \quad &\Rightarrow \quad B_y - H_y - R_{CD}\sin\theta = 0 \qquad (5.24) \\
\sum F_x = 0 \quad &\Rightarrow \quad B_x - H_x - R_{CD}\cos\theta = 0 \qquad (5.25) \\
\sum M_D = 0 \quad &\Rightarrow \quad (H_x + B_x)\frac{h}{2} + (H_y + B_y)\frac{d}{2} = 0 \qquad (5.26)
\end{aligned}$$

So, now we have seven independent equations, eqns. (5.20)–(5.26), in seven unknowns — $A_x$, $A_y$, $B_x$, $B_y$, $R_{CD}$, $G_x$, and $H_x$ (we have already solved for $G_y$ and $H_y$). We can solve these seven equations on a computer.

Before we go to the computer, let us find the undetermined geometric quantities $h_1$ and $h_2$. From Fig. 5.54, we see that

$$\begin{aligned} h_1 &= \frac{h}{2} - \Delta \\ h_2 &= \frac{h}{2} + \Delta \end{aligned}$$

where $\Delta = d' \sin\theta$, $d' = 3d/4$, and $\theta = \tan^{-1}(d/2h)$. Now, we are ready to solve the seven equations on a computer.

```
% input given quantities
d = 2; h = 3; F = 1; alpha = pi/3;
% Define other used quantities in the equations
Delta = 3*d^2/(8*h);
h1 = h/2 - Delta;  h2 = h/2 + Delta;
theta = arctan(0.5*d/h);
% Input equations
eqset = { Hx - Gx = F*cos(alpha)
          Ay + RCD*sin(theta) = -F*sin(alpha)
          Ax*h1 - Gx*h2 = 0
          Ax + Gx + RCD*cos(theta) = 0
          By - RCD*sin(theta) = 2*F*sin(alpha)
          Bx - Hx - RCD*cos(theta) = 0
          (Hx+Bx)*h/2 + By*d/2 = -F*d*sin(alpha) }

solve eqset for Ax, Ay, Bx, By, Gx, Hx, and RCD
```

Including the values of $G_y$ and $H_y$ obtained from the first two equations of equilibrium of bar GE, we get the following values for all unknown forces from the computer solution.

$$\begin{aligned} A_x &= -9.93\,\text{kN} & A_y &= -10.79\,\text{kN} \\ B_x &= 10.43\,\text{kN} & B_y &= 11.66\,\text{kN} \\ R_{CD} &= 31.40\,\text{kN} & & \\ G_x &= -19.86\,\text{kN} & G_y &= 0.87\,\text{kN} \\ H_x &= -19.36\,\text{kN} & H_y &= 1.73\,\text{kN} \end{aligned}$$

$$\boxed{R_{CD} = 31.4\,\text{kN}}$$

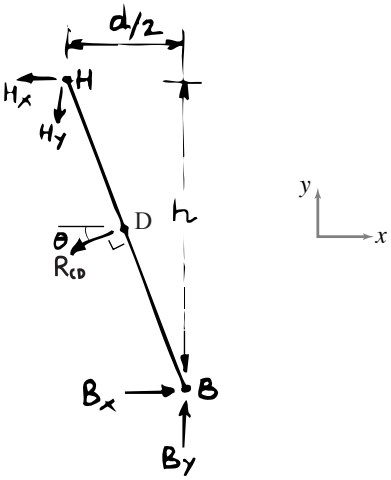



Figure 5.53:
Filename:sfig5-4frames-2c

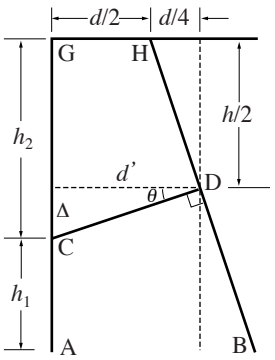



Figure 5.54: The geometry of $h_1$ and $h_2$. Since $d' = 3d/4$, $\Delta = d' \tan\theta$ and $\tan\theta = \frac{d/2}{h}$, we get $\Delta = \frac{3d}{4}\frac{d}{2h} = \frac{3d^2}{8h}$ and $h_1 = h/2 - \Delta$, $h_2 = h/2 + \Delta$.
Filename:sfig5-4frames-2d

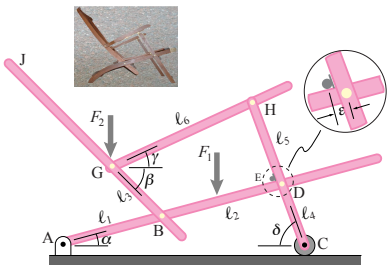



Figure 5.55: Dimensions: $\ell_1 = 45\,\text{cm}$, $\ell_2 = 60\,\text{cm}$, $\ell_3 = 30\,\text{cm}$, $\ell_4 = 30\,\text{cm}$, $\ell_5 = 70\,\text{cm}$, $\epsilon = 2.5\,\text{cm}$, $\alpha = 15°$, $\beta = 45°$, $\gamma = 25°$, and $\delta = 70°$. Loads: $F_1 = 500\,\text{N}$ and $F_2 = 200\,\text{N}$

Filename:sfig5-4-chair

**SAMPLE 5.12** An easy-chair uses a curved frame as shown in the small picture in Fig. 5.55. To simplify geometry, we can model the chair with straight bars as shown in the figure. Of special significance is the small pin at E that is attached to bar BDH and slides on bar AB (see inset). This pin is critical; it bears maximum load. Assume that it is 2.5 cm away from joint D along bar AD. The dimensions shown in the figure are $\ell_1 = 45\,\text{cm}$, $\ell_2 = 60\,\text{cm}$, $\ell_3 = 30\,\text{cm}$, $\ell_4 = 30\,\text{cm}$, $\ell_5 = 70\,\text{cm}$, $\alpha = 15°$, $\beta = 45°$, $\gamma = 25°$, and $\delta = 70°$. Find the support reactions and the load on the pin E for $F_1 = 500\,\text{N}$ and $F_2 = 200\,\text{N}$, where $F_1$ acts in the middle of bar segment BD and $F_2$ acts at G.

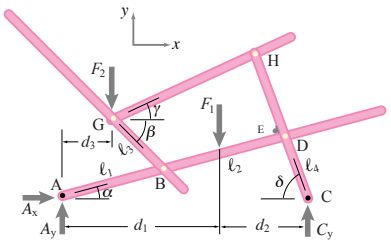



Figure 5.56: Free-body diagram of the whole chair. Note that $d_1 = (\ell_1 + \ell_2/2)\cos\alpha$, $d_2 = (\ell_2/2)\cos\alpha + \ell_3\cos\delta$, and $d_3 = \ell_1\cos\alpha - \ell_4\cos\beta$.

Filename:sfig5-4frames-3a

**Solution** Since the chair is supported by a hinge at A and a roller at B, there are three scalar support reactions. So, we can determine them from the static analysis of the whole chair frame. The free-body diagram of the chair is shown in Fig. 5.56. The moment and force equilibrium equations give

$$\begin{aligned}
\sum F_x = 0 \quad &\Rightarrow \quad A_x = 0 \\
\sum M_A = 0 \quad &\Rightarrow \quad C_y(d_1 + d_2) - F_1(d_1) - F_2(d_3) = 0 \\
&\Rightarrow \quad C_y = \frac{F_1 d_1 + F_2 d_3}{d_1 + d_2} \\
\sum F_y = 0 \quad &\Rightarrow \quad A_y = F_1 + F_2 - C_y.
\end{aligned}$$

From the given geometry,

$$\begin{aligned}
d_1 &= (\ell_1 + \ell_2/2)\cos\alpha = 72.44\,\text{cm} \\
d_2 &= (\ell_2/2)\cos\alpha + \ell_3\cos\delta = 39.24\,\text{cm} \\
d_3 &= \ell_1\cos\alpha - \ell_4\cos\beta = 22.25\,\text{cm}.
\end{aligned}$$

Substituting these dimensions above with their numerical values, we get

$$C_y = 364\,\text{N}, \quad \text{and} \quad A_y = 336\,\text{N}.$$

The support reactions are thus determined. To find the force on the pin E, we can use either bar ABD or bar CDH. In either case however, we have more unknown force on the bars that we can determine from the equilibrium equations of that bar alone. So, we will have to use equilibrium of some other bar as well. Note that bar GH is a two-force body. Therefore, the tension in this rod can be shown as a single scalar force $R_{GH}$. Let us now analyze the equilibrium of bar BGJ since it has only three unknown forces on it (see Fig. 5.57). The moment and force equilibrium equations give

$$\begin{aligned}
\sum M_G = 0 \quad &\Rightarrow \quad F_2(\ell_4\cos\beta) - R_{GH}(\ell_4\sin(\gamma + beta)) = 0 \\
&\Rightarrow \quad R_{GH} = \frac{F_2(\ell_4\cos\beta)}{\ell_4\sin(\gamma+\beta)} = 183\,\text{N}. \\
\sum F_x = 0 \quad &\Rightarrow \quad B_x = R_{GH}\cos\gamma = 166\,\text{N} \\
\sum F_y = 0 \quad &\Rightarrow \quad B_y = R_{GH}\sin\gamma - F_2 = -123\,\text{N}.
\end{aligned}$$

Now that we know $A_x$, $A_y$, $B_x$ and $B_y$, we can analyze bar ABD and determine the rest of the unknown forces on it including the force in the pin E, $R_E$ (see the free-body diagram in Fig. 5.58):

$$\begin{aligned}
\sum M_D = 0 \quad &\Rightarrow \quad -A_y(d_4+d_5) - B_y d_5 + B_x h + F_1 d_6 + R_E \epsilon = 0 \\
&\Rightarrow \quad R_E = \frac{A_y(d_4+d_5) + B_y d_5 - B_x h - F_1 d_6}{\epsilon} \\
\sum F_x = 0 \quad &\Rightarrow \quad B_x + D_x + R_E \sin\alpha = 0 \\
&\Rightarrow \quad D_x = -B_x - R_E \sin\alpha \\
\sum F_y = 0 \quad &\Rightarrow \quad D_y = R_E \cos\alpha - A_y - B_y.
\end{aligned}$$

From geometry,

$$\begin{aligned}
d_4 &= \ell_1 \cos\alpha \\
d_5 &= \ell_2 \cos\alpha \\
d_6 &= d_5/2 = (\ell_2/2)\cos\alpha \\
h &= \ell_2 \sin\alpha.
\end{aligned}$$

Substituting these variables with their numerical values above, we get

$$R_E = 3953\,\text{N}, \qquad D_x = -1189\,\text{N}, \qquad \text{and} \qquad D_y = 4106\,\text{N}.$$

$$\boxed{A_x = 0,\ A_y = 336\,\text{N},\ C_y = 364\,\text{N},\ R_E = 3953\,\text{N}}$$

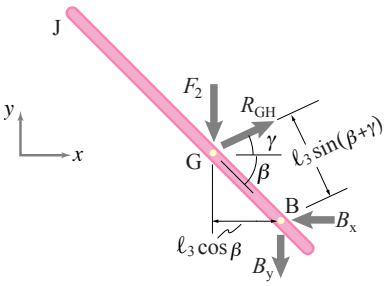



Figure 5.57:

Filename:sfig5-4frames-3b

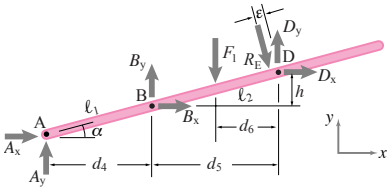



Figure 5.58: Free-body diagram of bar ABD. Note that $d_4 = \ell_1 \cos\alpha$, $d_5 = \ell_2 \cos\alpha$, $d_6 = d_5/2 = (\ell_2/2)\cos\alpha$, and $h = \ell_2 \sin\alpha$.

Filename:sfig5-4frames-3c

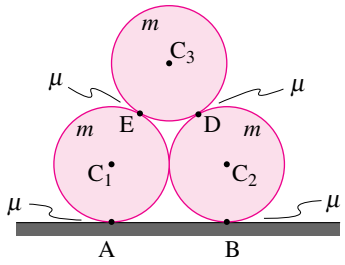



Figure 5.59:

Filename:sfig4-single-3balls

**SAMPLE 5.13 Can a stack of three balls be in static equilibrium?** Three identical spherical balls, each of mass $m$ and radius $R$, are stacked such that the top ball rests on the lower two balls. The two balls at the bottom do not touch each other. Let the coefficient of friction at each contact surface be $\mu$. Find the minimum value of $\mu$ so that the three balls are in static equilibrium.

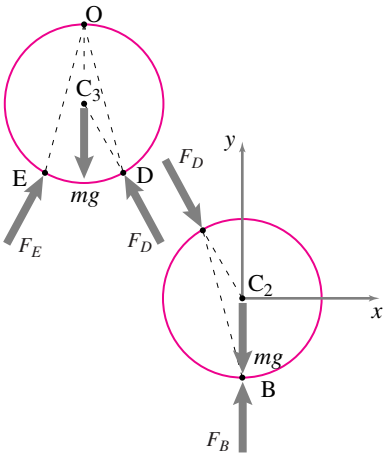



Figure 5.60:

Filename:sfig4-single-3balls-a

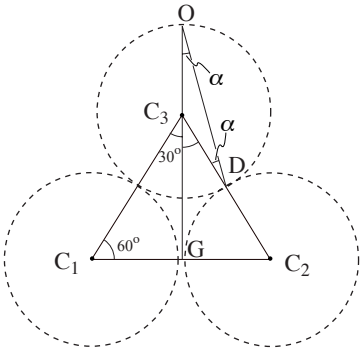



Figure 5.61:

Filename:sfig4-single-3balls-b

**Solution** Let us assume that the three balls are in equilibrium. We can then find the forces required on each ball to maintain the equilibrium. If we can find a plausible value of the friction coefficient $\mu$ from the required friction force on any of the balls, then we are done, otherwise our initial assumption of static equilibrium is wrong.

The free body diagrams of the upper ball and the lower right ball (why the right ball? No particular reason) are shown in Fig. 5.60. The contact forces, $\vec{\boldsymbol{F}}_{\mathrm{E}}$ and $\vec{\boldsymbol{F}}_{\mathrm{D}}$, act on the upper ball at points E and D, respectively. Each contact force is the resultant of a tangential friction force and a normal force acting at the point of contact. From the free body diagrams, we see that each ball is a three-force-body. Therefore, all the three forces — the two contact forces and the force of gravity — must be concurrent. This requires that the two contact forces must intersect on the vertical line passing through the center of the ball (the line of action of the force of gravity). Now, if we consider the free body diagram of the lower right ball, we find that force $\vec{\boldsymbol{F}}_{\mathrm{D}}$ has to pass through point B since the other two forces intersect at point B. Thus, we know the direction of force $\vec{\boldsymbol{F}}_{\mathrm{D}}$.

Let $\alpha$ be the angle between the contact force $\vec{\boldsymbol{F}}_{\mathrm{D}}$ and the normal to the ball surface at D. Now, from geometry, $\angle \mathrm{C_3DO} + \angle \mathrm{C_3OD} + \mathrm{OC_3D} = 180^\circ$. But, $\alpha = \angle \mathrm{C_3DO} = \angle \mathrm{C_3OD}$. Therefore,

$$\begin{aligned} \alpha &= \frac{1}{2}(180^\circ - \angle \mathrm{OC_3D}) = \frac{1}{2}(\angle \mathrm{GC_3D}) \\ &= \frac{1}{2}30^\circ = 15^\circ \end{aligned}$$

where $\angle \mathrm{GC_3D} = 30^\circ$ follows from the fact that $\mathrm{C_1C_2C_3}$ is an equilateral triangle and $\mathrm{C_3G}$ bisects $\angle\, \mathrm{C_1C_3C_2}$.

Now, from Fig. 5.61, we see that

$$\tan\alpha = \frac{F_s}{N}.$$

But, the force of friction $F_s \leq \mu N$. Therefore, it follows that

$$\mu \geq \tan\alpha = \tan 15^\circ = 0.27.$$

Thus, the friction coefficient must be at least 0.27 if the three balls have to be in static equilibrium.

$$\boxed{\mu \geq 0.27}$$

# 5.5 3D trusses and advanced truss concepts

After you have mastered the elementary 2D truss analysis of the previous section you might wonder

- **Do the ideas generalize to 3D?** Yes, with a only minor elaboration.
- **Does at least one of the methods presented always work?** Yes, if you just look at the homework problems for elementary truss analysis. And yes again for many practical structures. But some trusses trusses cannot be analyzed by the simple methods. In this section we classify trusses into types. One type of truss can be analyzed by simple methods, the others cannot.

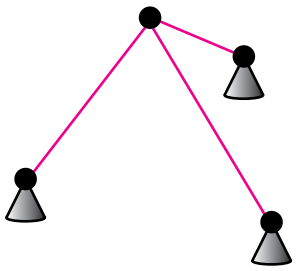


Figure 5.62: A tripod is the simplest rigid 3D truss.

Filename:tfigure-tripod

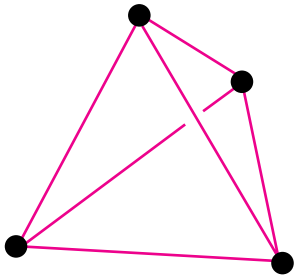


Figure 5.63: A tetrahedron is the simplest rigid truss in 3D that does not depend on grounding.

Filename:tfigure-tetrahedron

## 3D truss analysis

The concepts for 3D trusses are basically the same as for 2D trusses with these differences;

- In the method of joints each joint is associated with 3 scalar force balance equations instead of 2;
- In the method of sections, and in the free body diagram of the whole structure one has 6 scalar equations instead of 3;
- To hold the structure in place takes at least 6 reaction components instead of 3;
- The rule-of-thumb check for static determinacy of a grounded structure is $b + r = 3j$ instead of $b + r = 2j$;
- The rule of thumb for rigidity for a floating truss is $b + 6 = 3j$ instead of $b + 3 = 2j$.

There are various ways to think about the number six in the counts above. Assuming the structure is more than a point, six is the number of ways a rigid structure can move in three dimensional space (three translations and three rotations), six is the number of equilibrium equations for the whole structure (one 3D vector moment, and one 3D vector force, and six is the number of constraints needed to hold a structure in place.

Example: **A tripod**
A tripod is the simplest rigid 3D structure. With four joints ($j = 4$), three bars ($b = 3$), and nine unknown reaction components ($r = 3 \times 3 = 9$), it exactly satisfies the equation $3j = b + r$, a check for determinacy of rigidity of 3D structures.

A tripod is the 3D equivalent of the two-bar truss shown in Fig. 5.66a on page 271.

Example: **A tetrahedron**
The simplest 3D rigid floating structure is a tetrahedron. With four joints ($j = 4$) and six bars ($b = 6$) it exactly satisfies the equation $3j = b + 6$ which is a check for determinacy of rigidity of floating 3D structures.

A tetrahedron is thus, in some sense the 3D equivalent of a triangle in 2D.

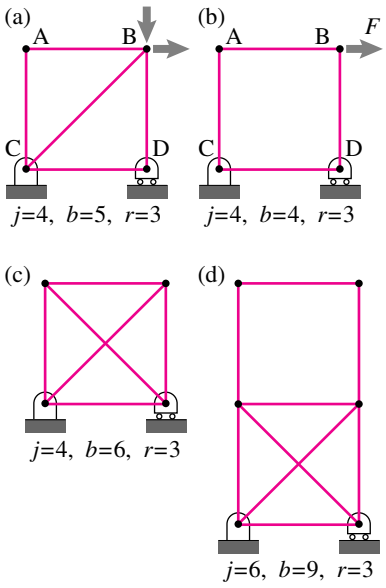



Figure 5.64: a) a statically determinate truss, b) a non-rigid truss, c) a redundant truss, and d) a non-rigid and redundant truss.

Filename:tfigure-4cases

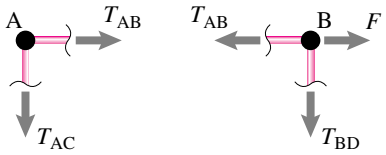



Figure 5.65: Free body diagrams of joints A and B from 5.64b. The solutions to the equilibrium equations are in conflict. That the equilibrium equations can't be satisfied means the structure cannot carry the given load. It is not rigid.

Filename:tfigure-squarejoints

* A curiosity. Notice that you could make the diagonals in fig. 5.64c both sticks and all of the outside square from cables and the truss would still carry all loads. This is the simplest 'tensegrity' structure. In a tensegrity structure no more than one bar in compression is connected to any one joint. (See fig. 5.11 for a more elegant example.). The label 'Tensegrity structure' was coined by the truss-pre-occupied designer Buckminster Fuller. Fuller is also responsible for re-inventing the "geodesic dome" a type of structure studied previously by Cauchy.

Example: **Geodesic domes**
Any closed polyhedron, with each face a triangle of rods, is a rigid structure. This includes a tetrahedron (above), an octahedron, a cube with a diagonal on each face, an icosohedron, and Buckminster Fuller's geodesic domes.

Well, so Cauchy thought. It turns out that there are some strange non-convex polyhedra that are not rigid. But, for practical purposes, if you see triangles all around the outside of a structure you can assume its rigid.

## Determinate, rigid, and redundant trusses

Your first concern when studying trusses is to develop the ability to solve a truss using free body diagrams and equilibrium equations. A truss that yields a solution, and only one solution, to such an analysis for all possible loadings is called *statically determinate* or just *determinate*. The braced box supported with one pin joint and one pin on rollers (see fig. 5.64a) is a classic statically determinate truss. A statically determinate truss is *rigid* and does not have *redundant* bars.

You should beware, however, that there are a few other possibilities.

Some trusses are *non-rigid*, like the one shown in fig. 5.64b, and can not carry arbitrary loads at the joints.

Example: **Joint equations and non-rigid structures**
Free body diagrams of joints A and B of fig. 5.64b are shown in fig. 5.65.

$$\begin{aligned} \text{jointB}: \quad \left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{\imath} \quad &\Rightarrow \quad T_{AB} = F \\ \text{jointA}: \quad \left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{\imath} \quad &\Rightarrow \quad T_{AB} = 0 \end{aligned}$$

The contradiction that $T_{AB}$ is both $F$ and 0 implies that the equations of statics have no solution for a horizontal load at joint B.

A non-rigid truss can carry some loads, and you can find the bar tensions using the joint equilibrium equations when these loads are applied. For example, the structure of fig. 5.64b can carry a vertical load at joint B. Engineers sometimes choose to design trusses that are not rigid, the simplest example being a single piece of cable hanging a weight. A more elaborate example is a suspension bridge which, when analyzed as a truss, is not rigid.

A *redundant* truss has more bars than needed for rigidity. As you can tell from inspection or analysis, the braced square of fig. 5.64a is rigid. None the less engineers will often choose to add extra redundant bracing as in fig. 5.64c for a variety of reasons.

- Redundancy is a safety feature. If one member brakes the whole structure holds up.
- Redundancy can increase a structure's strength.
- Redundancy can allow tensile bracing. In the structure of Fig. 5.64a top load to the left puts bar BC in compression. Thus bar BC can't be, say, a cable. But in structure fig. 5.64c both diagonals can be cables and neither need carry compression for any load*.

A property of redundant structures is that you can find more than one set of bar forces that satisfy the equilibrium equations. Even when the loads are all zero these structures can have non-zero *locked in* forces (sometimes called

('locked in stress', or 'self stress'). In the structure of fig. 5.64c, for example, if one of the diagonals got hot and stretched both it and the opposite diagonal would be put in compression while the outside was in tension. For structures whose parts are likely to expand or contract, or for which the foundation may shift, this locked in stress can be a contributor to structural failure. So redundancy is not all good.

Finally, a structure can be both non-rigid and redundant as shown in fig. 5.64d. This structure can't carry all loads, but the loads it can carry it can carry with various locked in bar forces.

More examples of statically determinate, non-rigid, and redundant truss are given on pages 274 and 275.

Note, one of the basic assumptions in elementary truss analysis which we have thus far used without comment is that motions and deformations of the structure are not taken into account when applying the equilibrium equations. If a bar is vertical in the drawing then it is taken as vertical for all joint equilibrium equations.

Example: **Hanging rope**
For elementary truss analysis, a hanging rope would be taken as hanging vertically even if side loads are applied to its end. This obviously ridiculous assumption manifests itself in truss analysis by the discovery that a hanging rope cannot carry any sideways loads (if it must stay vertical this is true).

## Determining determinacy: counting equations and unknowns

How can you tell if a truss is statically determinate? The only sure test is to write all the joint force balance equations and see if they have a unique solution for all possible joint loads. Because this is an involved linear algebra calculation (which we skip in this book), it is nice to have shortcuts, even if not totally reliable. Here are three:

- See, using your intuition, if the structure can deform without any of the bars changing length. You can see that the structures of fig. 5.64b and d can distort. If a structure can distort it is not rigid and thus is not statically determinate.
- See, using your intuition, if there are any redundant bars. A redundant bar is one that prevents a structural deformation that already is prevented. It is easy to see that the second diagonal in structures of fig. 5.64c and d is clearly redundant so these structures are not statically determinate.
- Count the total number of joint equations, two for each joint. See if this is equal to the number of unknown bar forces and reactions. If not, the structure is not statically determinate.

The counting formula in the third criterion above is:

$$2j = b + r \tag{5.27}$$

where $j$ is the number of joints, including joints at reaction points, $b$ is the number of bars, and $r$ is the number of reaction components that shows on a free body diagram of the whole structure (2 from pin joints, 1 from a pin on a roller).

If $2j > b + r$ the structure is necessarily not rigid because then there are more equations than unknowns*. For such a structure there are some loads for which there is no set of bar forces and reactions that can satisfy the joint equilibrium equations. A structure that is non-redundant and non-rigid always has $2j > b + r$ (see fig. 5.64b).

If $2j < b + r$ the structure is redundant because there are not as many equations as unknowns; if the equations can be solved there is more than one combination of forces that solve them. A structure that is rigid and redundant always has $2j < b + r$ (see fig. 5.64b).

But the possibility of structures that are both non-rigid and redundant makes the counting formulas an imperfect way to classify structures*.Non-rigid redundant structures can have $2j < b + r$, $2j = b + r$, or $2j > b + r$. The redundant non-rigid structure in fig. 5.64d has $2j = b + r$.

The discussion above can be roughly summarized by this table (refer to fig. 5.64 for a simple example of each entry and to pages 274 and 275 for several more examples).

| **Truss Type** | Rigid | Non-rigid |
|---|---|---|
| Non-redundant | a) $2j = b + r$ (Statically determinate) | b) $2j > b + r$ |
| Redundant | c) $2j < b + r$ | d) $2j < b + r$, $2j = b + r$, or $2j > b + r$ |

A basic summary is this:

If

- $2j = b + r$ *and*
- you cannot see any ways the structure can distort, *and*
- you cannot see any redundant bars

then the truss is likely statically determinate. But the only way you can know for sure is through either a detailed study of the joint equilibrium equations, or familiarity with similar structures.

On the other hand if

- $2j > b + r$, *or*
- $2j < b + r$, *or*
- you can see a way the structure can distort, *or*

* A non-rigid truss is sometimes called 'over-determinate' because there are more equations than unknowns. However, the term 'over-determinate' may incorrectly conjure up the image of there being too many bars (which we call *redundant*) rather than too many joints. So we avoid use of this phrase.

* In the language of mathematics we would say that satisfaction of the counting equation $2j = b + r$ is a *necessary* condition for static determinacy but it is not *sufficient*.

- you can see one or more redundant bars,

then the truss is *not* statically determinate.

Example: **The classic statically determinate structure**
A *triangulated truss* can be drawn as follows:

1. draw one triangle,
2. then another by adding two bars to an edge,
3. then another by adding two bars to an existent edge
4. and so on, but never adding a triangle by adding just one bar, and
5. you hold this structure in place with a pin at one joint and one pin on roller at another joint

then the structure is statically determinate. Many elementary trusses are of exactly this type. (*Note*: if you violate the 'but' in the 4th rule you can make a truss that looks 'triangulated' but is redundant, and therefore not statically determinate.)

## Floating trusses

Sometimes one wants to know if a structure is rigid and non-redundant when it is floating unconnected to the ground (but still in 2D, say). For example, a triangle is rigid when floating and a square is not. The truss of fig. 5.66a is rigid as connected but not when floating (fig. 5.66b).A way to find out if a floating structure is rigid is to connect one bar of the truss to the ground by connecting one end of the bar with a pin and the other with a pin on a roller, as in fig. 5.66c. All determinations of rigidity for the floating truss are the same as for a truss grounded this way. The counting formula eqn. 5.27, is reduced to

$$2j = b + 3$$

because this minimal way of holding the structure down uses $r = 3$ reaction force components.

## The principle of superposition for trusses

Say you have solved a truss with a certain load and have also solved it with a different load. Then if both loads were applied the reactions would be the sums of the previously found reactions and the bar forces would be the sums of the previously found bar forces.

This useful fact follows from the linearity of the equilibrium equations*.

Example: **Superposition and a truss**

* A careful derivation would also show that the linearity depends on the nature of the foundation. Linearity holds for pins and pins on rollers, but not for frictional contact.

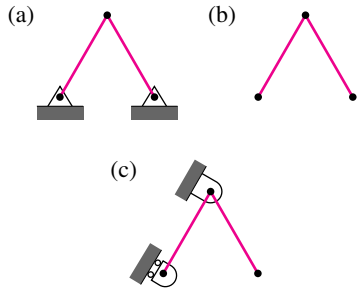
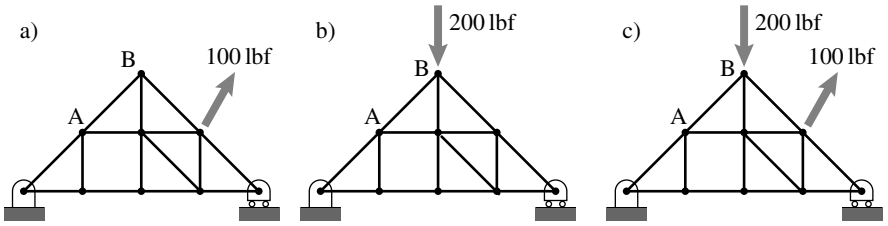



Figure 5.66: a) a determinate two bar truss connected to the ground, b) the same truss is not rigid when floating, which you can tell by seeing that c) it is not rigid when one bar is fixed to the ground.

Filename:tfigure-rigidonground




## 5.2 *THEORY*

### *Stuctural rigidity and geometric congruence*

This box is only for the curious. It will not help you solve truss homework problems.

In high school geometry one learns to prove that two shapes are congruent (the same shape and size) if they have enough in common. One proof is based on "side-side-side" (SSS); if two triangles have three sides with corresponding lengths then the corresponding angles must also be equal. High school geometry proofs are based on triangles.

Now, here, we claim that structures made of triangles tend to be rigid. Is there a relation between the central role of triangles in both geometry proofs and in structural rigidity? Yes, but more subtly than you may expect.

Consider one triangle. If the lengths are specified it is like three sticks connected with rubber bands. That two different triangles each with the same 3 side lengths are congruent means that one triangle whose side-lengths are given has no choice about its shape. So for one triangle the SSS proof corresponds exactly to structural rigidity.

More generally, imagine looking at a structure and thinking of certain aspects of it as fixed and others as not fixed. For example, think of of a collection of bars with the lengths fixed (each bar length is not changeable) and the angles between them as not-fixed (the angles are flexible). This would be a model, say, of bars connected with pin joints. If one could find a geometry proof that these two structures had identical shapes it would mean that each one of them had no choice about its shape. So a geometry proof of congruency, based on the aspects of a structure that are approximately fixed, is a proof of structural rigidity. So there *is* a connection between congruence proofs and structural rigidity.

Here's the subtlety. Neither one depends essentially on triangles. There are congruence proofs that make no use of any closed triangle so there are rigid structures that have no close triangles.

There is a whole arcane mathematics of rigidity. And the things mathematicians have learned about rigidity are incredible.

Example: $K_{33}$

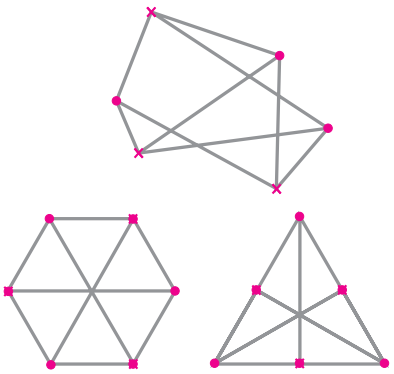
Take 3 points on a plane and mark them with dots. Take 3 more points on the plane and mark them with little x's. Connect each dot with each x. Thats 9 connection lines. In topology-speak they call this set of dots and lines "K three three" (*K*onnections between *three* dots and *three* x's).

Now think of that criss-crossed $K_{33}$ drawing as a structure made of sticks connected with hinges at the dots and x's. Note that, neglecting where the sticks cross but are not connected, there are no closed triangles. Yet, this structure is always rigid. Well, almost always. If all 6 points happen to lie on one circle, ellipse, parabola or hyperbola then the structure is not rigid.

If you take a regular hexagon made of sticks (length $\ell$ and hinges and brace it with three cross bars (each with length $2\ell$) you will see that you have K33; every-other corner is a dot and the alternate ones are x's. But the points on a hexagon are on a circle, so that structure is not rigid.

On the other hand, take an equilateral triangle and cut each side in half so you have six bars around the outside (each with length $\ell/2$). Now brace that hexagon (that is shaped like a triangle) with the three triangle altitudes (each with length $\sqrt{3}\ell$) and you again have K33. But this time it's rigid.

We have used these examples in the text and homework because they illustrate structures that don't lend themselves to the simple joint-by-joint method-of-joints, nor the method of sections.



The mathematical magic goes on.

Example: $K_{nn}$

If you take any $n$ dots and any $n$ x's and connect each dot to each x with a rigid rod ($K_{nn}$) you get a rigid structure. Unless all $2n$ dots happen to lie on a conic section.

The proofs of such rigidity theorems are way over our heads. But you can simply check such structures for rigidity with the computer program developed in section 5.3.

So yes, geometric congruence and structural rigidity are the same subject. But that subject does not totally depend on triangles. Triangles just provide the simple examples and what we vaguely think of as the essence.

If for the loading (a) you found $T_{AB} = 50\,\text{lbf}$ and for loading (b) you found $T_{AB} = -140\,\text{lbf}$ then for loading (c) $T_{AB} = 50\,\text{lbf} - 140\,\text{lbf} = -90\,\text{lbf}$

Example: **Spider web**
Any truss that only has bars in tension cannot be statically determinate. It has to have a locked-in pre-stress to be rigid.

## 5.3 Theory: Rigidity, redundancy, linear algebra and maps

This mathematical aside is only for people who have had a course in linear algebra. For definiteness this discussion is limited to 2D trusses, but the ideas also apply to 3D trusses.

For beginners trusses fall into two types, those that are uniquely solvable (statically determinate) and those that are not. Statically determinate trusses are rigid and non-redundant. However, a truss could be non-rigid and non-redundant, rigid and redundant, or non-rigid and redundant. These four possibilities are shown with a simple example each in figure 5.64 on page 268, as a simple table on page 270, and as a big table of examples on pages 274 and 275. The table below, which we now proceed to discuss in detail, is a more abstract mathematical representation of this same set of possibilities.

• • •

To start with we use the matrix form of the truss joint equations from page 250. To make contact with linear algebra here we take the unknowns as $[v]$ with $[v] \equiv [T]$ being the $n$ unknown tensions and reaction components. The set of lists of all conceivable tensions and reaction forces we call the "vector space" $V$ (it is also $R^n$).

The $m$ possible loads at the joints are written in the column vector $[w]$ (called $[L]$, for loads, in the numeric set up). The set of all possible loads we call the vector space $W$.

If we use the method of joints we can write two scalar equilibrium equations for each joint. These are linear algebraic equations. Thus we can write them in matrix form as (see eqn. (5.18) on page 250),

$$[A][v] = [w] \tag{5.28}$$

The classification of trusses is really a statement about the solutions of eqn. 5.28. This classification follows, in turn, from the properties of the matrix $[A]$.

Another point of view is to think of eqn. 5.28 as a function that maps one vector space onto another. For any $[v]$ eqn. 5.28 maps that $[v]$ to some $[w]$. That is, if one were given all the bar tensions and reactions one could uniquely determine the applied loads from eqn. 5.28. This map, from $V$ to $W$ we call $T$.

• • •

We can now discuss each of the truss categorizations in turn, with reference to the table at the end of this box.

The first column of the table corresponds to rigid trusses. These trusses have at least one set of bar forces that can equilibrate any particular load. This means that for every $[w]$ there is some $[v]$ that maps to (whose image is) $[w]$. In these cases the map $T$ is onto. And the column space of $[A]$ is $W$. Thus $[A]$ needs to have at least as many columns as the dimension of $W$ which is the number of rows of $[A]$.

On the other hand if the structure is not rigid there are some loads that cannot be equilibrated by any bar forces. This is the second column of the table. There is at least some $[w]$ with no pre-image $[v]$. Thus the map T is not onto and the column space of $[A]$ is less than all of $W$.

The first row of the table describes trusses which are not-redundant. Thus, any loads which can be equilibrated can be equilibrated with a unique set of bar tensions and reactions. Thus the columns of $[A]$ are linearly independent and the map $T$ is one-to-one. The matrix $[A]$ must have at least as many rows as columns.

If a truss is redundant, as in the second row of the table, then there are various ways to equilibrate loads which can be carried. Points in $W$ in the image of one, and the columns of A are linearly dependent.

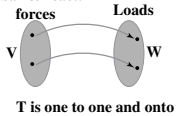
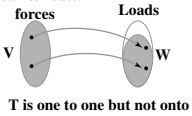
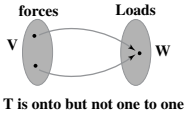
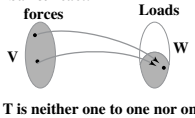
• • •

We can now look at the four entries in the table. The top left case is the statically determinate case where the structure is rigid and non-redundant. The map $T$ is one to one and onto, $V = W$, and the matrix $[A]$ is square and non-singular.

The bottom left case corresponds to a truss that is rigid and redundant. The map to is onto but not one to one. The columns of $[A]$ are linearly dependent and it has more columns than rows (it is wide).

The top right case is not rigid and not redundant. Some loads cannot be equilibrated and those that can be are equilibrated uniquely. $T$ is one to one but not onto. The columns of $[A]$ are linearly independent but they do not span $W$. The matrix $[A]$ has more rows than columns and is thus tall.

The bottom right case is the most perverse. The structure is not rigid but is redundant. Not all loads can be equilibrated but those that can be equilibrated are equilibrated non-uniquely. The matrix $[A]$ could have any shape but its columns are linearly dependent and do not span $W$. The map T is neither one to one nor onto.

| | **Rigid**<br>• T is onto<br>• col(A) = W | **Not rigid**<br>• T is not onto<br>• col(A) ≠ W |
|---|---|---|
| **Not redundant**<br>• T is one to one<br>• columns of A are linearly independent | **A is square and invertible**<br>bar & react. forces / Loads<br>  <br>V W<br>**T is one to one and onto** | **A is tall**<br>bar & react. forces / Loads<br>  <br>V W<br>**T is one to one but not onto** |
| **Redundant**<br>• T is not one to one<br>• columns of A are linearly dependent | **A is wide**<br>bar & react. forces / Loads<br>  <br>V W<br>**T is onto but not one to one** | **A can be wide, square, or tall**<br>bar & react. forces / Loads<br>  <br>V W<br>**T is neither one to one nor onto** |

**2D TRUSS CLASSIFICATION (page 1)**

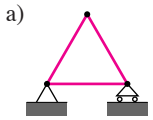
## Rigid

• loads can be equilibriated with bar forces

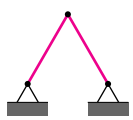
## Not redundant

• Not indeterminate
• If there are bar forces that can equilibriate the loads they are unique
• No locked in stresses

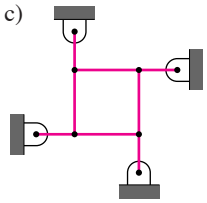
**Statically determinate**, rigid and not redundant, $b + r = 2j$,
One and only one set of bar forces can equilibriate any given load.



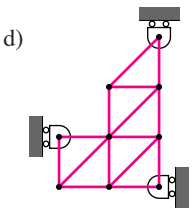

$j$=3, $b$=3, $r$=3



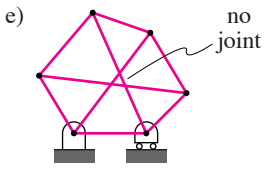

$j$=3, $b$=2, $r$=4




$j$=8, $b$=8, $r$=8




$j$=9, $b$=15, $r$=3

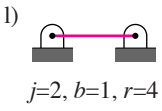



$j$=6, $b$=9, $r$=3

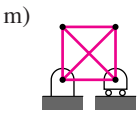
## Redundant

• indeterminate
• locked in stress possible
• solutions not unique if they exist

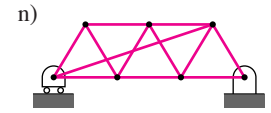
$b + r > 2j$, "too few equations", rigid and redundant,
Every possible load can be equilibrated
but the bar forces are not unique.



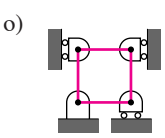

$j$=2, $b$=1, $r$=4



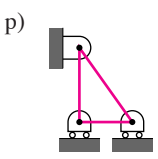

$j$=4, $b$=6, $r$=3



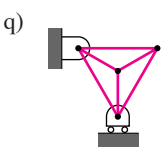

$j$=7, $b$=12, $r$=3




$j$=4, $b$=4, $r$=5




$j$=3, $b$=3, $r$=4




$j$=4, $b$=6, $r$=3

Figure 5.67: Examples of 2D trusses. These two pages concern the 2-fold system for identifying trusses. Trusses can be rigid or not rigid (the two columns) and they can be redundant or not redundant (the two rows). Elementary truss analysis is only concerned with rigid and not redundant trusses (*statically determinate* trusses). Note that the only difference between trusses (b) and (s) is a change of shape (likewise for the far more subtle examples (e) and (u)). Truss (e) is interesting as a rare example of a determinate truss with no triangles.

Filename:tfigure-trussclass1

**2D TRUSS CLASSIFICATION**
**(page 2)**

# Not rigid

$b + r < 2j$ , not rigid and not redundant, "too many equations"
Unique bar forces for some loads,
no solution for other loads.

## Not redundant

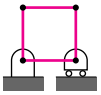
- Not indeterminate
- If there are bar forces that can equilibriate the loads they are unique
- No locked in stresses

f)



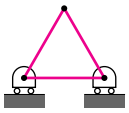
$j$=2
$b$=1
$r$=2

g)



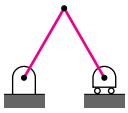
$j$=4, $b$=4, $r$=3

h)



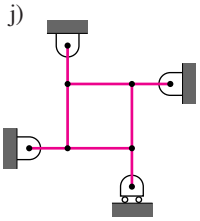
$j$=3, $b$=3, $r$=2

i)



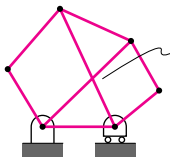
$j$=3, $b$=2, $r$=3

j)



$j$=8, $b$=8, $r$=7

k)




$j$=6, $b$=8, $r$=3

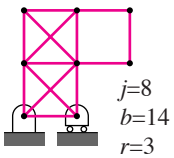
Not rigid and redundant

## Redundant

- indeterminate
- locked in stress possible
- solutions not unique if they exist

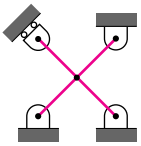
$b + r > 2j$

r)



$j$=8
$b$=14
$r$=3

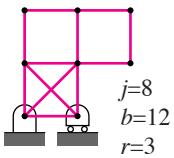
x)



$j$=5
$b$=4
$r$=7

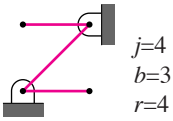
$b + r < 2j$

w)



$j$=8
$b$=12
$r$=3

z)



$j$=4
$b$=3
$r$=4

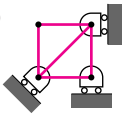
$b + r = 2j$

s)



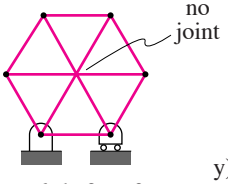
$j$=3, $b$=2, $r$=4

t)



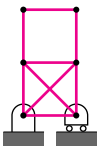
$j$=4
$b$=5
$r$=3

u) regular hexagon



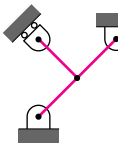

$j$=6, $b$=9, $r$=3

v)



$j$=6
$b$=9
$r$=3

y)



$j$=4
$b$=3
$r$=5

Figure 5.68: (Second page of a two page table.)

Filename:tfigure-trussclass2

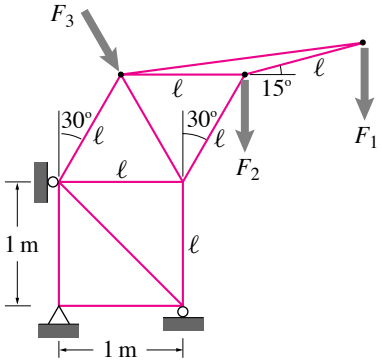



Figure 5.69:

Filename:sfig4-truss-over

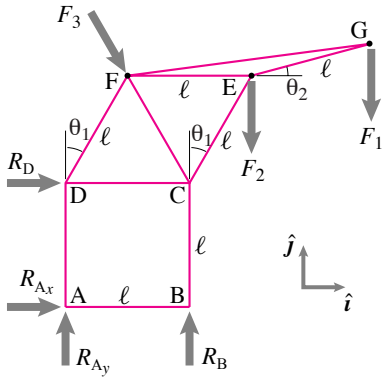



Figure 5.70:

Filename:sfig4-truss-over-a

**SAMPLE 5.14 An indeterminate truss:** For the truss shown in the figure, find all support reactions.

**Solution** The free-body diagram of the truss is shown in Fig. 5.70. We need to find the support reactions $R_{A_x}$, $R_{A_y}$, $R_B$, and $R_D$.

The $x$ and $y$ components of the force equilibrium, $\sum \vec{F} = \vec{0}$, give

$$\sum F_x = 0 \quad \Rightarrow \quad R_{A_x} + R_D = -F_3 \cos\theta_1 \tag{5.29}$$

$$\sum F_y = 0 \quad \Rightarrow \quad R_{A_y} + R_B = F_1 + F_2 + F_3 \sin\theta_1. \tag{5.30}$$

Now we apply moment balance about point A, $\sum \vec{M}_{\rm A} = \vec{0}$. Let A be the origin of our $xy$-coordinate system (so that we can write $\vec{r}_{\rm D/A} = \vec{r}_{\rm D}$, etc.).

$$\vec{r}_{\rm D} \times \vec{R}_D + \vec{r}_{\rm F} \times \vec{F}_3 + \vec{r}_{\rm G} \times \vec{F}_1 + \vec{r}_{\rm E} \times \vec{F}_2 + \vec{r}_{\rm B} \times \vec{R}_B = 0$$

where,

$$\begin{aligned}
\vec{r}_{\rm D} \times \vec{R}_D &= \ell\hat{\boldsymbol{j}} \times R_D\hat{\boldsymbol{i}} = -R_D\ell\hat{\boldsymbol{k}} \\
\vec{r}_{\rm F} \times \vec{F}_3 &= (\vec{r}_{\rm D} + \vec{r}_{\rm F/D}) \times \vec{F}_3 \\
&= [\ell\hat{\boldsymbol{j}} + \ell(\sin\theta_1\hat{\boldsymbol{i}} + \cos\theta_1\hat{\boldsymbol{j}})] \times F_3(\cos\theta_1\hat{\boldsymbol{i}} - \sin\theta_1\hat{\boldsymbol{j}}) \\
&= F_3\ell\cos\theta_1\hat{\boldsymbol{k}} - F_3\ell\hat{\boldsymbol{k}} = -F_3\ell(1+\cos\theta_1)\hat{\boldsymbol{k}} \\
\vec{r}_{\rm G} \times \vec{F}_1 &= (r_{G_x}\hat{\boldsymbol{i}} + r_{G_y}\hat{\boldsymbol{j}}) \times (-F_1\hat{\boldsymbol{j}}) = -r_{G_x}F_1\hat{\boldsymbol{k}} \\
&= -F_1\ell(1+\sin\theta_1+\cos\theta_2)\hat{\boldsymbol{k}} \\
\vec{r}_{\rm E} \times \vec{F}_2 &= -F_2(\ell + \ell\sin\theta_1\hat{\boldsymbol{k}} = -F_2\ell(1+\sin\theta_1)\hat{\boldsymbol{k}} \\
\vec{r}_{\rm B} \times \vec{R}_B &= \ell\hat{\boldsymbol{i}} \times R_B\hat{\boldsymbol{j}} = R_B\ell\hat{\boldsymbol{k}}.
\end{aligned}$$

Adding them together and dotting with $\hat{\boldsymbol{k}}$ we get

$$-R_D\ell - F_3\ell(1+\cos\theta_1) - F_1\ell(1+\sin\theta_1+\cos\theta_2) - F_2\ell(1+\sin\theta_1) + R_B\ell = 0$$

$$\Rightarrow \quad R_B - R_D = F_1(1+\sin\theta_1+\cos\theta_2) + F_2(1+\sin\theta_1) + F_3(1+\cos\theta_1). \tag{5.31}$$

We have three equations (5.29–5.31) containing four unknowns $R_{A_x}$, $R_{A_y}$, $R_B$, and $R_D$. So, we cannot solve for the unknowns uniquely. This was expected as the truss is indeterminate. However, if we assume a value for one of the unknowns, we can solve for the rest in terms of the assumed one. For example, let $R_D = \alpha$. For simplicity let the right hand sides of eqns. (5.29, 5.30, and 5.31) be $C_1$, $C_2$, and $C_3$ (computed values), respectively. Then, we get $R_{A_x} = C_1 - \alpha$, $R_{A_y} = C_2 - C_3 - \alpha$, and $R_B = C_3 + \alpha$. The equilibrium is satisfied for any value of $\alpha$. Thus there are infinite number of solutions! This is true for all indeterminate systems. However, when deformations of structures are taken into account (extra constraint equations), then solutions do turn out to be unique. You will learn about such things in courses dealing with strength of materials.

**SAMPLE 5.15 A simple 3-D truss:** The 3-D truss shown in the figure has 12 bars and 6 joints. Nine of the 12 bars that are either horizontal or vertical have length $\ell = 1\,\text{m}$. The truss is supported at A on a ball and socket joint, at B on a linear roller, and at C on a planar roller. The loads on the truss are $\vec{F}_1 = -50\,\text{N}\hat{k}$, $\vec{F}_2 = -60\,\text{N}\hat{k}$, and $\vec{F}_3 = 30\,\text{N}\hat{j}$. Find all support reactions and the tension in bar BC.

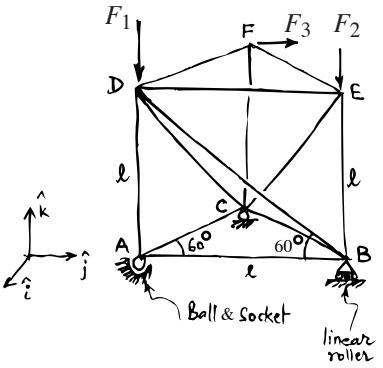



Figure 5.71:
Filename:sfig4-3d-truss1

**Solution** The free-body diagram of the entire structure is shown in Fig. 5.72. Let the support reactions at A, B, and C be $\vec{R}_A = R_{A_x}\hat{\imath} + R_{A_y}\hat{\jmath} + R_{A_z}\hat{k}$, $\vec{R}_B = R_{B_x}\hat{\imath} + R_{B_z}\hat{k}$, and $\vec{R}_C = R_C\hat{k}$. Then the moment balance about point A, $\sum \vec{M}_{\text{A}} = \vec{0}$, gives

$$\vec{r}_{\text{B/A}} \times \vec{R}_B + \vec{r}_{\text{C/A}} \times \vec{R}_C + \vec{r}_{\text{E/A}} \times \vec{F}_2 + \vec{r}_{\text{F/A}} \times \vec{F}_3 = \vec{0}. \tag{5.32}$$

Note that $\vec{F}_1$ passes through A and, therefore, produces no moment about A. Now we compute each term in the equation above.

$$\begin{aligned}
\vec{r}_{\text{B/A}} \times \vec{R}_B &= \ell\hat{\jmath} \times (R_{B_x}\hat{\imath} + R_{B_z}\hat{k}) &&= -R_{B_x}\ell\hat{k} + R_{B_z}\ell\hat{\imath},\\
\vec{r}_{\text{C/A}} \times \vec{R}_C &= \ell(\cos\theta\,\hat{\jmath} - \sin\theta\hat{\imath}) \times R_C\hat{k} &&= R_C\tfrac{\ell}{2}\hat{\imath} + R_C\tfrac{\sqrt{3}\ell}{2}\hat{\jmath}\\
\vec{r}_{\text{E/A}} \times \vec{F}_2 &= (\ell\hat{\jmath} + \ell\hat{k}) \times (-F_2\hat{k}) &&= -F_2\ell\hat{\imath},\\
\vec{r}_{\text{F/A}} \times \vec{F}_3 &= [\ell(\cos\theta\,\hat{\jmath} - \sin\theta\hat{\imath}) + \ell\hat{k}] \times F_3\hat{\jmath} &&= -F_3\ell\hat{\imath} - F_3\tfrac{\sqrt{3}\ell}{2}\hat{k}.
\end{aligned}$$

Substituting these products in eqn. (5.32), and dotting the resulting equation with $\hat{\jmath}$, $\hat{k}$, and $\hat{\imath}$, respectively, we get

$$\begin{aligned}
R_C &= 0\\
R_{B_x} &= -\frac{\sqrt{3}}{2}F_3 = -15\sqrt{3}\,\text{N}\\
R_{B_z} &= -\frac{1}{2}R_C + F_2 + F_3 = 90\,\text{N}.
\end{aligned}$$

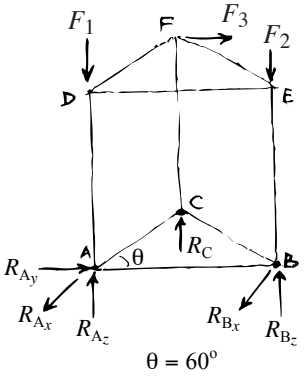



Figure 5.72:
Filename:sfig4-3d-truss1-a

Thus, $\vec{B} = R_{B_x}\hat{\imath} + R_{B_z}\hat{k} = -15\sqrt{3}\,\text{N}\hat{\imath} + 30\,\text{N}\hat{k}$ and $\vec{R}_C = \vec{0}$. Now from the force balance, $\sum \vec{F} = \vec{0}$, we find $\vec{R}_A$ as

$$\begin{aligned}
\vec{R}_A &= -\vec{R}_B - \vec{R}_C - \vec{F}_1 - \vec{F}_2 - \vec{F}_3\\
&= -(-15\sqrt{3}\,\text{N}\hat{\imath} + 90\,\text{N}\hat{k}) - (-50\,\text{N}\hat{k}) - (-60\,\text{N}\hat{k}) - (30\,\text{N}\hat{\jmath})\\
&= 15\sqrt{3}\,\text{N}\hat{\imath} - 30\,\text{N}\hat{\jmath} + 20\,\text{N}\hat{k}.
\end{aligned}$$

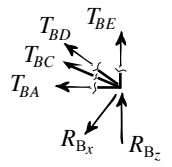



Figure 5.73:
Filename:sfig4-3d-truss1-b

To find the force in bar BC, we draw a free-body diagram of joint B (which connects BC) as shown in Fig. 5.73. Now, writing the force balance for the joint in the $x$-direction, *i.e.*, $[\sum \vec{F} = \vec{0}] \cdot \hat{\imath}$, gives

$$\begin{aligned}
R_{B_x} + \vec{T}_{\text{BC}} \cdot \hat{\imath} &= 0\\
\text{or} \qquad R_{B_x} + T_{BC}\sin\theta &= 0\\
\Rightarrow \quad T_{BC} &= -\frac{R_{B_x}}{\sin\theta}\\
&= -\frac{-15\sqrt{3}\,\text{N}}{\sqrt{3}/2} = 30\,\text{N}.
\end{aligned}$$

Thus, the force in bar BC is $T_{BC} = 30\,\text{N}$ (tensile force).

$$\boxed{\vec{R}_A = 15\sqrt{3}\,\text{N}\hat{\imath} - 30\,\text{N}\hat{\jmath} + 20\,\text{N}\hat{k},\ \vec{R}_B = -15\sqrt{3}\,\text{N}\hat{\imath} + 90\,\text{N}\hat{k},\ \vec{R}_C = \vec{0},\ T_{BC} = 30\,\text{N}}$$

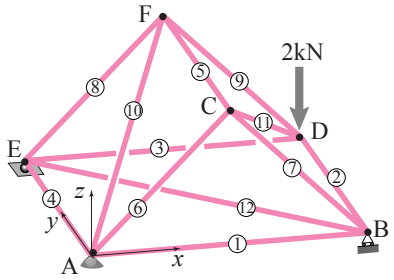



Figure 5.74:
Filename:sfig5-4-3d-truss2

**SAMPLE 5.16 A 3-D truss solved on the computer:** The 3-D truss shown in the figure is fabricated with 12 bars. Bars 1–5 are of length $\ell = 1\,\text{m}$, bars 6–9 have length $\ell/\sqrt{2}(\approx 0.71\,\text{m})$, and bars 10–12 are cut to size to fit between the joints they connect. The truss is supported at A on a ball and socket, at B on a linear roller, and at C on a planar roller. A load $F = 2\,\text{kN}$ is applied at D as shown. Write all equations required to solve for all bar forces and support reactions and solve the equations using a computer.

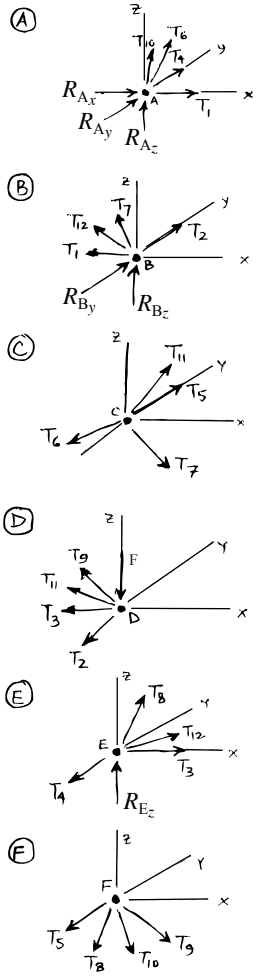



Figure 5.75:
Filename:sfig4-3d-truss2-a

**Solution** There are 12 bars and 6 joints in the given truss. The unknowns are 12 bar forces and six support reactions (3 at A ($R_{A_x}$, $R_{A_y}$, $R_{A_z}$), 2 at B ($R_{B_y}$, $R_{B_z}$), and 1 at E ($R_{E_z}$)). Therefore, we need 18 independent equations to solve for all the unknowns. Since the force equilibrium at each joint gives one vector equation in 3-D, *i.e.*, three scalar equations, the 6 joints in the truss can generate the required number ($6 \times 3 = 18$) of equations. Therefore, we go joint by joint, draw the free-body diagram of the joint, write the force equilibrium equation, and extract the 3 scalar equations from each vector equation. We switch from the letters to denote the bars in the force vectors to numbers in its scalar representation ($T_1$, $T_2$, *etc.*) to facilitate computer solution.

- **Joint A:**

$$T_1\hat{\boldsymbol{\imath}} + \frac{T_6}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}) + \frac{T_{10}}{\sqrt{6}}(\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}) + T_4\hat{\boldsymbol{\jmath}} + R_{A_x}\hat{\boldsymbol{\imath}} + R_{A_y}\hat{\boldsymbol{\jmath}} + R_{A_z}\hat{\boldsymbol{k}} = \vec{\boldsymbol{0}}.$$

- **Joint B:**

$$-T_1\hat{\boldsymbol{\imath}} + \frac{T_7}{\sqrt{2}}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}) + T_2\hat{\boldsymbol{\jmath}} + \frac{T_{12}}{\sqrt{2}}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}) + R_{B_y}\hat{\boldsymbol{\jmath}} + R_{B_z}\hat{\boldsymbol{k}} = \vec{\boldsymbol{0}}.$$

- **Joint C:**

$$-\frac{T_6}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}) - \frac{T_7}{\sqrt{2}}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}) + T_5\hat{\boldsymbol{\jmath}} + \frac{T_{11}}{\sqrt{6}}(\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}) = \vec{\boldsymbol{0}}.$$

- **Joint D:**

$$-T_2\hat{\boldsymbol{\jmath}} - \frac{T_{11}}{\sqrt{6}}(\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}) - T_3\hat{\boldsymbol{\imath}} + \frac{T_9}{\sqrt{2}}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}) - F\hat{\boldsymbol{k}} = \vec{\boldsymbol{0}}.$$

- **Joint E:**

$$-T_4\hat{\boldsymbol{\jmath}} + \frac{T_{12}}{\sqrt{2}}(\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}) + T_3\hat{\boldsymbol{\imath}} + \frac{T_8}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}) + R_{E_z}\hat{\boldsymbol{k}} = \vec{\boldsymbol{0}}.$$

- **Joint F:**

$$-T_5\hat{\boldsymbol{\jmath}} - \frac{T_8}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}) - \frac{T_{10}}{\sqrt{6}}(\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}) - \frac{T_9}{\sqrt{2}}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}) = \vec{\boldsymbol{0}}.$$

Now we can separate out 3 scalar equations from each of the joint vector equations by dotting them with $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$.

| Joint | [Eqn.] $\cdot\,\hat{\boldsymbol{\imath}}$ | [Eqn.] $\cdot\,\hat{\boldsymbol{\jmath}}$ | [Eqn.] $\cdot\,\hat{\boldsymbol{k}}$ |
|---|---|---|---|
| $A$ | $T_1 + \frac{1}{\sqrt{2}}T_6 + \frac{1}{\sqrt{6}}T_{10} + R_{A_x} = 0,$ | $\frac{2}{\sqrt{6}}T_{10} + T_4 + R_{A_y} = 0,$ | $\frac{1}{\sqrt{2}}T_6 + \frac{1}{\sqrt{6}}T_{10} + R_{A_z} = 0$ |
| $B$ | $-T_1 - \frac{1}{\sqrt{2}}T_7 - \frac{1}{\sqrt{2}}T_{12} = 0,$ | $T_2 + \frac{1}{\sqrt{2}}T_{12} + R_{B_y} = 0,$ | $\frac{1}{\sqrt{2}}T_7 + R_{B_z} = 0$ |
| $C$ | $-\frac{1}{\sqrt{2}}T_6 + \frac{1}{\sqrt{2}}T_7 + \frac{1}{\sqrt{6}}T_{11} = 0,$ | $T_5 + \frac{2}{\sqrt{6}}T_{11} = 0,$ | $\frac{1}{\sqrt{2}}T_6 + \frac{1}{\sqrt{2}}T_7 + \frac{1}{\sqrt{6}}T_{11} = 0$ |
| $D$ | $-\frac{1}{\sqrt{6}}T_{11} - T_3 - \frac{1}{\sqrt{2}}T_9 = 0,$ | $-T_2 - \frac{2}{\sqrt{6}}T_{11} = 0,$ | $\frac{1}{\sqrt{6}}T_{11} + \frac{1}{\sqrt{2}}T_9 = F$ |
| $E$ | $\frac{1}{\sqrt{2}}T_{12} + T_3 + \frac{1}{\sqrt{2}}T_8 = 0,$ | $-T_4 - \frac{1}{\sqrt{2}}T_{12} = 0,$ | $\frac{1}{\sqrt{2}}T_8 + R_{E_z} = 0$ |
| $F$ | $-\frac{1}{\sqrt{2}}T_8 - \frac{1}{\sqrt{6}}T_{10} + \frac{1}{\sqrt{2}}T_9 = 0,$ | $-T_5 - \frac{2}{\sqrt{6}}T_{10} = 0,$ | $\frac{1}{\sqrt{2}}T_8 + \frac{1}{\sqrt{6}}T_{10} + \frac{1}{\sqrt{2}}T_9 = 0.$ |

Thus, we have 18 required equations for the 18 unknowns. Before we go to the computer, we need to do just one more little thing. We need to order the unknowns in some way in a one-dimensional array. So, let

$$x = [R_{A_x} \quad R_{A_y} \quad R_{A_z} \quad R_{B_x} \quad R_{B_y} \quad R_{E_z} \quad T_1 \ldots T_{12}].$$

Thus $x_1 = R_{A_x},\ x_2 = R_{A_y},\ \ldots, x_7 = T_1,\ x_8 = T_2,\ \ldots, x_{18} = T_{12}$. Now we are ready to go to the computer, feed these equations, and get the solution. We enter each equation as part of a matrix [A] and a vector {b} such that `[A]{x} = {b}`. Here is the pseudocode:

```
sq2i = 1/sqrt(2)              % define a constant
sq6i = 1/sqrt(6)              % define another constant
F = 2                         % specify given load
A(1,[1 7 12 16]) = [1 1 sq2i sq6i]
A(2,[2 10 16]) = [1 1 2*sq6i]
.
.
A(18,[14 15 16]) = [sq2i sq2i sq6i]
b(12,1) = F
form A and b setting all other entries to zero
solve A*x = b for x
```

The solution obtained from the computer is the one-dimensional array `x` which after decoding according to our numbering scheme gives the following answer.

$$\begin{gathered} R_{A_x} = R_{A_y} = 0, \quad R_{A_z} = -2\,\text{kN}, \quad R_{B_y} = 0, \quad R_{B_z} = 2\,\text{kN}, \quad R_{E_z} = 2\,\text{kN}, \\ T_1 = T_3 = -2\,\text{kN}, \quad T_2 = T_4 = T_5 = -4\,\text{kN}, \quad T_6 = 0, \\ T_7 = T_8 = -2.83\,\text{kN}, \quad T_9 = 0, \quad T_{10} = T_{11} = 4.9\,\text{kN}, \quad T_{12} = 5.66\,\text{kN}, \end{gathered}$$

# Problems for Chapter 5

Trusses

## 5.1 Method of joints

### Preparatory Problems

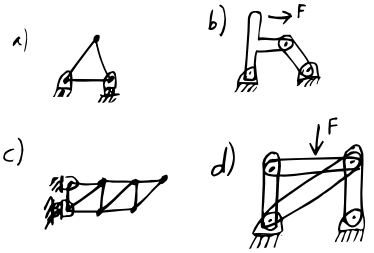
**5.1** Define these terms

a) truss
b) ideal truss
c) bar
d) joint
e) load
f) "bar force"
g) bar tension
h) bar compression
i) reaction
j) roller support
k) pin support

**5.2** Name as many positive attributes of trusses as you can.

**5.3** Name as many negative attributes of trusses as you can.

**5.4** Which of the structures below are trusses and which are not? Why not?




**problem 5.4:**

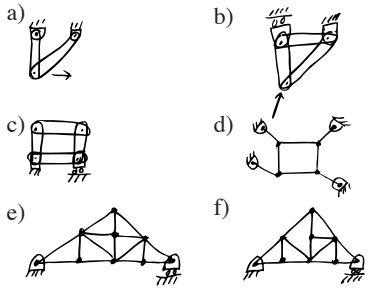
Filename:pfigure-whichistruss

**5.5** Consider this formula

$$b + r = 2j$$

a) What do $b$, $r$, and $j$ stand for?
b) What is the use of this formula?
c) What is the source of this formula?

**5.6** For each of the trusses below: **i)** What are $b$, $j$, and $r$? **ii)** What does the formula $b + r = 2j$ tell you ?



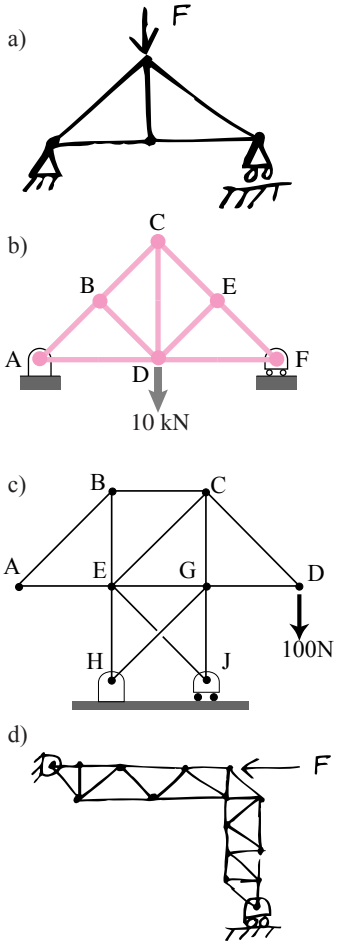

**problem 5.6:**

Filename:pfigure-trussbjr

**5.7** For a given truss you are told values for $b$, $j$, and $r$.

a) When solving the truss how many unknowns are you trying to solve for?
b) How many independent scalar equations do you have from using the method of joints on the whole structure?

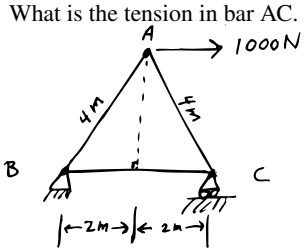
**5.8** Find the zero-force members in the trusses below.




**problem 5.8:**

Filename:pfigure-zeroforcetruss

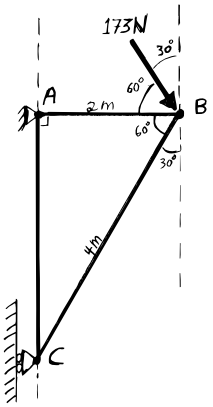
**5.9** What is the tension in bar AC.




**problem 5.9:**

Filename:S03final-truss1

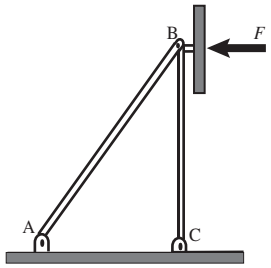
**5.10** The only force acting on the negligible-weight truss ABC is the 173 N force shown. Find the tension in the bar AB.




**problem 5.10:**

Filename:S03q3a-truss

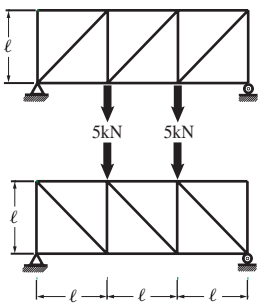
**5.11** A hoarding is supported by a two bar truss as shown in the figure. The two bars have pin joints at A, B, and C. If the total wind load on the board is estimated to be 300 N, find the forces in bars AB and BC.




**problem 5.11:**

Filename:pfigure4-1-rp11

**5.12** Find the support reactions for the two trusses without any (written) calculations. Should the support reactions be different? Why?

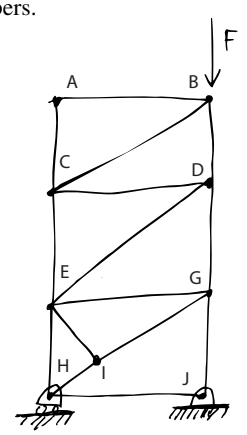



**problem 5.12:**

Filename:pfigure4-2-truss5

## More-Involved Problems

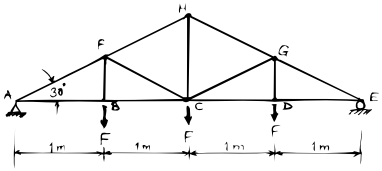
**5.13** Sketch the truss below. Write a big clear zero on top of each of the zero-force members.




**problem 5.13:**

Filename:S03final-zero

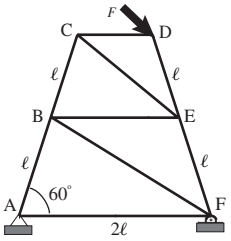
**5.14** Find the support reactions on the truss shown in the figure taking $F = 5$ kN.




**problem 5.14:**

Filename:pfigure4-2-truss1

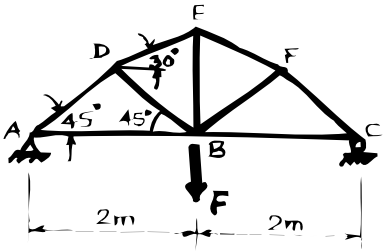
**5.15** Find the support reactions at A and F for a load $F = 3$ kN acting at D at 45° with respect to CD, if $\ell = 1$ m and $\theta = 60°$. How will the support reactions change if bar BF was removed and used to connect joints A and E instead of B and F?




**problem 5.15:**

Filename:pfigure4-2-truss4

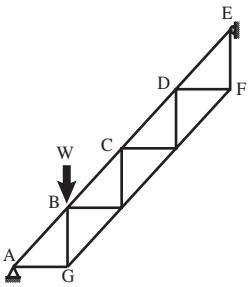
**5.16** How do the support reactions on the truss shown in the figure change if the load at point C is replaced by three equal loads, $F/3$ each, acting at points D, E, and F?




**problem 5.16:**

Filename:pfigure4-2-truss2

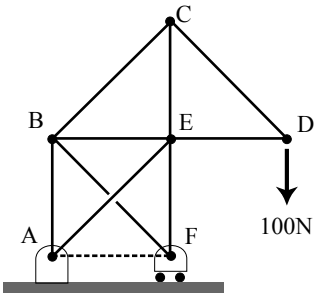
**5.17** The stairstep truss shown in the figure has 500 mm long horizontal and vertical bars. Find the support reactions at A and E when a load $W = 1$ kN is applied at (a) point B, (b) Point C, and (c) point D, respectively.




**problem 5.17:**

Filename:pfigure4-2-truss3

**5.18** In the truss shown in the figure, how does the force in bar EF change if the diagonal bar BF is removed and another bar AF (shown by dotted line) is introduced instead? You can assume any reasonable dimensions for the bars if needed.

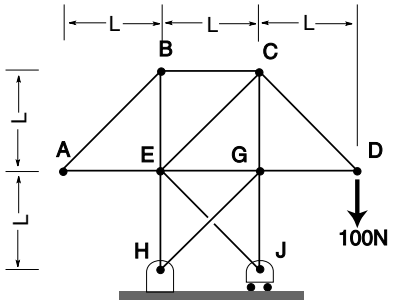



**problem 5.18:**

Filename:pfigure-zeroforce2

**5.19** For the truss shown, find:

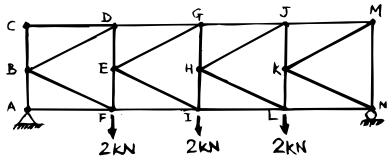
a) The reaction at J.
b) The bar force in BC (tension or compression).
c) The force in bar CG (tension or compression).




**problem 5.19:**

Filename:pfigure-s01-p1-1

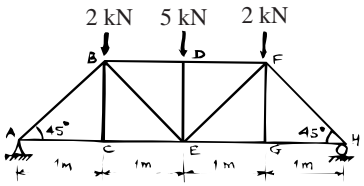
**5.20** The truss shown in the figure consists of-4 bays of 'K' structure. Each bay has 2 m long horizontal and vertical bars. Find the tensions in rods DE and DG.




**problem 5.20:**

Filename:pfigure5-1-truss7

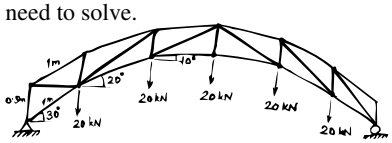
**5.21** Analyze the truss shown in the figure and find the forces in all the bars.




**problem 5.21:**

Filename:pfigure4-2-truss8

**5.22** Analyze the truss shown in the figure and find out forces in all bars. Use symmetry to reduce the number of equations you need to solve.




**problem 5.22:**

Filename:pfigure4-2-truss10

## 5.2 Method of sections

## Preparatory Problems

**5.23** What is the method of sections?

**5.24** When is the method of sections most useful?

**5.25** With the free body diagram associated with one section cut how many bar tensions can you hope to find?

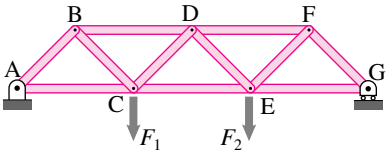
**5.26** Given a truss and a particular bar in that truss

a) Can you always find one section cut with which you can find the desired bar tension?

b) If so, how do you find that cut? If not, why not?

c) Whichever your answer above, give an example of a bar in a truss that illustrates your point.

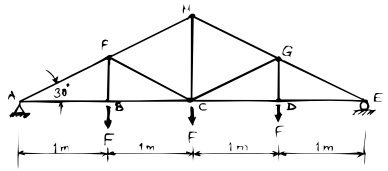
**5.27** This problem is exactly the same as Sample 5.2 where it was solved using method of joints. The truss is made up of five horizontal and six inclined rods. All inclined rods are 1 m long and at right angles to each other. The truss carries two vertical loads, $F_1 = 4\,\text{kN}$ and $F_2 = 1\,\text{kN}$ as shown. Find the tensions in rods CE, DE, and DF.




**problem 5.27:**

Filename:pfigure5-2-sample3truss

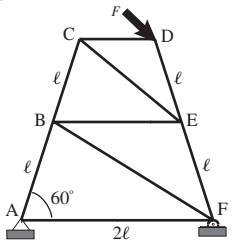
**5.28** For the truss shown in the figure, find the tensions in rods BC and FH, assuming $F = 10\,\text{kN}$.




**problem 5.28:**

Filename:pfigure5-2-truss1

**5.29** A force $F = 3\,\text{kN}$ acts at $45^\circ$ with the horizontal at joint D of the truss shown in the figure. Find the tension in rod BE.

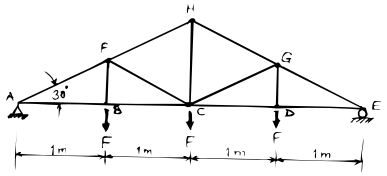



**problem 5.29:**

Filename:pfigure5-2-truss4

## More-Involved Problems

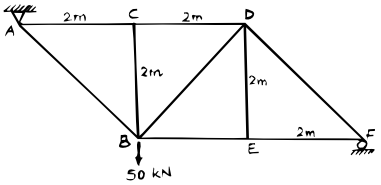
**5.30** Find the forces in bars FH, FB, and BC of the truss shown in the figure taking $F = 10\,\text{kN}$. Now pretend that bars FC and CG are removed and two new bars BH and HD are put in place. Find the forces in bars FH, FB, and BC again. Are the forces different now? Why?




**problem 5.30:**

Filename:pfigure4-2-truss6

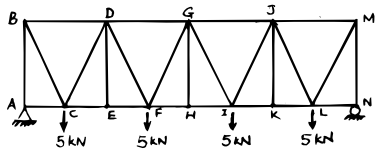
**5.31** Find the forces in bars BC and BD in the truss shown in the figure. How does the force change in each of these bars if the load is moved to joint B from joint E?




**problem 5.31:**

Filename:pfigure4-2-truss7

**5.32** For the truss shown in the figure, assume that AC=CE=1 m, and AB=BD=2 m. The rest of the bays are identical to bay ABDE. For the given loads, find the tensions in rods GH, GI, and GJ. [Hint: you can use information about zero force members.]

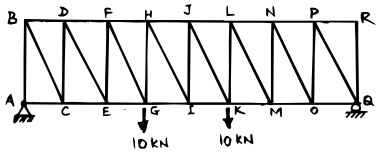



**problem 5.32:**

Filename:pfigure5-2-trussM

**5.33** Consider the truss shown in Problem 5.28. Find the tension in rod CH. [Hint: you may have to use multiple sections or solve Problem 5.28 first.]

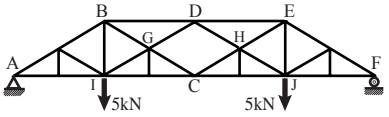
**5.34** The truss shown in the figure consists of 8 'N' bays. In each bay, the vertical rod is 2 m long and the horizontal rod is 1 m long. For the given loads, find the tensions in rods HJ, HI, and GH.




**problem 5.34:**

Filename:pfigure5-2-trussN

**5.35** A complex symmetric truss spanning a length of 16 m is shown in the figure. The outermost inclined rods make an angle of 30° with the horizontal. Find the tension in rod BD. [Note: you may have to use more than one section to get the answer.]




**problem 5.35:**

Filename:pfigure5-2-truss6

**5.36** The 2D truss shown consists of 12 diagonally braced rectangles (each $a$ high and $b$ wide). Thus the slope of the diagonal elements is $a/b$. The whole structure is supported by 4 bars (with lengths $c$, $d$, and $e$ as marked). The loading is idealized as 11 identical loads $F$ shown. Give your answers in terms of some or all of $a$, $b$, $c$, $d$, $e$ and $F$.

a) On a sketch of the figure below clearly mark all the zero-force members (put a '0' on the middle of each bar that has a 'bar force' of zero).

b) Find the 'bar-force' in bar EB.

c) Find the 'bar-force' in bar HI.

d) Find the 'bar-force' in bar JK. [Hint: Use the method of sections and, to reduce calculations, replace a group of the $F$ forces with a single equivalent force.]



**problem 5.36:**

Filename:F02p1p1bridge

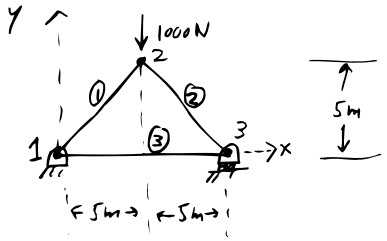
## 5.3 Solving trusses on computers

## Preparatory Problems

You should check your mastery of the method of joints problems before working on this section. **5.37** Define these matrices and column vectors used to define a truss, the loading on it, the bar tensions , the reactions, and the coefficients in the matrix form of the joint equilibrium equations:

a) $[J]$
b) $[B]$
c) $[R]$
d) $[F]$
e) $[T]$
f) $[L]$
g) $[A]$

**5.38** By hand, with no use of a computer, find all of the matrices and column vectors above for this truss.




**problem 5.38:**

Filename:pfigure-trussnumericsimple

**5.39** When does the numerical recipe presented here succeed and when does it fail? When it fails, how does it fail?

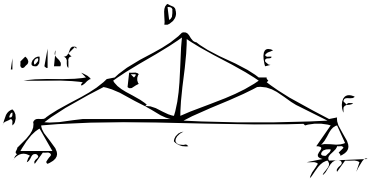
**5.40**

a) Write a computer program, using your preferred language or package, that takes as input the matrices $[J]$, $[B]$, $[R]$, and $[F]$ and calculates $[T]$.

b) Test this program on the truss of problem 5.38.

## More-Involved Problems

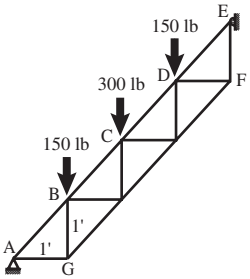
**5.41** All of the bars in the symmetric truss below are either level or at 30° from the horizontal. Find all the bar forces and reactions.




**problem 5.41:**

Filename:pfigure-simpletruss

**5.42** Find the force in each bar of the staircase truss shown in the figure by writing the required number of equilibrium equations and then solving them on a computer.



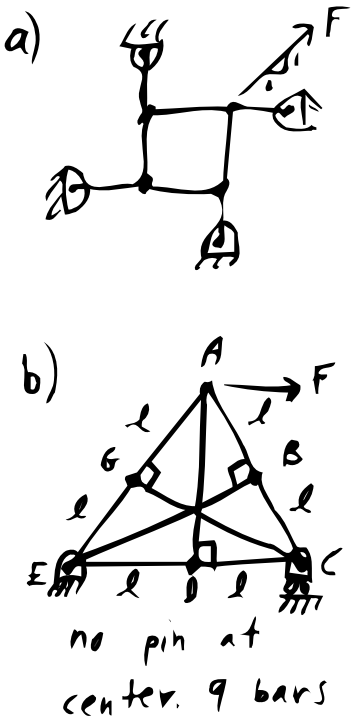

**problem 5.42:**

Filename:pfigure4-2-truss9

**5.43** Find the tensions in all the bars, and all the reactions for these structures.

a) A square supported by four bars. This is perhaps the simplest rigid structure that has no triangles.

b) The 9-bar structure shown. This structure also has no triangles in that there is no closed circuit that involves only three bars (for example, from D to A to B to C and back to D involves 4 bars).




**problem 5.43:**

Filename:pfigure-advtruss1

**5.44** Analyse the truss given in Problem 5.20 and solve for all bar tensions and support reactions.

**5.45** Solve Problem 5.21.

**5.46** Solve Problem 5.22.

**5.47** Using your program from problem 5.40 solve each of the following problems:

a) Problem 5.10
b) Problem 5.11
c) Problem 5.13
d) Problem 5.20
e) Problem 5.21
f) Problem 5.22
g) Problem 5.27
h) Problem 5.29
i) Problem 5.34
j) Problem 5.42

## 5.4 Frames

## Preparatory Problems

**5.48** In what way(s) is/are trusses different from more general frames?

**5.49** Consider a frame made of-3 pieces connected together. Assume that no free body diagram cut is within a part.

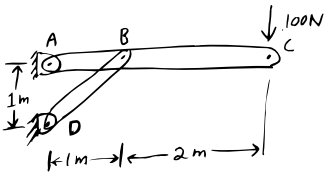
a) How many different free body diagrams can you draw?

b) For each free body diagram how many independent scalar equations can be extracted from the equilibrium relations?

c) In total, from all the free body diagrams, how many independent scalar equations can be extracted from the various equilibrium conditions.

**5.50** Consider the two-bar frame shown. Choose appropriate coordinate axes. Find

a) The reaction at D.
b) The tension in bar DB.
c) The reaction at A.
d) The force of DB on ABC.
e) The moment in ABC
   - just (an infinitesimal distance) to the right of A
   - just to the left of B
   - just to the right of B
   - just to the left of C

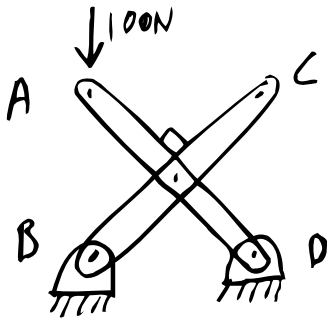



**problem 5.50:**

Filename:pfigure-simplestframe

**5.51** Two equal length bars are pinned together at right angles as shown. Find

a) the reactions at B and D
b) the force of BC on AD
c) the moment in BD just above the hinge

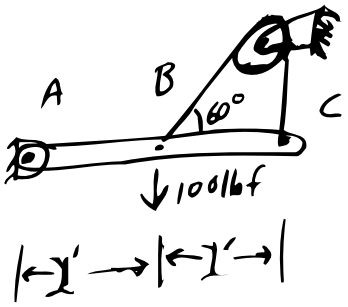



**problem 5.51:**

Filename:pfigure-xframe

**5.52** For the structure shown find

a) the tension in the string
b) the reaction at A
c) the moment in ABC just to the right of B




**problem 5.52:**

Filename:pfigure-cantileverpulley

## More-Involved Problems

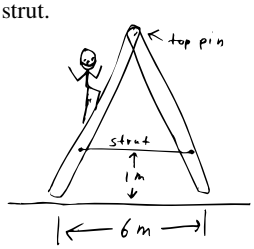
**5.53** An A-frame aluminum ladder consists of two uniform 5 m, 150 N sections that are pinned at the top and held from splitting by a massless strut $1m$ above the slippery floor. An 800 N person has climbed halfway up the left side.

a) Find the reactions (the forces of the ground on the two ladder sections).

b) Find the force of the left section on the right at the top pin.

c) Find the tension in the connection strut.

d) Find the moment in the right leg of the ladder just above the tension strut.

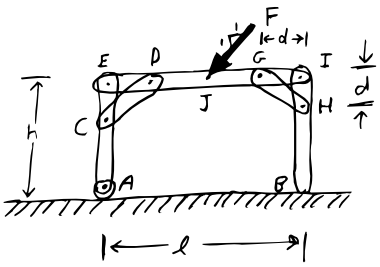



**problem 5.53:**

Filename:pfigureSoodak4-49

**5.54** To make a model of a table statically determinate we assume that one leg slides easily on the floor. Assume the other leg does not slip. Use $F = 200\,\text{N}$, $h = 1\,\text{m}$, $\ell = 2\,\text{m}$, and $d = .25\,\text{m}$. Find

a) the reactions at A and B,
b) the tension in GH,
c) the moment in IHB just below H.

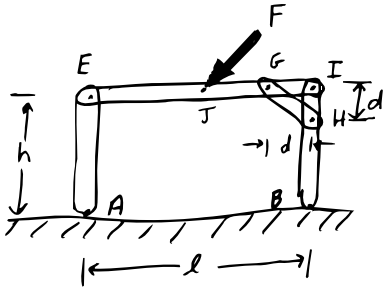



**problem 5.54:**

Filename:pfigure-frametable1

**5.55** To make a model of a table statically determinate we assume that one leg is not braced. Assume the other leg does not slip. Use $F = 200\,\text{N}$, $h = 1\,\text{m}$, $\ell = 2\,\text{m}$, and $d = .25\,\text{m}$. Find

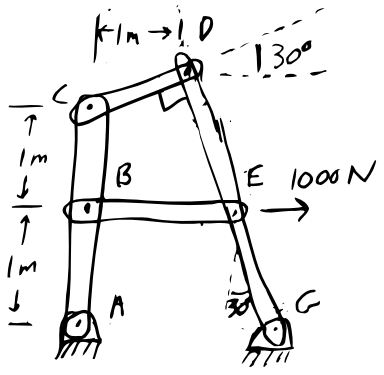
a) the reactions at A and B
b) the tension in GH
c) the moment in IHB just below H




**problem 5.55:**

Filename:pfigure-frametable2

**5.56** For the structure shown find the reaction at A.

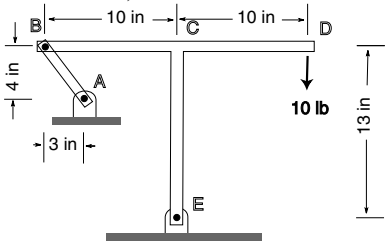



**problem 5.56:**

Filename:pfigure-frame-Aish

**5.57** The structure consists of two pieces: bar AB and 'T' EBCD. They are connected to each other with a hinge at B. They are connected to the ground with hinges at A and E. The force of gravity is negligible. Find

a) The reaction at A.
b) The reaction at E.
c) The moment in BCED just to the left of C.
d) Why are these forces so big or small? (Your answer should be in words).




**problem 5.57:**

Filename:pfigure-s01-p1-2

## 5.5 Advanced truss analysis: determinacy, rigidity, and redundancy

## Preparatory Problems

**5.58** Define these terms

a) statically determinate
b) rigid and non-rigid
c) redundant and non-redundant

**5.59** For each set of conditions below, find 2 trusses both of which fit the description

a) rigid and non-redundant
b) rigid and redundant
c) not rigid and not redundant
d) not rigid and redundant

**5.60** In 2D trusses we used the formula $b + r = 2j$. With what formula do we replace this for 3D trusses? Explain why.

**5.61** For the 3D method of joints, for a whole truss how many independent scalar equilibrium equations can one write?

**5.62** For one section cut in 3D how many bar tensions can you hope to find?

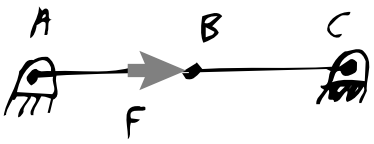
**5.63** For a 3D truss that is rigid when not grounded, how many independent reaction components do you need to make it a statically determinate structure for any loading.

## More-Involved Problems

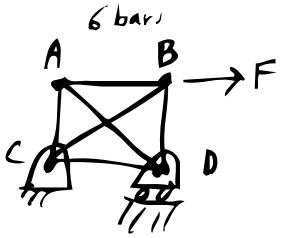
**5.64** For the following structures find at least 2 different sets of bar forces that can equilibrate the applied load shown.

a) Two bars in a line with a force in the same line.
b) A square with two diagonal braces.

a)




b)

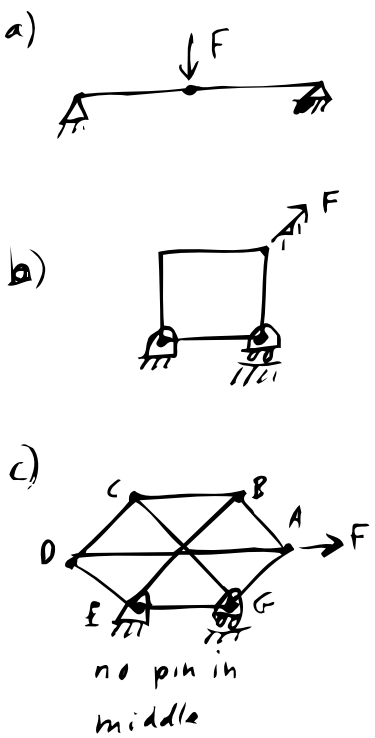



**problem 5.64:**

Filename:pfigure-advtruss3

**5.65** For the structures and loading shown show that there is no set of bar forces for which equilibrium is possible (at least with the geometry shown). All of these structures are not rigid, they require either infinite bar forces or some (or a lot of) deformation to withstand the load applied

a) Two bars in a straight line.
b) A square without a diagonal.
c) A regular hexagon with three diameters. [This problem is hard and might best be answered using linear algebra methods on the matrix form of the system of equilibrium equations.]

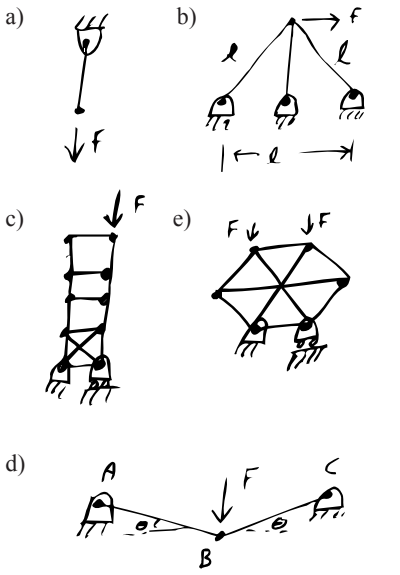



**problem 5.65:**

Filename:pfigure-advtruss2

**5.66** For each of the structures below and the shown loading answer these questions: i) Does a set of equilibrium bar forces and ground reactions exist? ii) If so, find one such set. iii) Are the solutions, if they exist, unique? iv) If not find at least two solutions. v) Is the structure rigid? vi) If not, how can it deform?

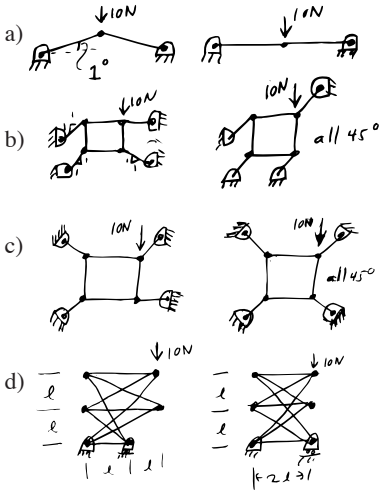
a) One hanging rod
b) A braced pole
c) A tower
d) Two bars holding a vertical load. Comment in your answers how they change in the limit $\theta \to 0$.
e) A regular hexagon with three diagonals (this is a hard problem).




**problem 5.66:**

Filename:pfigure-advtruss4

**5.67** Use your program from problem 5.40 to analyze each pair of structures shown. In each case the output of your program should be radically different for the right structure than for the superficially similar structure on the left. **i)** Describe the difference in your computer program behavior, **ii)** As well as you can, explain what it is about the structures that causes this difference in computer behavior.




**problem 5.67:**

Filename:pfigure-nonsingular-singular

CHAPTER 6

# Transmissions and mechanisms

*Some collections of solid parts are assembled so as to cause force or torque in one place given a different force or torque in another. These include levers, gear boxes, presses, pliers, clippers, chain drives, and crank-drives. Besides solid parts connected by pins, a few special-purpose parts are commonly used, including springs and gears. Tricks for amplifying force are usually based on principals idealized by pulleys, levers, wedges and toggles. Force-analysis of transmissions and mechanisms is done by drawing free body diagrams of the parts, writing equilibrium equations for these, and solving the equations for desired unknowns.*

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6.2 Force amplification . . . . . **310**
6.3 Mechanisms . . . . . **320**
Problems for Chapter 6 . . . . . **331**

Here we consider collections of parts assembled to transmit motion or force. We are not going to address the conversion of thermal, chemical (or biological) or electrical source to a useful force. Rather we discuss the transmission of that force. We are concerned with the passive parts of machines, or with passive machines that have no energy source within them. Most often there is an input force or torque and a desired output which does the machine's job.

The categorization of an assembly of parts as a structure or as a machine is mostly a matter of intent. Is the main job to hold or support something still (a structure) or to move something. There is no useful intrinsic aspect of an assembly of parts that well-defines the difference between a structure and a machine*. So the statics analysis of mechanisms and transmissions is the same as for frames. Our concern is as in the rest of statics:

> Given some information about the forces on or in a mechanism find out more about the forces.

The practice of mechanism design is often dominated by kinematic analysis, the study of the geometry of the interacting motions of the parts as the mechanism configuration changes. Such is not our concern here. Rather we focus on the relations between the various forces in a given configuration of the mechanism*.

## Building blocks

In the same way that machines and buildings are built from bricks, gears, beams, bolts and other standard pieces, elementary mechanics models of the world are made from a few elementary building blocks. Conspicuous so far, roughly categorized, are:

- Special objects:
  - Point masses.
  - Rigid bodies:
    - Two force bodies,
    - Three force bodies,
    - Pulleys, and
    - Wheels.
- Special connections:
  - Hinges,
  - Welds,
  - Sliding contact, and
  - Rolling contact.

* A candidate delineation between machine and structure might be whether the assembly allows any motion (mechanism) or not (structure). Even that concept is ambiguous. Common machines, like presses, wrenches and clamps, are as motion-restricted in their end-use as many structures. And, conversely, many structures are adjustable and thus are designed to move.

* The dynamics portion of this book is largely an introduction to the kinematics of mechanisms.

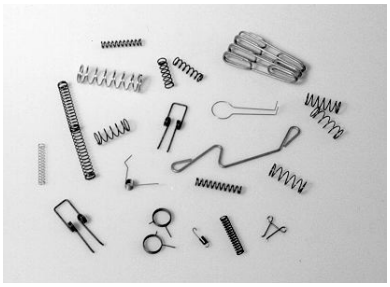


Figure 6.1: Some commercially available springs (Viking Products).
Filename:tfigure-SpringVikingProduct

**Products and models** Some of these things have dual lives, as products and as models. On the one hand a mechanical hinge corresponds to a product you can buy in a hardware store called a hinge. On the other hand a hinge in mechanics represents a constraint that restricts certain motions and freely allows others. A hinge in a mechanics model may or may not correspond to hardware called a hinge. For example, when considering a box balanced on an edge, we may *model* the contact as a hinge meaning we would use the same equations for the forces of contact as we would use for a hinge. Although you can buy a pulley, you might model a rope sliding around a post as a rope on a pulley even though there was no literal pulley in sight.

The connection between product and model can even sound contradictory. Although 'like a rock' means 'solid' in English, one may model a rock as a spring (which is done for foundation engineering, understanding waves in rocks, and understanding the energy of earthquakes). A coil spring may be modeled as a rigid rod for a simple structure-like study of a machine. And a hinge might be modeled as a spring if its deformation is important. The appropriate mechanics model for a thing and its name don't always correspond.

**What's new in this chapter** The new content in this chapter is

- Detailed discussion of a few components used in mechanisms and transmissions that are not used commonly in simple 'structures'. These include springs, pulleys, wheels, and gears.
- Introduction to a variety of design tricks to, say, cause a big force when only a small force is available.

We start the chapter by discussing a few special parts and assemblies of those parts. Then we consider more general assemblies.

## 6.1 Springs

A *spring* is a deformable solid that regains its original shape after being compressed, extended or otherwise deformed. The word spring has a dual personality as 1) a product and 2) a model.

1) **Spring as product.** Springs, in various forms, most characteristically as helices made of steel wire, can be purchased from hardware stores and mechanical parts suppliers(Fig. 6.1). Springs are used to hold things in place (in a clothes pin), to store energy (in a clock or wind-up toy), to reduce contact forces ( bumpers), to isolate something from vibrations (a car suspension), and to modulate the feel for human interaction (under keyboard keys). You will find springs in most any complicated machine. Take apart a disposable camera, a laser printer, a gas lawn mower, a bicycle, a cruise missile, or a washing machine and you will find springs.
2) **Spring as model.** On the other hand, springs are used in mechanical 'models' of many things that are not, by name, springs (see page 11 for discussion of 'models'). For much of this book we approximate solids

as rigid. But sometimes the flexibility or *elasticity* of an object is an important part of its mechanics. The simplest accounting for this is to think of the object as a spring. So a tire may be modeled as a spring as might be the near-contact-point material of a bouncing ball, a strut in a truss, the snapping-back part of the earth's crust in an earthquake, your achilles tendon, or the give of soil under a concrete slab. Engineer Tom McMahon idealized the give of a running track as that of a spring when he designed the record breaking track used in the Harvard stadium.

For simplicity we only concern ourselves with *tension and compression*

## 6.1 'Zero-length' springs

### Zero rest-length springs

A special case of linear springs that has remarkable mechanical consequences is a *zero-rest-length* spring (also called a 'zero-length' spring for short) with $\ell_0 = 0$. These ideas are useful for design, but not essential for basic understanding of statics.

The defining equations for a zero-rest-length spring, in scalar and vector form, are

$$T = k\ell \quad \text{and} \quad \vec{F}_B = k \cdot \vec{r}_{AB}.$$

The tension verses length curve for a zero-length spring is shown in Fig. 6.4b.

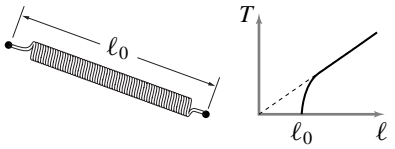
At first blush such a spring seems *non-physical*, meaning that it seems to represent something which you can't build. If you take a coil spring all the metal gets in the way of the spring collapsing to zero length, when the ends would coincide. In fact, however, there are many ways to make zero-rest-length springs springs. For example, the tension verses length curve of a rubber band (or piece of surgical tubing) looks something like that shown in Fig. 6.4c. Over some portion of the curve the zero-length spring approximation is reasonable (a sign of this is that the vibration frequency is almost independent of stretch for some range of stretch). For other physical implementations of zero-length springs see box 6.1 on page 291.

The mathematics in many mechanics problems is simpler for $\ell_0 = 0$ springs than for $\ell_0 \neq 0$ springs.

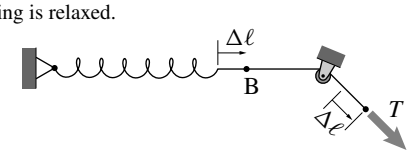
**Rubber bands.** As shown in Fig. 6.4c straps of rubber behave like zero-length springs over some of their length. If this is the working length of your mechanism then the zero-length spring approximation may be good.

**A stretchy conventional spring.** Some springs are stretched way beyond their rest lengths. Thus the approximation that $k(\ell - \ell_0) = k\ell\left(1 - \frac{\ell_0}{\ell}\right) \approx k\ell$ may be reasonable.

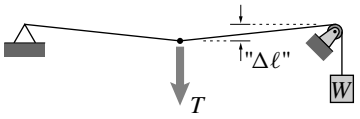
**A pre-stressed coil spring.** Some door springs and many springs used in desk lamps are made tightly wound so that each coil of wire is pressed against the next one. It takes some tension just to start to stretch such a spring. The tension verses length curve for such springs can look very much like a zero-length spring once stretch has started. In fact, in the original elegant 1930's patent, which commonly seen present-day parallelogram-mechanism lamps imitate, specifies that the spring should behave as a zero-length spring. Such a pre-stressed zero-length coil spring was a central part of the design of the long period seismometer featured on a 1959 Scientific American cover.



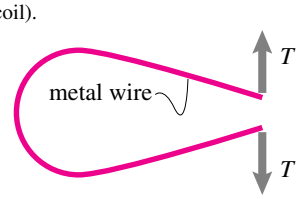

**A spring, string, and pulley.** If a spring is connected to a string that is wrapped around a pulley then the end of the string can feel like a zero force spring if the attachment point is at the pulley when the spring is relaxed.




**A string pulled from the side.** If a taught string is pulled from the side it acts like a zero-length spring in the plane orthogonal to the string.




**A 'U' clip.** If a springy piece of metal is bent so that its unloaded shape is a pinched 'U' then it acts very much like a zero length spring. This is perhaps the best example in that it needs no anchor (unlike the pulley) and can be relaxed to almost zero length (unlike a pre-stressed coil).



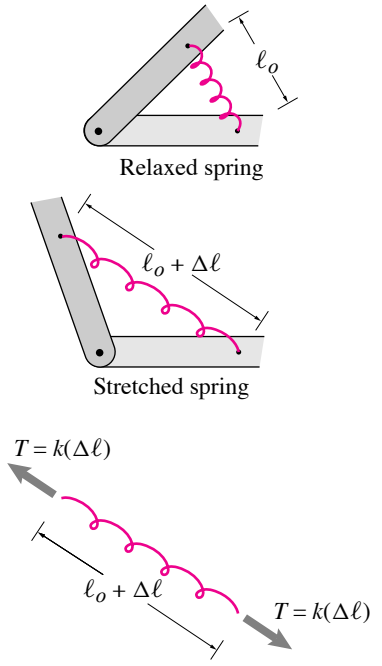



Figure 6.2: **Spring connection.** The tension in a spring is usually assumed to be proportional to its change in length, with proportionality constant $k$: $T = k\Delta\ell$.

Filename:tfigure2-spring

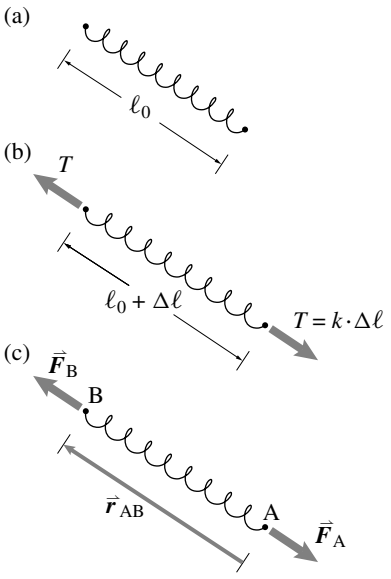



Figure 6.3: An ideal spring with rest length $\ell_0$ and stretched length $\ell_o + \Delta\ell$. The tension in the spring is $T$ and the vector forces at the ends are $\vec{F}_A$ and $\vec{F}_B$.

Filename:tfigure2-spring2

*springs* here. These are springs which only have axial loads applied and only at the ends.

If the tension in a spring is a function of its length alone, independent of its rate of lengthening, the spring is said to be 'elastic.' Many materials are well-modeled as elastic for small-enough deformation. If the tension in the spring is proportional to its stretch the spring is said to be 'linear.' Most elastic materials are close to linear in their behavior. Thus the word *spring* is often used as short for *linear elastic spring*. The stretch of a spring is the amount by which the spring is longer than when it is relaxed. This relaxed length is also called the *unstretched* length, the *rest* length, or the *reference* length. If the relaxed length (the length at zero tension) is $\ell_0$, and the present length $\ell$, then the stretch of the spring is

$$\Delta\ell = \ell - \ell_0 = \text{Increase in length from rest length}$$

An *ideal spring* is a massless two-force body characterized by its *rest length* $\ell_0$ (also called the *relaxed length*, or *reference length*), its *spring constant* $k$, and the defining equation (or constitutive law), *Hook's law*:

$$T = k \cdot (\ell - \ell_0) \quad \text{or} \quad T = k \cdot \Delta\ell$$

where $\ell$ is the present length and $\Delta\ell$ is the increase in length or *stretch* (see Fig. 6.3).

The spring constant $k$ is also sometimes called the spring *rate*, the spring *stiffness* or the spring *proportionality constant*.

The ideal spring is called *linear* because of the formula $k\Delta\ell$ and not, say, $k(\Delta\ell)^3$. The defining spring formula is sometimes, although we don't recommend this, memorized as '$F = kx$'

Note: the formula '$F = kx$' can lead to errors: the direction of the force is not evident, and some people are unclear about the meaning of $x$ in this formula. The safest way to avoid sign errors when dealing with springs is to

- Draw a free body diagram of the spring;
- Write the *increase* in length $\Delta\ell$ in terms of geometry variables in your problem (even if you know that this increase is going to be a negative number);
- Use $T = k\Delta\ell$ to find the tension in the spring (even if you know the tension will turn out negative); and then
- Use the principle of action and reaction to find the forces on the objects to which the spring is connected.

The main idea is to pick a sign convention (tension and lengthening are positive) and stick with it, accepting the arithmetic of negative numbers if

it arises. A plot of tension verses length for an ideal spring is shown in Fig. 6.4a.

**A comment on the notation** $\Delta\ell$ Often in engineering we write $\Delta(something)$ to mean the change of '*something*.' Most often one also has in mind a small change. In the context of springs, however, $\Delta\ell$ is allowed to be a rather large change. A useful way to think about springs is that increments of force are proportional to increments of length change, whether the force or length is already large or small:

$$\Delta T = k\Delta\ell \quad \text{or} \quad \frac{dT}{d\ell} = k$$

**Compliance.** A spring with a large stiffness is called *stiff* or *hard*. The reciprocal of stiffness $\frac{1}{k}$ is called the *compliance*. A spring with a small stiffness and large compliance is called *compliant* or *soft* and has a lot of 'give'.

**The force vector on one end of a spring.** Because the spring force is along the spring, a known direction, we can write a vector formula for the force on the B (say) end of the spring as ( see Fig. 6.3)

$$\vec{\boldsymbol{F}}_{\mathrm{B}} = k \cdot \underbrace{\left(|\vec{\boldsymbol{r}}_{\mathrm{AB}}| - \ell_0\right)}_{\Delta\ell} \underbrace{\left(\frac{\vec{\boldsymbol{r}}_{\mathrm{AB}}}{|\vec{\boldsymbol{r}}_{\mathrm{AB}}|}\right)}_{\hat{\boldsymbol{\lambda}}_{\mathrm{AB}}}. \tag{6.1}$$

where $\hat{\boldsymbol{\lambda}}_{\mathrm{AB}}$ is a unit vector along the spring. This explicit formula is useful for, say, numerical calculations. This formula becomes especially simple if the rest-length of the spring is zero ($\ell_0 = 0$) so

$$\vec{\boldsymbol{F}}_{\mathrm{B}} = k\vec{\boldsymbol{r}}_{\mathrm{B/A}}.$$

Absurd as this seems, how could a spring have zero rest length, the idea is useful both as a model and for engineering design (see box 6.1 on page 291.

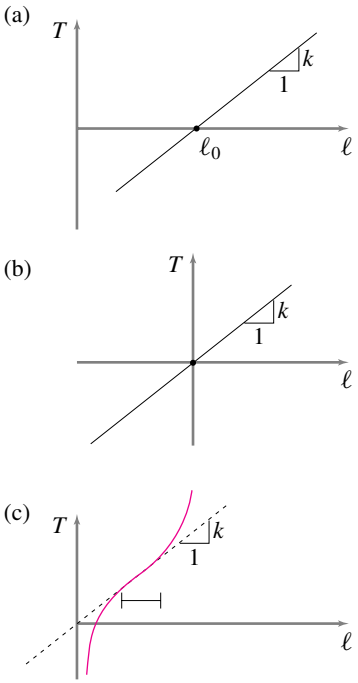



Figure 6.4: a) Tension verses length for an ideal spring, b) for a zero-length spring, and c) for a strip of rubber.

Filename:tfigure-tensionvslength

## Assemblies of springs

Here we see how springs are put together with other springs *in parallel* and *in series*. For starters we'll put together just two springs with rest lengths $\ell_{01}$ and $\ell_{02}$. The extensions and tensions of the two springs are $\Delta\ell_1$, $\Delta\ell_2$, $T_1$, and $T_2$.

The assembly of springs also acts like a single spring. The central issue is determination of the properties of the combined spring.

Much of what you need to know about the words 'in parallel' and 'in series' follows easily from these phrases:

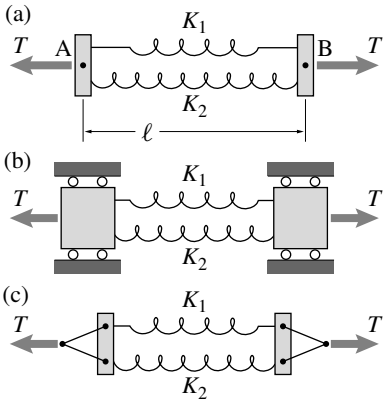



Figure 6.5: a) Schematic of parallel springs, b) genuinely parallel springs, c) a simple implementation that gives a reasonable approximation of parallel springs. The triangle of cables limits the rotation of the end bars if the triangles are long.

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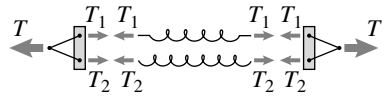



Figure 6.6: Free body diagrams of the components of a parallel spring arrangement.

Filename:tfigure-parallelfbds

| | | | | |
|---|---|---|---|---|
| In parallel, | forces | and | stiffnesses | add. |
| In series, | displacements | and | compliances | add. |

which we discuss in detail below.

## Springs in parallel

Two springs that share the burden of a load and stretch the same amount are said to be *in parallel*.

Fig. 6.5a shows the standard schematic for springs in parallel. This schematic is a non-physical cartoon because the applied tension would likely cause the end-bars to rotate. What is meant by the schematic in Fig. 6.5a is the somewhat clumsy constrained mechanism of Fig. 6.5b. In engineering practice one rarely builds such a structure. For the purposes of discussion here, we assume that any of Fig. 6.5abc represent a situation where the springs both stretch the same amount.

For each spring we have the defining constitutive relation:

$$T_1 = k_1 \Delta\ell_1 \quad \text{and} \quad T_2 = k_2 \Delta\ell_2. \tag{6.2}$$

Using the free body diagrams in Fig. 6.6), force balance for one of the end supports shows that

$$T = T_1 + T_2. \tag{6.3}$$

This is what is meant by the two springs sharing the load. Springs in parallel stretch the same amount thus we have the kinematic relation:

$$\Delta\ell_1 = \Delta\ell_2 = \Delta\ell. \tag{6.4}$$

For simplicity we have assumed that the two springs have the same rest length. Put the two results above together and we have

$$\begin{aligned} T &= T_1 + T_2 \\ &= k_1\Delta\ell_1 + k_2\Delta\ell_2 \\ &= k_1\Delta\ell + k_2\Delta\ell \\ &= \underbrace{(k_1 + k_2)}_{k}\Delta\ell \ . \end{aligned}$$

Thus w the effective spring constant of the pair of springs in parallel is, as you might guess:

$$k = k_1 + k_2. \tag{6.5}$$

The loads carried by the springs are

$$T_1 = \frac{k_1}{k_1 + k_2} T \quad \text{and} \quad T_2 = \frac{k_2}{k_1 + k_2} T$$

which add up to $T$ as they must.

Example: **Two springs in parallel.**
Take $k_1 = 99\,\mathrm{N/cm}$ and $k_2 = 1\,\mathrm{N/cm}$. The effective spring constant of the parallel combination is:

$$k = k_1 + k_2 = 99\,\mathrm{N/cm} + 1\,\mathrm{N/cm} = 100\,\mathrm{N/cm}.$$

Note that $T_1/T = .99$ so even though the two springs share the load, the stiffer one carries 99% of it. For practical purposes, or for the design of this system, it would be reasonable to remove the much less stiff spring.

The reasoning above with two springs in parallel is easy enough to reproduce with 3 or more springs. The result is:

$$k_{\text{tot}} = k_1 + k_2 + k_3 + \ldots \quad \text{and} \quad T_1 = T k_1/k_{\text{tot}}, \quad T_2 = T k_2/k_{\text{tot}} \ldots$$

That is,

- The net spring constant is the sum of the constants of the separate springs; and
- The load carried by springs is in proportion to their spring constants.

## Some comments on parallel springs

Once you understand the basic ideas and calculations for two side-by-side springs connected to common ends, there are a few things to think about for context.

**The simplest redundant truss** For the purposes of drawing pictures (*e.g.*, Fig. 6.5a) parallel springs are drawn side by side. But in the mechanics analysis we treated them as if they were on top of each other. A pair of parallel springs is like a two bar truss where the bars are on top of each other but connected at their ends. With 2 bars and 2 joints we have $2j < b+3$, and a redundant truss. In fact this is the simplest redundant truss, as one spring (read bar) does exactly the same job as the other (carries the same loads, resists the same motions). With statics alone we can not find the tensions in the springs since the statics equation $T_1 + T_2 = T$ has non-unique solutions.

**Statically indeterminate problems.** Calculating the forces in a set of parallel springs is solving (using more than just statics, namely the spring constitutive law) the simplest statically-indeterminate problem.

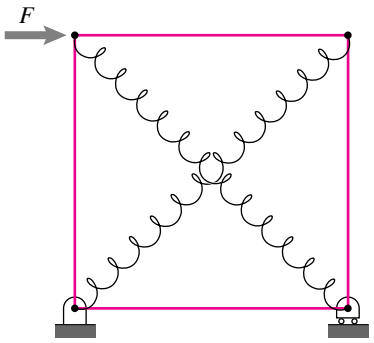



Figure 6.7: Parallel springs are not always geometrically parallel. The deformation of the structure above into a diamond is resisted by the two springs. They share the load and they have stretches that are linked by the kinematics. Thus these two perpendicular springs are 'in parallel.' (Detailed analysis of this structure is a little beyond the coverage here.)

Filename:tfigure-paralleljoke

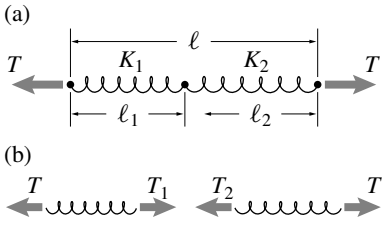



Figure 6.8: Schematic of springs in series.

Filename:tfigure-seriessprings

**Parallel springs and the three pillars of mechanics** The laws of statics allow multiple solutions to redundant problems. But a bar in a real physical structure has, at one instant of time, some unique bar tension determined by the deformations and material properties. This is the first, and perhaps most conspicuous, occasion in this book that you see a problem where the three pillars of mechanics (see page 3) are assembled in such clear harmony, namely, material properties (eq. 6.2), the laws of mechanics (eq. 6.3), and the geometry of motion and deformation (eq. 6.4). In strength of materials calculations, where the distribution of stress is not determinable by statics alone, this threesome (geometry of deformation, material properties and statics) clearly come together in almost every calculation.

**Parallel springs are not necessarily geometrically parallel** n the discussion above 'in parallel' corresponded to the springs being geometrically parallel. In common mechanics usage the words 'in parallel' are more general and mean that the net load is the sum of the loads carried by the two springs, and the stretches of the two springs are the same (or in a ratio restricted by kinematics). You will see cases where 'in parallel' springs are not the least bit parallel (*e.g.*, see Fig. 6.7).

## Springs in series

Two springs that share a displacement and carry the same load are *in series*.

A schematic of two springs in series is shown in Fig. 6.8a where the springs are aligned serially, one after the other. To determine the net stiffness of this simple spring network we again assemble the three pillars of mechanics, using the free body diagram of Fig. 6.8b.

$$\begin{array}{lccc} \text{Constitutive law:} & T_1 = k_1(\ell_1 - \ell_{1_0}), & T_2 = k_2(\ell_2 - \ell_{2_0}), & \\ \text{Kinematics:} & \ell_0 = \ell_{1_0} + \ell_{2_0}, & \ell = \ell_1 + \ell_2, & \\ \text{Force Balance:} & T_1 = T, \quad \text{and} & T_2 = T. & \end{array} \tag{6.6}$$

(where,*e.g.*, $\ell_{1_0}$ is the rest length of spring 1). We can manipulate these equations much as we did for the similar equations for springs in parallel. The manipulation differs in structure the same way the equations do. For springs in parallel the tensions add and the displacements are equal. For

springs in series the displacements add and the tensions are equal:

$$\begin{aligned}
\Delta\ell &= \ell - \ell_0 \\
&= (\ell_1 + \ell_2) - (\ell_{1_0} + \ell_{2_0}) \\
&= (\ell_1 - \ell_{1_0}) + (\ell_2 - \ell_{2_0}) \\
&= \Delta\ell_1 + \Delta\ell_2 \\
&= \frac{T_1}{k_1} + \frac{T_2}{k_2} \\
&= \frac{T}{k_1} + \frac{T}{k_2} \\
&= \underbrace{\left(\frac{1}{k_1} + \frac{1}{k_2}\right)}_{\frac{1}{k}} T \ .
\end{aligned}$$

Thus we get that the net compliance is the sum of the compliances:

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} \quad \text{or} \quad k = \frac{1}{1/k_1 + 1/k_2} = \frac{k_1 k_2}{k_1 + k_2},$$

which you should compare with the case of springs in parallel (Eqn. 6.5). The sharing of the net stretch is in proportion to the compliances:

$$\Delta\ell_1 = \frac{1/k_1}{1/k_1 + 1/k_2}\Delta\ell \quad \text{and} \quad \Delta\ell_2 = \frac{1/k_2}{1/k_1 + 1/k_2}\Delta\ell$$

which add up to $\Delta\ell$ as they must.

Example: **Two springs in series.**
Take $1/k_1 = 99\,\text{cm/N}$ and $1/k_2 = 1\,\text{cm/N}$. The effective compliance of the parallel combination is:

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} = 99\,\text{cm/N} + 1\,\text{cm/N} = 100\,\text{cm/N}.$$

Note that $\Delta\ell_1/\Delta\ell = .99$ so even though the two springs share the displacement, the more compliant one has 99% of it. For design purposes, or for modeling this system, it would be fair to replace the much more stiff spring with a rigid link.

## Consequences of series and parallel springs for modeling

As the previous two examples illustrate, springs can sometimes be replaced with 'air' (nothing) or with rigid links without changing the system or model behavior much. One way to think about this is that in the limit as $k \to \infty$ a spring becomes a rigid bar and in the limit $k \to 0$ a spring becomes air.

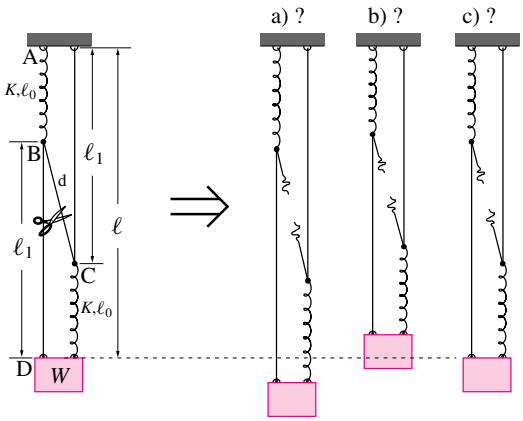
These ideas are used by engineers, often intuitively or even subconsciously and with no substantiating calculations, when making a model of a mechanical system.

- If one of several pieces in series is much stiffer than the others it is often replaced with a rigid link.
- If one of several pieces in parallel is much more compliant than the others it is often replaced with air (nothing, sailboat fuel).

## 6.2 A puzzle with two springs and three ropes.

This is a tricky puzzle whose study is not required in order to learn the basic concepts of this chapter.

Consider a weight hanging from 3 strings (BD,BC, and AC) and 2 springs (AB and CD) as in the left picture below. Point B is above point C and all ropes and springs are somewhat taught (none is slack).




When rope BC is cut does the weight go (a) down?, (b) up?, or (c) stay put?

Stop reading and try thinking, experimenting, or calculating each time you see three dots. Like now.

• • •

**Do the experiment.** In 15 minutes or so you can set up this experiment with 3 pieces of string, 2 rubber bands and a soda bottle. Hang the partially filled soda bottle from a door knob (or the top corner of a door, or a ruler cantilevered over the top of a refrigerator). Adjust the string lengths and amount of weight so that no strings or rubber bands are slack and make sure point B is above point C. The two points A can coincide as can the two points D. You might want to separate them a little with, say, a small wad of paper so you can see which string is which.

• • •

Looking at your experimental setup, but not pulling and poking at it, try to predict whether your bottle will go up down or not move when you cut the middle string. Ask your friends what they think.

• • •

**Result in small print here:** *When you cut the middle string the weight goes up a little.* This violates many peoples' intuitions. This puzzle was published as one for which people have bad intuitions.

• • •

Now try to figure out why the experiment comes out the way it does. Also, try to figure out the error in your thinking if you got it wrong (like most people do).

• • •

**Clue:** All simple explanations are based on the assumption that the lengths of the strings AC and BD are constant at $\ell_1$.

• • •

**Explanation 1:** To simplify the reasoning let's assume that springs AB and CD are identical and carry the same tension $T_s$ and that the ropes AC and BD carry the same tension $T_r$. As usual, we need free body diagrams. (With the symmetry we have assumed diagrams (a) and (c) provide identical information.) The three free body diagrams can be considered before and after the middle string removal by having $T_d > 0$ or $T_d = 0$, respectively. Vertical force balance gives (approximating $T_d$ as vertical):

$$T_s + T_r = W \quad \text{and} \quad 2T_r + T_d = W \quad \Rightarrow \quad T_s = (W + T_d)/2.$$

Because we approximate AC as rigid with length $\ell_1$, the downwards position of the weight is the string length $\ell_1$ plus the rest length of the spring $\ell_0$ plus the stretch of the spring $T_s/k$:

$$\ell = \ell_1 + \ell_0 + T_s/k = \ell_1 + \ell_0 + (W + T_d)/(2k).$$

In the course of this experiment $\ell_1$, $\ell_0$, $W$ and $k$ are constants. So as the tension $T_d$ goes from positive to zero (when the rope BC is cut) $\ell$ decreases. So the weight goes up.

• • •

**Explanation 2:** More intuitively, start with the configuration with the rope already cut and apply a small upwards force at C. It has no effect on the tension in spring CD thus the weight does not move. Now apply a small downwards force at B. This does stretch spring AB and thus lower point B, thus lowering the weight since $\ell_1$ is constant. Applying both simultaneously is like attaching the middle rope. Thus attaching the middle rope lowers the weight and cutting the middle rope raises it again.

• • •

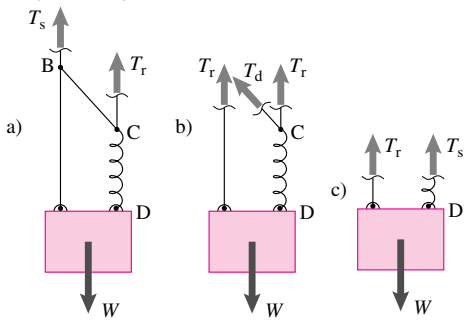
**Explanation 3:** Here is another intuitive approach. Point C can't move. Point B moves up and down just as much as the weight does. Point B is a distance $d$ above point C. Since the rope BC is taught, releasing it will allow B and C to separate, thus increasing $d$ and raising the weight.

• • •

**A wrong explanation:** What about springs in parallel and series? Here is a quick but wrong explanation for the experimental result, though it happens to predict the right direction of motion.

> "Before rope BC is cut the two springs are more or less in series because the load is carried from spring through BC to spring. Afterwards they are more or less in parallel because they have the same stretch and share the load. Two springs in parallel have 4 times the stiffness of the same two springs in series. So in the parallel arrangement the deflection is less. So the weight goes up when the springs switch from series to parallel."

What is the error in this thinking? The position of the weight comes from spring deflection added to the position when there is no weight. For the argument just presented to make sense, the rest-position of the mass (with gravity switched off) would have to be the same for the supposed 'series' and 'parallel' cases, which it is not ($\ell_1 + \ell_0 \neq \ell_0 + d + \ell_0$).




Another way to see the fallacy of this 'parallel verses series' argument is that the incremental stiffness of the system is, assuming inextensible ropes, infinite. That is, if you add or subtract a small load to the bottle it doesn't move. (The small deformation you do see has to do with the stretch of the ropes, something that none of the simple explanations take into account.) If the springs were in series or parallel we would expect an incremental stiffness that was related to spring stretch not rope stretch.

## 6.3 *THEORY*

### *How stiff a spring is a solid rod*

Here we derive the formula for stiffness of a rod:

$$k = EA/\ell$$

This foreshadowing of Strength of Materials concepts is not central to the study of statics.

Let's take a reference bar with cross sectional area $A_0$ and rest length $\ell_0$ and pull it with tension $T$ and measure the elongation $\Delta\ell_0$ (Fig. **??**ab). The stiffness of this reference rod is $k_0 = T_0/\Delta\ell_0$. Now put two such rods side by side and you have parallel springs. You might imagine this sequence: two bars are near each other, then side by side, then touching each other, then glued together, then melted together into one rod with twice the cross section. The same tension in each causes the same elongation, or it takes twice the tension to cause the same elongation when you have twice the cross sectional area. Likewise with three side by side bars and so on, so for bars of equal length

$$k = \frac{A}{A_0}k_0.$$

On the other hand we could put the reference rods end to end in series. Then the same tension causes twice the elongation. We could be three or more rods together in series thus for bars with equal cross sections:

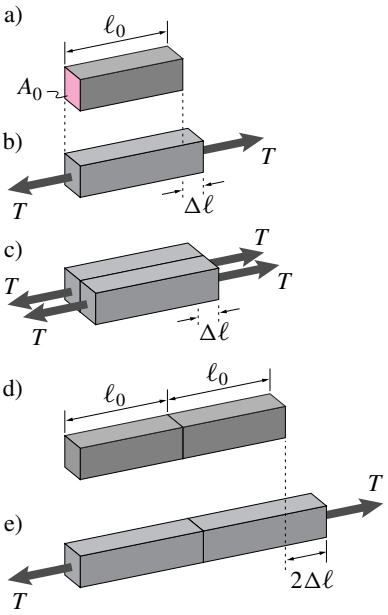
$$k = \frac{\ell_0}{\ell}k_0.$$

Putting these together we get:

$$k = \left(\frac{A}{A_0}\right)\left(\frac{\ell_0}{\ell}\right)k_0 = \left(\frac{k_0\ell_0}{A_0}\right)\frac{A}{\ell}.$$

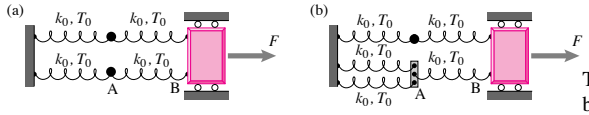
Now presumably if we took a rod with a given material, length, and cross section the stiffness would be $k$, no matter what the dimensions of the reference rod. So $\left(\frac{k_0\ell_0}{A_0}\right)$ has to be a material constant. It is called $E$, the *modulus of elasticity* or Young's modulus. For all steels $E \approx 30\cdot10^6$ lbf/in$^2$ $\approx 210\cdot10^9$ N/m$^2$ (consistent with Fig. 6.10c). Aluminum has about a third this stiffness. So, a solid bar is a linear spring, obeying the spring equations:

$$k = \frac{EA}{\ell} \quad \text{or} \quad \Delta\ell = \frac{T\ell}{EA} \quad \text{or} \quad T = \frac{\Delta\ell\, EA}{\ell}\,.$$




## 6.4 Stiffer but weaker

This is an aside for those who wonder about the fine points.




The structure on the left is made with 4 springs. The structure on the right is made with 5 springs. All 9 springs are identical with stiffness $k_0$ and break when the tension in them reaches $T_0$. We now want to compare the stiffness and strength of the two structures. Because of the mixture of parallel and series springs, the net stiffness of the structure in (a) is

$$k_{\text{net}} = k_0 \text{ and strength} = 2T_0$$

because none of the springs reaches its breaking tension until $F = T_0$.

By doubling up one of the springs in (a) to get (b) we get

$$k_{\text{net}} = 7k_0/6 \text{ and strength} = 21T_0/12.$$

The structure is made 16% stiffer but spring AB now reaches its breaking point $T_0$ when the applied load is 12.5% smaller.

**What's going on?** The second structure is made stiffer by reducing the deflection of point A. But this causes spring AB to stretch more and thus break at a smaller load. In some approximate sense, the load is thus concentrated in spring AB. This concentration of load into one part of structure is one reason that stiffness and strength need to be considered separately. Load concentration (or *stress concentration*) is a major cause of structural failure.

In common experience stiffness and strength do correlate. But this common correlation does not represent a trustworthy rule.

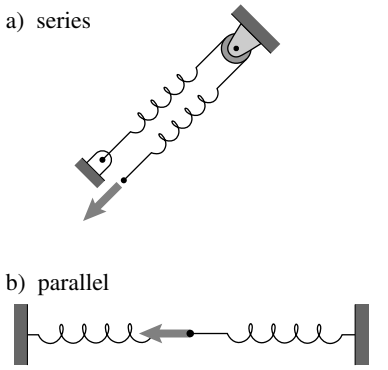



Figure 6.9: a) The two springs shown are in series because the carry the same load and their displacements add. b) These two springs are in parallel because the have a common displacement and their forces add.

Filename:tfigure-parallelconfusion

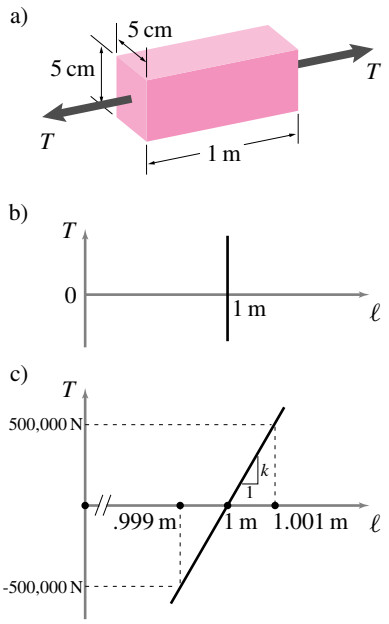



Figure 6.10: a) a steel rod in tension, b) tension verses length curve, c) zoom in on the tension verses length curve.

Filename:tfigure-steelrodspring

For example:

- When a coil spring is connected to a linkage, the other pieces in the linkage, though undoubtedly somewhat compliant, are typically modelled as rigid. They are stiffer than the spring and in series with it.
- A single hinge resists rotation about axes perpendicular to the hinge axis. But a door connected at two points along its edge is stiffly prevented against such rotations. Thus the hinge stiffness is in parallel with the greater rotational stiffness of the two connection points and is thus often neglected (see the discussion and figures in section 3.6 starting on page 129).
- Welded joints in a determinate truss are modeled as frictionless pins. The rotational stiffness of the welds is 'in parallel' with the axial stiffness of the bars. To see this look at two bars welded together at an angle. Imagine trying to break this weld by pulling the two far bar ends apart. Now imagine trying to break the weld if the two far ends are connected to each other with a third bar. The third bar is 'in parallel' with the weld material. See the first few sentences of section 5.1 for a do-it-yourself demonstration of the idea.
- Human bones are often modeled as rigid because, in part, when they interact with the world they are in series with more compliant flesh.

Note, again, that the mechanics usage of the words 'in parallel' and 'in series' don't always correspond to the geometric arrangement. For example the two springs in Fig. 6.9a are in series and the two springs in Fig. 6.9b are in parallel.

## Strength and stiffness

Most often when you build a structure you want to make it stiff and strong. The ideas of stiffness and strength are so intimately related that it is sometimes hard to untangle them. For example, you might examine a product in discount store by putting your hand on it, applying small forces and observing the motion. Then you might say: "pretty shaky, I don't think it will hold up" meaning that the stiffness is low so you think the thing may break if the loads get high.

Although stiffness and strength are often correlated, they are distinct concepts. Something is stiff if the force to cause a given motion is high. Something is strong if the force to cause any part of it to break is high. In fact, it is possible for a structure to be made weaker by making it stiffer (see box 6.4 on page )

## Why aren't springs in all mechanical models?

All things deform a little under load. Why don't we take this deformation into account in all mechanics calculations by, for example, modeling solids as elastic springs? Because many problems have solutions which would be little effected by such deformation. In particular, if a problem is statically

determinate then very small deformations only have a very small effect on the equilibrium equations and calculated forces.

## Linear springs are just one way to model 'give'

If it is important to consider the deformability of an object, the linear spring model is just one simple model. It happens to be a good model for the small deformation of many solids. But the linear spring model is defined by the two words 'linear' and 'elastic'. For some purposes one might want to model the force due to deformation as being *non-linear*, like $T = k_1(\Delta\ell) + k_2(\Delta\ell)^3$. And one may want to take account of the dissipative or *in-elastic* nature of something. The most common example being a linear dashpot $T = c\dot{\ell}$. Various mixtures of non-linearity and inelasticity may be needed to model the large deformations of a yielding metal, for example.

## Solid bars are linear springs

When a structure or machine is built with literal springs (*e.g.*, a wire helix) it is common to treat the other parts as rigid. But when a structure has no literal springs the small amount of deformation in rigid looking objects can be important, especially for determining how loads are shared in redundant structures.

Let's consider a 1 m (about a yard) steel rod with a 5 cm square (about $(2\,\text{in})^2$) cross section (Fig. 6.10a). If we plot the tension verses length we get a curve like Fig. 6.10b. The length just doesn't visibly change (unless the tension got so large as to damage the rod, not shown.) But, when you pull on anything, it does deform at least a little. If we zoom in on the tension verses length plot we get Fig. 6.10c. To change the length by one part in a thousand (a millimeter, a twenty fifth of an inch) we have to apply a tension of about 500, 000 N (about 60 tons). Nonetheless the plot reveals that the solid steel rod behaves like a (very stiff) linear spring.

Surprisingly perhaps this little bit of compliance is important to structural engineers. Modeling solid metal rods as linear springs is essential for finding internal forces in statically indeterminate structures. Because it is hard to picture steel deforming, your intuition may be helped by exaggerating the deformation. Think of all solids as being rubber. Or, if you want to look inside the solid in your mind, think of every solid as if it was a piece deforming Jello. (Jello is colored sugar water held together, jelled, by long springy gelatine molecules extracted from animal hooves. Vegetarians can use sea-weed based Agar jell for their deformation fantasies. )

How does a solid bar's stiffness depend on its shape and composition?In box 6.1 on page 299 we show that the stiffness of a solid elastic bar is

$$k = \frac{EA}{\ell}$$

where $E$ is a material property called the Young's modulus. It's that $E$ is big that keeps most solids from deforming visibly*.

* **Why aren't springs in all mechanical models?** All things deform a little under load. Why don't we take this deformation into account in all mechanics calculations by, for example, modeling solids as elastic springs? Because many problems have solutions which would be little effected by such deformation. In particular, if a problem is statically determinate then very small deformations only have a very small effect on the equilibrium equations and calculated forces.

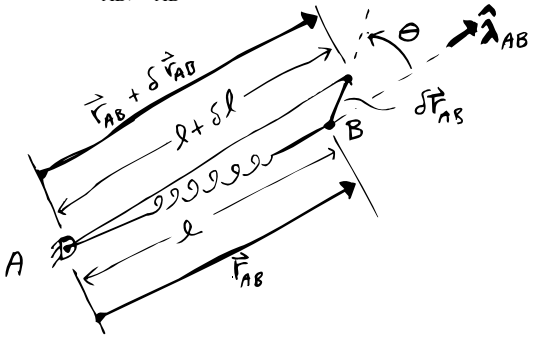
## 6.5 2D geometry of spring stretch

The material here is used in advanced sample **??** on page **??** and some of the later homework problems.

The key result concerns a spring with one end fixed at A and the other at moving point B. When point B moves from $\vec{r}_{\rm B}$ to $\vec{r}_{\rm B} + \Delta\vec{r}_{\rm B}$ then the spring length changes from $\ell$ to $\ell + \Delta\ell$ with

$$\Delta\ell \approx \hat{\boldsymbol{\lambda}}_{AB} \cdot \Delta\vec{r}_{\rm B} \tag{6.7}$$

where $\hat{\boldsymbol{\lambda}}_{AB} = \vec{r}_{\rm AB}/|\vec{r}_{\rm AB}|$ is a unit vector in the direction AB.



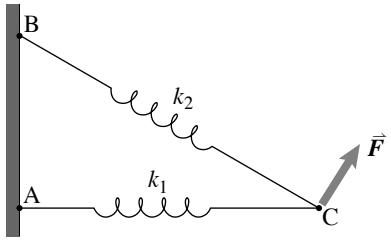

Before we derive this result a few ways, lets discuss its relevance.

**The Usual fixed-configuration statics.** The forces and moments on a system in static equilibrium satisfy force and moment balance. In these equations the force magnitudes and directions, the moments and the locations of points of application of these are those *in the equilibrium configuration*. The equilibrium of the deformed state is expressed in terms of the geometry of that deformed state. Where the structure was before loading doesn't appear in the equilibrium equations.

However, often we know the geometry of a structure before the loads are applied, not after. To avoid calculation and confusion, we assume that the deformations cause negligible changes in positions. This is one reason people mistakenly think of statics as being limited to rigid bodies. Rather, for bodies that don't deform much, we can use the before-load geometry of a structure for reasonably accurate estimation of the deformed geometry.

**Statics of deformable solids.** In principal, the statics of deformable solids is the same as for rigid solids. You just need to use the deformed geometry in the statics calculations. Unfortunately, to find that geometry one needs the forces and their points of application. And one can't find all the locations without finding the deformation which depends on the forces, etc. This dizzying circle is escapable using the 'three pillars' (page 3).

Example: **A structure made of springs.**




Assume all the lengths and geometry of the two-bar truss are known when there is no load at C. We can find all the tensions and deflections as follows (See page xiii for the general strategy):

1. Assume that the equilibrium loaded location of C is displaced from the rest location by $\delta\vec{r}_{\rm C} = \delta x_C\hat{\boldsymbol{\imath}} + \delta y_C\hat{\boldsymbol{\jmath}}$ where $\delta x_C$ and $\delta y_C$ are *unknowns*;
2. Calculate the lengths of the springs in terms of $\delta x_C$ and $\delta y_C$ (this will be a complex expression with squares and square roots);
3. Find the tensions in the springs in terms of their new lengths and thus in terms of $\delta x_C$ and $\delta y_C$;
4. Draw a free body diagram of C, using the spring orientations and tensions you have found (still in terms of unknowns $\delta x_C$ and $\delta y_C$);
5. Write the force balance equations. These are two equations for two unknowns $\delta x_C$ and $\delta y_C$.
6. Solve for $\delta x_C$ and $\delta y_C$.
7. Use $\delta x_C$ and $\delta y_C$ to find the lengths and thus the tensions in the springs.

The trap — having to know the deflection to find the tensions but having to know the tensions to find the deflection — is avoided by setting up and solving non-linear equations.

Although this approach is correct, it is generally not used in structural mechanics because:

- The equations are a mess.
- It is hard to solve non-linear equations, sometimes even hard on a computer.
- There may be more than one solution. For example in the math problem above, if $F$ is not too large, there will be two solutions. One solution with C deflected up and to the right, and another with C way to the left of the wall. To get rid of such off-the-wall solutions you need to either use judgment after you find them, or further specify your math problem to eliminate them.
- There are simpler methods that give almost-as-accurate an answer.

**The structural-mechanics approach.** So long as $F$ is not too large, the motion of point C will be small compared to the lengths of the springs. Especially since, in practice, those springs are often solid metal rods. The usual *small deformation* assumption is that

- The deflection is small enough so that the spring angle changes have negligible effect on the equilibrium equations, and
- The deflection is small enough for the approximate formula for spring length change, eqn. (6.7), to be adequate.

The recipe for finding the deflection of C in the example above is greatly simplified with these approximations:

1. Assume that the equilibrium loaded location of C is displaced from the rest location by $\delta\vec{r}_{\rm C} = \delta x_C\hat{\boldsymbol{\imath}} + \delta y_C\hat{\boldsymbol{\jmath}}$ where $\delta x_C$ and $\delta y_C$ are *unknowns* (unchanged);
2. Calculate the lengths of the springs in terms of $\delta x_C$ and $\delta y_C$ using eqn. (6.7) (simplified);
3. Find the tensions in the springs in terms of their new lengths (unchanged) and thus in terms of $\delta x_C$ and $\delta y_C$ (much simpler expressions);

4. Draw a free body diagram of C, using the original undeflected geometry (much simplified);
5. Write the force balance equations. These are two equations for two unknowns $\delta x_C$ and $\delta y_C$. (These will now be linear equations instead of a non-linear mess.)
6. Solve for $\delta x_C$ and $\delta y_C$. (This is now the solution of linear instead of non-linear equations.)
7. (simplified) Use $\delta x_C$ and $\delta y_C$ to find the lengths and thus the tensions in the springs. (This now uses eqn. (6.7) instead of complicated relations with square roots, etc.)

This simplified recipe depends on the simplified formula for the spring length change eqn. (6.7).

**Derivation 1 of eqn. (6.7).** The law of cosines (page 77) says

$$(\ell + \delta\ell)^2 = \ell^2 + |\delta\vec{\boldsymbol{r}}_{\rm AB}|^2 + 2\ell|\delta\vec{\boldsymbol{r}}_{\rm AB}|\cos\theta$$

($\theta$ here is negative of that used in the statement of the law of cosines). Expanding the left side and dropping terms in $\delta\ell^2$ and $|\delta\vec{\boldsymbol{r}}_{\rm AB}|^2$ on both sides (assuming $\delta\ell/\ell \ll 1$ and $|\delta\vec{\boldsymbol{r}}_{\rm AB}|/\ell \ll 1$), and dividing both sides by $\ell$ we get

$$\delta\ell \approx |\delta\vec{\boldsymbol{r}}_{\rm AB}|\cos\theta = \hat{\boldsymbol{\lambda}}_{AB}\cdot\delta\vec{\boldsymbol{r}}_{\rm AB}$$

where the last equality comes from the definition of the dot product (Section 2.2).

**Derivation 2 of eqn. (6.7).** Use the pythogorean theorem to determine the lengths of $\vec{\boldsymbol{r}}_{\rm AB}$ and of $\vec{\boldsymbol{r}}_{\rm AB} + \delta\vec{\boldsymbol{r}}_{\rm AB}$:

$$\begin{aligned}\ell &= \sqrt{\left(x_{AB}^2 + y_{AB}^2\right)}\\ \ell + \delta\ell &= \sqrt{\left((x_{AB}+\delta x_{AB})^2 + (y_{AB}+\delta y_{AB})^2\right)}\end{aligned}$$

Subtracting the first from the second, dividing both sides by $\ell$, and expanding the contents of the square root we get

$$\delta\ell/\ell = \sqrt{1 + 2\left(x_{AB}\delta x_{AB} + y_{AB}\delta y_{AB}\right)/\ell^2 + \delta x_{AB}^2\delta x_{AB}^2/\ell^2} - 1.$$

Neglecting $\delta x_{AB}^2$ and $\delta y_{AB}^2$ (assuming $\delta\ell \ll \ell$) and expanding the square root ($\sqrt{1+\epsilon} \approx 1 + \epsilon/2$), and multiplying through by $\ell$, we get

$$\delta\ell \approx (x_{AB}/\ell)\delta x_{AB} + (y_{AB}/\ell)\delta y_{AB})$$

which is eqn. (6.7) because $\hat{\boldsymbol{\lambda}}_{AB} = (x_{AB}/\ell)\hat{\boldsymbol{\imath}} + (y_{AB}/\ell)\hat{\boldsymbol{\jmath}}$.

**Derivation 3 of eqn. (6.7).** Using vector notation throughout:

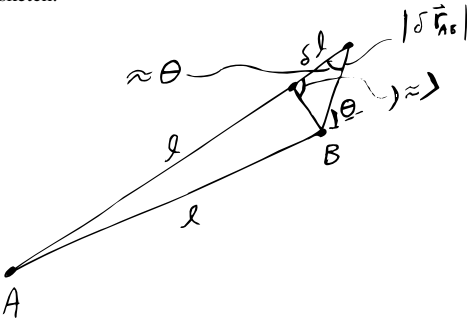
$$\begin{aligned}\ell^2 &= \vec{\boldsymbol{r}}_{\rm AB}\cdot\vec{\boldsymbol{r}}_{\rm AB}\\ (\ell+\delta\ell)^2 &= (\vec{\boldsymbol{r}}_{\rm AB} + \delta\vec{\boldsymbol{r}}_{\rm AB})\cdot(\vec{\boldsymbol{r}}_{\rm AB} + \delta\vec{\boldsymbol{r}}_{\rm AB})\end{aligned}$$

Expanding the second equation, neglecting second order terms and subtracting the first we get

$$\ell\delta\ell \approx \vec{\boldsymbol{r}}_{\rm AB}\cdot\delta\vec{\boldsymbol{r}}_{\rm AB}$$

dividing by $\ell$ and noting that $\hat{\boldsymbol{\lambda}}_{AB} = \vec{\boldsymbol{r}}_{\rm AB}/\ell$ we again get eqn. (6.7).

**Derivation 4 of eqn. (6.7).** Finally, and most intuitively, look at this sketch.




The line AB and its deflected self are nearly parallel. Thus the triangle at the end is nearly a right triangle. So, approximately, $\delta\ell = |\delta\vec{\boldsymbol{r}}_{\rm AB}|\cos\theta$ which is $\delta\vec{\boldsymbol{r}}_{\rm AB}\cdot\hat{\boldsymbol{\lambda}}_{AB}$, again giving eqn. (6.7).

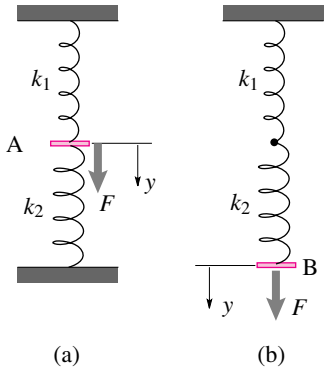



Figure 6.11:

Filename:sfig4-2springs

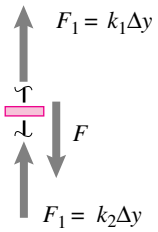



Figure 6.12: Free body diagram of point A.

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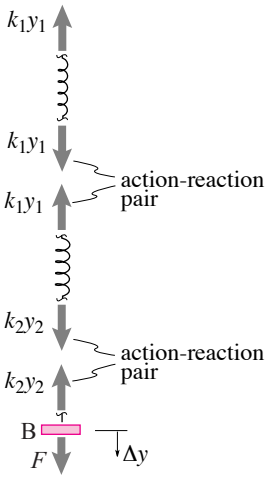



Figure 6.13: Free body diagrams

Filename:sfig4-2springs-b

**SAMPLE 6.1 Springs in series versus springs in parallel:** Two springs with spring constants $k_1 = 100\,\text{N/m}$ and $k_2 = 150\,\text{N/m}$ are attached together as shown in Fig. 6.11. In case (a), a vertical force $F = 10\,\text{N}$ is applied at point A, and in case (b), the same force is applied at the end point B. Find the force in each spring for static equilibrium. Also, find the equivalent stiffness for (a) and (b).

**Solution** In static equilibrium, let $\Delta y$ be the displacement of the point of application of the force in each case. We can figure out the forces in the springs by writing force balance equations in each case.

- **Case (a):** The free body diagram of point A is shown in Fig. 6.12. As point A is displaced downwards by $\Delta y$, spring 1 gets stretched by $\Delta y$ whereas spring 2 gets compressed by $\Delta y$. Therefore, the forces applied by the two springs, $k_1\Delta y$ and $\Delta k_2 y$, are in the same direction. Then, the force balance in the vertical direction, $\hat{\boldsymbol{j}} \cdot (\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}})$, gives:

$$\begin{aligned} F &= F_1 + F_2 = (k_1 + k_2)\Delta y \\ \Rightarrow \quad \Delta y &= \frac{F}{k_1 + k_2} = \frac{10\,\text{N}}{(100 + 150)\,\text{N/m}} = 0.04\,\text{m} \\ \Rightarrow \quad F_1 &= k_1\Delta y = 100\,\text{N/m} \cdot 0.04\,\text{m} = 4\,\text{N} \\ \Rightarrow \quad F_2 &= k_2\Delta y = 150\,\text{N/m} \cdot 0.04\,\text{m} = 6\,\text{N}. \end{aligned}$$

The equivalent stiffness of the system is the stiffness of a single spring that will undergo the same displacement $\Delta y$ under $F$. From the equilibrium equation above, it is easy to see that,

$$k_e \equiv \frac{F}{\Delta y} = k_1 + k_2 = 250\,\text{N/m}.$$

$$\boxed{F_1 = 4\,\text{N}, \quad F_2 = 6\,\text{N}, \quad k_e = 250\,\text{N/m}}$$

- **Case (b):** The free body diagrams of the two springs is shown in Fig. 6.13 along with that of point B. In this case both springs stretch as point B is displaced downwards. Let the net stretch in spring 1 be $y_1$ and in spring 2 be $y_2$. $y_1$ and $y_2$ are unknown, of course, but we know that

$$y_1 + y_2 = \Delta y.$$

Now, using the free body diagram of point B and writing the force balance equation in the vertical direction, we get $F = k_2 y_2$, and from the free-body diagram of spring 2, we get $k_2 y_2 = k_1 y_1$. Thus the force in each spring is the same and equals the applied force, *i.e.*,

$$F_1 = k_1 y_1 = F = 10\,\text{N} \quad \text{and} \quad F_2 = k_2 y_2 = F = 10\,\text{N}.$$

The springs in this case are in series. Therefore, their equivalent stiffness, $k_e$, is

$$k_e = \left(\frac{1}{k_1} + \frac{1}{k_2}\right)^{-1} = \left(\frac{1}{100\,\text{N/m}} + \frac{1}{150\,\text{N/m}}\right)^{-1} = 60\,\text{N/m}.$$

Note that the displacements $y_1$ and $y_2$ are different in this case. They can be easily found from $y_1 = F/k_1$ and $y_2 = F/k_2$.

$$\boxed{F_1 = F_2 = 10\,\text{N}, \quad k_e = 60\,\text{N/m}}$$

**Comments:** Although the springs attached to point A do not visually seem to be in parallel, from mechanics point of view they are parallel. Springs in *parallel* have the *same displacement* but *different forces*. Springs in *series* have *different displacements* but the *same force*.

**SAMPLE 6.2 Stiffness of three springs:** For the spring networks shown in Fig. 6.14(a) and (b), find the equivalent stiffness of the springs in each case, given that each spring has a stiffness of $k = 20\,\text{N/m}$.

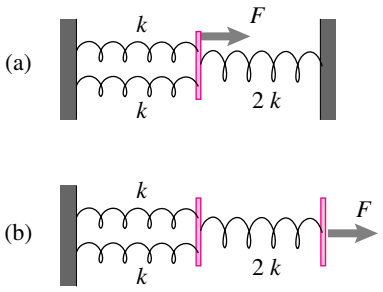



Figure 6.14:
Filename:sfig4-3springs

**Solution**

1. In Fig. 6.14(a), all springs are in parallel since all of them undergo the same displacement $\Delta x$ in order to balance the applied force $F$. Each of the two springs on the left stretches by $\Delta x$ and the spring on the right compresses by $\Delta x$. Therefore, the equivalent stiffness of the three springs is

$$k_p = k + k + 2k = 4k = 80\,\text{kN/m}.$$

Pictorially,

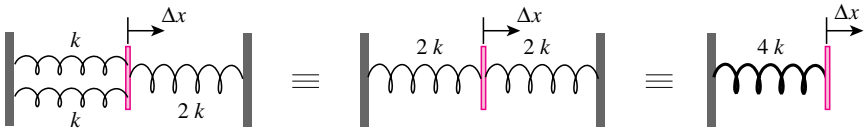



Figure 6.15:
Filename:sfig4-3springs-a

$$\boxed{k_{\text{equiv}} = 80\,\text{kN/m}}$$

2. In Fig. 6.14(b), the first two springs (on the left) are in parallel but the third spring is in series with the first two. To see this, imagine that for equilibrium point A moves to the right by $\Delta x_A$ and point B moves to the right by $\Delta x_B$. Then each of the first two springs has the same stretch $\Delta x_A$ while the third spring has a net stretch $= \Delta x_B - \Delta x_A$. Therefore, to find the equivalent stiffness, we can first replace the two parallel springs by a single spring of equivalent stiffness $k_p = k + k = 2k$. Then the springs with stiffnesses $k_p$ are $2k$ are in series and therefore their equivalent stiffness $k_s$ is found as follows:

$$\begin{aligned} \frac{1}{k_s} &= \frac{1}{k_p} + \frac{1}{2k} = \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k} \\ \Rightarrow \quad k_s &= k = 20\,\text{kN/m}. \end{aligned}$$

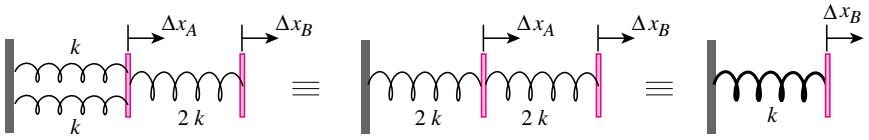



Figure 6.16:
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$$\boxed{k_{\text{equiv}} = 20\,\text{kN/m}}$$

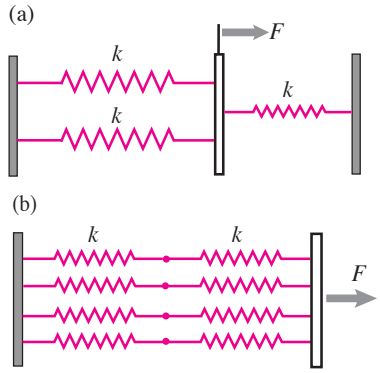



Figure 6.17:
Filename:sfig4-manysprings

**SAMPLE 6.3 Stiffness vs strength:** Which of the two structures (network of springs) shown in the figure is stiffer and which one has more strength if each spring has stiffness $k = 10\,\text{kN/m}$ and strength $T_0 = 10\,\text{kN}$.

**Solution** In structure (a), all the three springs are in parallel. Therefore, the equivalent stiffness of the three springs is

$$k_a = k + k + k = 3k = 30\,\text{kN/m}.$$

For figuring out the strength of the structure, we need to find the force in each spring. From the free-body diagram in Fig. 6.18 we see that,

$$\begin{aligned} k\Delta x + k\Delta x + k\Delta x &= F \\ \Rightarrow \quad \Delta x &= \frac{F}{3k}. \end{aligned}$$

Therefore, the force in each spring is

$$F_s = k\Delta x = \frac{F}{3}.$$

But the maximum force that a spring can take is $(F_s)_{\max} = T_0 = 10\,\text{kN}$. Therefore, the maximum force that the structure can take ( *i.e.*, the strength of the structure), is

$$F_{\max} = 3T_0 = 30\,\text{kN}.$$

Stiffness = 30 kN/m, Strength = 30 kN

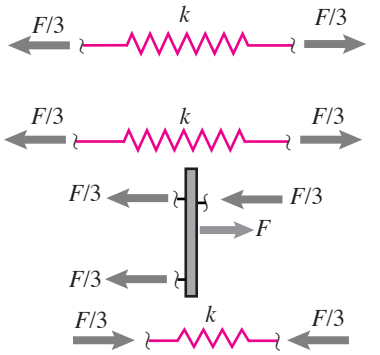



Figure 6.18:
Filename:sfig4-manysprings-a

Now we carry out a similar analysis for structure (b). There are four parallel chains in this structure, each chain containing two springs in series. The stiffness of each chain, $k_c$, is found from

$$\begin{aligned} \frac{1}{k_c} &= \frac{1}{k} + \frac{1}{k} = \frac{2}{k} \\ \Rightarrow \quad k_c &= \frac{k}{2} = 5\,\text{kN/m}. \end{aligned}$$

So, the stiffness of the entire structure is

$$k_b = k_c + k_c + k_c + k_c = 4k_c = 20\,\text{kN/m}.$$

From the free-body diagram shown in Fig. 6.19, we find the force in each spring to be F/4 . Therefore, the maximum force that the structure can take is

$$F_{\max} = 4T_0 = 40\,\text{kN}.$$

Stiffness = 20 kN/m, Strength = 40 kN

Thus, structure (a) is stiffer but structure (b) is stronger (higher strength).

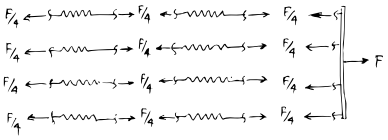



Figure 6.19:
Filename:sfig4-manysprings-b

**SAMPLE 6.4 Zero length springs are special.** A rigid and massless rod OAB of length 2 m supports a weight $W = 100\,\text{kg}$ hung from point B. The rod is pinned at O and supported by a zero length (in relaxed state) spring attached at mid-point A and point C on the vertical wall. Find the equilibrium angle $\theta$ and the force in the spring.

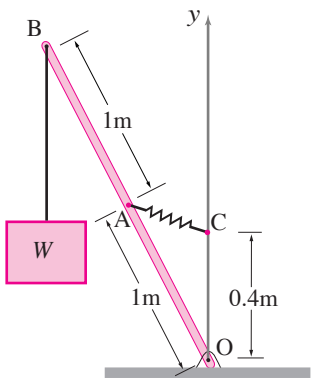



Figure 6.20:
Filename:sfig4-zerospring

**Solution** The free-body diagram of the rod is shown in Fig. 6.21 in an assumed equilibrium state. Let $\hat{\boldsymbol{\lambda}} = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}$ be a unit vector along OB. The spring force can be written as $\vec{\boldsymbol{F}}_{\text{s}} = k\vec{\boldsymbol{r}}_{\text{C/A}}$ (since AC is a *zero-length* spring, the stretch in the spring is $|\vec{\boldsymbol{r}}_{\text{C/A}}|$). We need to determine $\theta$ and $F_s$.

Let us write moment equilibrium equation about point O, *i.e.*, $\sum \vec{\boldsymbol{M}}_{\text{O}} = \vec{\boldsymbol{0}}$,

$$\vec{\boldsymbol{r}}_{\text{B/O}} \times \vec{\boldsymbol{W}} + \vec{\boldsymbol{r}}_{\text{A/O}} \times \vec{\boldsymbol{F}}_{\text{s}} = \vec{\boldsymbol{0}}.$$

Noting that

$$\begin{aligned} \vec{\boldsymbol{r}}_{\text{B/O}} &= \ell\hat{\boldsymbol{\lambda}}, \quad \vec{\boldsymbol{r}}_{\text{A/O}} = \frac{\ell}{2}\hat{\boldsymbol{\lambda}}, \\ \vec{\boldsymbol{F}}_{\text{s}} &= k\vec{\boldsymbol{r}}_{\text{C/A}} = k(\vec{\boldsymbol{r}}_{\text{C}} - \vec{\boldsymbol{r}}_{\text{A}}) \\ &= k\left(h\hat{\boldsymbol{\jmath}} - \frac{\ell}{2}\hat{\boldsymbol{\lambda}}\right), \end{aligned}$$

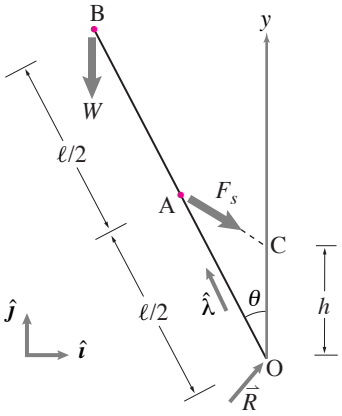



Figure 6.21:
Filename:sfig4-zerospring-a

we get,

$$\begin{aligned} \ell\hat{\boldsymbol{\lambda}} \times (-W\hat{\boldsymbol{\jmath}}) + \frac{\ell}{2}\hat{\boldsymbol{\lambda}} \times k\left(h\hat{\boldsymbol{\jmath}} - \frac{\ell}{2}\hat{\boldsymbol{\lambda}}\right) &= \vec{\boldsymbol{0}} \\ -W\ell(\hat{\boldsymbol{\lambda}} \times \hat{\boldsymbol{\jmath}}) + kh\frac{\ell}{2}(\hat{\boldsymbol{\lambda}} \times \hat{\boldsymbol{\jmath}}) &= \vec{\boldsymbol{0}}. \end{aligned}$$

Dotting this equation with $(\hat{\boldsymbol{\lambda}} \times \hat{\boldsymbol{\jmath}})$, we get,

$$\begin{aligned} -W\ell + kh\frac{\ell}{2} &= 0 \\ \Rightarrow \quad kh &= 2W. \end{aligned}$$

Thus the result is independent of $\theta$! As long as the spring stiffness $k$ and the height $h$ of point C are such that their product equals $2W$, the system will be in equilibrium at any angle. This, however, is in general not possible if AC is not a zero-length spring.

Equilibrium is satisfied at any angle if $kh = 2W$

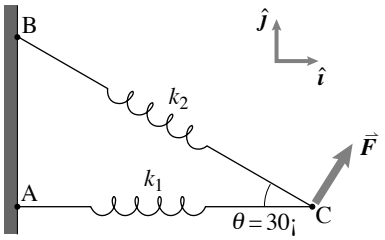



Figure 6.22:
Filename:sfig4-springs-compl

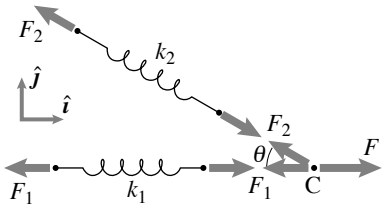



Figure 6.23:
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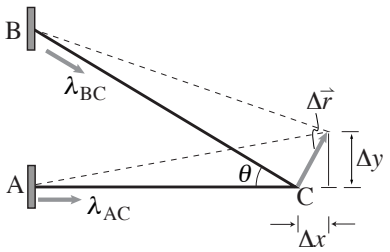



Figure 6.24:
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**SAMPLE 6.5 Deflection of an elastic structure:** For the two-spring structure shown in the figure, find the deflection of point C when

1. $\vec{F} = 1\,\text{N}\hat{\imath}$,
2. $\vec{F} = 1\,\text{N}\hat{\jmath}$,
3. $\vec{F} = 30\,\text{N}\hat{\imath} + 20\,\text{N}\hat{\jmath}$,

The spring stiffnesses are $k_1 = 10\,\text{kN}/m$ and $k_2 = 20\,\text{kN}/m$.

**Solution** Let $\Delta\vec{r} = \Delta x\hat{\imath} + \Delta y\hat{\jmath}$ be the displacement of point C of the structure due to the applied load. We can figure out the deflections in each spring as follows. Let $\hat{\boldsymbol{\lambda}}_{AC}$ and $\hat{\boldsymbol{\lambda}}_{BC}$ be the unit vectors along AC and BC, respectively (see Fig. 6.24). Then, the change in the length of spring AC due to the (assumed small) displacement of point C is (see page 302 for a discussion)

$$\begin{aligned}\Delta_{AC} &= \hat{\boldsymbol{\lambda}}_{AC}\cdot\Delta\vec{r} \quad \text{(this is the key equation)}\\ &= \hat{\imath}\cdot(\Delta x\hat{\imath} + \Delta y\hat{\jmath}) = \Delta x.\end{aligned}$$

Similarly, the change in the length of spring BC is

$$\begin{aligned}\Delta_{BC} &= \hat{\boldsymbol{\lambda}}_{BC}\cdot\Delta\vec{r}\\ &= (\cos\theta\hat{\imath} - \sin\theta\hat{\jmath})\cdot(\Delta x\hat{\imath} + \Delta y\hat{\jmath}) = \Delta x\cos\theta - \Delta y\sin\theta.\end{aligned}$$

Now we can find the force in each spring since we know the deflection in each spring.

$$\text{Force in spring AC} = F_1 = k_1\Delta x \tag{6.8}$$

$$\text{Force in spring BC} = F_2 = k_2(\Delta x\cos\theta - \Delta y\sin\theta). \tag{6.9}$$

The forces in the springs, however, depend on the applied force, since they must satisfy static equilibrium. Thus, we can determine the deflection by first finding $F_1$ and $F_2$ in terms of the applied load and substituting in the equations above to solve for the deflection components.

1. *Deflections with unit force in the x-direction:*

   Let $\vec{F} = f_x\hat{\imath} = 1\,\text{N}\hat{\imath}$, (we have adopted a special symbol $f_x$ for the unit load). Then, from the free-body diagram of the springs and the end pin shown in Fig. 6.23 and the force equilibrium ($\sum\vec{F} = \vec{0}$), we have,

   $$f_x\hat{\imath} - F_1\hat{\imath} + F_2(-\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) = \vec{0}.$$

   Dotting this eqn. with $\hat{\jmath}$ and $\hat{\imath}$, respectively, we get,

   $$\begin{aligned}F_2 &= 0\\ F_1 &= f_x = 1\,\text{N}.\end{aligned}$$

   Substituting these values of $F_1$ and $F_2$ in eqns. (6.8) and (6.9), and solving for $\Delta x$ and $\Delta y$ we get,

   $$\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}_{\vec{F}=f_x\hat{\imath}} = \begin{pmatrix}\frac{1}{k_1}\\ \frac{1}{k_1}\cot\theta\end{pmatrix} f_x. \tag{6.10}$$

   Substituting the given values of $\theta$, $k_1$, and $f_x = 1\,\text{N}$, we get

   $$\Delta\vec{r} = \Delta x\hat{\imath} + \Delta y\hat{\jmath} = (100\hat{\imath} + 173\hat{\jmath})\times 10^{-6}\,\text{m}.$$

   $$\boxed{\Delta\vec{r} = (100\hat{\imath} + 173\hat{\jmath})\times 10^{-6}\,\text{m}}$$

2. *Deflections with unit force in the y-direction:* We carry out a similar analysis for this case. We again assume the displacement of point C to be $\Delta\vec{r} = \Delta x\hat{\imath} + \Delta y\hat{\jmath}$. Since the geometry of deformation and the associated results are the same, eqns. (6.8) and

(6.9) remain valid. We only need to find the spring forces from the static equilibrium under the new load. From the free-body diagram in Fig. 6.25 we have,

$$(-F_1 - F_2\cos\theta)\hat{\boldsymbol{\imath}} + (F_2\sin\theta + F)\hat{\boldsymbol{\jmath}} = \vec{\mathbf{0}} \tag{6.11}$$

$$\{\text{eqn. (6.11)}\}\cdot\hat{\boldsymbol{\jmath}} \quad \Rightarrow \quad F_2 = -\frac{F}{\sin\theta}$$

$$\{\text{eqn. (6.11)}\}\cdot\hat{\boldsymbol{\imath}} \quad \Rightarrow \quad F_1 = -F_2\cos\theta = F\cot\theta.$$

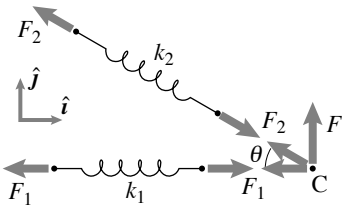



Figure 6.25:
Filename:sfig4-springs-compl-c

Substituting these values of $F_1$ and $F_2$ in terms of $F = f_y$ in eqns. (6.8) and (6.9), we get

$$\begin{aligned} f_y\cot\theta &= k_1\Delta x \quad \Rightarrow \quad \Delta x = \frac{f_y}{k_1}\cot\theta \\ -\frac{f_y}{\sin\theta} &= k_2(\Delta x\cos\theta - \Delta y\sin\theta) \\ \Rightarrow \quad \Delta y &= \frac{1}{\sin\theta}\left(\Delta x\cos\theta + \frac{f_y}{k_2\sin\theta}\right) \\ &= f_y\left(\frac{1}{k_1}\cot^2\theta + \frac{1}{k_2}\csc^2\theta\right). \end{aligned}$$

Thus,

$$\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}_{\vec{\boldsymbol{F}}=f_y\hat{\boldsymbol{\jmath}}} = \begin{pmatrix}\frac{1}{k_1}\cot\theta\\ \frac{1}{k_1}\cot^2\theta + \frac{1}{k_2}\csc^2\theta\end{pmatrix} f_y. \tag{6.12}$$

Substituting the values of $\theta, k_1, k_2$, and $f_y = 1\,\text{N}$, we get

$$\Delta\vec{\boldsymbol{r}} = \Delta x\hat{\boldsymbol{\imath}} + \Delta y\hat{\boldsymbol{\jmath}} = (173\hat{\boldsymbol{\imath}} + 500\hat{\boldsymbol{\jmath}})\times 10^{-6}\,\text{m}.$$

$$\boxed{\Delta\vec{\boldsymbol{r}} = (173\hat{\boldsymbol{\imath}} + 500\hat{\boldsymbol{\jmath}})\times 10^{-6}\,\text{m}}$$

3. *Deflection under general load:* Since we have already got expressions for deflections in the $x$ and $y$-directions under unit loads in the $x$ and $y$-directions, we can now combine the results (using superposition, see page 178) to find the deflection under any general load $\vec{\boldsymbol{F}} = F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}}$ as follows.

$$\begin{aligned} \Delta\vec{\boldsymbol{r}} &= \begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} = F_x\cdot\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}_{\vec{\boldsymbol{F}}=1\hat{\boldsymbol{\imath}}} + F_y\cdot\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}_{\vec{\boldsymbol{F}}=1\hat{\boldsymbol{\jmath}}} \\ &= \begin{bmatrix} k_1^{-1} & k_1^{-1}\cot\theta \\ k_1^{-1}\cot\theta & k_1^{-1}\cot^2\theta + k_2^{-1}\csc^2\theta \end{bmatrix}\begin{pmatrix}F_x\\ F_y\end{pmatrix}. \end{aligned}$$

Once again, substituting all given values and $F_x = 30\,\text{N}$ and $F_y = 20\,\text{N}$, we get

$$\Delta\vec{\boldsymbol{r}} = (6.4\hat{\boldsymbol{\imath}} + 15.2\hat{\boldsymbol{\jmath}})\times 10^{-3}\,\text{m}.$$

$$\boxed{\Delta\vec{\boldsymbol{r}} = (6.4\hat{\boldsymbol{\imath}} + 15.3\hat{\boldsymbol{\jmath}})\times 10^{-3}\,\text{m}}$$

**Note:** The matrix obtained above for finding the deflection under general load is called the *compliance matrix* of the structure. Its inverse is known as the *stiffness matrix* of the structure and is used to find forces given deflections.

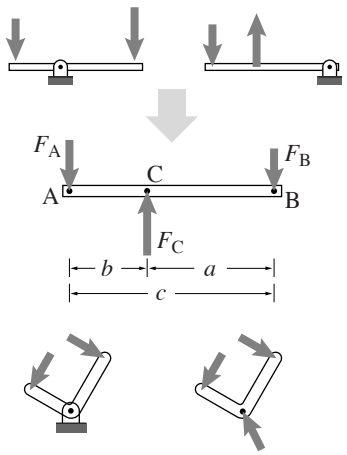



Figure 6.26: A lever can have the pivot in various places. The free body diagram looks the same in any case.

Filename:tfigure-lever

*People studied levers before mechanics was a quantitative subject. So they have a specialized antiquated vocabulary, classifying them according to which of A, B, or C is the pivot point and which of the other two forces you think of as input and which as the output: A 'class one' lever has the pivot in between; a 'class two' lever has the pivot at one end and the input force at the other; and a 'class three' lever has the pivot at one end and the input force in the middle.

# 6.2 Force amplification devices: Levers, wedges, toggles, gears, and pulleys

Simple objects can be connected in various arrangements for various purposes. Here we describe 5 machine fragments that can be used to amplify force. Most machines use these ideas in combination. It might help intuitive understanding of machines to recognize one of these methods in use, although precise categorization of every machine part as one or another of these devices is not possible.

## A lever

One of the simplest machines, long understood longer used by humans, is a lever (fig. 6.26). Although now we think of statics as a special case of dynamics, the statics of a lever was well understood 50 generations before Newton and Euler.

> An ideal lever is a rigid body held in place with a frictionless hinge and with two other applied loads.

The free body diagram Fig. 6.26 is the same whether the hinge is at point A, B or C*.

Lots of things can be viewed as levers including, for example, a wheelbarrow, a hammer pulling a nail, a boat oar, one half of a pair of tweezers, a break lever, a gear, and, most generally, any three-force body. Using the equilibrium relations on the free body diagram in fig. 6.26 you can find that

$$\frac{F_A}{a} = \frac{F_B}{b} = \frac{F_C}{c}$$

from which you could find the relation between any pair of the forces. In practice it is easier to use moment balance about an appropriate point than to memorize and recall this formula.

## An ideal wedge

Wedges are a kind of machine. For an ideal wedge one neglects friction, effectively replacing sliding contact with rolling contact (see Fig. 6.27ab). Although this approximation may not be accurate, it is helpful for building intuition. For the free body diagrams of Fig. 6.27c we have not fussed over the exact location of the contact forces since the key idea depends on force balance and not moment balance. Neglecting gravity,

$$\begin{aligned} \text{For block A, } \left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{j} \quad &\Rightarrow \quad -F_A + F\sin\theta = 0 \\ \text{For block B, } \left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{\imath} \quad &\Rightarrow \quad -F_B + F\cos\theta = 0 \end{aligned}$$

$$\text{eliminating } F \quad \Rightarrow \quad F_B = \frac{1}{\tan\theta} F_A .$$

The force amplification of a wedge is about the reciprocal of the wedge angle (in radians).

To multiply the force $F_A$ by 10 takes a wedge with a taper of $\theta = \tan^{-1} 0.1 \approx 6^\circ$. With this taper, an ideal wedge could also be viewed as a device to attenuate the force $F_B$ by a factor of 10, although wedges are never used for force attenuation in practice, as we now explain.

## A wedge with friction

In the real world frictionless things are hard to find. In the case of wedges, neglecting friction is not generally an accurate model.

Consideration of friction qualitatively changes the behavior of the machine. For simplicity we still take the wall and floor interactions to be frictionless.

Figure 6.28 shows free body diagrams of wedge blocks. We draw separate free body diagrams for the case when (a) block A is sliding down and block B to the right, and (b) block A is sliding up and block B to the left. In both cases the friction resists relative slip and obeys the sliding friction relation

$$F_f = \underbrace{\tan\phi}_{\mu} N$$

where Fig. 6.28 shows the resultant contact force (normal component plus frictional component) and its angle $\phi$ to the surface normal.

Assuming block A is sliding down we get from free body diagram 6.28a that

$$\begin{aligned} \text{For block A, } \left\{\sum \vec{\boldsymbol{F}}_i = \vec{\boldsymbol{0}}\right\} \cdot \hat{\boldsymbol{\jmath}} \quad &\Rightarrow \quad -F_A + F\sin(\theta+\phi) = 0 \\ \text{For block B, } \left\{\sum \vec{\boldsymbol{F}}_i = \vec{\boldsymbol{0}}\right\} \cdot \hat{\boldsymbol{\imath}} \quad &\Rightarrow \quad -F_B + F\cos(\theta+\phi) = 0 \end{aligned}$$

$$\text{eliminating } F \quad \Rightarrow \quad F_B = \frac{1}{\tan(\theta+\phi)} F_A. \tag{6.13}$$

If we take a taper of $6^\circ$ and a friction coefficient of $\mu = .3$ ($\Rightarrow \phi \approx 17^\circ$) we get that $F_B/F_A \approx 2.5$ instead of 10 as we got when neglecting friction. The wedge still serves as a way to multiply force, but substantially less so than the frictionless idealization led us to believe. Now lets consider the case when force $F_B$ is pushing block B to the left, pinching block A, and forcing it up. The only change in the calculation is the change in the direction of the friction interaction force. From free body diagram 6.28b

$$\begin{aligned} \text{For block A, } \left\{\sum \vec{\boldsymbol{F}}_i = \vec{\boldsymbol{0}}\right\} \cdot \hat{\boldsymbol{\jmath}} \quad &\Rightarrow \quad -F_A + F\sin(\theta-\phi) = 0 \\ \text{For block B, } \left\{\sum \vec{\boldsymbol{F}}_i = \vec{\boldsymbol{0}}\right\} \cdot \hat{\boldsymbol{\imath}} \quad &\Rightarrow \quad -F_B + F\cos(\theta-\phi) = 0 \end{aligned}$$

$$\text{eliminating } F \quad \Rightarrow \quad F_A = \tan(\theta-\phi) F_B. \tag{6.14}$$

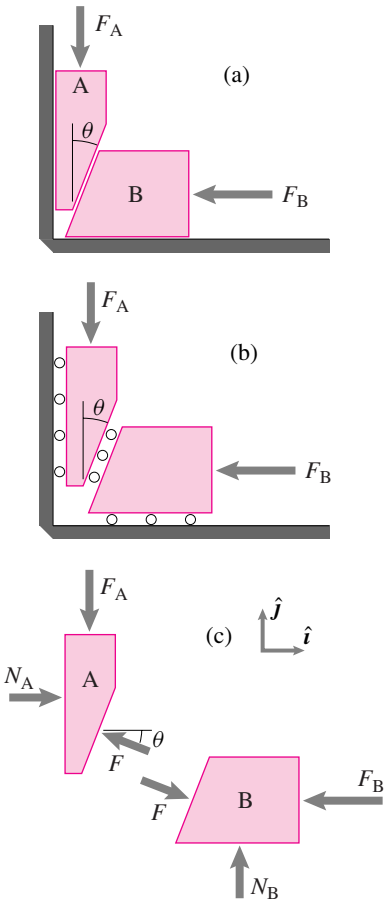



Figure 6.27: a) A wedge, b) treated as frictionless in the ideal case, c) free body diagrams

Filename:tfigure-wedge

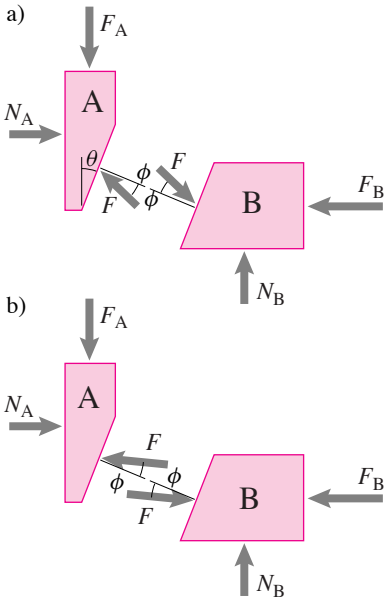



Figure 6.28: Free body diagrams of a wedge a) assuming A slides down, b) assuming A slides up

Filename:tfigure-wedgefriction

*Standard car transmissions are backdriven when they are push-started and when a driver downshifts to slow the car instead of using the brakes. On the other hand most electric hand-mixers cannot be backdriven; you can't turn the motor by forcing the beater blade (*Unplug before trying.*)

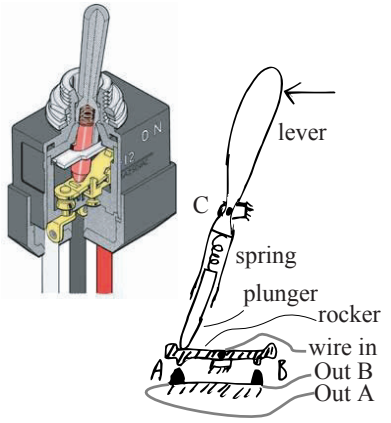



Figure 6.29: Cutaway drawing of a toggle switch and a simplified schematic of the idea. The lever, spring and plunger rotate about C. The spring is prestressed so that the plunger presses hard against the rocker. The plunger slides on the rocker (input line) which is pressed against contact at outputs A or B. The relevant idea here is that the force of the plunger on the rocker can be much bigger than the input force on the lever (cutaway from NKK Switches online catalogue).

Filename:tfigure-NKKtoggleswitch

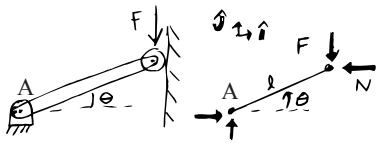



Figure 6.30: A simple toggle mechanism. Moment balance about point A gives that $N = F/\tan\theta$ so if $\theta$ is small $N$ is large.

Filename:tfigure-simpletoggle

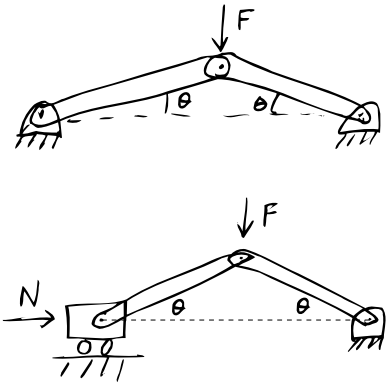



Figure 6.31: Two bars in a toggle mechanism.

Filename:tfigure-toggle2bar

Again using $\theta = 6^\circ$ and $\phi = 17^\circ$ we see that if $F_B = 100\,\text{lbf}$ then $F_A = \tan(-11^\circ)\cdot 100\,\text{lbf} \approx -20\,\text{lbf}$. That is, the 100 pounds doesn't push block A up at all, but even with no gravity you need to pull up with a 20 pound force to get it to move. If we insist that the downwards force $F_A$ is positive or zero, that the pushing force $F_B$ is positive, and that block A is sliding up then there is no solution to the equilibrium equations whenever $\phi > \theta$. (Actually we didn't need to do this second calculation at all. Eqn 6.13 shows the same paradox when $\theta + \phi > 90^\circ$. Trying to squeeze block B to the right for large $\theta$ is exactly like trying to squeeze block A up for small $\theta$.)

This *self locking* situation is intuitive. In fact it's hard to picture the contrary, that pushing a block like B would lift block A. If you view this wedge mechanism as a transmission, it is said to be *non-backdrivable* whenever $\phi > \theta$. Even though pushing down on A can 'drive' block B to the right, but pushing to the left on block B cannot 'back-drive' block B up. Non-backdrivability is a feature or a defect depending on context*.

The borderline case of backdrivability is when $\theta = \phi$ and $F_B = F_A/\tan 2\theta$. Assuming $\theta$ is a fairly small angle we get

$$F_B = \frac{F_A}{\tan 2\theta} \approx \frac{F_A}{2\theta} \approx \frac{1}{2}\frac{F_A}{\tan\theta} \approx \frac{1}{2}\cdot(\text{the value of } F_B \text{ had there been no friction}).$$

Thus the design guideline:

> Non-back-drivable transmissions are generally 50% or less efficient, they transmit 50% or less of the force they would transmit if they were frictionless.

To use a wedge in this backwards way requires very low friction. A rare case where a narrow wedge is back drivable is with a fresh wet watermelon seed squeezed between two pinched fingers.

## A toggle

The classic *toggle* mechanism for amplifying force tends to have a 'snap-through' or bi-stable aspect which is used in the design of some electrical switches. Hence, perhaps, the two dictionary meanings of the word toggle: 1) a force amplifying mechanism, 2) a switch between two states (see Fig. 6.29).

The simplest version of the toggle mechanism is shown in Fig. 6.30. The force amplification is $N/F = 1/\tan\theta$.

Usually the toggle concept is not used with a wall but with a pair of bars (Fig. 6.31) Simple truss analysis shows the bar compressions are $-T = N/2\sin\theta$ and $N = F/2\tan\theta$. The toggle-like force amplification occurs for tension as well as compression. But, because of the oft-desirable snap-through and because the amplification increases as the applied force $F$ moves down, the toggle is most often used in compression.

**A toggle as lever and wedge.** The distinction between toggles and wedges and levers is not precise. On the one hand the toggle is a lever where the lever arm of $F$ is $\ell\cos\theta$ and the lever arm of $N$ is $\ell\sin\theta$. On the other hand the toggle is sort of a rotary wedge with wedge angle $\theta$.

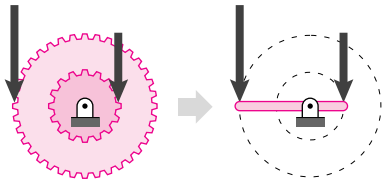


Figure 6.32: One gear may be thought of as a lever.

Filename:tfigure-gearisalever

## General force amplification concepts

If a mechanism generates a large force ratio (output/input) this usually corresponds to a large geometric ratio. For a lever we have the ratio of two lever arms. For a wedge the small wedge angle, and for a toggle also a small angle.

More precisely

> For a frictionless transmission the ratio of the input force to output force is the reciprocal of the ratio of input motion to output motion.

For a high-gain lever the handle moves much further than the load. For a narrow wedge the slip distance is much bigger than the spreading distance. For a toggle the motion of the compressed end is much smaller than that of the applied load. That the force amplification is identical to the motion attenuation follows from energy conservation. The work in to the mechanism is the work out. sectionPulleys and Gears Here we discuss a few more common machine components which are used to transmit and amplify or attenuate a force or moment.

## Gears

One type of transmission is based on gears (Fig. 6.33a). If we think of the input and output as the moments on the two gears, we find from the free body diagram in Fig. 6.33b that

$$\begin{aligned} \text{For gear A, } \left\{\sum \vec{M}_{\text{i/A}} = \vec{0}\right\}\cdot\hat{k} \quad &\Rightarrow \quad -R_{\text{A}}F + M_{\text{A}} = 0 \\ \text{For gear B, } \left\{\sum \vec{M}_{\text{i/B}} = \vec{0}\right\}\cdot\hat{k} \quad &\Rightarrow \quad -R_{\text{B}}F + M_{\text{B}} = 0 \end{aligned}$$

$$\text{eliminating } F \quad \Rightarrow \quad M_B = \frac{R_B}{R_A}M_A \quad \text{or} \quad M_A = \frac{R_A}{R_B}M_B$$

depending on which you want to think of input and which as output. The force amplification or attenuation ratio is just the radius ratio, just like for a lever.

Because the spacing of gear teeth for both of a meshed pair of gears is the same, a gears circumference, and hence its radius is proportional to the number of teeth. And formulas involving radius ratios can just as well be expressed in terms of ratios of numbers of teeth. The tooth ratio is not just used as an approximation to the radius ratio. Averaged over the passage of several teeth, it is exactly the reciprocal ratio of the turning rates of the meshed gears.

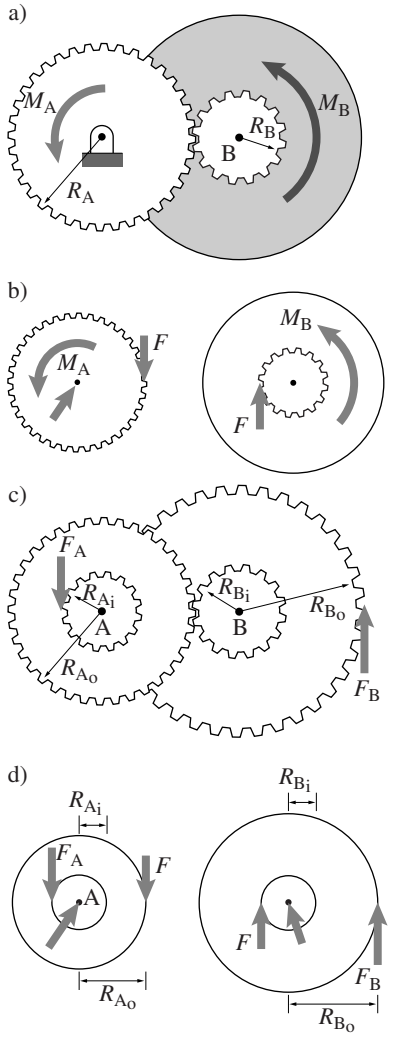



Figure 6.33: a) Two gear pairs pulled out of a transmission with forces on the teeth, b) Free body diagrams, c) The same gear pair, but loaded with tooth-forces from unseen gears, d) the consequent free body diagrams.

Filename:tfigure-gears

Two gears pulled out of a bigger transmission are shown in Fig. 6.33c. Gear A has an inner part with radius $R_{A_i}$ welded to an outer part with radius $R_{A_o}$. Gear B also has an inner part welded to an outer part.

Moment balance about A in the first free body diagram in Fig. 6.33d gives that $R_{A_i} F_A = R_{A_o} F$. You can think of the one gear as a lever (see Fig. 6.32). Moment balance about B in the second free body diagram gives that $R_{B_i} F = R_{B_o} F_B$. Combining we get

$$F_B = \frac{R_{A_i}}{R_{A_o}} \frac{R_{B_i}}{R_{B_o}} F_A \quad \text{or} \quad F_A = \frac{R_{A_o}}{R_{A_i}} \frac{R_{B_o}}{R_{B_i}} F_B$$

depending on which force you want to find in terms of the other. The transmission attenuates the force if you think of $F_A$ as the input and amplifies the force if you think of $F_B$ as the input. If the inner gears have one tenth the radius of the outer gears than the multiplication or attenuation is a factor of 100.

Trains of gears can build up large net gear ratios. The ratio of the fastest to slowest gear in a common clock or mechanical watch is on the order of 10,000.

In some gear trains, like the example above, large torque amplification comes from a large ratio of concentrically welded gears. A large amplification can also come from differences rather than ratios. The designs based primarily on differences rather than ratios are called 'differentials', 'harmonic drives', or 'planetary gears'.

Example: **Planetary gear with a large ratio**

Fig. **??** shows a gear design where the ratio of the input torque on the drive gear, to the output torque, on the spider can be huge. In particular, for the design shown the torque ratio is approximately:

$$\frac{M_{out}}{M_{in}} \approx \frac{2}{R_D/R_R - 1}$$

where $R_D$ is the ratio of the inner drive gear to outer drive gear radius and $R_R$ is the ratio of the inner ring gear to outer ring gear radius. Thus if the inner and outer drive gears have 49 and 50 teeth, respectively, and the inside and outside of the ring gear have 50 and 51 teeth then the torque multiplication is nearly 5000. (See homework 6.36).

## Pulleys

We have already studied a pulley as a single object (see page 176. Now we show, as you probably have learned a few times before in school, how to use pulleys to amplify or attenuate force. We assume pulleys are round, massless, and have frictionless bearings.

The classic problem is shown in Fig. 6.35a where you would like to use a pulley to make the task easier. Figures 6.35b-c show three possible uses of pulleys. If, at a glance, you can't see that these three designs are quite different in their effects you should puzzle them out slowly now.

Because the two tension in the rope that wraps around the pulley is the same on both sides, the central rope has twice the tension. Design (b) gives no mechanical advantage but does allow one to pull down in order to lift the weight. Design (b) halves the effort. Design (c), which might look superfi-

cially similar to (b) doubles the required pulling force, requiring 4 times the force of (b).

By using pulleys in combination one can get various force attenuations and gains. The design in Fig. 6.36 multiplies the force by about 1000.

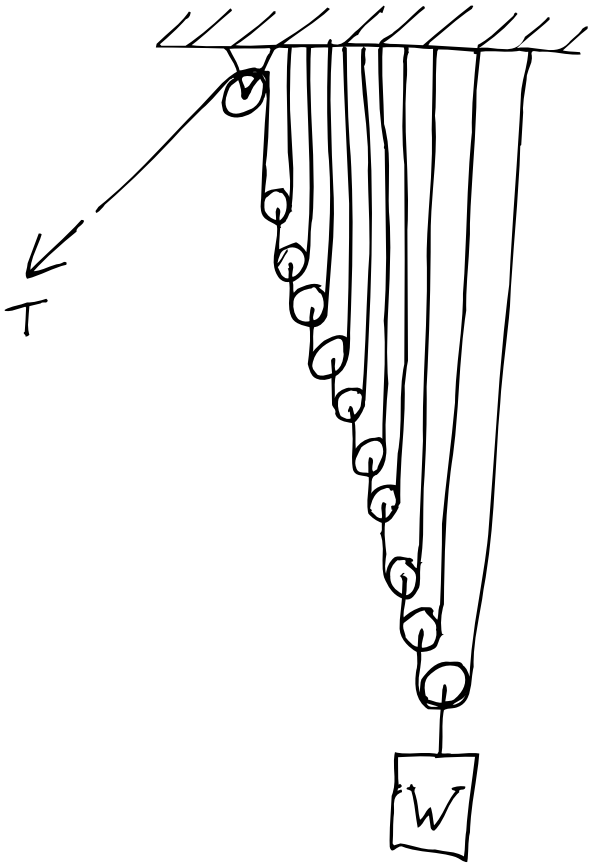



Figure 6.36: A pulley arrangement sometimes attributed to Archimedes. A weight of $W$ can be lifted with a pull of about $T \approx W/1000$.

Filename:tfigure-ArchimedesPulley

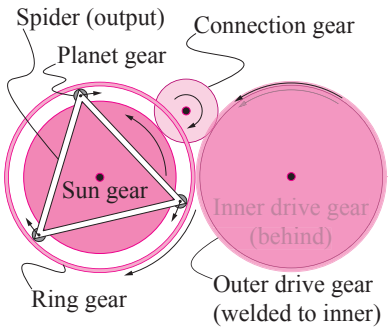



Figure 6.34: A high gain planetary gear system, so named because the 'planet' gears go around the 'sun' gear. Often planetary gears depend on the sun or the spider or the ring not rotating. In this design all three rotate. A CCW torque is applied to the Inner drive gear which is welded to the outer drive gear. The outer drive gear turns the ring gear CW. The inner drive gear turns the the connection gear CW which turns the sun gear CCW. The three planet gears spin between the inside of the ring gear and the outside of the sun gear. If the inside of the ring gear moves a little faster down than the outside of the sun gear moves up, then the spider turns CW with the center of the planet gears.

Filename:tfigure-planetaryhighgain

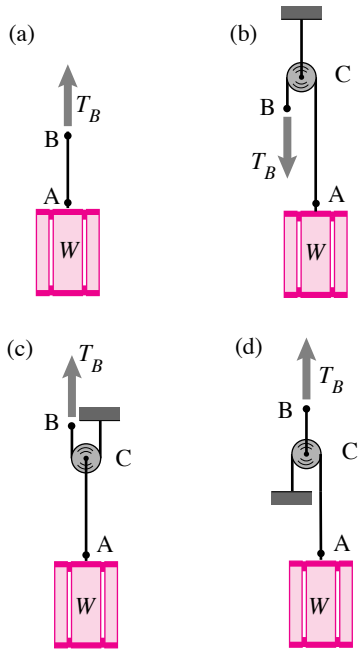



Figure 6.35: **a)** Lifting a weight, **b)** a pulley lets you pull down instead of up, **c)** a pulley halves the needed pull, **d)** a pulley doubles the needed pull.

Filename:tfigure-pulleystatics

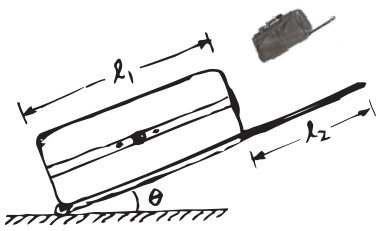



Figure 6.37:
Filename:sfig6-2-suitcase

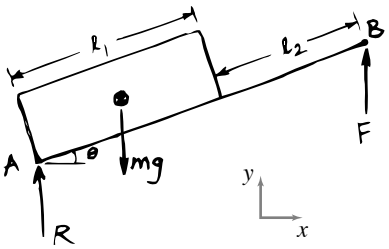



Figure 6.38:
Filename:sfig6-2-suitcase-a

**SAMPLE 6.6** A wheeled suitcase of length 60 cm and 'weighing' 20 kg on the airport check-in counter, has a telescopic handle of length 40 cm. The suitcase is dragged at an angle $\theta = 30°$. Assuming good wheels (negligible friction), find the force applied on the handle to wheel the suitcase steadily. (Take $g \approx 10\,\text{m/s}^2$).

**Solution** The free-body diagram of the suitcase is shown in Fig. 6.38. The reaction force at the wheel is almost vertical because of negligible friction. So, we can also assume the force $F$ applied at the handle to be almost vertical. Now the moment balance equation about point A, $\sum M_A = 0$, gives

$$\begin{aligned} F(\ell_1+\ell_2)\cos\theta - mg(\ell_1/2)\cos\theta &= 0 \\ \Rightarrow \quad F &= \frac{\ell_1}{2(\ell_1+\ell_2)}mg \\ &= \frac{60\,\text{cm}}{200\,\text{cm}}(200\,\text{N}) = 60\,\text{N}. \end{aligned}$$

$$\boxed{F = 60\,\text{N}}$$

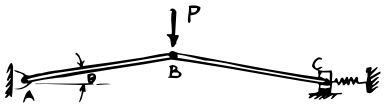



Figure 6.39:
Filename:sfig6-2-togglemech

**SAMPLE 6.7** The figure shows a basic toggle mechanism. If the applied force is $P = 20\,\text{N}$ and the mechanism is in equilibrium at $\theta = 5°$, find the force applied by the spring. If doubling of load P ($P = 40\,\text{N}$) decreases *theta* by 1° ($\theta = 4°$), does the spring force at C double too?.

**Solution** The free-body diagrams of the pin connecting the two rods and the BC are shown in Fig. 6.40. From the static equilibrium of the pin B, we have

$$\sum F_x = 0 \quad \Rightarrow \quad T_2\cos\theta - T_1\cos\theta = 0 \quad \Rightarrow \quad T_1 = T_2$$

$$\sum F_y = 0 \quad \Rightarrow \quad -(T_1+T_2)\sin\theta - P = 0 \quad \Rightarrow \quad T_2 = -\frac{P}{2\sin\theta}.$$

which follows from setting $T_1 + T_2 = 2T_2$ since $T_1 = T_2$. Now, we consider the free-body diagram of rod BC. The force balance equation in the $x$-direction ($\sum F_x = 0$) gives

$$-T_2\cos\theta - F = 0 \quad \Rightarrow \quad F = -T_2\cos\theta = \frac{P\cos\theta}{2\sin\theta}.$$

Since $\theta$ is very small, we have $\sin\theta \approx \theta$ and $\cos\theta \approx 1$. Thus $F = P/2\theta$ where $\theta$ is in radians. Substituting $P = 20\,\text{N}$ and $\theta = 5\pi/180$, we get

$$F = \frac{20\,\text{N}}{2\pi/36} = 115\,\text{N}$$

which is almost 6 times $P$.

If $P$ is doubled, we do not expect $F$ to double if $\theta$ also changes because $F = P/2\theta$. Thus, by repeating the calculation above for $\theta = 4°$ with $P = 40\,\text{N}$, we get $F = 286\,\text{N}$ which is 2.5 times the previous spring force.

$$\boxed{\text{For } P = 20\,\text{N}, \theta = 5°, F = 115\,\text{N}; \text{ and for } P = 40\,\text{N}, \theta = 4°, F = 286\,\text{N}}$$

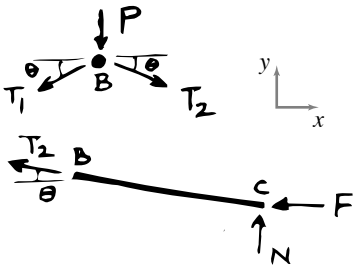



Figure 6.40:
Filename:sfig6-2-togglemech-a

**SAMPLE 6.8 A gear train:** In the compound gear train shown in the figure, the various gear radii are: $R_A = 10\,\text{cm}$, $R_B = 4\,\text{cm}$, $R_C = 8\,\text{cm}$ and $R_D = 5\,\text{cm}$. The input load $F_i = 50\,\text{N}$. Assuming the gears to be in static equilibrium find the machine load $F_o$.

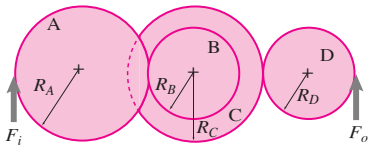



Figure 6.41:
Filename:sfig4-mech-gear

**Solution** You may be tempted to think that a free-body diagram of the entire gear train will do since we only need to find $F_o$. However, it is not so because there are unknown reactions at the axle of each gear and, therefore, there are too many unknowns. On the other hand, we can find the load $F_o$ easily if we go gear by gear from the left to the right.

The free-body diagram of gear A is shown in Fig. 6.42. Let $F_1$ be the force at the contact tooth of gear A that meshes with gear B. From the moment balance about the axle-center O, $\sum \vec{M}_{\text{O}} = \vec{0}$, we have

$$\begin{aligned} \vec{r}_{\text{M}} \times \vec{F}_{\text{i}} + \vec{r}_{\text{N}} \times \vec{F}_1 &= \vec{0} \\ -F_i R_A \hat{k} + F_1 R_A \hat{k} &= \vec{0} \\ \Rightarrow \quad F_1 &= F_i . \end{aligned}$$

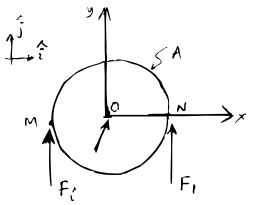



Figure 6.42:
Filename:sfig4-mech-gear-a

Similarly, from the free-body diagram of gear B and C (together) we can write the moment balance equation about the axle-center P as

$$\begin{aligned} F_1 R_B \hat{k} + F_2 R_C \hat{k} &= \vec{0} \\ \Rightarrow \quad F_2 &= \frac{R_B}{R_C} F_1 \\ &= \frac{R_B}{R_C} F_i . \end{aligned}$$

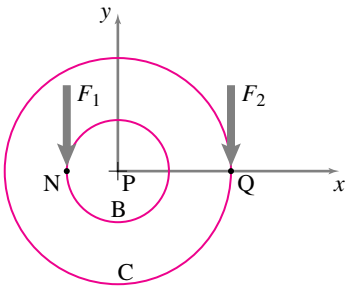



Figure 6.43:
Filename:sfig4-mech-gear-b

Finally, from the free-body diagram of the last gear D and the moment equilibrium about its center R, we get

$$\begin{aligned} -F_2 R_D \hat{k} + F_o R_D \hat{k} &= \vec{0} \\ \Rightarrow \quad F_o &= F_2 \\ &= \frac{R_B}{R_C} F_i \\ &= \frac{4\,\text{cm}}{8\,\text{cm}} \cdot 50\,\text{N} = 25\,\text{N}. \end{aligned}$$

$$\boxed{F_o = 25\,\text{N}}$$

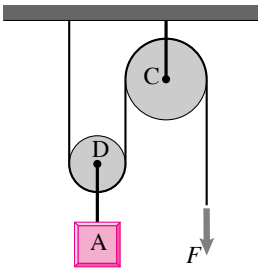



Figure 6.44:

Filename:sfig6-2-pulleys1

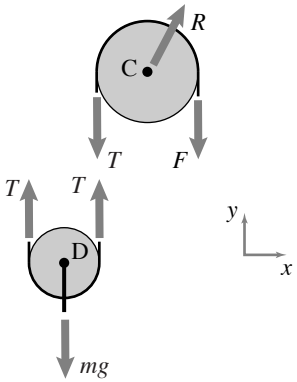



Figure 6.45: Free-body diagram of the two pulleys. We use C and D to designate both the pulleys and their respective centers.

Filename:sfig6-2-pulleys1a

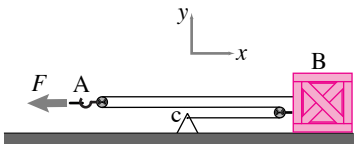



Figure 6.46:

Filename:sfig6-2-pulleys2

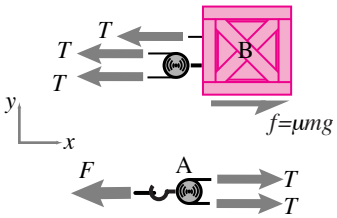



Figure 6.47:

Filename:sfig6-2-pulleys2a

**SAMPLE 6.9** Find the force $F$ to hold the 100 kg box shown in the figure in equilibrium. Assume $g \approx 10\,\text{m/s}^2$.

**Solution** The free-body diagrams of the two pulleys are shown in Fig. 6.45 where the tension in the rope running over the two pulleys has been assumed as $T$. For the lower pulley D, the force balance in the $y$-direction, $\sum F_y = 0$, requires

$$2T - mg = 0 \quad \Rightarrow \quad T = \frac{mg}{2}.$$

The free-body diagram of the upper pulley C contains an unknown reaction force $R$ at the attachment point C. However, if we write moment balance about point C, $\sum M_C = 0$, this unknown force contributes nothing. Let the radius of pulley C be $r$. Thus, the moment balance equation about C gives

$$\begin{aligned} Tr - Fr &= 0 \\ \Rightarrow \quad F &= T = \frac{mg}{2} \approx 500\,\text{N}. \end{aligned}$$

$$\boxed{F \approx 500\,\text{N}}$$

**SAMPLE 6.10** A container box weighing 1 kN is dragged slowly and steadily along the floor with force $F$ as shown in the figure. The coefficient of friction between the box and the floor is 0.6. Find the force required to pull the box and the force amplification obtained by the pulley arrangement.

**Solution** It is clear from the figure that the same rope passes over the two pulleys used in the arrangement to pull the box. Let the tension in the rope be $T$. A partial free-body diagram (that includes forces acting only in the $x$-direction) of the box along with the pulley attached to it is shown in Fig. 6.47. The same figure also shows the free-body diagram of pulley A at the force end. From the force balance equation for the box in the $x$-direction, we get

$$f - 3T = 0 \quad \Rightarrow \quad T = \frac{f}{3} = \frac{\mu mg}{3}.$$

Now, from the force balance of pulley A in the $x$-direction, we get

$$\begin{aligned} 2T - F &= 0 \\ \Rightarrow \quad F &= 2T = \frac{2\mu mg}{3} \\ &= \frac{2 \cdot (0.6) \cdot (1\,\text{kN})}{3} = 400\,\text{N}. \end{aligned}$$

Since the force of friction on the box while sliding is $f = \mu mg = 0.6(1\,\text{kN}) = 600\,\text{N}$ and the force applied at A to overcome this friction is 400 N, the force amplification is 1.5. That is, the pulley arrangement amplifies the input force (400 N) 1.5 times at the output end.

$$\boxed{F = 400\,\text{N},\ \text{Force amplification} = 1.5}$$

**SAMPLE 6.11** A differential hoist is used to lift a crate of mass 500 kg. The hoist pulley uses two discs of radius 30 cm and 25 cm. Find the force $F$ required to lift the crate steadily. Take $g \approx 10\,\text{m/s}^2$.

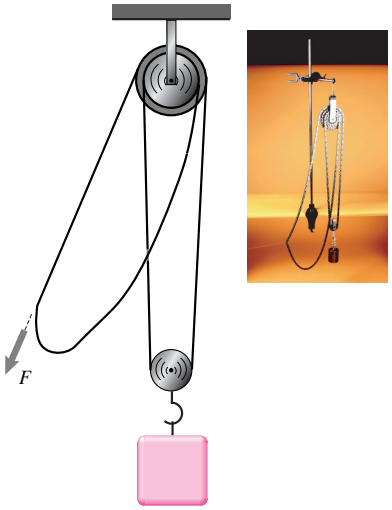



Figure 6.48:
Filename:sfig6-2-pulleys3

**Solution** The free-body diagrams of the upper pulley and the lower pulley are shown in Fig. 6.49. Since the lower pulley is slightly smaller than the upper pulley, the chain passing over the two pulleys is not exactly vertical but makes a small angle with the vertical. Thus the tension forces shown in the free-body diagrams are slightly off from the vertical direction. However, since the angle is very small, we can treat $T$ to be essentially vertical.

For the lower pulley, the force balance in the $y$ direction gives

$$\begin{aligned} 2T - mg &= 0 \\ \Rightarrow \quad T &= \frac{mg}{2}. \end{aligned}$$

Now the moment balance about point C, $\sum M_C = 0$, for the upper pulley gives

$$\begin{aligned} Fr_o + Tr_i - Tr_o &= 0 \\ \Rightarrow \quad F &= \left(\frac{r_o - r_i}{r_o}\right) T \\ &= \left(1 - \frac{r_i}{r_o}\right) \frac{mg}{2} \\ &= \left(1 - \frac{25\,\text{cm}}{30\,\text{cm}}\right) \frac{5000\,\text{N}}{2} \\ &= 417\,\text{N} \end{aligned}$$

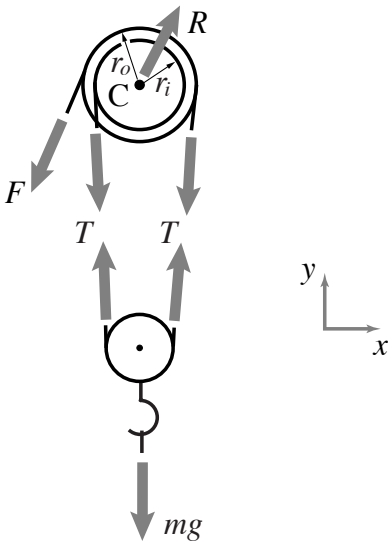



Figure 6.49:
Filename:sfig6-2-pulleys3a

Thus the force amplification in this case is about 12 (5000 N/417 N). From the analysis above, it is also clear that the ratio of the radii of the two disks used in the upper pulley decide this force amplification. One can get a big force amplification, at least theoretically, by making $r_i \approx r_o$. In this problem, for example, if $r_i = 29\,\text{cm}$ rather than the given 25 cm, we get $r_i/r_o \approx 0.97$ giving $F \approx 83\,\text{N}$ which corresponds to a force amplification of 60.

$$\boxed{F = 417\,\text{N}}$$

## 6.3 Mechanisms

We would now like to analyze things built of pieces that are connected in a way that amplifies, attenuates or redirects a force or moment.

For completeness, we present the statics recipe for machines, although it is an exact repeat of the recipe used for frames.

- Draw free body diagrams of
  - the whole machine; and
  - the separate parts of the machine; and
  - collections of parts of the machine if such seems likely to be fruitful;
  - Use the principal of action and reaction in the free body diagrams so that there is only one unknown force at a point where two bodies contact;
- for each free body diagram write equilibrium conditions. These should yield three independent scalar equations for each non-point part (in 2D)
- solve some or all of the equilibrium equations for desired unknowns

Some useful tricks and shortcuts include:

- for any two force bodies assign an equal valued tension to each end (thus eliminating any need or use for equilibrium equations for that object)
- To minimize calculation, look for a subset of the equilibrium equations that
  - contains your unknowns of interest, and
  - has as many uknowns as scalar equations, and
  - contains as few equations as possible.

Example: **Stamp machine**

Pulling on the handle (below) causes the stamp arm to press down with a force $N$ at D. We can find $N$ in terms of $F_h$ by drawing free body diagrams of the handle and stamp arm, writing three equilibrium equations for each piece and then solving these 6 equations for the 6 unknowns ($A_x$, $A_y$, $F_C$, $N$, $B_x$, and $B_y$).

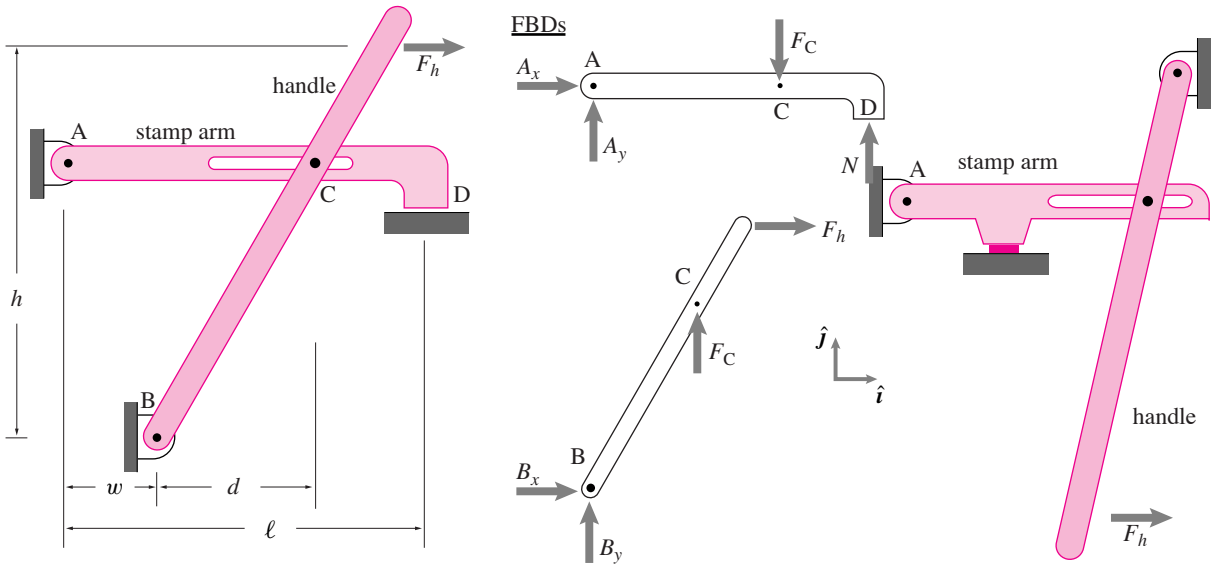



Figure 6.50: An improved stamp machine.

Filename:tfigure-stampmachine2

For this problem, the answer can be found more quickly with a judicious choice of equilibrium equations.

$$\text{For the handle, } \left\{\sum \vec{M}_{/\mathrm{B}} = \vec{0}\right\} \cdot \hat{k} \quad \Rightarrow \quad -hF_h + dF_c = 0$$

$$\text{For the stamp arm, } \left\{\sum \vec{M}_{/\mathrm{A}} = \vec{0}\right\} \cdot \hat{k} \quad \Rightarrow \quad -(d+w)F_c + \ell N = 0$$

$$\text{eliminating } F_c \quad \Rightarrow \quad N = \frac{h(d+w)}{d\ell} F_h .$$

Note that the stamp force $N$ can be made very large by making $d$ small and thus the handle nearly vertical. Often in structural or machine design one or another force gets extremely large or small as the design is changed to put pieces in near alignment.

Example: **Improved stamp machine**

Fig. 6.50 shows a stamp machine with all the same components. The method of analysis is identical. However the design represents an improvement 2 ways:

- The lever in the stamp arm amplifies rather then attenuates the stamp force.
- In the previous design it gets harder and harder to generate a given stamp force as the stamped object compresses. In this design the toggle mechanism associated with the lever arm and sliding pin is in compression. Thus as the stamping progresses and the handle becomes more vertical the stamping force increases for a fixed hand-force.

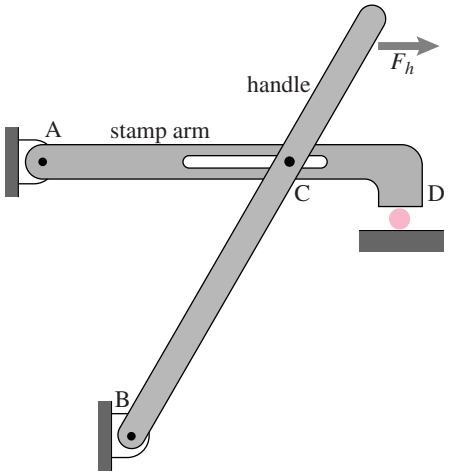
## Non-rigid structures are mechanisms

A non-rigid structure cannot carry all loads and, if not also redundant, has more equilibrium equations than unknown reaction or interaction force components. Such a structure is also called a mechanism. The stamp machine above is a mechanism if there is assumed to be no contact at D. In particular the equilibrium equations cannot be satisfied unless $F_h = 0$. Mechanisms have variable configurations. That is, the constraints still allow relative motion.

An attempt to design a rigid structure that turns out to be a mechanism is a design failure. But for machine design, the mechanism aspect of a structure is essential. Even though mechanisms are called ‘statically indeterminate’

because they cannot carry all possible loads, the desired forces can often be determined using statics. For the stamp machine above the equilibrium equations are made solvable by treating one of the applied forces, say $N$, as an unknown, and the other, $F$ in this case, as a known. This is a common situation in machine design where you want to determine the loads at one part of a mechanism in terms of loads at another part. For the purposes of analysis, a trick is to make a mechanism determinate by putting a pin on rollers connection to ground at the location of any forces with unknown magnitudes but known directions.

Example: **Stamp machine with roller**
Putting a roller at D, the location of the unknown stamp force, turns the stamp machine into a determinate structure.




## Pulley and chain drives

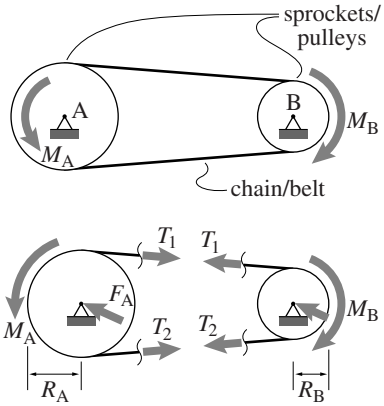



Figure 6.51: a) A chain or pulley drive involving two sprockets or pulleys and one chain or belt, b) free body diagrams of each of the sprockets/pulleys.

Filename:tfigure-chainpulley

Chain and pulley drives are kind of like spread out gears (Fig. 6.51). The rotation of two shafts is coupled not by the contact of gear teeth but by a belt around a pulley or a chain around a sprocket. For simple analysis one draws free body diagrams for each sprocket or pulley with a little bit of chain as in Fig. 6.51b. Note that $T_1 \neq T_2$, unlike the case of an ideal undriven pulley. Applying moment balance we find,

$$\begin{aligned} \text{For gear A, } \left\{\sum \vec{M}_{\text{i/A}} = \vec{0}\right\} \cdot \hat{k} \quad &\Rightarrow \quad -R_{\text{A}}(T_2 - T_1) + M_{\text{A}} = 0 \\ \text{For gear B, } \left\{\sum \vec{M}_{\text{i/B}} = \vec{0}\right\} \cdot \hat{k} \quad &\Rightarrow \quad R_{\text{B}}(T_1 - T_2) - M_{\text{B}} = 0 \end{aligned}$$

$$\text{eliminating } (T_2 - T_1) \quad \Rightarrow \quad M_B = \frac{R_B}{R_A} M_A \quad \text{or} \quad M_A = \frac{R_A}{R_B} M_B$$

exactly as for a pair of gears. Note that we cannot find $T_2$ or $T_1$ but only their difference. Typically in design if, say, $M_A$ is positive, one would try to keep $T_1$ as small as possible without the belt slipping or the chain jumping teeth. If $T_1$ grows then so must $T_2$, to preserve their difference. This increase in tension increases the loads on the bearings as well as the chain or belt itself.

## 4-bar linkages

Four bar linkages often, confusingly, have 3 bars, the fourth piece is the something bigger. A planar mechanism with four pieces connected in a loop by hinges is a four bar linkage. Four bar linkages are remarkably common. After a single body connected at a hinge (like a gear or lever) a four bar linkage is one of the simplest mechanisms that can move in just one way (have just one degree of freedom).

A reasonable model of seated bicycle pedaling uses a 4-bar linkage (Fig. 6.52a). The whole bicycle frame is one bar, the human thigh is the second, the calf is the third, and the bicycle crank is the fourth. The four hinges are the hip joint, the knee joint, the pedal axle, and the bearing at the bicycle crank axle. A more sophisticated model of the system would include the ankle joint and the foot would make up a fifth bar.

A standard door closing mechanism is part of a 4-bar linkage (Fig. 6.52b). The door jamb and door are two bars and the mechanism pieces make up the other two.

A standard folding ladder design is, until locked open, a 4-bar linkage (Fig. 6.52c).

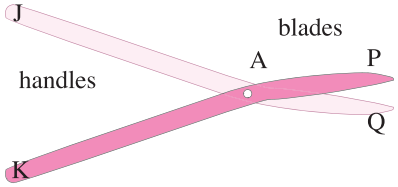
An abstracted 4-bar linkage with two loads is shown in Fig. 6.52d with free body diagrams in Fig. 6.52e. If one of the applied loads is given, then the other applied load along with interaction and reaction forces make up nine unknown components (after using the principle of action and reaction). With three equilibrium equations for each of the three bars, all these unknowns can be found.

## Slider crank

A mechanism closely related to a four bar linkage is a slider crank (Fig. 6.53a). An umbrella is one example (rotated $90^\circ$ in Fig. 6.53b). If the sliding part is replaced by a bar, as in Fig. 6.53c, the point C moves in a circle instead of a straight line. If the height $h$ is very large then the arc traversed by C is nearly a straight line so the motion of the four-bar linkage is almost the same as the slider crank. For this reason, slider cranks are sometimes regarded as a special case of a four-bar linkage in the limit as one of the bars gets infinitely long.

## 6.6 Shears with gears

Many cutters, pliers and shears are essentially two levers pivoting against each other. For example these shears

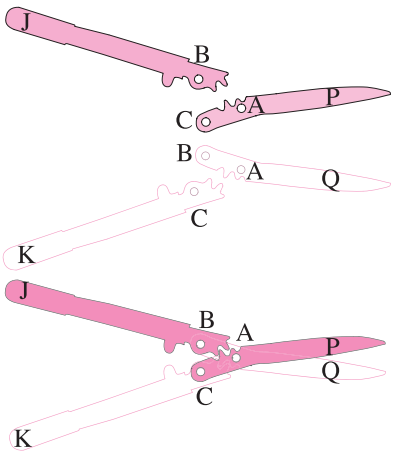



consist of two levers, JAQ and KAP, pivoted at A. The hands squeeze the handles at J and K causing a cutting force on an object between the blades at P and Q. The force at P, say, is $|KA|/|AP|$ times the force at K (from moment balance about A using a free body diagram of KAP). Two possible deficiencies of this bi-lever design are that

- One may want more mechanical advantage but not longer handles, and
- For a given hand strength (available force at J and K) the force at the cutting edge gets less and less as the location of the cut force at P and Q moves farther out on the blade, away from A.

The Fiskars company, known mostly for scissors using the basic design above, has some designs that address these defficiencies. The loppers in problem **??** use a 3 piece mechanism to address these issues. Here, even more elaborately, are Fiskars shears using 4 moving parts.




The two identical blades AP and AQ are hinged at A. The two identical handles JB and and KC are hinged to the blades at B and C. Each handle also has gear teeth at the end that engage gear teeth on the opposite blade. Lets take P and Q to be the point of contact of the object being cut.

Lets try to understand the mechanism without detailed analysis (see homework problem **??**). To start, forget handle KC and assume that blade BAQ is held firmly by something outside. Blade CAP is attached at A about which it is free to spin. Handle JB is attached at B about which it is free to spin. But JB and CAP roll against each other with engaged gear teeth. So if handle JB rotates counter-clockwise about B then CAP rotates clockwise about A.

Although the gear teeth are complex looking, there is always an *effective* contact point G between handle JB and blade CAP on the line segment AB. G is effectively a hinge between JB and CAP. You can think of the handle as a lever with force points at J, B and G. Thus blade CAP is closed by the force on the gear teeth at G. The shorter BG the bigger the forces at B and G.

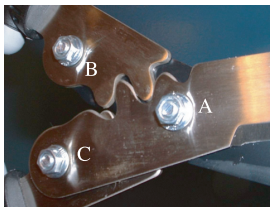
Simultaneously you could think of blade CAP as fixed with blade BAQ and handle KC hinged to it and geared to each other at G' (not shown). Thus blade CAP is also closed by an upwards force at C from handle KC. Similarly blade BAQ is closed by a downwards force at B from handle JB and a downwards force at G' from handle KC.

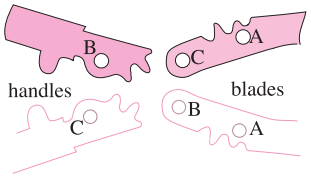
The effective hinges G and G' have locations which change as the blades close. When blades are wide open G and G' are near A. When the blades are closed G and G' have moved to about the midpoint between B and A and C and A, respectively.

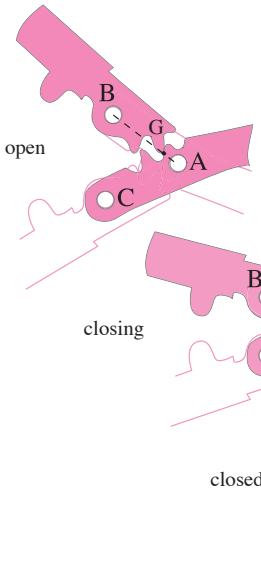
If G was *at* A then this 4-piece design would be equivalent to a standard 2-piece cutter. Because BG is shorter than BA this design gives a bigger downwards force at B.

The shape of the geared curves makes the distance BG decrease, and the distance AG increase, as the blades close. Thus for given forces acting at J and Q, as the blades close the force at B increases, the force at G increases, and the lever-arm AG increases. These three effects partially compensate for the standard scissors problem, the decreasing mechanical advantage from the distance AP increasing as the blades close.

Another way to see the mechanical advantage of this design compared to the 2-piece design is to see that during a cut the handle angle decrease is greater than the blade angle decrease. Following the general rule for mechanisms, a motion attenuation is a force gain.




Mechanism details. The effective hinge point G, between one handle and the opposite blade, moves towards the handle as the handles and blades close.

**SAMPLE 6.12 A slider crank:** A torque $M = 20\,\text{N}\cdot\text{m}$ is applied at the bearing end A of the crank AD of length $\ell = 0.2\,\text{m}$. If the mechanism is in static equilibrium in the configuration shown, find the load $F$ on the piston.

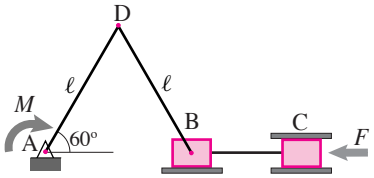



Figure 6.54:
Filename:sfig4-mech-slider

**Solution** The free-body diagram of the whole mechanism is shown in Fig. 6.55. From the moment equilibrium about point A, $\sum \vec{M}_{\text{A}} = \vec{0}$, we get

$$\begin{aligned} \vec{M} + \vec{r}_{\text{B/A}} \times (\vec{B} + \vec{F}) &= \vec{0} \\ -M\hat{k} + 2\ell\cos\theta\hat{\imath} \times (B_y\hat{\jmath} - F\hat{\imath}) &= \vec{0} \\ (-M + 2B_y\ell\cos\theta)\hat{k} &= \vec{0} \\ \Rightarrow \quad B_y &= \frac{M}{2\ell\cos\theta}. \end{aligned}$$

The force equilibrium, $\sum \vec{F} = \vec{0}$, gives

$$\begin{aligned} (A_x - F)\hat{\imath} + (A_y + B_y)\hat{\jmath} = 0 \\ A_x = F \\ A_y = -B_y \end{aligned}$$

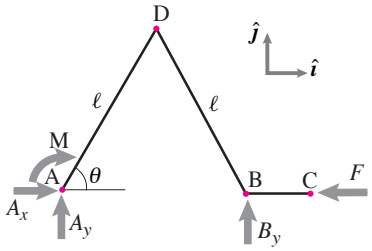



Figure 6.55:
Filename:sfig4-mech-slider-a

Note that we still need to find $F$ or $A_x$. So far, we have had only three equations in four unknowns ($A_x$, $A_y$, $B_y$, $F$). To solve for the unknowns, we need one more equation. We now consider the free-body diagram of the mechanism without the crank, that is, the connecting rod DB and the piston BC together. See Fig. 6.56. Unfortunately, we introduce two more unknowns (the reactions) at D. However, we do not care about them. Therefore, we can write the moment equilibrium equation about point D, $\sum \vec{M}_{\text{D}} = \vec{0}$ and get the required equation without involving $D_x$ and $D_y$.

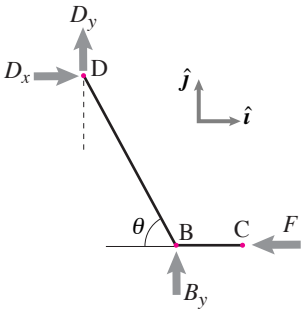



Figure 6.56:
Filename:sfig4-mech-slider-b

$$\begin{aligned} \vec{r}_{\text{B/D}} \times (-F\hat{\imath} + B_y\hat{\jmath}) &= \vec{0} \\ \ell(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) \times (-F\hat{\imath} + B_y\hat{\jmath}) &= \vec{0} \\ B_y\ell\cos\theta\hat{k} - F\ell\sin\theta\hat{k} &= \vec{0}. \end{aligned}$$

Dotting the last equation with $\hat{k}$ we get

$$\begin{aligned} F &= B_y\frac{\cos\theta}{\sin\theta} \\ &= \frac{M}{2\ell\cos\theta}\cdot\frac{\cos\theta}{\sin\theta} \\ &= \frac{M}{2\ell\sin\theta} \\ &= \frac{20\,\text{N}\cdot\text{m}}{2\cdot 0.2\,\text{m}\cdot\sqrt{3}/2} \\ &= 57.74\,\text{N}. \end{aligned}$$

$$\boxed{F = 57.74\,\text{N}}$$

Note that the force equilibrium carried out above is not really useful since we are not interested in finding the reactions at A. We did it above to show that just one free-body diagram of the whole mechanism was not sufficient to find $F$. On the other hand, writing moment equations about A for the whole mechanism and about D for the connecting rod plus the piston is enough to determine $F$.

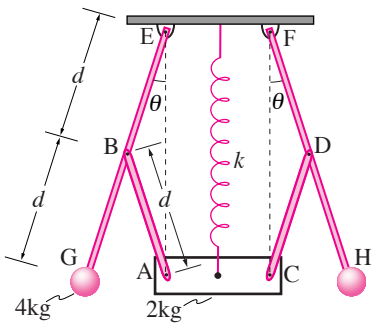



Figure 6.57:
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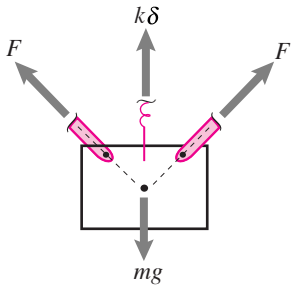



Figure 6.58:
Filename:sfig4-mech-gov-a

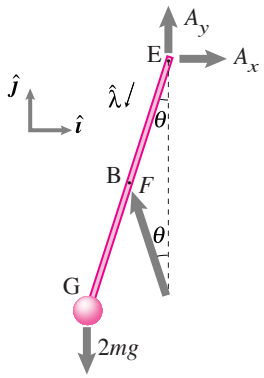



Figure 6.59:
Filename:sfig4-mech-gov-b

**SAMPLE 6.13 A flyball governor:** A flyball governor is shown in the figure with all relevant masses and dimensions. The relaxed length of the spring is 0.15 m and its stiffness is 500 N/m.

1. Find the static equilibrium position of the center collar.
2. Find the force in the strut AB or CD.
3. How does the spring force required to hold the collar depend on $\theta$?

**Solution** Let $\ell_0 (= 0.15\,\text{m})$ denote the relaxed length of the spring and let $\ell$ be the stretched length in the static equilibrium configuration of the flyball, *i.e.*, the collar is at a distance $\ell$ from the fixed support EF. Then the net stretch in the spring is $\delta \equiv \Delta\ell = \ell - \ell_0$. We need to determine $\ell$, the spring force $k\delta$, and its dependence on the angle $\theta$ of the ball-arm.

The free-body diagram of the collar is shown in Fig. 6.58. Note that the struts AB and CD are two-force bodies (forces act only at the two end points on each strut). Therefore, the force at each end must act along the strut. From geometry (AB = BE = $d$), then, the strut force $F$ on the collar must act at angle $\theta$ from the vertical. Now, the force balance in the vertical direction, *i.e.*, $[\sum \vec{F} = \vec{0}] \cdot \hat{\jmath}$, gives

$$-2F\cos\theta + k\delta = mg. \tag{6.15}$$

Thus to find $\delta$ we need to find $F$ and $\theta$. Now we draw the free-body diagram of arm EBG as shown in Fig. 6.59. From the moment balance about point E, we get

$$\begin{aligned}
\vec{r}_{G/E} \times (-2mg\hat{\jmath}) + \vec{r}_{B/E} \times \vec{F} &= \vec{0} \\
2d\hat{\boldsymbol{\lambda}} \times (-2mg\hat{\jmath}) + d\hat{\boldsymbol{\lambda}} \times F(-sin\theta\hat{\imath} + \cos\theta\hat{\jmath}) &= \vec{0} \\
-4mgd(\underbrace{\hat{\boldsymbol{\lambda}} \times \hat{\jmath}}_{-\sin\theta\hat{\boldsymbol{k}}}) + Fd[-\sin\theta(\underbrace{\hat{\boldsymbol{\lambda}} \times \hat{\imath}}_{\cos\theta\hat{\boldsymbol{k}}}) + \cos\theta(\underbrace{\hat{\boldsymbol{\lambda}} \times \hat{\jmath}}_{-\sin\theta\hat{\boldsymbol{k}}})] &= \vec{0} \\
4mgd\sin\theta\hat{\boldsymbol{k}} + Fd(-\sin\theta\cos\theta\hat{\boldsymbol{k}} - \cos\theta\sin\theta\hat{\boldsymbol{k}}) &= \vec{0} \\
(4mgd\sin\theta - 2Fd\sin\theta\cos\theta)\hat{\boldsymbol{k}} &= \vec{0}.
\end{aligned}$$

Dotting this equation with $\hat{\boldsymbol{k}}$ and assuming that $\theta \neq 0$, we get

$$2F\cos\theta = 4mg. \tag{6.16}$$

Substituting eqn. (6.16) in eqn. (6.15) we get

$$\begin{aligned}
k\delta &= mg + 2F\cos\theta = mg + 4mg = 5mg \\
\Rightarrow \quad \delta &= \frac{5mg}{k} = \frac{5 \cdot 2\,\text{kg} \cdot 9.81\,\text{m/s}^2}{500\,\text{N/m}} = 0.196\,\text{m}.
\end{aligned}$$

1. The equilibrium configuration is specified by the stretched length $\ell$ of the spring (which specifies $\theta$). Thus,

   $$\ell = \ell_0 + \delta = 0.15\,\text{m} + 0.196\,\text{m} = 0.346\,\text{m}.$$

   Now, from $\ell = 2d\cos\theta$, we find that $\theta = 30.12°$.

2. The force in strut AB (or CD) is

   $$F = 2mg/\cos\theta = 45.36\,\text{N}.$$

3. The force in the spring $k\delta = 5mg$ as shown above and thus, it does not depend on $\theta$! In fact, the angle $\theta$ is determined by the relaxed length of the spring.

| (a) $\ell = 0.346\,\text{m}$, (b) $F = 45.36\,\text{N}$, (c) $k\delta \neq f(\theta)$ |
|---|

**SAMPLE 6.14 : A motor housing support:** A slotted arm mechanism is used to support a motor housing that has a belt drive as shown in the figure. The motor housing is bolted to the arm at B and the arm is bolted to a solid support at A. The two bolts are tightened enough to be modeled as welded joints (*i.e.*, they can also take some torque). Find the support reactions at A.

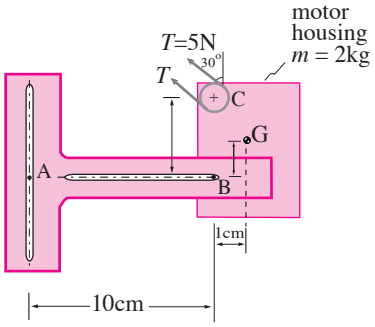



Figure 6.60:
Filename:sfig4-mech-motor

**Solution** Although the mechanism looks complicated, the problem is straightforward. We cut the bolt at A and draw the free-body diagram of the motor housing plus the slotted arm. Since the bolt, modeled as a welded joint, can take some torque, the unknowns at A are $\vec{A}(= A_x\hat{\imath} + A_y\hat{\jmath})$ and $\vec{M}_{\rm A}$. The free-body diagram is shown in Fig. 6.61. Note that we have replaced the tension at the two belt ends by a single equivalent tension $2T$ acting at the center of the axle. Now taking moments about point A, we get

$$\vec{M}_{\rm A} + \vec{r}_{\rm C/A} \times 2\vec{T} + \vec{r}_{\rm G/A} \times m\vec{g} = \vec{0}$$

where

$$\begin{aligned} \vec{r}_{\rm C/A} \times 2\vec{T} &= (\ell\hat{\imath} + h\hat{\jmath}) \times 2T(-\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) \\ &= 2T(\ell\sin\theta + h\cos\theta)\hat{\boldsymbol{k}} \\ \vec{r}_{\rm G/A} \times m\vec{g} &= [(\ell + d)\hat{\imath} + (\text{anything})\hat{\jmath}] \times (-mg\hat{\jmath}) \\ &= -mg(\ell + d)\hat{\boldsymbol{k}}. \end{aligned}$$

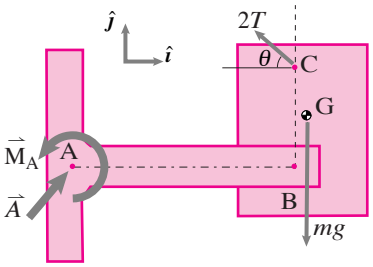



Figure 6.61:
Filename:sfig4-mech-motor-a

Therefore,

$$\begin{aligned} \vec{M}_{\rm A} &= -\vec{r}_{\rm C/A} \times 2\vec{T} - \vec{r}_{\rm G/A} \times m\vec{g} \\ &= -2T(\ell\sin\theta + h\cos\theta)\hat{\boldsymbol{k}} + mg(\ell + d)\hat{\boldsymbol{k}} \\ &= -2(5\,\text{N})(0.1\,\text{m}\cdot\sin 60^\circ + 0.04\,\text{m}\cdot\cos 60^\circ)\hat{\boldsymbol{k}} \\ &\quad + 2\,\text{kg}\cdot 9.81\,\text{m/s}^2\cdot(0.1 + 0.01)\,\text{m}\hat{\boldsymbol{k}} \\ &= 1.092\,\text{N·m}\hat{\boldsymbol{k}}. \end{aligned}$$

The reaction force $\vec{A}$ can be determined from the force balance, $\sum \vec{F} = \vec{0}$ as follows.

$$\begin{aligned} \vec{A} + 2\vec{T} + m\vec{g} &= \vec{0} \\ \Rightarrow \quad \vec{A} &= -2\vec{T} - m\vec{g} \\ &= -10\,\text{N}(-\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath}) - (-19.62\,\text{N}\hat{\jmath}) \\ &= 5\,\text{N}\hat{\imath} + 10.96\,\text{N}\hat{\jmath}. \end{aligned}$$

$$\boxed{\vec{M}_{\rm A} = 1.092\,\text{N·m}\hat{\boldsymbol{k}} \text{ and } \vec{A} = 5\,\text{N}\hat{\imath} + 10.96\,\text{N}\hat{\jmath}}$$

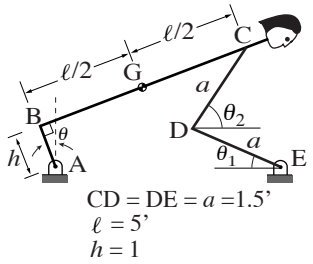



Figure 6.62:
Filename:sfig4-mech-pushup

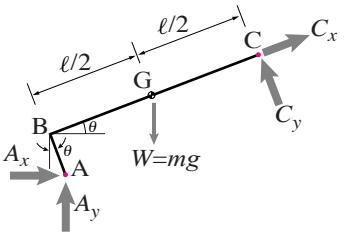



Figure 6.63:
Filename:sfig4-mech-pushup-a

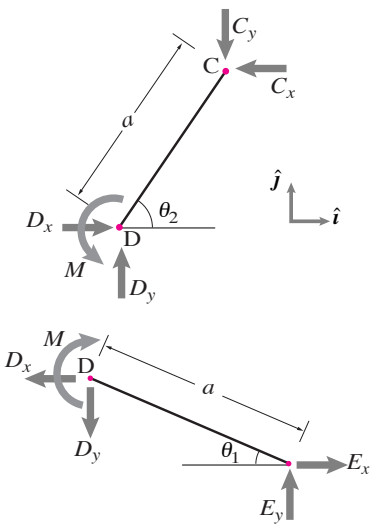



Figure 6.64:
Filename:sfig4-mech-pushup-b

**SAMPLE 6.15 Push-up mechanics:** During push-ups, the body including the legs, usually moves as a single rigid unit; the ankle is almost locked, and the push-up is powered by the shoulder and the elbow muscles. A simple model of the body during push-ups is a four-bar linkage ABCDE shown in the figure. In this model, each link is a rigid rod, joint B is rigid (thus ABC can be taken as a single rigid rod), joints C, D, and E are hinges, but there is a motor at D that can supply torque. The weight of the person, $W = 150\,\text{lbf}$, acts through G. Find the torque at D for $\theta_1 = 30°$ and $\theta_2 = 45°$.

**Solution** The free-body diagram of part ABC of the mechanism is shown in Fig. 6.63. Writing moment balance equation about point A, $\sum \vec{M}_A = \vec{0}$, we get

$$\vec{r}_C \times \vec{C} + \vec{r}_G \times \vec{W} = \vec{0}.$$

Let $\vec{r}_C = r_{C_x}\hat{\imath} + r_{C_y}\hat{\jmath}$ and $\vec{r}_G = r_{G_x}\hat{\imath} + r_{G_y}\hat{\jmath}$ for now (we can figure it out later). Then, the moment equation becomes

$$\begin{aligned}(r_{C_x}\hat{\imath} + r_{C_y}\hat{\jmath}) \times (C_x\hat{\imath} + C_y\hat{\jmath}) + (r_{G_x}\hat{\imath} + r_{G_y}\hat{\jmath}) \times (-W\hat{\jmath}) &= \vec{0} \\ [(C_y r_{C_x} - C_x r_{C_y})\hat{k} - W r_{G_x}\hat{k} &= \vec{0}] \\ [\,]\cdot\hat{k} \quad \Rightarrow \qquad C_y r_{C_x} - C_x r_{C_y} &= W r_{G_x}. \end{aligned} \tag{6.17}$$

We now draw free-body diagrams of the links CD and DE separately (Fig. 6.64) and write the moment and force balance equations for them.

For link CD, the force equilibrium $\sum \vec{F} = \vec{0}$ gives

$$(-C_x + D_x)\hat{\imath} + (D_y - C_y)\hat{\jmath} = \vec{0}.$$

Dotting with $\hat{\imath}$ and $\hat{\jmath}$ gives

$$\begin{aligned} D_x &= C_x \\ D_y &= C_y \end{aligned} \tag{6.18}$$

and the moment equilibrium about point D, gives

$$\begin{aligned} M\hat{k} - a(\cos\theta_2\hat{\imath} + \sin\theta_2\hat{\jmath}) \times (-C_x\hat{\imath} - C_y\hat{\jmath}) &= \vec{0} \\ M\hat{k} + (C_y a\cos\theta_2 - C_x a\sin\theta_2)\hat{k} &= \vec{0}. \end{aligned} \tag{6.19}$$

Similarly, the force equilibrium for link DE requires that

$$\begin{aligned} E_x &= D_x \\ E_y &= D_y \end{aligned} \tag{6.20}$$

and the moment equilibrium of link DE about point E gives

$$-M + D_x a\sin\theta_1 + D_y a\cos\theta_1 = 0. \tag{6.21}$$

Now, from eqns. (6.18) and (6.21)

$$-M + C_x a\sin\theta_1 + C_y a\cos\theta_1 = 0. \tag{6.22}$$

Adding eqns. (6.19) and (6.22) and solving for $C_x$ we get

$$C_x = \frac{\cos\theta_1 + \cos\theta_1}{\sin\theta_2 - \sin\theta_1} C_y.$$

For simplicity, let

$$f(\theta_1, \theta_2) = \frac{\cos\theta_1 + \cos\theta_1}{\sin\theta_2 - \sin\theta_1}$$

so that

$$C_x = f(\theta_1, \theta_2) C_y. \tag{6.23}$$

Now substituting eqn. (6.23) in (6.17) we get

$$C_y = \frac{r_{G_x}}{r_{C_x} - r_{C_y} f} W.$$

Now substituting $C_y$ and $C_x$ into eqn. (6.22) we get

$$M = \frac{r_{G_x} a(\cos\theta_1 + f \sin\theta_1)}{r_{C_x} - r_{C_y} f} W$$

where

$$\begin{aligned} r_{G_x} &= (\ell/2)\cos\theta - h\sin\theta \\ r_{C_x} &= \ell\cos\theta - h\sin\theta \\ r_{C_y} &= \ell\sin\theta + h\cos\theta. \end{aligned}$$

Now plugging all the given values: $W = 160\,\text{lbf}$, $\theta_1 = 30°$, $\theta_2 = 45°$, $\ell = 5\,\text{ft}$, $h = 1\,\text{ft}$, $a = 1.5\,\text{ft}$, and, from simple geometry, $\theta = 9.49°$,

$$\begin{aligned} f &= 7.60 \\ r_{C_x} &= 4.77\,\text{ft}, \quad r_{C_y} = 1.81\,\text{ft}, \quad r_{G_x} = 2.30\,\text{ft} \\ \Rightarrow \quad M &= -269.12\,\text{lb·ft}. \end{aligned}$$

$$\boxed{M = -269.12\,\text{lb·ft}}$$

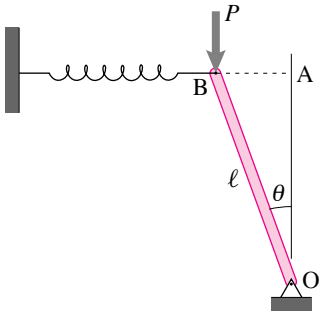



Figure 6.65:
Filename:sfig4-mech-buckling

**SAMPLE 6.16 A spring and rod buckling model:** A simple model of sideways buckling of a flexible (elastic) rod can be constructed with a spring and a *rigid* rod as shown in the figure. Assume the rod to be in static equilibrium at some angle $\theta$ from the vertical. Find the angle $\theta$ for a given vertical load $P$, spring stiffness $k$, and bar length $\ell$. Assume that the spring is relaxed when the rod is vertical.

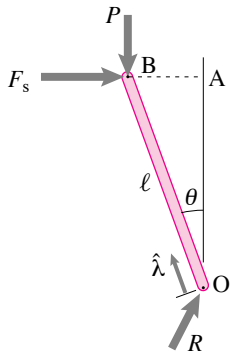



Figure 6.66:
Filename:sfig4-mech-buckling-a

**Solution** When the rod is displaced from its vertical position, the spring gets compressed or stretched depending on which side the rod tilts. The spring then exerts a force on the rod in the opposite direction of the tilt. The free-body diagram of the rod with a counterclockwise tilt $\theta$ is shown in Fig. 6.66. From the moment balance $\sum \vec{M}_{\rm O} = \vec{0}$ (about the bottom support point O of the rod), we have

$$\vec{r}_{\rm B} \times \vec{P} + \vec{r}_{\rm B} \times \vec{F}_{\rm s} = \vec{0}.$$

Noting that

$$\begin{aligned} \vec{r}_{\rm B} &= \ell\hat{\boldsymbol{\lambda}}, \\ \vec{P} &= -P\hat{\boldsymbol{\jmath}}, \\ \text{and } \vec{F}_{\rm s} &= k(\vec{r}_{\rm A} - \vec{r}_{\rm B}) \\ &= k(\ell\hat{\boldsymbol{\jmath}} - \ell\hat{\boldsymbol{\lambda}}), \end{aligned}$$

we get

$$\begin{aligned} \ell\hat{\boldsymbol{\lambda}} \times (P\hat{\boldsymbol{\jmath}}) + \ell\hat{\boldsymbol{\lambda}} \times k\ell(\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{\lambda}}) &= \vec{0} \\ -P\ell(\hat{\boldsymbol{\lambda}} \times \hat{\boldsymbol{\jmath}}) + k\ell^2(\hat{\boldsymbol{\lambda}} \times \hat{\boldsymbol{\jmath}}) &= \vec{0}. \end{aligned}$$

Dotting this equation with $(\hat{\boldsymbol{\lambda}} \times \hat{\boldsymbol{\jmath}})$ we get

$$\begin{aligned} -P\ell + k\ell^2 &= 0 \\ \Rightarrow \quad P &= k\ell. \end{aligned}$$

Thus the equilibrium only requires that $P$ be equal to $k\ell$ and it is independent of $\theta$! That is, the system will be in static equilibrium at any $\theta$ as long as $P = k\ell$.

If $P = k\ell$, any $\theta$ is an equilibrium position.

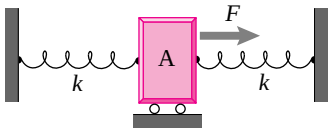
# Problems for Chapter 6

Machines

## 6.1 Springs

### Preparatory Problems

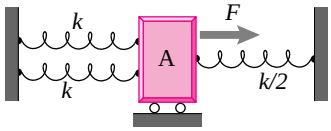
**6.1** Find the force $F$ required to push the massless block by 1 cm to the right if $k = 500\,\text{N/m}$.




**problem 6.1:**

Filename:pfig-springs-2p

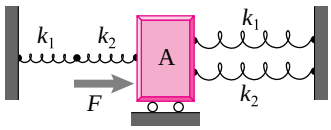
**6.2** A force $F = 20\,\text{N}$ is applied on the massless block shown in the figure. Find the displacement of the block for equilibrium if $k = 100\,\text{N/cm}$.




**problem 6.2:**

Filename:pfig-springs-3p

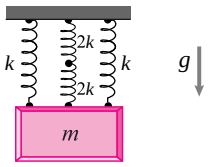
**6.3** A network of relaxed springs holds a massless block as shown in the figure where $k_1 = 100\,\text{N/cm}$ and $k_2 = 400\,\text{N/cm}$. If the block is pushed to the right by 2 cm, find the force $F$ to hold the block in equilibrium.




**problem 6.3:**

Filename:pfig-springs-3p1s

**6.4** A block of mass $m = 300\,\text{kg}$ hangs from the ceiling with the help of a network of springs in series and parallel as shown. Taking $k = 20\,\text{kN/m}$ and $g = 10\,\text{m/s}^2$, find the stretch in the two side (the left and right) springs.



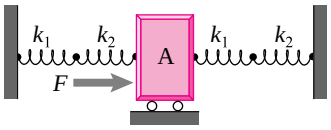

**problem 6.4:**

Filename:pfig-springs-3p1sv

**6.5** For the arrangement of springs shown in the figure, $k_1 = 50\,\text{N/cm}$ and $k_2 = 100\,\text{N/cm}$. Find

a) the equivalent spring stiffness of the arrangement,

b) the displacement of the block if a force $F = 50\,\text{N}$ acts on the block.



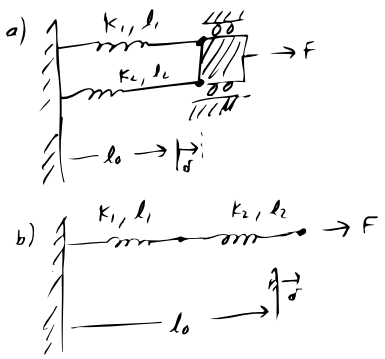

**problem 6.5:**

Filename:pfig-springs-2p2s

**6.6** Find $F$ in terms of some or all of $k_1$,$\ell_1$, $k_2$, $\ell_2$, $\ell_0$ and $\delta$. Note that $F$ is generally not zero even if $\delta$ is zero.

a) Springs in parallel.

b) Springs in series.

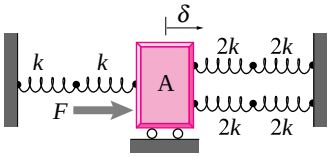



**problem 6.6:**

Filename:pfigure-spring2

## More-Involved Problems

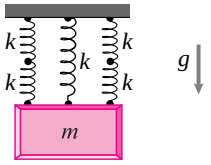
**6.7** A massless block is held in position by a network of springs shown in the figure. If the blcok is displaced to the right by 1 cm from the relaxed position of the springs, a force of $F = 50\,\text{N}$ is required to keep the block in equilibrium. Find the value of $k$.




**problem 6.7:**

Filename:pfig-springs-3p3s

**6.8** A box weighing 1000 N is hung from the ceiling using a network of springs, each with stiffness $k = 500\,\text{N/cm}$. Find the stretch in each spring.

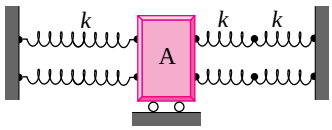



**problem 6.8:**

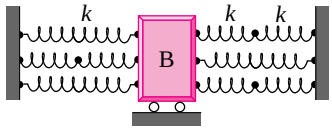
Filename:pfig-springs-3p2sv

**6.9** For the netwrok of springs shown below, find the stiffness and strength of each netwrok if the stiffness and strength of individual springs are $k = 10\,\text{kN/m}$ and $T_0 = 2\,\text{kN}$, respectively.

(a)



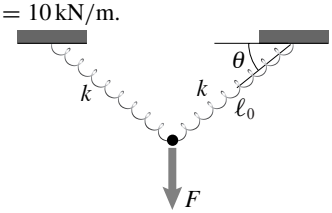

(b)




**problem 6.9:**

Filename:pfig-springs-strstf

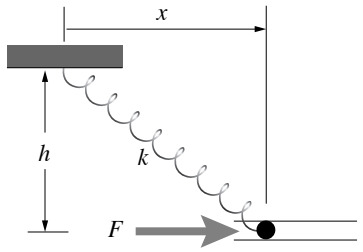
**6.10** Find the stretch in each spring to hold the pin in equilibrium for $F = 10\,\text{kN}$ if the relaxed length (in the horizontal position) of each spring is $\ell_0 = 15\,\text{cm}$ and $k = 10\,\text{kN/m}$.




**problem 6.10:**

Filename:pfig-springs-angle2p

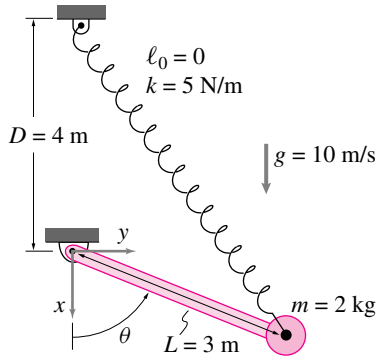
**6.11** A pin is held in a horizontal track with a zero-length spring ($\ell_0 = 0$) of stiffness 50 kN/m. Find the horizontal position $x$ of the pin if it is in equilibrium with an applied force $F = 1000\,\text{N}$.




**problem 6.11:**

Filename:pfig-springs-angle1h

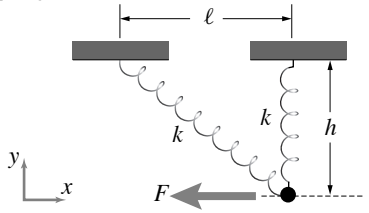
**6.12** A zero length spring (relaxed length $\ell_0 = 0$) with stiffness $k = 5\,\text{N/m}$ supports the pendulum shown. Assume $g = 10\,\text{N/m}$. Find $\theta$ for static equilibrium.




**problem 6.12:**

Filename:pfigure2-blue-80-2-a

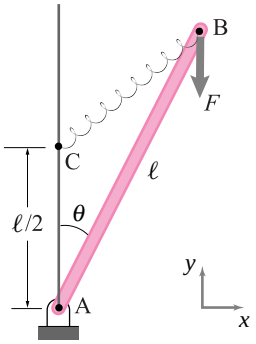
**6.13** In the figure shown, the two springs with $k_1 = 50\,\text{N/cm}$ and $k_2 = 100\,\text{N/cm}$ are in relaxed position when $h = 30\,\text{cm}$ and $\ell = 40\,\text{cm}$ (and, of course, $F = 0$). Find the position of the pin on the horizontal track and change in length of each spring if $F = 200\,\text{N}$.




**problem 6.13:**

Filename:pfig-springs-angle2ph

**6.14** In the mechanism shown, the relaxed length of the spring is $\ell/2$ and the length of the bar AB is $\ell = 2\,\text{m}$. For $F = 500\,\text{N}$, find the equilibrium angle $\theta$ of the rod and the stretch in the spring.



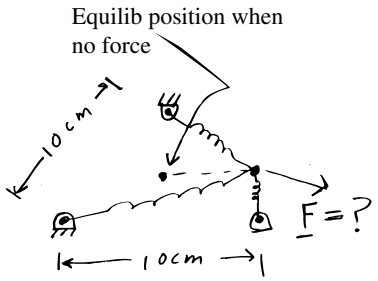

**problem 6.14:**

Filename:pfig-springs-inclinedbar

**6.15** The ends of three identical springs are rooted at the corners of a 10 cm equilateral triangle with base that is in the $\hat{\imath}$ direction. Find the force $\vec{F}$ needed to hold the ends of the springs 5 cm to the right of the triangle center if

a) $\ell_0 = 0, k = 10\,\text{N/cm}$?

b) $\ell_0 = 10/\sqrt{3}\,\text{cm}, k = 10\,\text{N/cm}$?



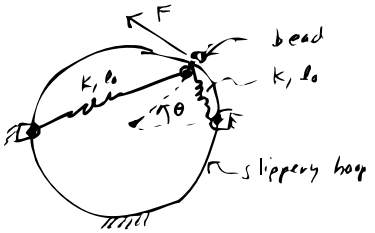

**problem 6.15:**

Filename:pfigure-threesprings

**6.16**

a) in terms of some or all of $k$, $\ell_0$,$r$, and $\theta$ find $F$. The hoop is rigid, round and frictionless and the force is tangent to the hoop.

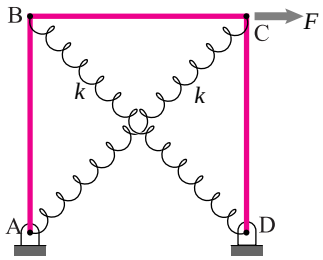
b) How does the answer above simplify in the special case that $\ell_0 = 0$? [You can do this by simplifying the expression above, or by doing the problem from scratch assuming $\ell_0 = 0$. In the latter case, an answer can be generated quickly if vector methods are used.]




**problem 6.16:**

Filename:pfigure-spring4

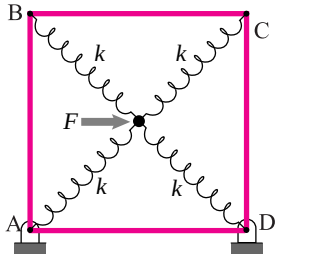
**6.17** The square box mechanism shown consists of three identical bars and two identical diagonal springs in their relaxed configuration. Each bar is 0.4 m long. A horizontal force $F = 100\,\text{N}$ acts at C. Find the change in length of each spring if $k = 10\,\text{kN/m}$.




**problem 6.17:**

Filename:pfig-springs-kinematics1

**6.18** In the mechanism shown, the pin is held in the center of the square frame of side 1 m with relaxed springs of stiffness $k = 5\,\text{kN/m}$ in the absence of any force. Find the change in length of each spring when an applied horizontal force $F = 50\,\text{N}$ keeps the pin in equilibrium at a position slightly to the right of the center.




**problem 6.18:**

Filename:pfig-springs-kinematics2

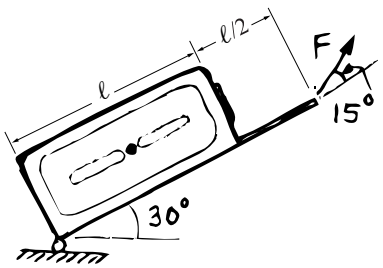
## 6.2 Levers, Wedges, Toggles, Gears and Pulleys

**6.19** A suitcase of length $\ell = 0.5\,\text{m}$ is pulled along steadily with a force $F = 100\,\text{N}$ as shown in the figure.

a) Find the weight $W$ of the suitcase.

b) Find the ground force on the wheel (both magnitude and direction).

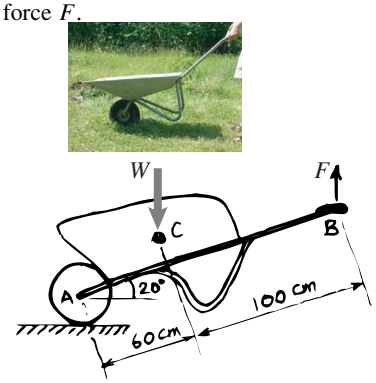
c) What is force amplification if you consider $F$ as the input and $W$ as the output.




**problem 6.19:**

Filename:pfig6-lever-suitcase

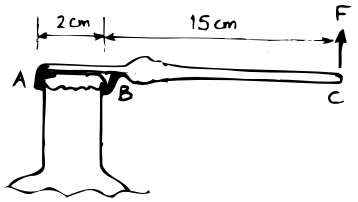
**6.20** A wheelbarrow containing 100 kg of this-n-that is wheeled steadily with a force $F$ as shown in the figure. For the given geometry and $g \approx 10\,\mathrm{m/s^2}$, find the required force $F$.




**problem 6.20:**

Filename:pfig6-lever-wheelbarrow

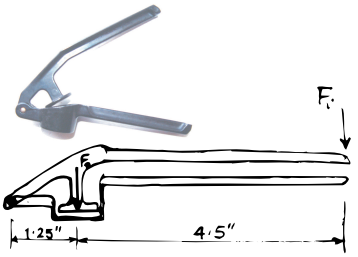
**6.21** A bottle-opener ABC contains a cut-out AB of approximate diameter 2 cm that clamps on the bottle cap. The arm BC is approximately 15 cm long. If the cap is opened by applying a vertical force $F = 10\,\mathrm{N}$ at C, find the force on the cap at B.




**problem 6.21:**

Filename:pfig6-lever-bottleopener

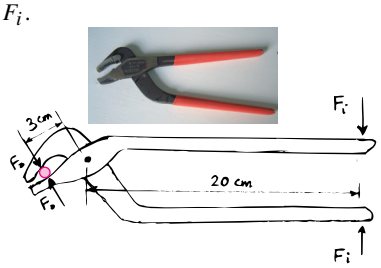
**6.22** A cut-out view of a garlic press is shown in the figure. For an input force $F_i = 10\,\mathrm{lbf}$, find the output force $F_o$ at the site of the press. What is the force amplification?




**problem 6.22:**

Filename:pfig6-lever-garlicpress

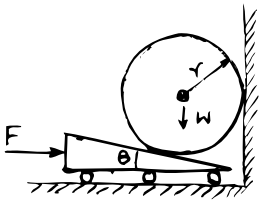
**6.23** A simple wrench is shown in the figure along with the relevant dimensions. If the torque required on the approximately circular bolt of diameter 1 cm is $2\,\mathrm{N{\cdot}m}$ and the coefficient of friction between the bolt and wrench is $\mu = 0.2$, find the input force $F_i$.




**problem 6.23:**

Filename:pfig6-lever-wrench

**6.24** Assuming all frictionles contacts, find the force $F$ on the wedge required to lift the the sphere weighing 500 N if the wedge angle $\theta = 10°$.



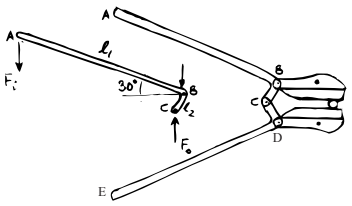

**problem 6.24:**

Filename:pfig6-wedge

**6.25** A cutter, shown in the figure, uses a toggle mechanism BCD to get a big force amplification at the cutting edge. A partial free-body diagram of one of the arms of the cutter is shown in the figure. Assuming an input force of $F_i = 20\,\mathrm{N}$ at A, find the intermediate output force $F_o$ at C when

a) $\theta = 30°$,

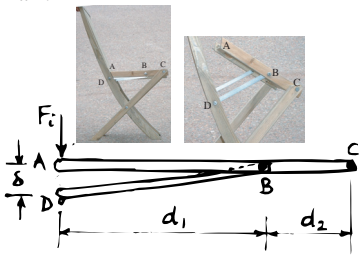
b) $\theta = 10°$.




**problem 6.25:**

Filename:pfig6-toggle-cutter

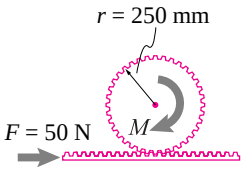
**6.26** A toggle-like mechanism is used in a folding chair shown in the pictures here. The metallic link DB gets almost parallel to the seating plank AC when the chair is open. Given the dimensions $d_1 = 30\,\mathrm{cm}$, $d_2 = 10\,\mathrm{cm}$, $\delta = 2\,\mathrm{cm}$ and the force at A, $F_i = 500\,\mathrm{N}$, find the tension in the link DB. Why is this force so big or small?




**problem 6.26:**

Filename:pfig6-toggle-chair

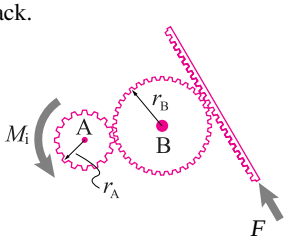
**6.27** A gear of radius 250 mm is meshed in with a rack that carries a horizontal load $F = 50\,\mathrm{N}$. Find the torque $M$ on the gear that is require for equilibrium.




**problem 6.27:**

Filename:pfig6-gears1

**6.28** The input gear A of radius $r_A = 10\,\mathrm{cm}$ drives gear B that is one and a half times bigger than gear A. Gear B, in turn, drives a rack. If the input torque on gear A is $M_i = 30\,\mathrm{N{\cdot}m}$, find the load $F$ on the rack.



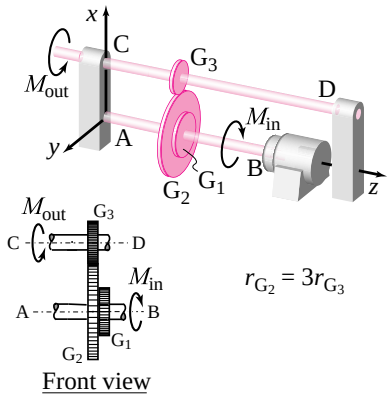

**problem 6.28:**

Filename:pfig6-gears2

**6.29** In the gear arrangement shown, gears $G_1$ and $G_2$ are welded together. The output gear $G_3$ is one third the size of gear $G_2$.

a) Is this gear train for torque amplification or for torque reduction?

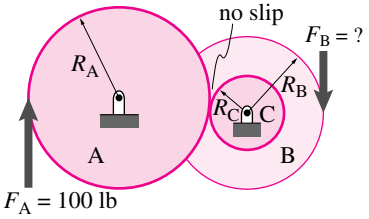
b) If the input torque $M_{\text{in}}$ on gear $G_1$ is 300 N·m, find the output torque $M_{\text{out}}$.




**problem 6.29:**

Filename:pfig6-gears3

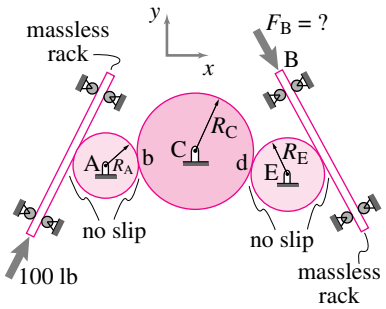
**6.30** At the input to a gear box, a 100 lbf force is applied to gear A. At the output, the machinery (not shown) applies a force of $F_B$ to the output gear. Assume the system of gears is at rest. What is $F_B$?




**problem 6.30:**

Filename:pg131-3-a

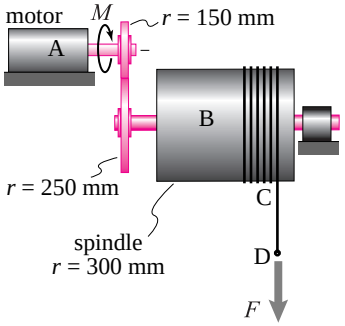
**6.31** A 100 lbf force is applied to one rack. At the output, the machinery (not shown) applies a force of $F_B$ to the other rack. Assume the gear-train is at rest. What is $F_B$?




**problem 6.31:**

Filename:ch4-5-a

**6.32** The gear train and spindle shown in the figure are used for hoisting heavy loads. For the dimensions given, if the load $F = 2$ kN, find the torque $M$ that the motor A must apply for equilibrium.




**problem 6.32:**

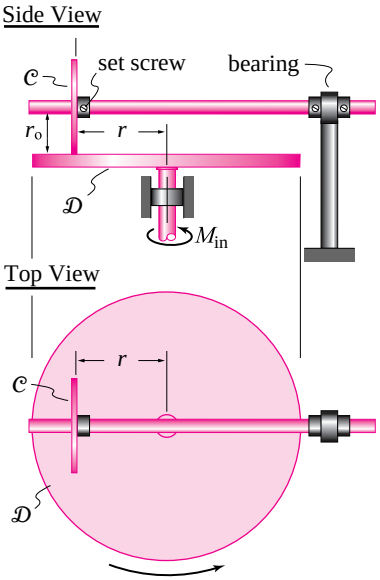
Filename:pfig6-gears4

**6.33** The figure shows a brush gear (also called a crown wheel) where wheel $\mathcal{C}$ of radius $r_o$, rolls on the surface of wheel $\mathcal{D}$ without slipping. In addition, the position $r$ of wheel $\mathcal{C}$ from the center of wheel $D$ can be varied. Let the input torque on wheel $\mathcal{D}$ be $M_i$.

a) Find the output torque $M_o$ on wheel $\mathcal{C}$ as a function of $r$.

b) Find the output torque $M_o$ when $r = r_o$ and when $r = r_o/4$.

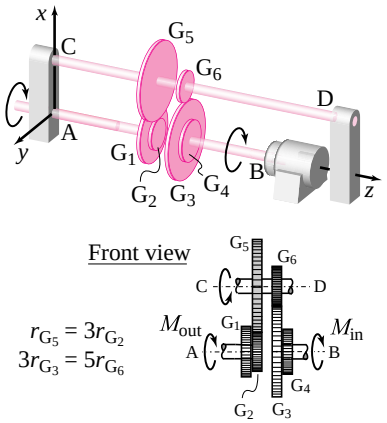
c) If the output torque were not to exceed 100 times the input torque, where will you put safety latches on the axle of wheel $\mathcal{C}$?




**problem 6.33:**

Filename:pfig6-gears5

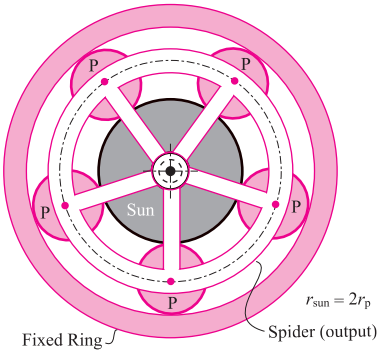
**6.34** For the gear train shown in the figure, find the torque amplification $M_{\text{out}}/M_{\text{in}}$.




**problem 6.34:**

Filename:pfig6-gears6

**6.35** A torque amplifying planetary gear is shown in the figure where the sun-gear is free to rotate but the ring-gear is fixed. The sun-gear drives five planet-gears that drive the spider-gear through their axles housed in bearings in the spider. The radius of the planet-gears $r_p = 50$ mm and the radius of the sun-gear is twice as big. If the input torque on the sun-gear is 2000 N·m, find the output torque on the spider.



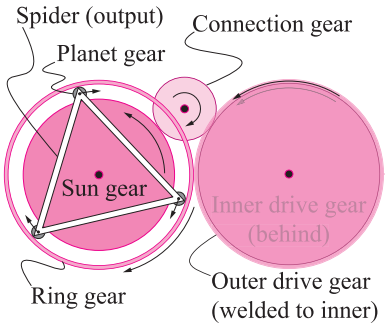

**problem 6.35:**

Filename:pfig6-gears-planetary1

**6.36** Consider the high gear ratio planetary gear discussed on page 314 of the text. Let $r_{D_i}$ and $r_{D_o}$ be the inner and outer radii, respectively, of the drive gear, and $r_{R_i}$ and $r_{R_o}$ be the inner and outer radii, respectively, of the ring-gear. Let $r_s$ and $r_p$ denote the radii of the sun and the planet-gear respectively. SHow that the ratio of the output torque $M_{\text{out}}$ on the spider-gear to the input torque $M_{\text{in}}$ on the drive gear is approximately given by

$$\frac{M_{\text{out}}}{M_{\text{in}}} \approx \frac{2}{\frac{R_D}{R_R} - 1}$$

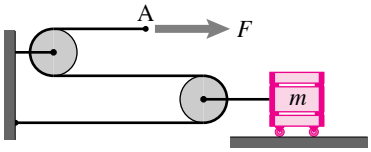
where $R_D = r_{D_i}/r_{D_o}$ and $R_R = r_{R_i}/r_{R_o}$.




**problem 6.36:**

Filename:pfig6-gears-planetary2

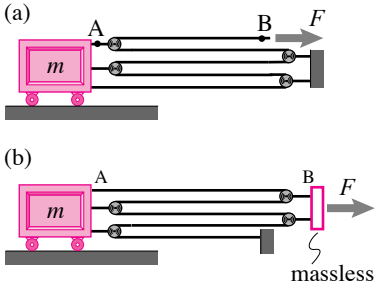
**6.37** A force $F = 100\,\text{N}$ acts at A. The pulleys are frictionless. Find the force on the box applied by the pulley.




**problem 6.37:**

Filename:pfig6-pulley-simple

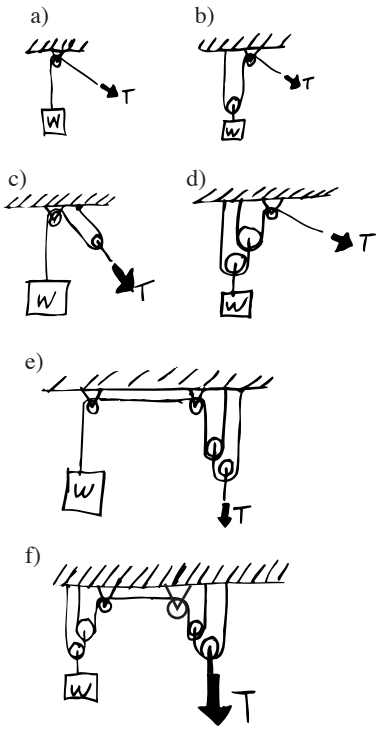
**6.38** A force $F$ is applied as shown in the pulley arrangements shown in (a) and (b). Which arrangement gives a bigger force amplification on the box?




**problem 6.38:**

Filename:pfig6-pulley-friction

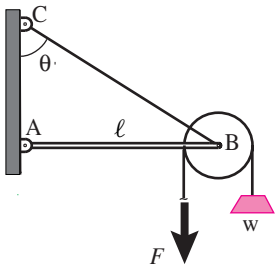
**6.39** Given $W$ and the frictionless pulleys shown find the tension $T$ needed to lift the weight in the situations shown.




**problem 6.39:**

Filename:pfigure-simplepulleys

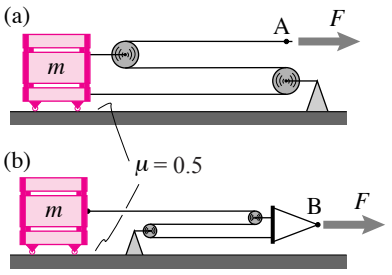
**6.40** A weight $W$ is held in place with a force $F = 100\,\text{N}$ applied through a massless pulley as shown in the figure. The pulley is attached to a rod AB which, in turn, is held horizontal with the help of a string CB. Find the tension (or compression) in rod AB.




**problem 6.40:**

Filename:pfigure4-1-rp9

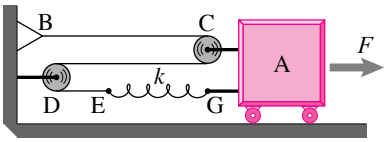
**6.41** In the two cases shown in (a) and (b), find the maximum force $F$ that can be applied before the box starts skidding on the ground. Take $m = 50\,\text{kg}$ and $g \approx 10\,\text{m/s}^2$. Which arrangement requires smaller force an why?




**problem 6.41:**

Filename:pfig6-pulley-friction1

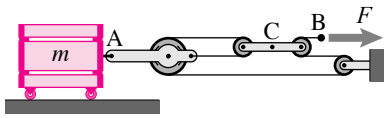
**6.42** The pulley arrangement shown in the figure uses a spring EG of stiffness $k = 200\,\text{N/cm}$. If the spring is stretched by 1.5 cm under the application of force $F$ for equilibrium, find $F$.




**problem 6.42:**

Filename:pfig6-pulley-spring

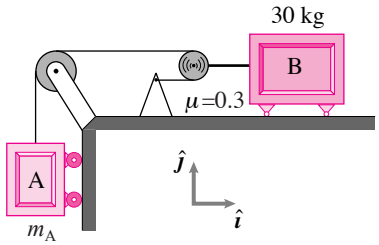
**6.43** Find the force on the mass at A in terms of $F$ and thus find the force amplification provided by the pulley arrangement used.




**problem 6.43:**

Filename:pfig6-pulley-complex

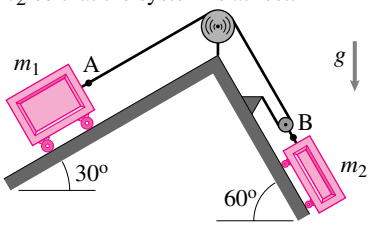
**6.44** In the figure shown, there is no friction between block A and the vertical wall but there is friction ($\mu = 0.3$) between block B and the floor. If $m_B = 30\,\text{kg}$, find the mass of block A for equilibrium.




**problem 6.44:**

Filename:pfig6-pulley-gravity1

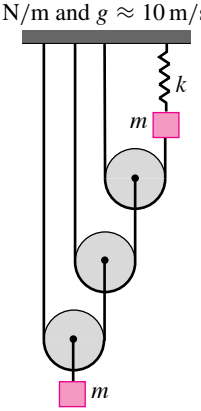
**6.45** Find the ratio of the masses $m_1$ and $m_2$ so that the system is at rest.




**problem 6.45:**

Filename:pulley4-c

**6.46** If the mass and pulley system shown in the figure is in equilibrium when the spring is stretched by 3 cm, find $m$, given $k = 500\,\text{N/m}$ and $g \approx 10\,\text{m/s}^2$.

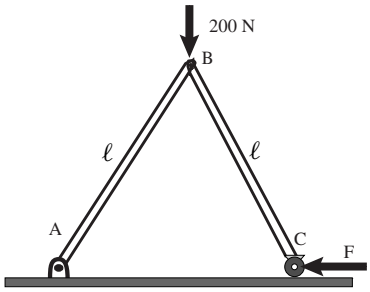



**problem 6.46:**

Filename:pfig6-pulley-gravity3

## 6.3 Machines

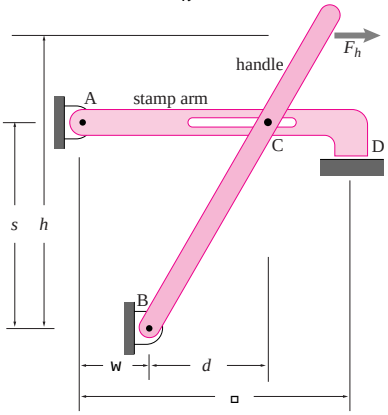
**6.47** A simply supported two bar mechanism supports a load of 200 N at joint B with the help of a horizontal force $F$ applied at joint C. Find $F$.




**problem 6.47:**

Filename:pfigure4-1-rp12

**6.48** Pulling on the handle causes the stamp arm to press down at D. Neglect gravity and assume that the hinges at A and B, as well as the roller at C, are frictionless. Find the force $N$ that the stamp machine causes on the support at D in terms of some or all of $F_h$, $w$, $d$, $\ell$, $h$, and $s$.




**problem 6.48:**

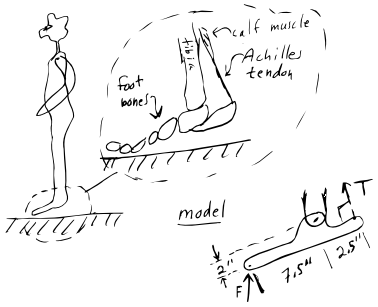
Filename:F01final3-stamp

## More-Involved Problems

**6.49** See Problem 4.41 on page 217. A person who weighs $W$ stands on tiptoes on one foot. Assume the weight of the foot is negligible.

a) Draw a free-body diagram of the whole person and find the force of the ground on the foot front.

b) Draw a free body diagram of the foot and find the force of the calf on the foot at the ankle and the tension in the Achilles Tendon.



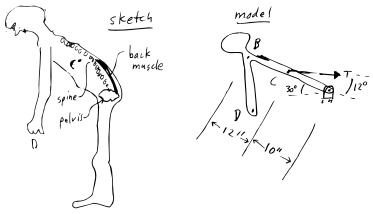

**problem 6.49:**

Filename:pfigureSoodak4-25

**6.50** See Problem 4.41 on page 217. A person with weight $W = 140\,\text{lbf}$ has an upper body with weight $0.7W$ with center of mass at C. The back muscles are idealized as a single muscle with one end (the muscle *origin* also at C. Use the idealization and geometry shown.

a) Find the back-muscle tension and the force of the lower body on the upper body at the hips.

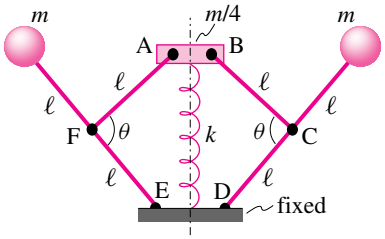
b) Repeat the problem but assume that the person is lifting a 30 lbf load at D.




**problem 6.50:**

Filename:pfigureSoodak4-26

**6.51** In the flyball governor shown, the mass of each ball is $m = 5\,\text{kg}$, and the length of each link is $\ell = 0.25$ m. There are frictionless hinges at points $A$, $B$, $C$, $D$, $E$, $F$ where the links are connected. The central collar has mass $m/4$. Assuming that the spring of constant $k = 500$ N/m is uncompressed when $\theta = \pi$ radians, what is the compression of the spring?



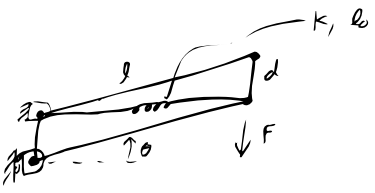

**problem 6.51:**

Filename:summer95p2-2-a

**6.52**

a) Find $F$ for equilibrium for the parallelogram structure shown assuming the rest length of the spring is zero.

b) Comment on how your answer above depends on $\theta$.

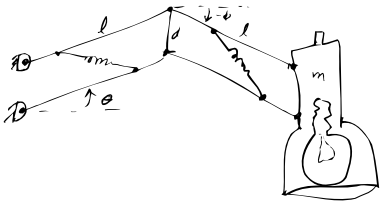



**problem 6.52:**

Filename:pfigure-spring5

**6.53** A common lamp design is shown. In principle the lamp should be in equilibrium in all positions. According to the original patent from the 1930s it can be, even with

no friction in the joints. Unfortunately, the recent manufacturers of this lamp seem to have lost the wisdom of the original patent. Show how to place what springs so this lamp is in equilibrium for all $\theta < \pi/2$ and $\phi < \pi/2$. [Hint: use springs with zero rest length.]



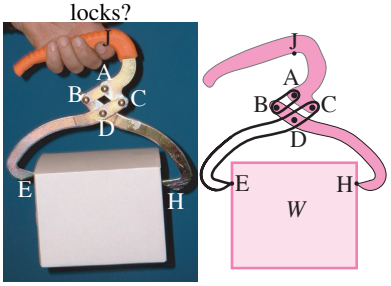
**problem 6.53:** Lamp.

Filename:pfigure-spring6

**6.54 Log carrier.** This self-locking scissors-mechanism gadget is used to pick up logs and blocks of ice. The 5 cm wide and 3 cm*high* diamond arrangement of hinges ABCD makes up a 4-bar linkage. The grips E and H are 16 cm*apart* and 9 cm below D. The block weighs 250 N. Neglect the weight of the mechanism.

a) What is the horizontal component of the force on the block at E?

b) What is the minimum coefficient of friction $\mu$ for which this device self locks?




**problem 6.54:**

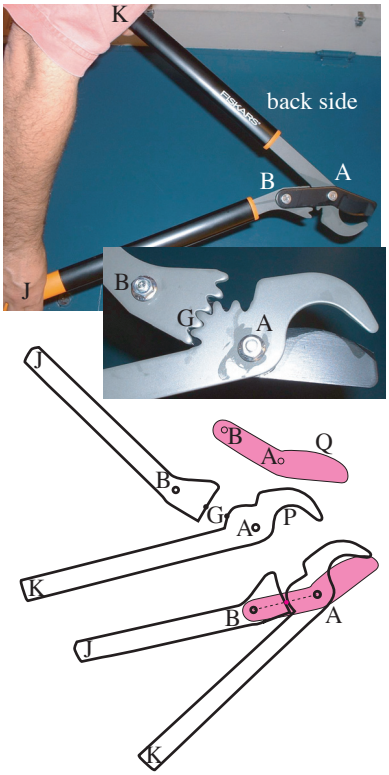
Filename:pfigure-loghanger

**6.55** Gear teeth on handle JB mesh with teeth on handle-and-blade KAP at point G midway between hinges A and B. Assume that in the configuration of interest J, B and A are co-linear, that K, A and Q are co-linear and that the cutting contact points Q and P are effectively coincident, that angle JAK = 20°, $JB = 40\,\text{cm}$, $BA = 6\,\text{cm}$, $KA = 46\,\text{cm}$, $AP = AQ = 3\,\text{cm}$, and that the co-linear squeezing forces at J and K are 100 N.

a) Find the cutting force at Q and P.

b) Replace this design with one that has no tooth engagement at G. But instead handle JB and blade BAQ are welded together as one piece. Assuming the same geometry as before, what then is the cutting force at Q and P?

c) Without detailed calculations, explain the ratio of the two answers above.




**problem 6.55:**

Filename:pfigure-FiskarsLoppers

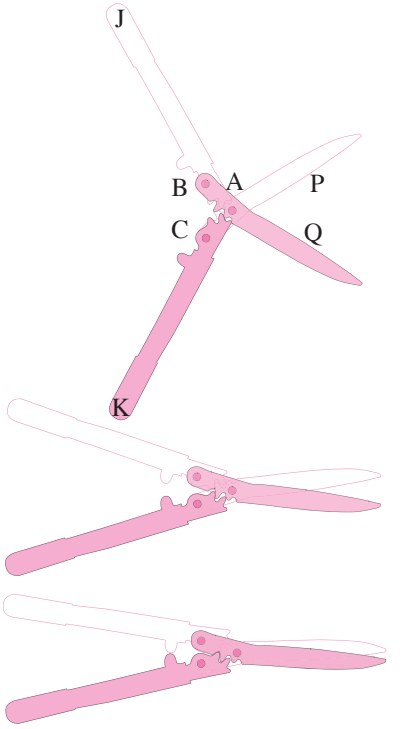
**6.56** These 4 piece shears use the mechanism in problem 6.55 twice over. Co-linear hand forces $F_{JK}$ are applied to handles JB and KC at J and K. Handle JB is hinged to blade BAQ at B. Handle KC is hinged to blade CAP at C. The blades are hinged to each other at A. Handle JB is effectively hinged to CAP, by means of gear teeth, at G, a point on the line segment AB. Similarly KC is effectively hinged to BAQ at point G' on the segment AC. The cut object presses with colinear forces $F_{PQ}$ on the blades at P and Q. See box 6.6 on page 324 for more pictures of these shears.

Assume $F_{JK} = 100\,\text{N}$, $JB = KC = 30\,\text{cm}$, $AB = AC = 6\,\text{cm}$, $AG = AG' = 3\,\text{cm}$, and $AP = AQ = 20\,\text{cm}$. Assume AB, BJ, AC,and CK all make angles of $\pm 20°$ with a horizontal line. Assume P and Q are coincident and on a horizontal line extending from A.

a) Find $F_{PQ}$.

b) Replace this design with one where JB is welded to BAQ at B, KC is wleded to CAP at C, and there are no contacting gears. In this same geometry what is $F_{PQ}$?

c) Give a quantitative estimate, but not a detailed calculation that tells you the ratio of the forces in the above two problems?




**problem 6.56:**

Filename:pfigure-mech-Fiskars

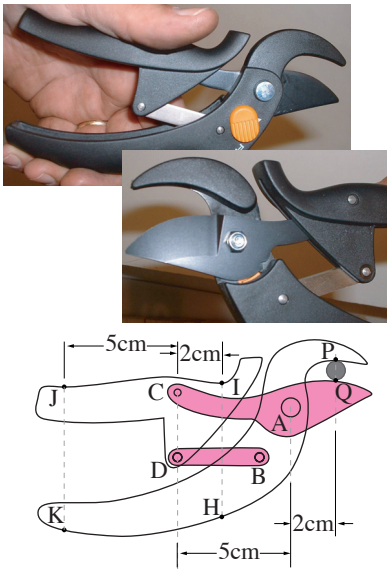
**6.57** The garden cutters shown are a 4-bar linkage. Estimate the locations of points, as needed, using the given dimensions as a scale (the drawn clippers are shrunk slightly from reality to simplify the numbers).

a) If the handle is squeezed with a pair of 50 N forces at J and K what is the cutting force at P and Q?

b) If the handle is squeezed with a pair of 50 N forces at I and H what is the cutting force at P and Q?

c) If this design was changed by eliminating link DB and welding handle JCDI to the blade CAQ, what would be the answers to the two questions above.

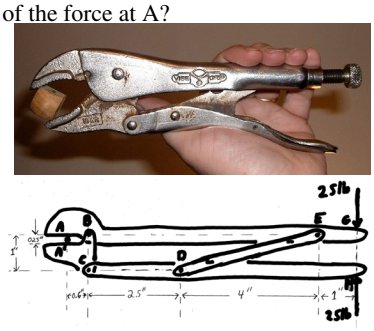
d) Describe in words, the reasons for the similarities and differences between the answers above.




**problem 6.57:**

Filename:pfigure-FiskarsCutter

**6.58** For simplicity the vice grips shown in the photo are approximated as in the drawing. Round piece AA' is gripped between the upper handle/jaw ABEG and the lower jaw A'BC. The upper handle ABEG is pinned to the lower jaw A'BC at B. Handle CDH is pinned to the lower jaw at C and to the bar DE at D. Bar DE is pinned to the upper handle ABEG at E. The 25 lbf forces act at G and H as shown. Dimensions are as shown. What is the magnitude of the force at A?

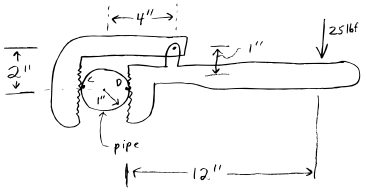



**problem 6.58:**

Filename:S03q4p8vicedrawing

**6.59 Pipe wrench.** A wrench is used to turn a pipe as shown in the figure. Neglecting the weight of the pipe, find

a) the torque of the pipe wrench forces about the center of the pipe
b) the forces on the pipe at C and D
c) the needed friction coefficient between the wrench and pipe for the wrench not to slip.
d) what design change would reduce this needed coefficient of friction (what change of dimensions)?
e) given that the design change above is possible, why isn't it used? [hint: implement the design change and calculate the forces on the pipe.]

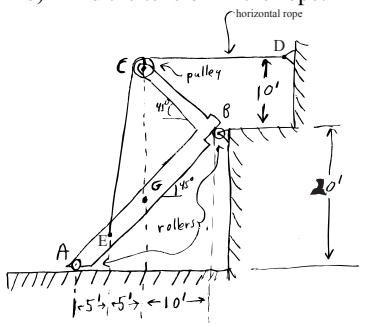



**problem 6.59:**

Filename:pfigure-wrench

**6.60** The center of mass of 200 pound structure AEGBC is at G. It is held by rollers at A and B as well as with the rope which starts at E, wraps around the pulley at C, and ends at D.

a) Find the force of the ground on the structure at A.
b) Find the tension in the rope.



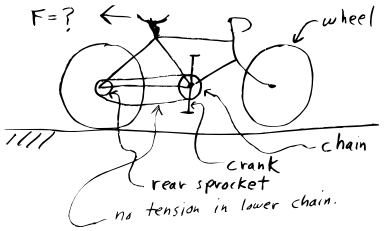

**problem 6.60:**

Filename:S03q2b-bentbar

**6.61** Consider a bike on level ground that is held from falling sideways with forces that don't push it forward or back. Assume that all the bearings are ideal and that the wheels don't slip.
$R_r$ = radius of rear wheel,
$R_s$ = radius of rear sprocket,
$R_p$ = crank length from crank-axle to pedal, and
$R_c$ = radius of chain wheel (front sprocket).
What backwards force $F$ on the seat is required to keep the bike from going forward (*i.e.*, to maintain static equilibrium) if

a) A person sits on the bike and pushes back on the bottom pedal with a force $F_p$? (is $F > 0$?)
b) A person standing next to the bike pushes back on the pedal with force $F_p$? (is $F > 0$?)

Your answer should be in terms of some or all of $R_r$, $R_s$, $R_p$, $R_c$, and $F_p$. Of great interest is whether $F$ is bigger or less than zero. So pay close attention to signs.
To solve this problem you have to draw several free body diagrams: 1) of the whole bike and rider (if the rider is on the bike), 2) of the crank-pedal-chain-wheel system, with a little bit of chain, 3) The rear wheel and rear sprocket, with a little bit of chain.

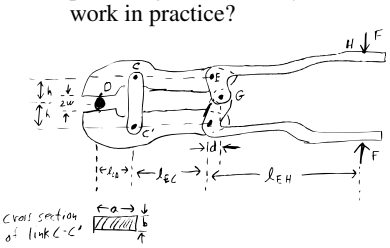



**problem 6.61:**

Filename:pfigure-bikepedalback

**6.62** The pliers shown are made of five pieces modeled as rigid: HEG and its mirror image, DCE and its mirror image, and link CC'. You may assume that the goemetry is symmetric about a horizontal line (the top is a mirror image of the bottom). The load $F$ and dimensions shown are given.

a) Find the force squeezing the piece at D;
b) Find the tension in CC';
c) What happens to the squeezing force if $d$ is made smaller, approaching zero? Why can't this work in practice?




**problem 6.62:**

Filename:F01p2-2-plyers

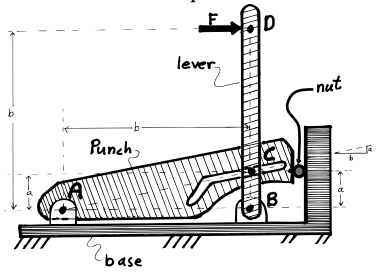
**6.63** The proposed nutcracker design consists of two moving parts: a lever hinged to the fixed base at B and a punch hinged to the fixed base at A. All joints and slots are assumed to have negligible friction.
**Mechanism and geometry clarifications:** The vertical lever has a pin at C and a horizontal force $F$ applied at D. The punch has a slot in which the lever pin slides at C. The slot is parallel to the line AC. The spherical nut is cracked by being squeezed between the vertical surface of the punch at

N and the vertical surface attached to the base. Point N at the left edge of of the nut is level with the sliding pin at C. The horizontal distance from C to N does not enter the solution, but assume it is $c$ if you need it for an intermediate calculation.

**Quantities:** $F = 10\,\text{lbf}$, $a = 2\,\text{in}$, $b = 10\,\text{in}$.

a) Find the force acting on the nut at N. A number is desired (i.e., so many lbf force). [Hint: Only substitute in numbers when you have a formula for your answer in terms of $a$, $b$ and $F$.]

b) The answer to (a) is conspicuous in its being either much smaller than $F$, very similar to $F$, or much bigger than $F$. Which is it? Explain, in words, why. The best possible answer will generate an approximate formula for the force at N using next-to-no equations.




**problem 6.63:**

Filename:F02p1-3Nut

CHAPTER 7

# Hydrostatics

*Hydrostatics concerns the equivalent force and moment due to distributed pressure on a surface from a still fluid. Pressure increases with depth. With constant pressure the equivalent force has magnitude = pressure times area, acting at the centroid. For linearly-varying pressure on a rectangular plate the equivalent force is the average pressure times the area acting 2/3 of the way down. The net force acting on a totally submerged object in a constant density fluid is the displace weight acting at the centroid.*

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*Hydrostatics* is primarily concerned with finding the net force and moment of still fluid on a surface. The surfaces are typically the sides of a pool, dam, container, or pipe, or the outer surfaces of a floating object such as a boat or of a submerged object like a toilet bowl float. Finally, one is sometimes concerned with the force on an imagined surface that separates some water of interest from the other water. Although the hydrostatics of air helps explain the floating of hot air balloons, dirigibles, and chimney smoke; and the hydrostatics of oil is important for hydraulics (hydraulic brakes for example), often the fluid of concern for engineers is water. So, as in the title of the chapter ('hydro'), we often use the word 'water' as an informal synonym for 'fluid.'

Besides the utility of the subject in applications, hydrostatics is also a good introduction to distributed forces and continuum mechanics.

## 7.1 What is pressure? Constant pressure.

Besides the basic laws of mechanics that you already know, elementary hydrostatics is based on the following two constitutive assumptions (see page 3):

1) The force of water on a surface is perpendicular to the surface; and
2) The density of water, $\rho$ (pronounced 'row') is a constant (doesn't vary with depth or pressure),

Sometimes we use the weight density $\gamma = g\rho$ (pronounced 'gammuh equals gee row'), the weight per unit volume . The first assumption, that all static water forces are perpendicular to surfaces on which they act, can be restated:

Still water cannot carry any shear stress.

For near-still water this constitutive assumption is abnormally accurate (compared to most constitutive assumptions for materials), approximately as good as the laws of mechanics.

The assumption of constant density is called *incompressibility* because it corresponds to the idea that water does not change its volume (compress)

much under pressure. This assumption is reasonable for most purposes. At the bottom of the deepest oceans, for example, the extreme pressure (about 800 atmospheres) causes water to increase its density only about 4% from that of water at the surface*

We also assume that the direction and magnitude of the local gravitational constant is, well, constant. This assumption becomes inaccurate when considering, say, the hydrostatics of whole oceans (the direction of the gravity force changes as you go around the word, this helps keep the Australians in place), or of the upper atmosphere (the magnitude of the gravity decays with distance from the center of the earth).

*That fluid density does depend on salinity, temperature and pressure is sometimes important in hydrostatics. In particular for determining which water floats on which other water. This is important in the ecology of lakes, the effects of the oceans on climate, and in air for the stability of the atmosphere, and the mechanics of fireplace chimneys.

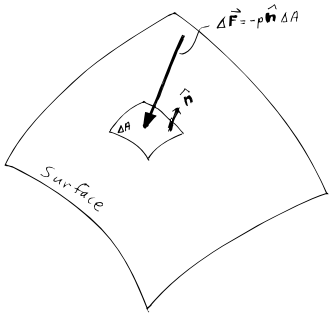



Figure 7.1: A bit of area $\Delta A$ on a surface on which pressure $p$ acts. The outward (into the water) normal of the surface is $\hat{\boldsymbol{n}}$ so the force is $\Delta\vec{\boldsymbol{F}} = -p\hat{\boldsymbol{n}}\Delta A$.

Filename:tfigure-deltaA

## Surface area $A$, outward normal $\hat{n}$, pressure $p$, and force $\vec{F}$

We are going to be generalizing the high-school physics fact

$$\text{force} = \text{pressure} \times \text{area}$$

to take account that force is a vector, that pressure varies with position, and that not all surfaces are flat. So we need a clear notation and sign convention. The area of a surface is $A$ which we can think of as being the sum of the bits of area $\Delta A$ that compose it:

$$A = \int dA.$$

Every bit of surface area has an *outer* normal $\hat{\boldsymbol{n}}$ that points from the surface out into the fluid. The (scalar) force per unit area on the surface is called the pressure $p$, so that the force on a small bit of surface is

$$\Delta\vec{\boldsymbol{F}} = p\,(-\hat{\boldsymbol{n}})\,(\Delta A)$$

pointing into the surface, assuming positive pressure, and with magnitude proportional to both pressure and area. Thus the total force and moment due to pressure forces on a surface :

$$\begin{aligned} \vec{\boldsymbol{F}} &= \int d\vec{\boldsymbol{F}} &&= -\int_A p\,\hat{\boldsymbol{n}}\; dA \\ \vec{\boldsymbol{M}}_{\mathrm{C}} &= \int_A d\vec{\boldsymbol{M}}_{/\mathrm{C}} &&= -\int_A \vec{\boldsymbol{r}}_{/\mathrm{C}} \times (p\,\hat{\boldsymbol{n}})\; dA \end{aligned} \tag{7.1}$$

Hydrostatics is the evaluation of the (intimidating-at-first-glance) integrals 7.1 and their role in equilibrium equations. In the rest of this section we consider a variety of important special cases.

## Water in equilibrium with itself

Before we worry about how water pushes on other things, let's first understand what it means for water to be in static equilibrium. These first important

facts about hydrostatics follow from drawing free body diagrams of various chunks of water and assuming static equilibrium (see box 7.2 on page 345).

1. Pressure is the same in every direction, $p_x = p_y = p$.
2. Pressure doesn't vary with side to side position, $p(x, y, z) = p(y)$.
3. Pressure varies linearly with depth, $p = \rho g h = \gamma h$.

## The buoyant force of water on water.

In a place under water in a still swimming pool where there is nothing but water, imagine a chunk of water the shape of a sea monster. Now draw a free body diagram of that water. Because your sea monster is in equilibrium, force balance and moment balance must apply. The only forces are the complicated distribution of pressure forces and the weight of water. The pressure forces must exactly cancel the weight of the water and, to satisfy moment balance, must pass through the center-of-mass of the water monster. So, in static equilibrium:

The pressure forces acting on a surface enclosing a volume of water is equivalent to the negative weight passing through the center-of-mass of the water.

## The force of water on submerged and floating objects

The net pressure force and moment on a still object surrounded by still water can be found by a clever argument credited to Archimedes. The pressure at any one point on the outside of the object does not depend on what's inside. The pressure is determined by how far the point of interest is below the surface by eqn. 7.2*. So if you can find the resultant force on any object that is the shape of the submerged object, but replacing the submerged object, it tells you what you want to know.

The clever idea is to replace your object with water. In this new system the water is in equilibrium, so the pressure forces exactly balance the weight. We thus obtain Archimedes' Principle:

The resultant of all pressure forces on a totally submerged object is an upwards force with the same magnitude as the weight of the displaced water. The resultant acts at the centroid of the displaced volume:

$$\vec{F}_{\text{buoyancy}} = \gamma V \hat{j} \qquad \text{acting at} \qquad \vec{r} = \frac{\int \vec{r}_{/0}\, dV}{V}.$$

* If there is no column of water from the point up to the surface it is still true that the pressure is $\gamma h$, as you can figure out by tracking the pressure changes along on a staircase-like path from the surface to that point.

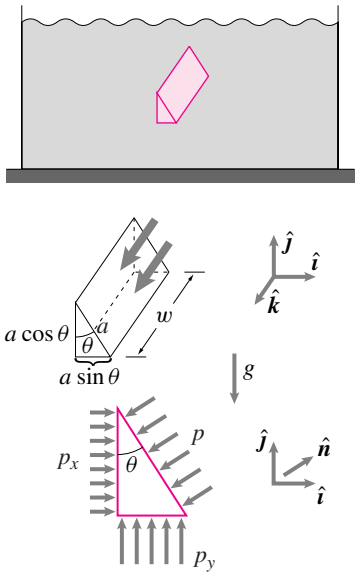



Figure 7.2: A small prism of water is isolated from some water in equilibrium. The free body diagram does not show the forces in the $z$ direction. Force balance applied to this free body diagram shows that $p_x = p_y = p$, pressure is the same in all directions.

Filename:tfigure-waterprism

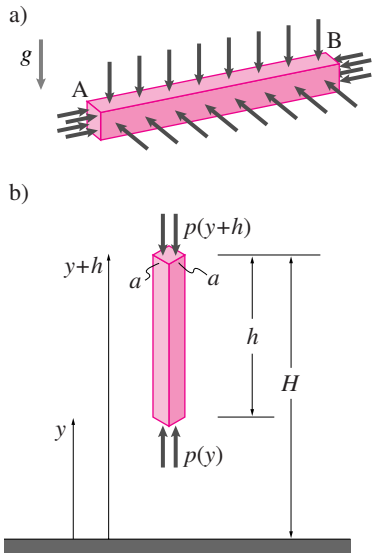



Figure 7.3: Free body diagrams of aligned boxes of water cut out of a bigger body of water. a) a horizontally aligned box, b) a vertically aligned box. Force balance on applied to these free body diagrams shows that $p = p(y)$.

Filename:tfigure-waterbox

## 7.1 *THEORY*

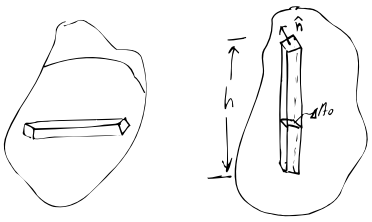
### *Adding forces to derive Archimedes' principle*

(One can do most hydrostatics calculations, say typical homework problems, without being able to reproduce the derivations here.)

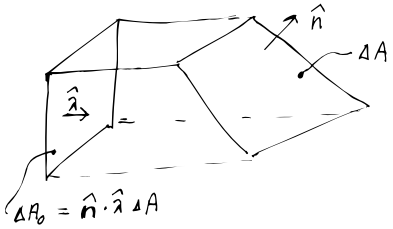
Archimedes' principle follows from adding up all the pressure forces on the outer surfaces of an arbitrarily shaped submerged solid, say something potato shaped.

First we find the answer by cutting the potato into french fries. This approach is effectively a derivation of a theorem in vector calculus. After that, for those who have the appropriate math background, we quote the vector calculus directly.

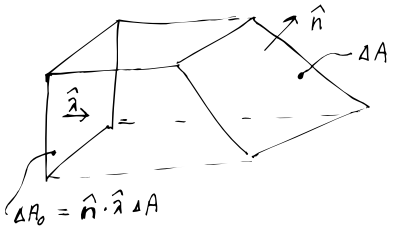
First cut the potato into horizontal french-fries (horizontal prisms) and look at the forces on the end caps (there are no water forces on the sides since those are inside the potato).




The pressure on two ends is the same (because they have the same water depth). The areas on the two ends are probably different because your potato is probably not box shaped. But the area is bigger at one end if the normal to the surface is more oblique compared to the axis of the prism. If the cross sectional area of the prism is $\Delta A_0$ then the area of one of the prism caps is




$\Delta A = \Delta A_0/(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\lambda}})$ where $\hat{\boldsymbol{\lambda}}$ is along the axis of the prism and $\hat{\boldsymbol{n}}$ is the outer unit normal to the end cap (Note $\Delta A \geq \Delta A_0$ because $\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\lambda}} \leq 1$).

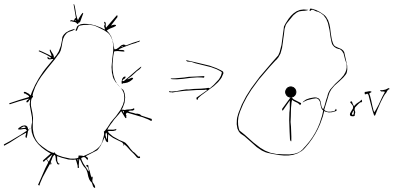



So the net force on the cap is $-p\Delta A_0 \hat{\boldsymbol{n}}/(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\lambda}})$. The component of the force along the prism is $\left[-p\Delta A_0 \hat{\boldsymbol{n}}/(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\lambda}})\right] \cdot \hat{\boldsymbol{\lambda}}$ which is $-p\Delta A_0$. An identical calculation at the other end of the french fry gives minus the same answer. So the net force of the water pressure along the prism is zero for this and every prism and thus the whole potato. Likewise for prisms with any horizontal orientation. Thus the net sideways force of water on any submerged object is zero.

To find the net vertical force on the potato we cut it into vertical french fries. The net forces on the end caps are calculated just as in the above paragraph but taking account that the pressure on the bottom of the french fry is bigger than at the top. The sum of the forces of the top and bottom caps is an upwards force that is

$$\begin{array}{rcl} \text{net upwards force on vertical french fry} & = & \Delta p \Delta A_0 \\ & = & (\gamma h)\Delta A_0 \\ & = & \gamma(h \Delta A_0) \\ & = & \gamma\, \Delta V_0 \end{array}$$

where $\Delta V_0$ is the volume of the french fry. Adding up over all the french fries that make up the potato one gets that the net upwards force is $\gamma V$ The net result, summarized by the figure below, is that the resultant of the pressure forces on a submerged solid is an upwards force whose magnitude is the weight of the displaced water. The location of the force is the centroid of the displaced volume. (Note that the centroid of the displaced volume is not necessarily at the center of mass of the submerged object.)




• • •

*A vector calculus derivation*

Here is a derivation of Archimedes' principle, at least the net force part, using multi-variable integral calculus. Only read on if you have taken a math class that covers the divergence theorem. The net pressure force on a submerged object is

$$\begin{array}{rcccccl} \vec{\boldsymbol{F}}_{\text{buoyancy}} & = & -\int_A & p & \hat{\boldsymbol{n}} & dA & \\ & = & -\int_S & p & \hat{\boldsymbol{n}} & dS & \\ & = & -\int_S & (H-z)\gamma & \hat{\boldsymbol{n}} & dS & \\ & = & -\int_V & \vec{\nabla}\left((H-z)\gamma\right) & & dV & \\ & = & -\int_V & (-\hat{\boldsymbol{k}})\,\gamma & & dV & \\ & = & \int_V & \gamma & & dV & \hat{\boldsymbol{k}} \\ & = & \multicolumn{3}{c}{\text{(weight of displaced water)}} & & \hat{\boldsymbol{k}}. \end{array}$$

In this derivation we first changed from calling bits of surface area $dA$to $dS$ because that is a common notation in calculus books. The depth from the surface, of a point with vertical component $z$ from the bottom, is $H - z$. The $\vec{\nabla}$ symbol indicates the gradient and its place in this equation is from the divergence theorem:

$$\int_S (\text{any scalar})\hat{\boldsymbol{n}}\, dS = \int_V \vec{\nabla}(\text{the same scalar})\, dV.$$

The gradient of $(H - z)\gamma$ is $-\hat{\boldsymbol{k}}\gamma$ because $H$ and $\gamma$ are constants. Note, where we write $\int_S$ some books would write $\iint_S$, and where we write $\int_V$ some books would write $\iiint_V$.

The result can also be found by adding the effects of all the pressure forces on the outside surface (see box 7.1 on page 344).

For floating objects, the same argument can be carried out, but since the replaced fluid has to be in equilibrium we cannot replace the whole object with fluid, but only the part which is below the level of the water surface.

## 7.2 *THEORY*

### *Pressure doesn't depend on direction or horizontal position and increases linearly with depth*

We assume that the pressure $p$ does not vary too wildly from point to point, thus if we look at a small enough region we can think of the pressure as constant in that region. If we draw a free body diagram of a little triangular prism of water the net forces on the prism must add to zero (see Fig. 7.2 on page 7.2). For each surface the magnitude of the force is the pressure times the area of the surface and the direction is minus the outward normal of the surface. We assume, for the time being, that the pressure is different on the differently oriented surfaces. So, for example, because the area of the left surface is $a\cos\theta w$ and the pressure on the surface is $p_x$, the net force is $a\cos\theta w p_x \hat{\boldsymbol{\imath}}$. Calculating similarly for the other surfaces:

$$\begin{aligned}\vec{\mathbf{0}} &= \sum \vec{F}_i \\ &= \underbrace{(a\cos\theta)w\, p_x\, \hat{\boldsymbol{\imath}} + (a\sin\theta)w\, p_y\, \hat{\boldsymbol{\jmath}} - a\, w\, p\, \hat{\boldsymbol{n}}}_{\text{pressure terms}} \\ &\qquad - \underbrace{\frac{a^2\cos\theta\sin\theta w}{2}\rho g\, \hat{\boldsymbol{\jmath}}}_{\text{weight}} \\ &= aw\left(\cos\theta\, p_x\, \hat{\boldsymbol{\imath}} + \sin\theta\, p_y\, \hat{\boldsymbol{\jmath}} - p\,\underbrace{(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}})}_{\hat{\boldsymbol{n}}} - \tfrac{a\cos\theta\sin\theta}{2}\rho g\hat{\boldsymbol{\jmath}}\right)\end{aligned}$$

If $a$ is arbitrarily small, the weight term drops out compared to the pressure terms. Dividing through by $aw$ we get

$$\vec{\mathbf{0}} = \cos\theta p_x\hat{\boldsymbol{\imath}} + \sin\theta\, p_y\, \hat{\boldsymbol{\jmath}} - p\,(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta)\,\hat{\boldsymbol{\jmath}}.$$

Taking the dot product of both sides of this equation with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ gives that

$$p = p_x = p_y.$$

Since $\theta$ could be anything, force balance for the free body diagram of a small prism tells us that for a fluid in static equilibrium

pressure is the same in every direction.

[Other free body diagrams can be used. That pressure has to be the same in any pair of directions could also be found by drawing a prism with a cross section which is an isosceles triangle. The prism is oriented so that two surfaces of the prism have equal area and have the desired orientations. Force balance along the base of the triangle gives that the pressures on the equal area surfaces are equal. The argument that pressure must not depend on direction in 3D is generally based on equilibrium of a small tetrahedron.]

**Pressure doesn't vary with side to side position** Consider the equilibrium of a horizontally aligned box of water cut out of a bigger body of water (Fig. 7.3a on page 343). The forces on the end caps at A and B are the only forces along the box. Therefor they must cancel. Since the areas at the two ends are the same, the pressure must be also. This box could be anywhere and at any length and any horizontal orientation. Thus for a fluid in static equilibrium

pressure doesn't depend on horizontal position.

If we take the $\hat{\boldsymbol{\jmath}}$ or $y$ direction to be up (Fig. 7.3 on page 343), then we have

$$p(x, y, z) = p(y).$$

**Pressure increases linearly with depth** Consider the vertically aligned box of Fig. 7.3b.

$$\begin{aligned}\left\{\sum \vec{F}_i = \vec{\mathbf{0}}\right\}\cdot\hat{\boldsymbol{\jmath}} \quad &\Rightarrow \quad \underbrace{p(y)a^2 - p(y+h)a^2}_{\text{pressure terms}} - \underbrace{\rho g a^2 h}_{\text{weight}} = 0 \\ &\Rightarrow \quad p_{\text{bottom}} - p_{\text{top}} = \rho g h.\end{aligned}$$

So the pressure increases linearly with depth. If the top of a lake, say, is at atmospheric pressure $p_a$ then we have that

$$p = p_a + \rho g h = p_a + \gamma h = p_a + (H - y)\gamma$$

where $h$ is the distance down from the surface, $H$ is the depth to some reference point underwater and $y$ is the distance up from that reference point (so that $h = H - y$). Neglecting atmospheric pressure at the top surface we have the useful and easy to remember formula:

$$p = \gamma h. \tag{7.2}$$

Because the pressure at equal depths must be equal and because the pressure at the top surface must be equal to atmospheric pressure, the top surface must be flat and level. Thus waves and the like are a definite sign of static disequilibrium as are any bumps on the water surface even if they don't seem to move (as for a bump in the water where a stream goes steadily over a rock).

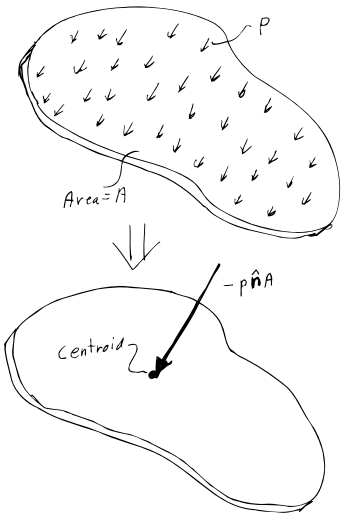



Figure 7.4: The resultant force from a constant pressure $p$ on a flat plate is $\vec{\boldsymbol{F}} = -pA\hat{\boldsymbol{n}}$ acting at the centroid of the plate.

Filename:tfigure-centroidpressure

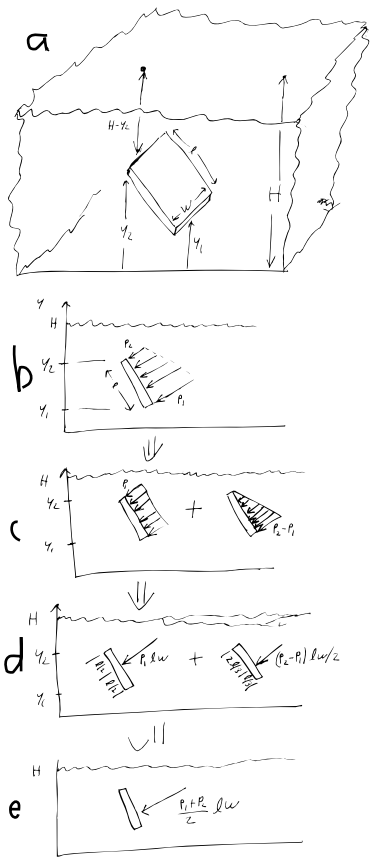



Figure 7.5: The resultant force from a constant depth-increasing pressure on a rectangular plate.

Filename:tfigure-plateunder

## Displaced fluid

Sometimes people discuss Archimedes' principle in terms of the displaced fluid. A floating object in equilibrium displaces an amount of fluid with the same weight as the object; this is also the amount of volume of the floating object that is below the water level. On the other hand an object that is totally under water, for whatever reason (it is resting on the bottom, or it is being held underwater by a string, etc), displaces as much fluid as the space it occupies. Putting these two ideas together one can remember that

> A floating object displaces its weight, a submerged object displaces its volume.

## The force of constant pressure on a totally immersed object

When there is no gravity, or gravity is neglected, the pressure in a static fluid is the same everywhere. Exactly the same argument we have just used shows that the resultant of the pressure forces is zero. We could derive this result just by setting $\gamma = 0$ in the formulas above.

## The force of constant pressure on a flat surface

The net force of constant pressure on one flat surface (not all the way around a submerged volume) is the pressure times the area acting normal to the surface at the centroid of the surface:

$$\begin{aligned} \vec{\boldsymbol{F}}_{\text{net}} &= \int_A -p\,\hat{\boldsymbol{n}}\,dA \\ &= -pA\hat{\boldsymbol{n}}. \end{aligned}$$

That this force acts at the centroid can be checked by calculating the moment of the pressure forces relative to the centroid C,

$$\begin{aligned} \vec{\boldsymbol{M}}_{\text{/C,net}} &= \int_A \vec{\boldsymbol{r}}_{\text{/C}} \times \left(-p\,\hat{\boldsymbol{n}}\,dA\right) \\ &= \underbrace{\left(\int_A \vec{\boldsymbol{r}}_{\text{/C}} dA\right)}_{0} \times \left(-p\,\hat{\boldsymbol{n}}\right) \\ &= 0\ . \end{aligned}$$

where the zero follows from the position of the center-of-mass relative to the center-of-mass being zero.

## The force of water on a rectangular plate

Consider a rectangular plate with width into the page $w$ and length $\ell$. Assume the water-side normal to the plate is $\hat{\boldsymbol{n}}$ and that the top edge of the plate is

horizontal. Take $\hat{\boldsymbol{\jmath}}$ to be the up direction with $y$ being distance up from the bottom and the total depth of the water is $H$. Thus the area of the plate is $A = \ell w$. If the bottom and top of the plate are at $y_1$ and $y_2$ the net force on the plate can be found as:

$$\begin{aligned}
\vec{\boldsymbol{F}}_{\text{net}} &= -\int_A p\hat{\boldsymbol{n}}\, dA \\
&= -\int_A \gamma(H-y)\hat{\boldsymbol{n}}\, dA \\
&= -w\int_0^\ell \gamma(H-y(s))\hat{\boldsymbol{n}}\, ds \\
&= -w\int_0^\ell \gamma\left(H-(y_1+\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\jmath}}\, s)\right)\hat{\boldsymbol{n}}\, ds \\
&= -w\gamma\left(H\ell - y_1\ell - \hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\jmath}}\,\ell^2/2\right)\hat{\boldsymbol{n}} \\
&= -w\ell\gamma\left(H-(y_1+\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\jmath}}\,\ell/2)\right)\hat{\boldsymbol{n}} \\
&= -w\ell\gamma\left(H-(y_1+(y_2-y_1)/2)\right)\hat{\boldsymbol{n}} \\
&= -w\ell\left(\gamma(H-y_1)/2+\gamma(H-y_2)/2\right)\hat{\boldsymbol{n}}
\end{aligned}$$

So

$$\begin{aligned}
\vec{\boldsymbol{F}}_{\text{net}} &= -w\ell\frac{p_1+p_2}{2}\,\hat{\boldsymbol{n}}. \\
&= -(\text{area})(\text{average pressure})(\text{outwards normal direction}).
\end{aligned}$$

The net water force is the same as that of the average pressure acting on the whole surface. To find where it acts it is easiest to think of the pressure distribution as the sum of two different pressure distributions. One is a constant over the plate at the pressure of the top of the plate. The other varies linearly from zero at the top to $\gamma(y_2 - y_1)$ at the bottom.

$$p = \gamma(H-y) = \underbrace{\gamma(H-y_2)}_{\text{Constant pressure, the pressure at the top edge.}} + \underbrace{\gamma(y_2-y)}_{\text{Varies linearly from 0 at the top to } \gamma(y_2-y_1) \text{ at the bottom.}}$$

The first corresponds to a force of $w\ell\gamma(H - y_2)$ acting at the middle of the plate. The second corresponds to a force of $w\ell\gamma\frac{y_2-y_1}{2}$ acting a third of the way up from the bottom of the plate.

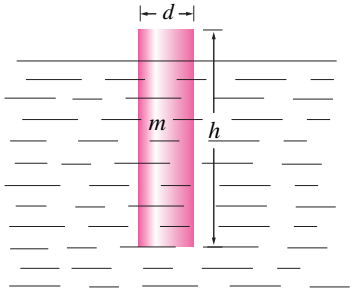



Figure 7.6:
Filename:sfig4-hydro-buoy

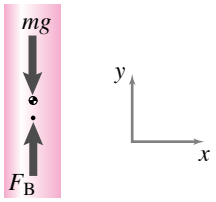



Figure 7.7:
Filename:sfig4-hydro-buoy-a

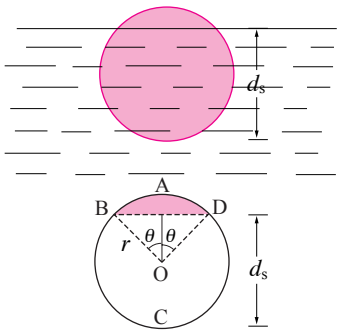



Figure 7.8:
Filename:sfig4-hydro-buoy-b

**SAMPLE 7.1** A uniform solid cylinder of mass $m = 12\,\text{kg}$, diameter $d = 0.1\,\text{m}$ and height $h = 2\,\text{m}$ floats in water (density $\rho = 1000\,\text{kg/m}^3$).

1. Assuming the cylinder floats vertically, find the submerged height of the cylinder.
2. If the cylinder floats longitudinally (its longitudinal axis parallel to the water surface), what will be the submerged section of the cylinder?

**Solution**

1. **Cylinder floating vertically:** Let $h_s$ be the submerged height of the cylinder and $r = d/2$ be its radius. Then the force of buoyancy $F_B$ is equal to the weight of water replaced by the submerged volume of the cylinder. Thus,

$$\vec{F}_\text{B} = \overbrace{\pi r^2 h_s}^{\text{volume}} \gamma\, \hat{j}.$$

From the force balance on the cylinder (see the free-body diagram in Fig. 7.7),

$$\begin{aligned} \vec{F}_\text{B} - mg\hat{j} &= \vec{0} \\ \Rightarrow \quad (\pi r^2 h_s \gamma - mg)\hat{j} &= \vec{0} \\ \Rightarrow \quad h_s &= \frac{mg}{\pi r^2 \gamma} = \frac{m}{\pi r^2 \rho} \\ &= \frac{12\,\text{kg}}{\pi \cdot (0.05\,\text{m})^2 \cdot 1000\,\text{kg/m}^3} = 1.53\,\text{m}. \end{aligned}$$

$$\boxed{h_s = 1.53\,\text{m}}$$

2. **Cylinder floating horizontally:** No matter how the cylinder floats, the force of buoyancy has to equal the weight of the cylinder. This force is equal to the weight of the displaced water. Thus, the volume of displaced water has to be the same no matter what the orientation of the cylinder is with respect to the water surface. Therefore, the submerged volume of the cylinder while floating longitudinally must equal the volume submerged while floating vertically. That is (see Fig. 7.8),

$$\text{area of BCD} \cdot h = \pi r^2 h_s \quad \Rightarrow \quad \text{area of BCD} = \pi r^2 (h_s/h) = 0.006\,\text{m}^2.$$

Now we can figure out what $d_s$ should be so that the submerged area is 76% of the total cross sectional area. This is an exercise in geometry. Since, area of BCD $=$ $\pi r^2 -$ area of ABD, area of ABD $= \pi r^2 -$ area of BCD $= 0.018\,\text{m}^2$. But the area of ABD is the area of the circular sector OBAD ($r^2\theta$) minus the area of triangle OBD ($\frac{1}{2} \cdot r\cos\theta \cdot 2r\sin\theta$). Thus,

$$\begin{aligned} \text{area of ABD} \equiv r^2\theta - \frac{1}{2}r^2 \sin 2\theta &= 0.018\,\text{m}^2 \\ \Rightarrow \quad \theta - \frac{1}{2}\sin 2\theta - 0.738 &= 0. \end{aligned}$$

We need to solve this nonlinear equation. Using trial and error or root finding on a computer or a graphical method, we find $\theta = 1.126\text{rad} = 64.5°$. Using this value, we get, $d_s = r + r\cos\theta = 0.07\,\text{m}$.

$$\boxed{d_s = 0.07\,\text{m}}$$

**SAMPLE 7.2 The force due to varying hydrostatic pressure:** The hydrostatic pressure distribution on the face of a wall submerged in water up to a height $h = 10\,\text{m}$ is shown in the figure. Find the net force on the wall from water. Take the length of the wall (into the page) to be 1 m.

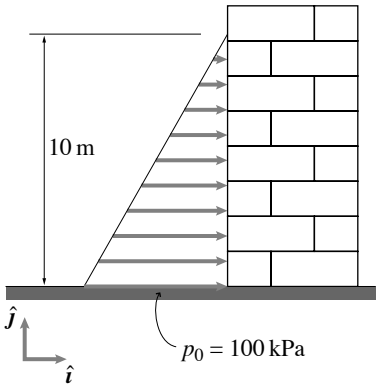



Figure 7.9:
Filename:sfig4-hydro-force1

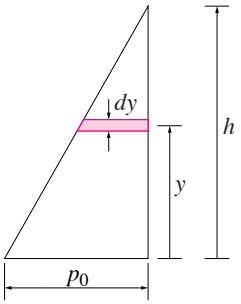
**Solution** Since the pressure varies across the height of the submerged part of the wall, let us take an infinitesimal strip of height $dy$ along the full length $\ell$ of the wall as shown in Fig. 7.10. Since the height of the strip is infinitesimal, we can treat the water pressure on this strip to be essentially constant and equal to $p_0\frac{y}{h}$. Then the force on the strip (of area $\ell dy$) due to the constant water pressure $p(y) = p_0 y/h$ is

$$d\vec{F} = (p(y)\cdot \overbrace{\ell dy}^{\text{area}})\,\hat{\imath} = p_0\frac{y}{h}\,\ell\,dy\,\hat{\imath}.$$

The net force due to the pressure distribution on the whole wall can now be found by integrating $d\vec{F}$ along the wall.

$$\begin{aligned}
\vec{F} &= \int d\vec{F} = \int_0^h p_0\frac{y}{h}\,\ell\,dy\,\hat{\imath} \\
&= \left(p_0\frac{\ell}{h}\int_0^h y\,dy\right)\hat{\imath} = p_0\frac{\ell}{h}\frac{h^2}{2}\hat{\imath} \\
&= \frac{1}{2}p_0 h\ell\,\hat{\imath} \\
&= \frac{1}{2}\cdot(100\frac{\text{kN}}{\text{m}^2})\cdot(10\,\text{m})\cdot(1\,\text{m})\hat{\imath} \\
&= (500\,\text{kN})\,\hat{\imath}.
\end{aligned}$$

$$\boxed{\vec{F} = 500\,\text{kN}\,\hat{\imath}}$$



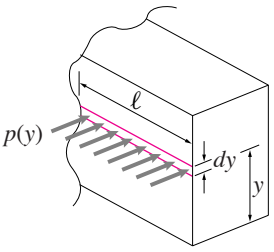




Figure 7.10:
Filename:sfig4-hydro-force1-a

**Alternatively,** the net force can be computed by calculating the area of the pressure triangle and multiplying by the unit length ($\ell = 1\,\text{m}$), *i.e.*,

$$\begin{aligned}
\vec{F} &= \overbrace{\frac{1}{2}\cdot h\cdot p_0}^{\text{triangle area}}\ \ell\,\hat{\imath} \\
&= \left(\frac{1}{2}\cdot 10\,\text{m}\cdot 100\frac{\text{kN}}{\text{m}^2}\cdot 1\,\text{m}\right)\hat{\imath} \\
&= 500\,\text{kN}\,\hat{\imath}.
\end{aligned}$$

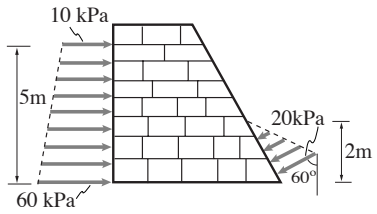



Figure 7.11:
Filename:sfig4-hydro-force2

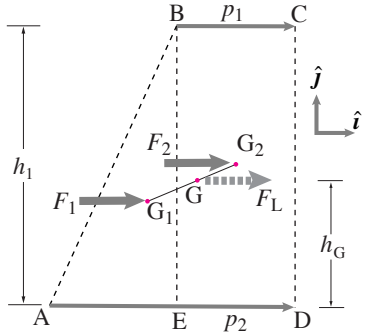



Figure 7.12:
Filename:sfig4-hydro-force2-a

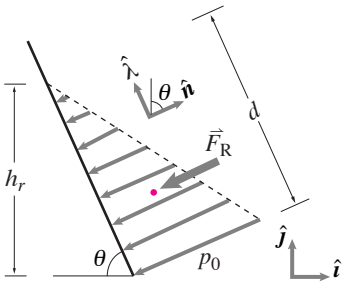



Figure 7.13:
Filename:sfig4-hydro-force2-b

**SAMPLE 7.3 The equivalent force due to hydrostatic pressure:** Find the net force and its location on each face of the dam due to the pressure distributions shown in the figure. Take unit length of the dam (into the page).

**Solution** We can determine the net force on each face of the dam by considering the given pressure distribution on one face at a time and finding the net force and its point of action.

On the left face of the dam we are given a trapezoidal pressure distribution. We break the given distribution into two parts — a triangular distribution given by ABE, and a rectangular distribution given by EBCD. We find the net force due to each distribution by finding the area of the distribution and multiplying by the unit length of the dam.

$$\begin{aligned}
\vec{F}_1 &= (\text{area of ABE}) \cdot \ell\hat{\boldsymbol{i}} = \frac{1}{2}(p_2 - p_1)h_l\ell\hat{\boldsymbol{i}} \\
&= \frac{1}{2}(60\text{kPa} - 10\text{kPa}) \cdot 5\,\text{m} \cdot 1\,\text{m}\hat{\boldsymbol{i}} \\
&= 125\,\text{kN}\hat{\boldsymbol{i}}, \\
\vec{F}_2 &= (\text{area of EBCD}) \cdot \ell\hat{\boldsymbol{i}} = p_1 h_l \ell\hat{\boldsymbol{i}} \\
&= 10\text{kPa} \cdot 5\,\text{m} \cdot 1\,\text{m}\hat{\boldsymbol{i}} \\
&= 50\,\text{kN}\hat{\boldsymbol{i}}.
\end{aligned}$$

The two forces computed above act through the centroids of the triangle ABE and the rectangle EBCD, respectively. The centroid are marked in Fig. 7.12. Now the net force on the left face is the vector sum of these two forces, *i.e.*,

$$\vec{F}_L = \vec{F}_1 + \vec{F}_2 = 175\,\text{kN}\hat{\boldsymbol{i}}.$$

The net force $\vec{F}_L$ acts through point G which is determined by the moment balance of the two forces $\vec{F}_1$ and $\vec{F}_2$ about point G:

$$\begin{aligned}
\vec{r}_{G_1/G} \times \vec{F}_1 &= -\vec{r}_{G_2/G} \times \vec{F}_2 \\
F_1(h_G - \frac{h_l}{3})\hat{\boldsymbol{k}} &= -F_2(\frac{h_l}{2} - h_G)(-\hat{\boldsymbol{k}}) \\
\Rightarrow \quad h_G &= \frac{F_1\frac{h_l}{3} + F_2\frac{h_l}{2}}{F_1 + F_2} \\
&= \frac{125\,\text{kN} \cdot 1.667\,\text{m} + 50\,\text{kN} \cdot 2.5\,\text{m}}{175\,\text{kN}} \\
&= 1.905\,\text{m}.
\end{aligned}$$

Similarly, we compute the force on the right face of the dam by calculating the area of the triangular distribution shown in Fig. 7.13.

$$\begin{aligned}
\vec{F}_R &= \frac{1}{2}p_0(\underbrace{h_r/\sin\theta}_{d})(\underbrace{-\sin\theta\hat{\boldsymbol{i}} - \cos\theta\hat{\boldsymbol{j}}}_{-\hat{\boldsymbol{n}}}) \\
&= \frac{1}{2}p_0 h_r(-\hat{\boldsymbol{i}} - \tan\theta\hat{\boldsymbol{j}}) \\
&= -20(\hat{\boldsymbol{i}} + \sqrt{3}\hat{\boldsymbol{j}})\,\text{kN}
\end{aligned}$$

and this force acts though the centroid of the triangle as shown in Fig. 7.13.

$$\boxed{\vec{F}_L = 175\,\text{kN}\hat{\boldsymbol{i}},\ \text{and}\ \vec{F}_R = -20(\hat{\boldsymbol{i}} + \sqrt{3}\hat{\boldsymbol{j}}\,\text{kN}}$$

**SAMPLE 7.4 Forces on a submerged sluice gate:** A rectangular plate is used as a gate in a tank to prevent water from draining out. The plate is hinged at A and rests on a frictionless surface at B. Assume the width of the plate to be 1 m. The height of the water surface above point A is $h$. Ignoring the weight of the plate, find the forces on the hinge at A as a function of $h$. In particular, find the vertical pull on the hinge for $h = 0$ and $h = 2\,\text{m}$.

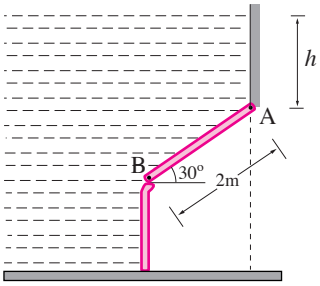



Figure 7.14:
Filename:sfig4-hydro-gate

**Solution** Let $\gamma = \rho g$ be the weight density (weight per unit volume) of water. Then the pressure due to water at point A is $p_A = \gamma h$ and at point B is $p_B = \gamma(h + \ell \sin\theta)$. The pressure acts perpendicular to the plate and varies linearly from $p_A$ at A to $p_B$ at B. The free-body diagram of the plate is shown in Fig. 7.15. Let $\hat{\boldsymbol{\lambda}}$ be a unit vector along BA and $\hat{\boldsymbol{n}}$ be a unit vector normal to BA. For computing the reaction forces on the plate at points A and B, we first replace the distributed pressure on the plate by two equivalent concentrated forces $F_1$ and $F_2$ by dividing the pressure distribution into a rectangular and a triangular region and finding their resultants.

$$F_1 = p_A \ell = \gamma h \ell, \quad F_2 = (p_B - p_A)\,\frac{\ell}{2} = \frac{1}{2}\gamma \ell^2 \sin\theta.$$

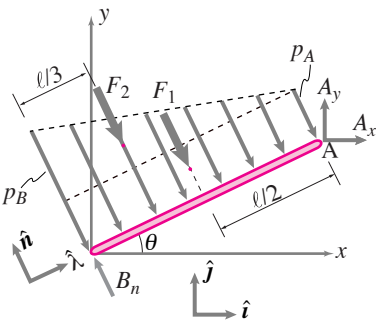



Figure 7.15:
Filename:sfig4-hydro-gate-a

Now, we carry out moment balance about point A, $\sum \vec{\boldsymbol{M}}_{\text{A}} = \vec{\boldsymbol{0}}$, which gives

$$\begin{aligned}
\vec{\boldsymbol{r}}_{\text{B/A}} \times \vec{\boldsymbol{B}} + \vec{\boldsymbol{r}}_{\text{D/A}} \times \vec{\boldsymbol{F}}_2 + \vec{\boldsymbol{r}}_{\text{C/A}} \times \vec{\boldsymbol{F}}_1 &= \vec{\boldsymbol{0}} \\
-\ell\hat{\boldsymbol{\lambda}} \times B_n\hat{\boldsymbol{n}} - \frac{2\ell}{3}\hat{\boldsymbol{\lambda}} \times (-F_1\hat{\boldsymbol{n}}) - \frac{\ell}{2}\hat{\boldsymbol{\lambda}} \times (-F_2\hat{\boldsymbol{n}}) &= \vec{\boldsymbol{0}} \\
-B_n\ell\hat{\boldsymbol{k}} + F_1\frac{2\ell}{3}\hat{\boldsymbol{k}} + F_2\frac{\ell}{2}\hat{\boldsymbol{k}} &= \vec{\boldsymbol{0}} \\
\Rightarrow \quad B_n = \frac{2F_1}{3} + \frac{F_2}{2} &= \gamma\ell(\frac{2}{3}h + \frac{1}{4}\ell\sin\theta)
\end{aligned}$$

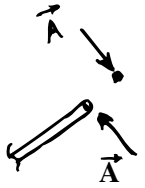



Figure 7.16:
Filename:sfig4-hydro-gate-b

and, from force balance, $\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$, we get

$$\begin{aligned}
\vec{\boldsymbol{A}} &= -B_n\hat{\boldsymbol{n}} + F_1\hat{\boldsymbol{n}} + F_2\hat{\boldsymbol{n}} \\
&= \left(-\gamma\ell(\frac{2}{3}h + \frac{1}{4}\ell\sin\theta) + \gamma h\ell + \frac{1}{2}\gamma\ell^2\sin\theta\right)\hat{\boldsymbol{n}} \\
&= \left(\frac{1}{3}\gamma h\ell + \frac{1}{2}\gamma\ell^2\sin\theta\right)\hat{\boldsymbol{n}} = \gamma\ell(\frac{1}{3}h + \frac{1}{2}\ell\sin\theta)\hat{\boldsymbol{n}}.
\end{aligned}$$

The force $\vec{\boldsymbol{A}}$ computed above is the force exerted by the hinge at A on the plate. Therefore, the force on the hinge, exerted by the plate, is $-\vec{\boldsymbol{A}}$ as shown in Fig. 7.16. From the expression for this force, we see that it varies linearly with $h$.

Let the vertical pull on the hinge be $A_{\text{hinge}_y}$. Then

$$A_{\text{hinge}_y} = -\vec{\boldsymbol{A}} \cdot \hat{\boldsymbol{\jmath}} = -\gamma\ell(\frac{1}{3}h + \frac{1}{2}\ell\sin\theta)\overbrace{\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\jmath}}}^{\cos\theta} = \frac{1}{4}\gamma\ell\sin 2\theta + (\frac{1}{3}\gamma\ell\cos\theta)h.$$

Now, substituting $\gamma = 9.81\,\text{kN/m}^3$, $\ell = 2\,\text{m}$, $\theta = 30°$, the two specified values of $h$, and multiplying the result (which is force per unit length) with the width of the plate (1 m) we get,

$$A_{\text{hinge}_y}(h = 0) = 4.25\,\text{kN}, \quad A_{\text{hinge}_y}(h = 2\,\text{m}) = 15.58\,\text{kN}.$$

$$\boxed{A_{\text{hinge}_y}|_{h=0} = 4.25\,\text{kN}, \quad A_{\text{hinge}_y}|_{h=2\,\text{m}} = 15.58\,\text{kN}}$$

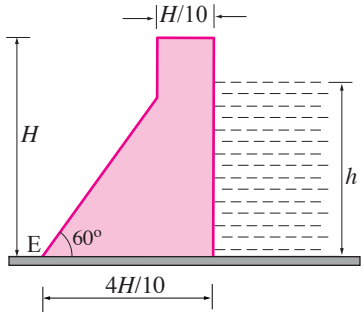



Figure 7.17:
Filename:sfig4-hydro-dam1

**SAMPLE 7.5 Tipping of a dam:** The cross section of a concrete dam is shown in the figure. Take the weight-density $\gamma(=\rho g)$ of water to be $10\,\text{kN/m}^3$ and that of concrete to be $25\,\text{kN/m}^3$. For the given design of the cross-section, find the ratio $h/H$ that is safe enough for the dam to not tip over (about the downstream edge E).

**Solution** Let us imagine the critical situation when the dam is just about to tip over about edge E. In such a situation, the dam bottom would almost lose contact with the ground except along edge E. In that case, there is no force along the bottom of the dam from the ground except at E.* With this assumption, the free-body diagram of the dam is shown in Fig. 7.18.

* This assumption is valid only if water does not leak through the edge B to the bottom of the dam. If it does, there would be some force on the bottom due to the water pressure. See the following sample where we include the water pressure at the bottom in the analysis.

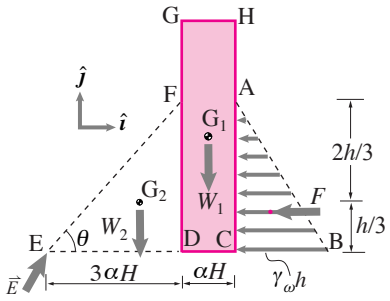



Figure 7.18:
Filename:sfig4-hydro-dam1-a

To compute all the forces acting on the dam, we assume the width $w$ (into the paper) to be unit (*i.e.*, $w = 1\,\text{m}$). Let $\gamma_w$ and $\gamma_c$ denote the weight-densities of water and concrete, respectively. Then the resultant force from the water pressure is

$$F = \frac{1}{2}\gamma_w h \cdot h \cdot w = \frac{1}{2}\gamma_w h^2 w.$$

This is the horizontal force (in the $-\hat{\boldsymbol{\imath}}$ direction) that acts through the centroid of triangle ABC.

To compute the weight of the dam, we divide the cross-section into two sections — the rectangular section CDGH and the triangular section DEF. We compute the weight of these sections separately by computing their respective volumes:

$$\begin{aligned} W_1 &= \underbrace{\alpha H^2 \cdot w}_{\text{volume}} \cdot \gamma_c = \gamma_c \alpha H^2 w \\ W_2 &= \underbrace{\frac{1}{2} \cdot 3\alpha H \cdot 3\alpha H \tan\theta \cdot w}_{\text{volume}} \cdot \gamma_c = \frac{9}{2}\gamma_c \alpha^2 H^2 w \tan\theta. \end{aligned}$$

Now we apply moment balance about point E, $\sum \vec{\boldsymbol{M}}_{\text{E}} = \vec{\boldsymbol{0}}$, which gives

$$\begin{aligned} \vec{\boldsymbol{r}}_{\text{G}_1} \times \vec{\boldsymbol{W}}_1 + \vec{\boldsymbol{r}}_{\text{G}_2} \times \vec{\boldsymbol{W}}_2 + \vec{\boldsymbol{r}}_{\text{G}_3} \times \vec{\boldsymbol{F}} &= \vec{\boldsymbol{0}} \\ -(3\alpha H + \frac{1}{2}\alpha H) W_1 \hat{\boldsymbol{k}} - \frac{2}{3}(3\alpha H) W_2 \hat{\boldsymbol{k}} + \frac{h}{3} F \hat{\boldsymbol{k}} &= \vec{\boldsymbol{0}}. \end{aligned}$$

Dotting this equation with $\hat{\boldsymbol{k}}$, we get

$$\begin{aligned} \frac{h}{3}F &= (3\alpha H + \frac{1}{2}\alpha H) \cdot \gamma_c \alpha H^2 w + \frac{2}{3}(3\alpha H) \cdot \frac{9}{2}\gamma_c \alpha^2 H^2 w \tan\theta \\ \frac{1}{2}\gamma_w \frac{h^3}{3} &= 9\gamma_c \alpha^3 H^3 \tan\theta + \frac{7}{2}\gamma_c \alpha^2 H^3 \\ \Rightarrow \quad \left(\frac{h}{H}\right)^3 &= \frac{\gamma_c}{\gamma_w}(54\alpha^3 \tan\theta + 21\alpha^2) \\ &= 2.5(54 \cdot 0.1^3 \cdot \sqrt{3} + 21 \cdot 0.1^2) = 0.7588 \\ \Rightarrow \quad \frac{h}{H} &= 0.91. \end{aligned}$$

Thus, for the dam to not tip over, $h \le 0.91H$ or 91% of $H$.

$$\boxed{\frac{h}{H} \le 0.91}$$

**SAMPLE 7.6 Dam design:** You are to design a dam of rectangular cross section ($b \times H$), ensuring that the dam does not tip over even when the water level $h$ reaches the top of the dam ($h = H$). Take the specific weight of concrete to be 3. Consider the following two scenarios for your design.

1. The downstream bottom edge of the dam is plugged so that there is no leakage underneath.
2. The downsteram edge is not plugged and the water leaked under the dam bottom has full pressure across the bottom.

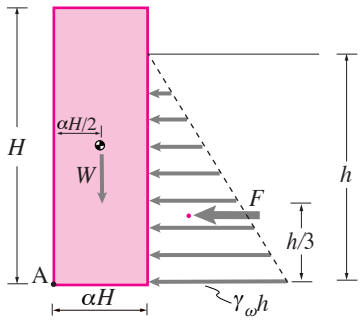



Figure 7.19:
Filename:sfig4-hydro-dam2-a

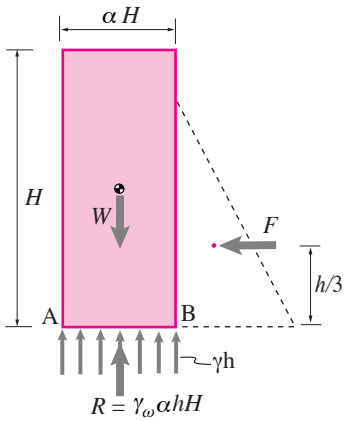



Figure 7.20:
Filename:sfig4-hydro-dam2-b

**Solution** Let $\gamma_c$ and $\gamma_w$ denote the weight densities of concrete and water, respectively. We are given that $\gamma_c/\gamma_w = 3$. Also, let $b/H = \alpha$ so that $b = \alpha H$. Now we consider the two scenarios and carry out analysis to find appropriate cross-section of the dam. In the calculations below, we consider unit length (into the paper) of the dam.

1. *No water pressure on the bottom:* When there is no water pressure on the bottom of the dam, then the water pressure acts only on the downstream side of the dam. The free-body diagram of the dam, considering critical tipping (just about to tip), is shown in Fig. 7.19 in which $F$ is the resultant force of the triangular water pressure distribution. The known forces acting on the dam are $W = \gamma_c \alpha H^2$, and $F = (1/2)\gamma_w h^2$. The moment balance about point A gives

$$\begin{aligned} F \cdot \frac{h}{3} &= W \cdot \frac{\alpha H}{2} \\ \frac{1}{2}\gamma_w \frac{h^3}{3} &= \gamma_c \frac{\alpha^2 H^3}{2} \\ \Rightarrow \quad \alpha^2 &= (1/3)(\gamma_w/\gamma_c)(h/H)^3. \end{aligned}$$

Considering the case of critical water level up to the height of the dam, *i.e.*, $h/H = 1$, and substituting $\gamma_c/\gamma_w = 3$, we get

$$\alpha^2 = 1/9 \quad \Rightarrow \quad \alpha = 1/3 = 0.333.$$

Thus the width of the cross-section needs to be at least one-third of the height. For example, if the height of the dam is 9 m then it needs to be at least 3 m wide.

$$\boxed{b/H = 0.33}$$

2. *Full water pressure on the bottom:* In this case, the water pressure on the bottom is uniformly distributed and its intensity is the same as the lateral pressure at B, *i.e.*, $p = \gamma_w h$. The free-body diagram diagram is shown in Fig. 7.20 where the known forces are $W = \gamma_c \alpha H^2$, $F = (1/2)\gamma_w h^2$, and $R = \gamma_w \alpha h H$. Again, we carry out moment balance about point A to get

$$\begin{aligned} F \cdot \frac{h}{3} &= (W - R) \cdot \frac{\alpha h}{2} \\ \gamma_w h^3 &= 3(\gamma_c \alpha H^2 - \gamma_w \alpha h H)\alpha H \\ \alpha^2 &= \frac{(h/H)^3}{3(\gamma_c/\gamma_w - h/H)}. \end{aligned}$$

Once again, substituting the given values and $h/H = 1$, we get

$$\alpha^2 = 1/6 \quad \Rightarrow \quad \alpha = 0.408.$$

Thus the width in this case needs to be at least 0.41 times the height $H$, slightly wider than the previous case.

$$\boxed{b/H \geq 0.41}$$

# Problems for Chapter 7

Statics

## 7.1 Net force and moments in hydrostatics

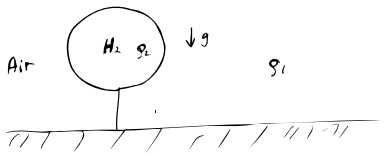
## Preparatory Problems

## More-Involved Problems

**7.1** A balloon with volume $V$, whose membrane has negligible mass, holds a gas with density $\rho_2$. It is surrounded by a gas with density $\rho_1$.

a) In terms of $\rho_1$, $\rho_2$, $g$, and $V$, find the tension in the string.

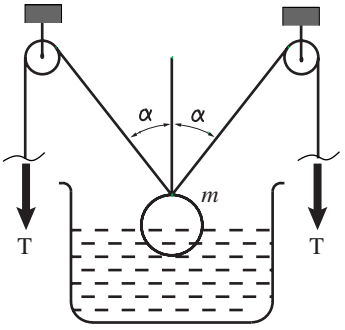
b) By some means look up the density of Helium and air at atmospheric temperature and pressure and calculate the volume, in cubic feet and in cubic meters, of a helium balloon that could lift 75 kg.




**problem 7.1:** Balloon

Filename:pfigure-hydro10

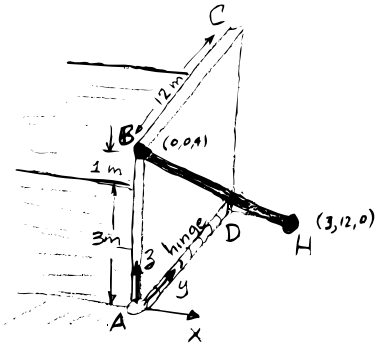
**7.2** A spherical body of mass $m = 10\,\text{kg}$ and radius $R = 100\,\text{mm}$ hangs from a continuous string as shown in the figure. The body is partially submerged in water and angle $\alpha = 45^\circ$ (fixed). If the force of buoyancy is $\rho V g$ where $\rho = 1000\,\text{kg/m}^3$ = density of water, $V$ is the submerged volume of the body, and $g$ is the usual $g$; find the tension in the string as a function of the submerged volume $V$. Find the maximum and the minimum tension corresponding to fully and zero submerged volume of the body respectively.




**problem 7.2:**

Filename:pfigure4-1-rp8

**7.3** A 4-meter-high 'door' holds back a stream ($\gamma = 1000\,\text{N/m}^3$) that is 3m deep and 12m wide. The door is hinged along its bottom and is propped up by a thin rod B that goes from a ball joint at H at () 3,12,0) to a boll joint at the upper left corner B of the door at (0,0,4). Neglect the mass of the door. Find the axial-force in the rod BH.



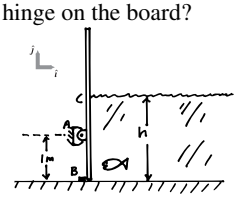

**problem 7.3:**

Filename:F02final2-dam

**7.4** Water is held in a reservoir by a board with negligible weight that is 5 meters long. It is hinged 1 meter off the bottom at A and kept from leaking by a seal at B. Assume $\rho = 1000\,\text{kg/m}^3$, $g = 10\,\text{N/kg}$.

a) What is $h$ when the board starts to pull away from the stop at B?

b) At that $h$ what is the force of the hinge on the board?



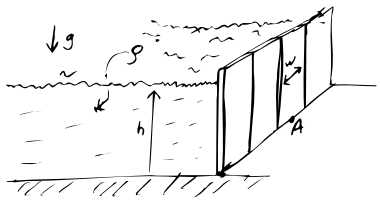

**problem 7.4:**

Filename:S03q5-dam

**7.5** The side of a pool is made of vertical boards which are stuck in the ground. Assuming that the boards, on average, get no support from their neighbors, and neglect the weight of the board itself,

a) calculate the force and moment from the ground on one board (answer in terms of some or all of $w$, $h$, $\rho$, and $g$.

b) For a one foot board and 8 foot deep pool, find the size of a force, and its location, so the force is equivalent to the water pressure on the board (answer in lbf).



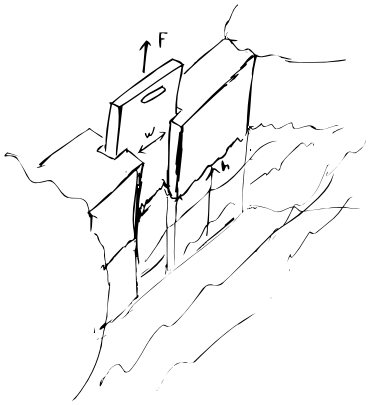

**problem 7.5:**

Filename:pfigure-hydro1

**7.6** A sluice gate is a dam that can be opened. Sometimes it is just a board in a slot that is opened by pulling up the board. For water with density $\rho$ and depth $h$ pressing against a board with width $w$ pressing against one face of the slot (the face away from the water) with coefficient of friction $\mu$

a) find the force $F$ needed to pull up the board in terms of $g$, $\rho$, $h$, and $w$.

b) Find the force in pounds force and Newtons assuming $g = 10\,\text{m/s}^2$, $h = 1\,\text{m}$, $w = 1\,\text{m}$, and $\rho = 1000\,\text{kg/m}^3$.




**problem 7.6:**

Filename:pfigure-hydro2

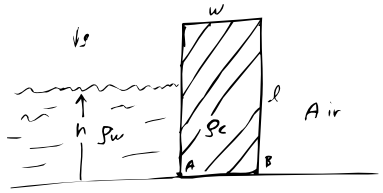
**7.7** A concrete (density = $\rho_c$) wall with height $\ell$, width $w$ and length (into the paper) $d$ rests on a flat rigid floor and serves as a damn for water with depth $h$ and density $\rho_w$. Assume the wall only makes contact at edges A and B.

a) Assume there is a seal at A, so no water gets under the damn. What is the coefficient of friction needed to keep the block from sliding?

b) What is the maximum depth of water before the block tips?

c) Assume that there is a seal at B and that water gets under the block. What is the coefficient of friction needed to keep the block from sliding?

d) What is the maximum depth of water before the block tips?



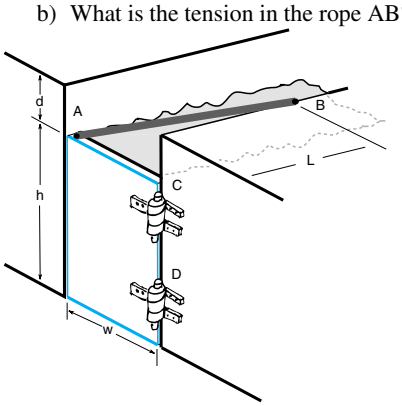

**problem 7.7:**

Filename:pfigure-hydro8

**7.8** A door holds back the water at a lock on a canal. The water surface is at the top of the door. The rope AB keeps it from swinging open. The door has hinges at C and D. The height of the door is $h$, the width $w$. The point B is a distance $d$ above the top of the door and is set back a distance $L$. The weight density of the water is $\gamma$.

a) What is the total force of the water on the door?

b) What is the tension in the rope AB?



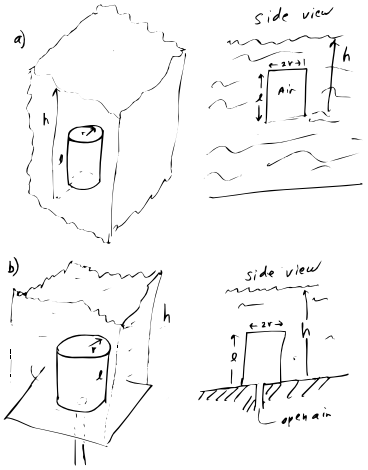

**problem 7.8:**

Filename:pfigure-s01-p1-3

**7.9** This problem somewhat explains the workings of some toilet valves. Open the tank of a toilet and look at the rubber piece at the bottom that sits on the bottom but then floats after initially lifted by the turning of the flush lever. The puzzle this problem solves is this: Why does the valve stick to the bottom, but then float when lifted.

a) A hollow cylinder with an open bottom (like an upside down but open can) is filled with air but is under water. What force is required to hold it under water (in terms of $\rho$, $r$, $h$, and $g$?

b) The same can is on the bottom of a tank of water and its edges are sealed. The bottom is open to atmospheric air. How much force is needed to hold the can down now (so there is no force from the bottom of the tank onto the edges of the cylinder)?




**problem 7.9:**

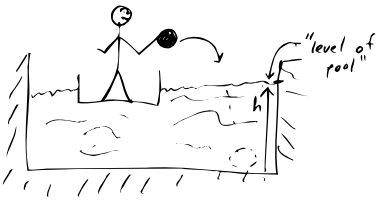
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• • •

**Some puzzles.** The following puzzles are sometimes presented as brain teasers. You should be able to reason the answers carefully and irrefutably. **7.10** A person is in a boat in a pool with surface area $A$. She is holding a ball with volume $V$ and mass $m$ in a still pool. The ball is then thrown into the pool, no water is splashed out and the pool comes to rest again.

a) Assuming the ball floats, by how much does the pool level go up or down?

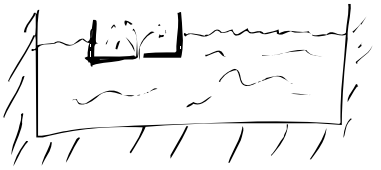
b) Assuming the ball sinks to the bottom by how much does the pool level go up or down?




**problem 7.10:**

Filename:pfigure-hydro5

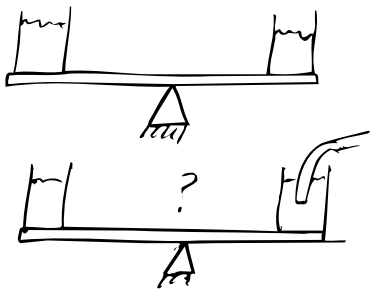
**7.11** A steel boat with mass $m$ and density $\rho_s$ is floating in a pool of water with density $\rho_w$ and cross sectional area $A$. By how much does the pool level go up or down when the boat sinks to the bottom?



**problem 7.11:**

Filename:pfigure-hydro6

**7.12** Two cups of water are balanced. You then gently stick your finger into one of them. Does this upset the balance? This experiment can be set up with two cups and a hexagonal-cross-section pencil. The cups need not be identical, they just need to be balanced at the start.



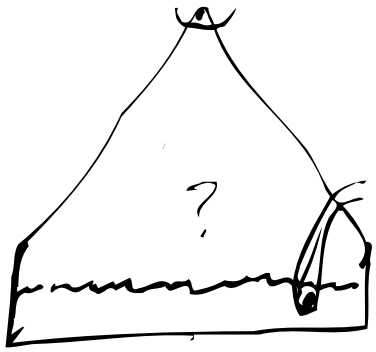

**problem 7.12:**

Filename:pfigure-hydro4

**7.13** A tray of water is suspended and level.

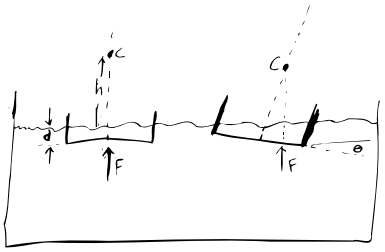
a) A hand is gently placed in the tray but does not touch the edges or bottom. Is the level of the tray upset.

b) *Challenge:* Assuming the tray is massless with width $w$ and water depth $h$, how high must be the hinge so the equilibrium is stable. That is, imagine the tray is rotated slightly about the hinge, the water pressure should cause a torque which tends to restore the vertical orientation shown.



**problem 7.13:**

Filename:pfigure-hydro7




**problem 7.14:** Ship stability.

Filename:pfigure-hydro9

**7.14** *Challenge:* This challenge problem is closely related to the challenge problem above, but is much more famous. It seems to have been first solved by Leonard Euler and Pierre Bouguer in about 1735. This solution seems to be the first mechanics problem in which the significance of the area moment of inertia was appreciated [this is a hint].

For simplicity assume that a boat is shaped like a box with width $h$ and and length into the paper of $b$. Assume that the boat floats with its bottom a depth $d$ under water. Now rotate the boat about an axis at the surface of the water and along its length (into the paper). Imagine that giant hands hold the boat in this position. In this rotated position the effect of the water pressure on the boat is a buoyant force and moment. This is equivalent to a force that is displaced slightly sideways.

Your goal is to find the height of the point that the line of action of this force intersects a mast of the boat. For small angles of boat tip the location is independent of the amount of tip.

This point is called the *metacenter* of the hull, and its distance up from the centroid of the boat's submerged volume is the hull's *metacentric height*. The condition of boat stability is that the metacenter be above the center of mass of the boat (thus the moment of the buoyant forces about the center of mass will tend to restore the boat to level).

Euler and Bouguer did the calculation you are asked to do here *after*, *e.g.*, , the launching of the 'great' Swedish ship Vasa, which capsized in the harbor on day 1. This was unfortunate for Sweden at the time, but fortunate now, because the brand-new 375 year old ship is a see-worthy tourist attraction in Stockholm.

CHAPTER 8

# Tension, shear and bending moment

*The 'internal forces' tension, shear and bending moment can vary from point to point in long narrow objects. Here we introduce the notion of graphing this variation and noting the features of these graphs.*

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8.1 Arbitrary cuts . . . . . **359**
Problems for Chapter 8 . . . . . **374**

In Section 4.4 starting on page 198 we defined the notion of *'internal forces'*, especially *tension* $T$, *shear* $V$, and *bending moment* $M$. A common issue in structural mechanics is keeping track of how these internal forces, and other more advance internal force concepts (ie stress) vary from point to point in a structure. Commonly this understanding comes from 'finite-element-method' programs*. However, there are a variety of important engineering problems for which accurate and useful estimation of internal forces can be found using methods at the level of this book. These are problems where the structure of interest is long and narrow. For reasons like those discussed in the introductory paragraphs about trusses (e.g. the discussion of 'swiss cheese' page 228), long narrow objects are surprisingly common in engineered objects as well as in biologically evolved designs. Despite the availability of computers for analysis of these, the elementary methods we will introduce here are useful because,

- For simple problems, they are easier than logging on to a computer;
- The methods here help build understanding and intuition;
- The methods here can give formulas from which a design can be controlled more easily than by numerical parameter studies;
- For very narrow things, the methods here are often more accurate than the computer solutions;
- To understand the vocabulary used in the output of the computer programs you need to understand the concepts associated with the methods here.

As for elementary truss analysis, the methods here are easily learned and pleasingly useful. For example, the formulas for bending moment in a simply-supported overhanging beam on page **??** not only tell you the 'internal moment' for a giving loading, but how to space the wing supports on a human-powered hydrofoil. And the capstain formula on page **??** isn't just a way to calculate cable tensions. It tells you how to make a simple modification to your bicycle to improve the performance of your brakes and derailleurs.

* Part of the finite element method is the dividing of an object into a grid; dividing the object of interest into 'finite elements'. Film and brochure makers are magnetically attracted to these grids, like a bee to a flower. So even if you have never used one of these programs, you have seen signs of them, grids superposed on objects, in advertising, science, and science-fiction videos.

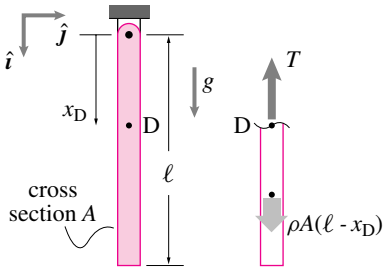



Figure 8.1: a) Rod hanging with gravity. b) free body diagram with cut at $x_D$.

Filename:tfigure-tensioncut

# 8.1 Free body cuts at arbitrary locations

## Tension, shear force, and bending moment diagrams

Engineers often want to know how the internal forces vary from point to point in a structure. If you want to know the internal forces at a variety of

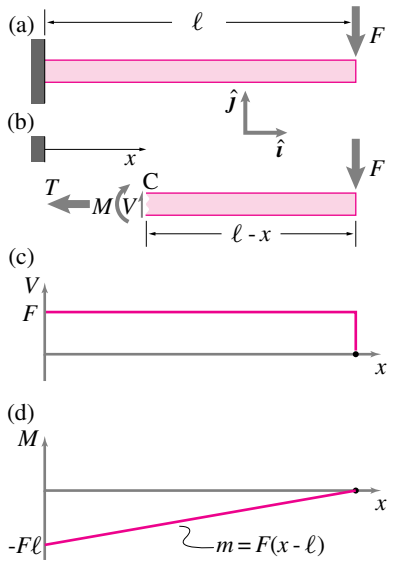



Figure 8.2: a) Cantilever beam, b) free body diagram, c) Shear force diagram, d) Bending moment diagram

Filename:tfigure-bendandsheardiag

points you can draw a variety of free body diagrams with cuts at those points of interest. Another approach, which we present now, is to leave the position of the free body diagram cut a variable, and then calculate the internal forces in terms of that variable.

Example: **Tension in a two-force body**
Recall that in the first example of this section we found $T$ without ever using information about the location of the free body diagram cut. So the location does not effect the tension. For a two force body the tension is a constant along the length.

Example: **Tension in a rod from its own weight.**
The uniform 1 cm$^2$ steel square rod with density $\rho = 7.7\,\text{gm/cm}^3$ and length $\ell = 100\,\text{m}$ has total weight $W = mg = \rho\ell Ag$ (see fig. 8.1). What is the tension a distance $x_\text{D}$ from the top? Using the free body diagram with cut at $x_\text{D}$ we get:

$$\begin{aligned}
&\left\{\sum \vec{F}_\text{i} = \vec{0}\right\}\cdot\hat{\imath} \\
\Rightarrow \quad T &= \rho Ag(\ell - x_\text{D}) \\
&= (7.7\,\text{gm/cm}^3)(1\,\text{cm}^2)(9.8\,\text{N/kg})(100\,\text{m} - x_\text{D}) \\
&= 7.7\cdot 9.8\,\tfrac{\text{gm N m}}{\text{cm kg}}\left(100 - \tfrac{x_\text{D}}{\text{m}}\right)\underbrace{\left(\frac{1\,\text{kg}}{1000\,\text{gm}}\right)}_{1}\underbrace{\left(\frac{100\,\text{cm}}{1\,\text{m}}\right)}_{1} \\
&= 7.5\left(100 - \tfrac{x_\text{D}}{\text{m}}\right)\,\text{N}.
\end{aligned}$$

So, at the bottom end at $x_\text{D} = 100\,\text{m}$ we get $T = 0$ and at the top end where $x_\text{D} = 0\,\text{m}$ we get $T = 750\,\text{N}$ and in the middle at $x_\text{D} = 50\,\text{m}$ we get $T = 375\,\text{N}$.

Because the free body diagram cut location is variable, we can plot the internal forces as a function of position. This is most useful in civil engineering where an engineer wants to know the internal forces in a horizontal beam carrying vertical loads. Common examples include bridge platforms and floor joists.

Example: **Cantilever $M$ and $V$ diagram**
A *cantilever* beam is mounted firmly at one end and has various loads orthogonal to its length, in this case a downwards load $F$ at the end (fig. 8.2a). By drawing a free body diagram with a cut at the arbitrary point C (fig. 8.2b) we can find the internal forces as a function of the position of C.

$$\begin{aligned}
\left\{\sum \vec{F}_\text{i} = \vec{0}\right\}\cdot\hat{\jmath} \quad &\Rightarrow \quad V = F \\
\left\{\sum \vec{F}_\text{i} = \vec{0}\right\}\cdot\hat{\imath} \quad &\Rightarrow \quad T = 0 \\
\left\{\sum \vec{M}_\text{C} = \vec{0}\right\}\cdot\hat{k} \quad &\Rightarrow \quad M(x) = F(x - \ell).
\end{aligned}$$

That the tension is zero in these problems is so well known that the tension is often not drawn on the free body diagram and not calculated. We can now plot $V(x)$ and $M(x)$ as in figs. 8.2c and 8.2d. In this case the shear force is a constant and the bending moment varies from its maximum magnitude at the wall ($M = -F\ell$) to 0 at the end. It is the big value of $|M|$ at the fixed support that makes cantilever beams typically break there.

Often one is interested in distributed loads from gravity on the structure itself or from a distribution (say of people on a floor). The method is the same.

Example: **Distributed load**
A cantilever beam has a downwards uniformly distributed load of $w$ per unit length (fig. 8.3a).

Using the free body diagram shown (fig. 8.3b) we can find:

$$\begin{aligned}
\left\{\sum \vec{F}_i = \vec{0}\right\}\cdot\hat{\jmath} \quad &\Rightarrow \quad \left\{V(x)\hat{\jmath} \quad + \quad \int d\vec{F}\right\}\cdot\hat{\jmath} \;=\; 0 \\
&\Rightarrow \quad V(x) \;=\; \int_x^\ell w\,dx' \\
&\qquad\qquad\;\; =\; w\cdot(\ell - x)
\end{aligned}$$

$$\begin{aligned}
\left\{\sum \vec{M}_C = \vec{0}\right\}\cdot\hat{k} \quad &\Rightarrow \quad \left\{M(x)(-\hat{k}) \quad + \quad \int \vec{r}_{/C}\times d\vec{F}\right\}\cdot\hat{k} \;=\; 0 \\
&\Rightarrow \quad M(x) \;=\; \int_x^\ell (x'-x)w\,dx' \\
&\qquad\qquad\;\; =\; w\cdot(x'^2/2 - x'x)\Big|_x^\ell \\
&\qquad\qquad\;\; =\; (\ell^2/2 - \ell x) - (x^2/2 - x^2) \\
&\qquad\qquad\;\; =\; -w\cdot(\ell - x)^2/2.
\end{aligned}$$

The integrals were used because of their general applicability for distributed loads. For this problem we could have avoided the integrals by using an equivalent downwards force $w\cdot(\ell - x)$ applied a distance $(\ell - x)/2$ to the right of the cut. Shear and bending moment diagrams are shown in figs. 8.3a and 8.3b.

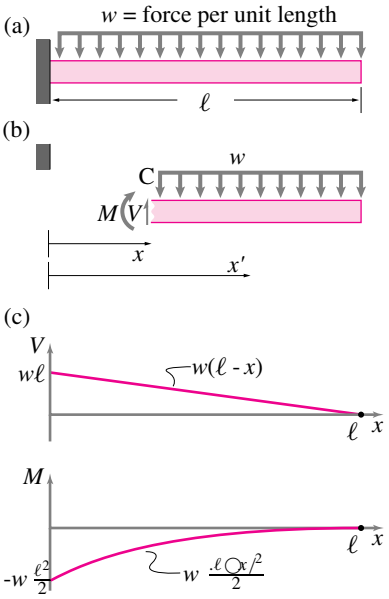



Figure 8.3: a) Cantilever beam, b) free body diagram, c) Shear force diagram, d) Bending moment diagram

Filename:tfigure-uniformcant

As for all problems based on the equilibrium equations and a given geometry, the principle of superposition applies.

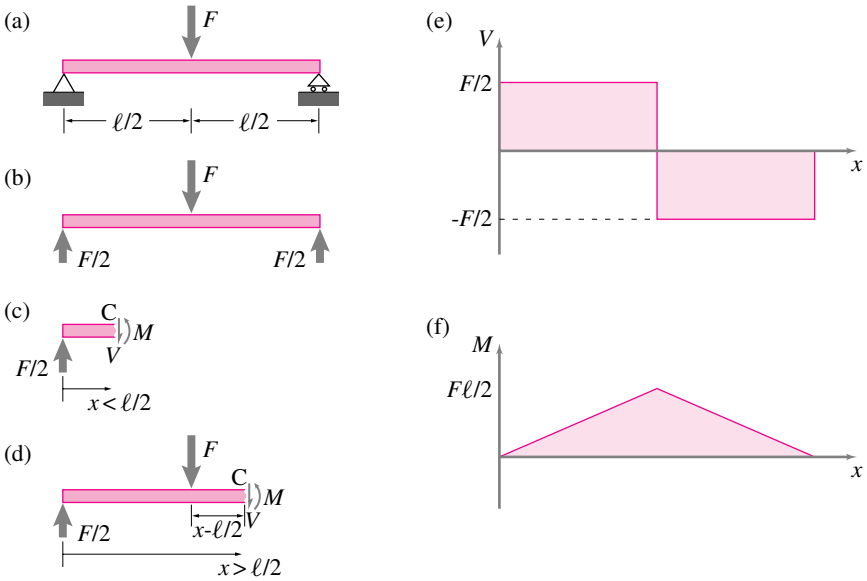
Example: **Superposition**
Consider a cantilever beam that simultaneously has both of the loads from the previous two examples. By the principle of superposition:

$$\begin{aligned}
V \quad &= \quad F \quad + \quad w(\ell - x) \\
M(x) \quad &= \quad F(x-\ell) \quad + \quad -w(\ell - x)^2/2.
\end{aligned}$$

The shear force at every point is the sum of the shear forces from the previous examples. The bending moment at every point is the sum of the bending moments.

If there are concentrated loads in the middle of the region of interest the calculation gets more elaborate; the concentrated force may or may not show up on the free body diagram of the cut bar, depending on the location of the cut.

Example: **Simply supported beam with point load in the middle**



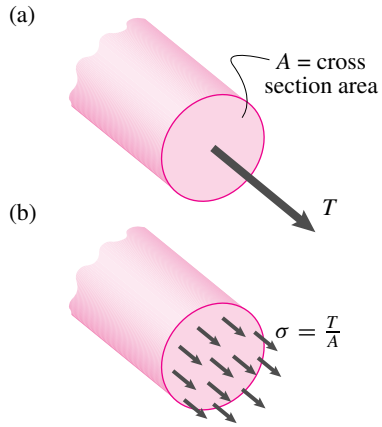



Figure 8.5: a) Tension on a free body diagram cut is equivalent to b) uniform tension stress.

Filename:tfigure-tension

Figure 8.4: a) Simply supported beam, b) free body diagram of whole beam, c) free body diagram with cut to the left of the applied force, d) free body diagram with cut to the right of the applied force e) Shear force diagram, f) Bending moment diagram

Filename:tfigure-simplesup

A *simply supported* beam is mounted with pivots at both ends (fig. 8.4a). First we draw a free body diagram of the whole beam (fig. 8.4a) and then two more, one with a cut to the left of the applied force and one with a cut to the right of the applied force (figs. 8.4c and 8.4d). With the free body diagram 8.4c we can find $V(x)$ and $M(x)$ for $x < \ell/2$ and with the free body diagram 8.4d we can find $V(x)$ and $M(x)$ for $x > \ell/2$.

$$\begin{aligned}
\left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{j} \quad &\Rightarrow \quad V = F/2 \quad \text{for } x < \ell/2 \\
&\quad\;\; = -F/2 \quad \text{for } x > \ell/2 \\
\left\{\sum \vec{M}_C = \vec{0}\right\} \cdot \hat{k} \quad &\Rightarrow \quad M(x) = Fx/2 \quad \text{for } x < \ell/2 \\
&\quad\;\; = F(\ell - x)/2 \quad \text{for } x > \ell/2
\end{aligned}$$

These relations can be plotted as in figs. 8.4e and 8.4f. Some observations: For this beam the biggest bending moment is in the middle, the place where simply supported beams often break. Instead of the free body diagram shown in (c) and (d) we could have drawn a free body diagrams of the bar to the right of the cut and would have got the same $V(x)$ and $M(x)$. We avoided drawing a free body diagram cut *at* the applied load where $V(x)$ has a discontinuity.

## How to find $T$, $V$, and $M$

Here are some guidelines for finding internal forces and drawing shear and bending moment diagrams.

- Draw a free body diagram of the whole bar.
- Using the free body diagram above find the reaction forces .
- Draw a free body diagram(s) of the cut bar of interest.
  - For each region between concentrated loads draw one free body diagram.
  - Show the piece from the cut to one or the other end (So that all but the internal forces are known).
  - Don't make cuts *at* intermediate points of connection or load application.
- Use the equilibrium equations to find $T$, $V$, or $M$ (Moment balance about a point at the cut is a good way to find $M$.)
- Use the results above to plot $V(x)$ and $M(x)$ ($T(x)$ is rarely plotted).
  - Use the same $x$ scale for this plot as for the free body diagram of the whole bar.
  - Put the plots directly under the free body diagram of the bar (so you can most easily relate features of the loads to features of the $V$ and $M$ diagrams).

## Stress is force per unit area

For a given load, if you replace one bar in tension with two bars side by side you would imagine the tension in each bar would go down by a factor of 2. Thus the pair of bars should be twice as strong as a single bar. If you glued these side by side bars together you would again have one bar but it would be twice as strong as the original bar. Why? Because it has twice the cross sectional area.

What makes a solid break is the force per unit area carried by the material. For an applied tension load $T$, the force per unit area on an interior free body diagram cut is $T/A$. Force per unit area normal to an internal free body diagram cut is called *tension stress* and denoted $\sigma$ (lower case 'sigma', the Greek letter s).

$$\sigma = T/A$$

Example: **Stress in a hanging bar**
Look at the hanging bar in the example on page 360. We can find the tension stress in this bar as a function of position along the bar as:

$$\sigma = \frac{T}{A} = \frac{\rho g A(\ell - x)}{A} = \rho g(\ell - x).$$

Note that the stress for this bar doesn't depend on the cross sectional area. The bigger the area the bigger the volume and hence the load. But also, the bigger the area on which to carry it.

For reasons that are beyond this book, the tension stress tends to be uniform in homogeneous (all one material) bars, no matter what their cross sectional shape, so that the average tension stress $\frac{T}{A}$ is actually the tension stress all across the cross section.

We can similarly define the average *shear stress* $\tau_{\text{ave}}$ ('tau') on a free body diagram cut as the average force per unit area tangent to the cut,

$$\tau_{\text{ave}} = \frac{V}{A}.$$

For reasons you may learn in a strength of materials class, shear stress is not so uniformly distributed across the cross section. But the average shear stress $\tau_{\text{ave}}$ does give an indication of the actual shear stress in the bar (*e.g.*, for a rectangular elastic bar the peak shear stress is 50% larger than $\tau_{\text{ave}}$).

The biggest stresses typically come from bending moment. Motivating formulas for these stresses here is too big a digression. The formulas for the stresses due to bending moment are a key part of elementary strength of materials. But just knowing that these stresses tend to be big, gives you the important notion that bending moment is a common cause of structural failure.

## Internal force summary

'Internal forces' are the scalars which describe the force and moment on potential internal free body diagram cuts. They are found by applying the equilibrium equations to free body diagrams that have cuts at the points of interest. The internal forces are intimately associated with the internal stresses (force per unit area) and thus are important for determining the strength of structures.

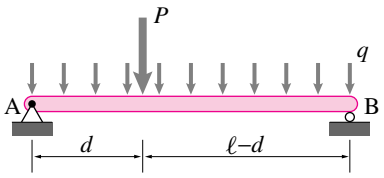



Figure 8.6:

Filename:sfig4-intern-ssup

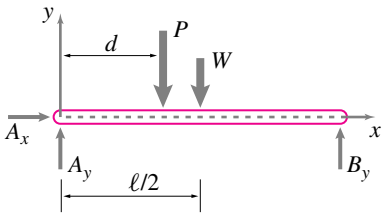



Figure 8.7:

Filename:sfig4-intern-ssup-a

**SAMPLE 8.1 Support reactions on a simply supported beam:** A uniform beam of length 3 m is simply supported at A and B as shown in the figure. A uniformly distributed vertical load $q = 100\,\text{N/m}$ acts over the entire length of the beam. In addition, a concentrated load $P = 150\,\text{N}$ acts at a distance $d = 1\,\text{m}$ from the left end. Find the support reactions.

**Solution** Since the beam is supported at A on a pin joint and at B on a roller, the unknown reactions are

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}, \qquad \vec{B} = B_y\hat{\jmath}.$$

The uniformly distributed load $q$ can be replaced by an equivalent concentrated load $W = q\ell$ acting at the center of the beam span. The free-body diagram of the beam, with the concentrated load replaced by the equivalent concentrated load is shown in Fig. 8.7. The moment equilibrium about point A, $\sum \vec{M}_\text{A} = \vec{0}$, gives

$$\begin{aligned} (-Pd - W\frac{\ell}{2} + B_y\ell)\hat{k} = \vec{0} \\ \Rightarrow \quad B_y &= P\frac{d}{\ell} + \frac{1}{2}W \\ &= 150\,\text{N}\cdot\frac{1}{3} + \frac{1}{2}\cdot 300\,\text{N} = 200\,\text{N}. \end{aligned}$$

The force equilibrium, $\sum \vec{F} = \vec{0}$, gives

$$\begin{aligned} \vec{A} + B_y\hat{\jmath} - P\hat{\jmath} - W\hat{\jmath} = \vec{0} \\ \Rightarrow \quad \vec{A} &= (-B_y + P + W)\hat{\jmath} \\ &= (-200\,\text{N} + 150\,\text{N} + 300\,\text{N})\hat{\jmath} = 250\,\text{N}\hat{\jmath}. \end{aligned}$$

$$\boxed{\vec{A} = 250\,\text{N}\hat{\jmath}, \quad \vec{B} = 200\,\text{N}\hat{\jmath}}$$

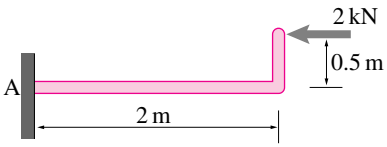



Figure 8.8:

Filename:sfig4-intern-cant

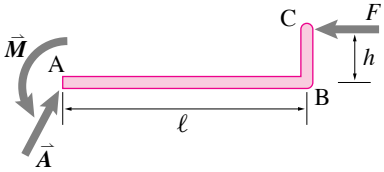



Figure 8.9:

Filename:sfig4-intern-cant-a

**SAMPLE 8.2 Support reactions on a cantilever beam:** A 2 kN horizontal force acts at the tip of an 'L' shaped cantilever beam as shown in the figure. Find the support reactions at A.

**Solution** The free-body diagram of the beam is shown in Fig. 8.9. The reaction force at A is $\vec{A}$ and the reaction moment is $\vec{M} = M\hat{k}$. Writing moment balance equation about point A, $\sum \vec{M}_\text{A} = \vec{0}$, we get

$$\begin{aligned} \vec{M} + \vec{r}_{\text{C/A}} \times \vec{F} &= \vec{0} \\ \vec{M} + (\ell\hat{\imath} + h\hat{\jmath}) \times (-F\hat{\imath}) &= \vec{0} \\ \Rightarrow \quad \vec{M} &= -Fh\,\hat{k} \\ &= -2\,\text{kN}\cdot 0.5\,\text{m}\,\hat{k} \\ &= -1\,\text{kN}\cdot\,\text{m}\,\hat{k}. \end{aligned}$$

The force equilibrium, $\sum \vec{F} = \vec{0}$, gives

$$\begin{aligned} \vec{A} + \vec{F} &= \vec{0} \\ \Rightarrow \quad \vec{A} &= -\vec{F} = -(-2\,\text{kN}\,\hat{\imath}) = 2\,\text{kN}\,\hat{\imath}. \end{aligned}$$

$$\boxed{\vec{A} = 2\,\text{kN}\,\hat{\imath}, \quad \vec{M} = -1\,\text{kN}\cdot\,\text{m}\,\hat{k}}$$

**SAMPLE 8.3 Net force of a uniformly distributed system:** A uniformly distributed vertical load of intensity 100 N/macts on a beam of length $\ell = 2\,\text{m}$ as shown in the figure.

1. Find the net force acting on the beam.
2. Find an equivalent force-couple system at the mid-point of the beam.
3. Find an equivalent force-couple system at the right end of the beam.

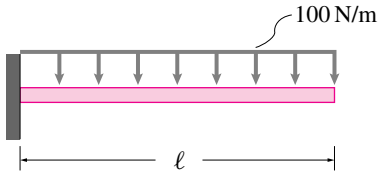



Figure 8.10:
Filename:sfig2-vec3-uniform

**Solution**

1. **The net force:** Since the load is uniformly distributed along the length, we can find the total or the net load by calculating the load on an infinitesimal segment of length $dx$ of the beam and then integrating over the entire length of the beam. Let the load intensity (load per unit length) be $q$ ($q = 100\,\text{N/m}$, as given). Then the vertical load on segment $dx$ is (see Fig. 8.11),

$$d\vec{F} = q\,dx(-\hat{j}).$$

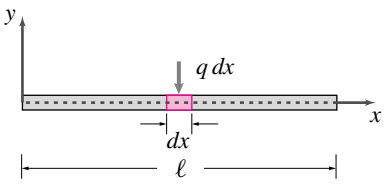



Figure 8.11:
Filename:sfig2-vec3-uniform-a

Therefore, the net force is,

$$\vec{F}_{\text{net}} = \int_0^{\ell} q\,dx(-\hat{j}) = q\,\ell\,\hat{j} = -100\,\text{N/m}\cdot 2\,\text{m}\,\hat{j} = -200\,\text{N}\,\hat{j}.$$

$$\boxed{\vec{F}_{\text{net}} = -200\,\text{N}\,\hat{j}}$$

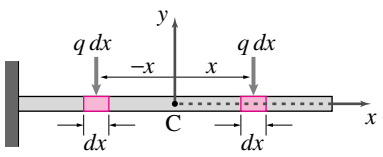



Figure 8.12:
Filename:sfig2-vec3-uniform-b

2. **The equivalent system at the mid-point:** We have already calculated the net force that can replace the uniformly distributed load. Now we need to calculate the couple at the mid-point of the beam to get the equivalent force-couple system. Again, consider a small segment of the beam of length $dx$ located at distance $x$ from the mid-point C (see Fig. 8.12). The moment about point C due to the load on $dx$ is $(q\,dx)x(-\hat{k})$. But, we can find a similar segment on the other side of C with exactly the same length $dx$, at exactly the same distance $x$, that produces a moment of $(q\,dx)x(+\hat{k})$. The two contributions cancel each other and we have a net zero moment about C. Now, you can imagine the whole beam made up of these pairs that contribute equal and opposite moment about C and thus the net moment about the mid-point is zero. You can also find the same result by straight integration:

$$\vec{M}_{\text{C}} = \int_{-\ell/2}^{+\ell/2} qx\,dx(-\hat{k}) = \frac{qx^2}{2}\bigg|_{-\ell/2}^{+\ell/2}(-\hat{k}) = \vec{0}.$$

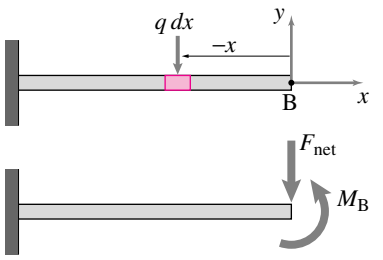



Figure 8.13:
Filename:sfig2-vec3-uniform-c

$$\boxed{\vec{F}_{\text{net}} = -200\,\text{N}\,\hat{j},\ \text{and}\ \vec{M}_{\text{C}} = \vec{0}}$$

3. **The equivalent system at the end:** The net force remains the same as above. We compute the net moment about the end point B, referring to Fig. 8.13, as follows.

$$\begin{aligned}\vec{M}_{\text{B}} &= \int_0^{\ell}(-x\hat{\imath})\times(-q\,dx\,\hat{\jmath}) = -q\int_0^{\ell} x\,dx\hat{k}\\ &= -\frac{q\ell^2}{2}\hat{k} = -\frac{100\,\text{N/m}\cdot 4\,\text{m}^2}{2}\hat{k} = -200\,\text{N·m}\hat{k}.\end{aligned}$$

$$\boxed{\vec{F}_{\text{net}} = -200\,\text{N}\,\hat{j}\ \text{and}\ \vec{M}_{\text{B}} = -200\,\text{N·m}\hat{k}}$$

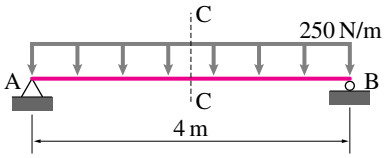



Figure 8.14:
Filename:sfig4-intern-ssvm

**SAMPLE 8.4** For the uniformly loaded, simply supported beam shown in the figure, find the shear force and the bending moment at the mid-section c-c of the beam.

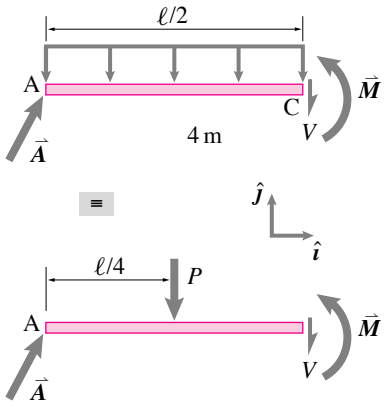



Figure 8.15:
Filename:sfig4-intern-ssvm-a

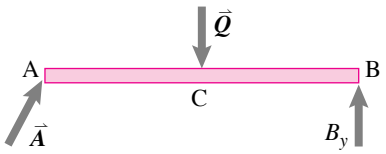



Figure 8.16:
Filename:sfig4-intern-ssvm-b

**Solution** To determine the shear force $V$ and the bending moment $M$ at the mid-section c-c, we cut the beam at c-c and draw its free-body diagram as shown in Fig. 8.15. For writing force and moment balance equations we use the second figure where we have replaced the distributed load with an equivalent single load $F = (q\ell)/2$ acting vertically downward at distance $\ell/4$ from end A.

The force balance, $\sum \vec{F} = \vec{0}$, implies that

$$A_x\hat{\imath} + A_y\hat{\jmath} - V\hat{\jmath} - F\hat{\jmath} = \vec{0}.$$

Dotting with $\hat{\imath}$ and $\hat{\jmath}$, respectively, we get

$$\begin{aligned} A_x &= 0 \\ V &= A_y - F \qquad (8.1) \\ &= A_y - \frac{q\ell}{2}. \qquad (8.2) \end{aligned}$$

From the moment equilibrium about point A, $\sum \vec{M}_{\mathrm{A}} = \vec{0}$, we get

$$M\hat{k} - \left(\frac{q\ell}{2}\cdot\frac{\ell}{4}\right)\hat{k} - V\ell\hat{k} = 0$$

$$\Rightarrow \quad M = \frac{q\ell^2}{8} + V\ell. \qquad (8.3)$$

Thus, to find $V$ and $M$ we need to know the support reaction $\vec{A}$. From the free-body diagram of the beam in Fig. 8.16 and the moment equilibrium equation about point B, $\sum \vec{M}_{\mathrm{B}} = \vec{0}$, we get

$$\begin{aligned} \vec{r}_{\mathrm{A/B}} \times \vec{A} + \vec{r}_{\mathrm{C/B}} \times \vec{Q} &= \vec{0} \\ (-A_y\ell + q\ell\,\frac{\ell}{2})\hat{k} &= \vec{0} \\ \Rightarrow \quad A_y = \frac{q\ell}{2} &= 500\,\mathrm{N}. \end{aligned}$$

Thus $\vec{A} = 500\,\mathrm{N}\,\hat{\jmath}$. Substituting $\vec{A}$ in eqns. (8.2) and (8.3), we get

$$\begin{aligned} V &= 500\,\mathrm{N} - 500\,\mathrm{N} = 0 \\ M &= 250\,\mathrm{N}\cdot\frac{(4\,\mathrm{m})^2}{8} + 0 \\ &= 500\,\mathrm{N{\cdot}m}. \end{aligned}$$

$$\boxed{V = 0, \quad M = 500\,\mathrm{N{\cdot}m}}$$

**SAMPLE 8.5** The cantilever beam AD is loaded as shown in the figure where $W = 200\,\text{lbf}$. Find the shear force and bending moment on a section just left of point B and another section just right of point B.

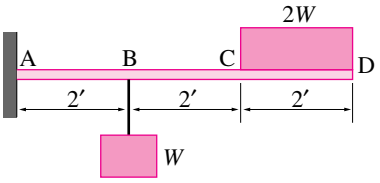



Figure 8.17:
Filename:sfig4-intern-cantvm

**Solution** To find the desired internal forces, we need to make a cut at a section just to the left of B and one just to the right of B. We first take the one that is to the right of point B. The free-body diagram of the right part of the cut beam is shown in Fig. 8.18. Note that if we selected the left part of the beam, we would need to determine support reactions at A. The uniformly distributed load $2W$ of the block sitting on the beam can be replaced by an equivalent concentrated load $2W$ acting at point E, at distance $a/2$ from the end D of the beam.

Let us denote the the shear force by $V^+$ and the bending moment by $M^+$ at the section of our interest. Now, from the force equilibrium of the part-beam BD we get

$$\begin{aligned} V^+\hat{\boldsymbol{\jmath}} - 2W\hat{\boldsymbol{\jmath}} &= \vec{\mathbf{0}} \\ \Rightarrow \quad V^+ &= 2W \\ &= 400\,\text{lbf}. \end{aligned}$$

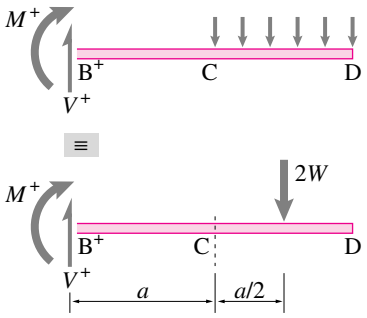



Figure 8.18:
Filename:sfig4-intern-cantvm-a

The moment equilibrium about point B, $\sum \vec{\boldsymbol{M}}_{\text{B}} = \vec{\mathbf{0}}$, gives

$$\begin{aligned} -M^+\hat{\boldsymbol{k}} - 2W \cdot \frac{3a}{2}\hat{\boldsymbol{k}} &= \vec{\mathbf{0}} \\ \Rightarrow \quad M^+ &= -3Wa \\ &= -1200\,\text{lb·ft}. \end{aligned}$$

Now, we determine the internal forces at a section just to the left of point B. Let the shear and bending moment at this section be $V^-$ and $M^-$, respectively, as shown in the free-body diagram (Fig. 8.19). Note that load $W$ acting at B is now included in the free-body diagram since the beam is now cut just a teeny bit left of this load.

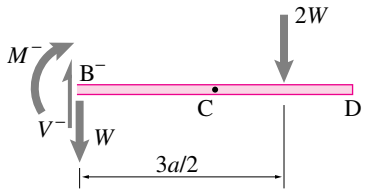



Figure 8.19:
Filename:sfig4-intern-cantvm-b

From the force equilibrium of the part-beam, we have

$$\begin{aligned} V^-\hat{\boldsymbol{\jmath}} - W\hat{\boldsymbol{\jmath}} - 2W\hat{\boldsymbol{\jmath}} &= \vec{\mathbf{0}} \\ \Rightarrow \quad V^- &= 3W \\ &= 600\,\text{lbf} \end{aligned}$$

and, from moment equilibrium about point B, $\sum \vec{\boldsymbol{M}}_{\text{B}} = \vec{\mathbf{0}}$, we get

$$\begin{aligned} -M^-\hat{\boldsymbol{k}} - 2W \cdot \frac{3a}{2}\hat{\boldsymbol{k}} &= \vec{\mathbf{0}} \\ \Rightarrow \quad M^- &= -3Wa \\ &= -1200\,\text{lb·ft}. \end{aligned}$$

$$\boxed{M^+ = M^- = -1200\,\text{lb·ft}, \quad V^+ = 400\,\text{lbf}, \quad V^+ = 600\,\text{lbf}}$$

Note that the bending moment remains the same on either side of point B but the shear force jumps by $V^+ - V^- = 200\,\text{lbf} = W$ as we go from right to the left. This jump is expected because a concentrated load $W$ acts at B, in between the two sections we consider. Concentrated external forces cause a jump in shear, and concentrated external moments cause a jump in the bending moment.

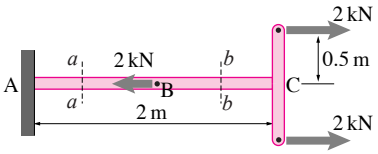



Figure 8.20:

Filename:sfig4-intern-axial1

**SAMPLE 8.6 Tension in a bar:** A T-shaped bar is fixed in a wall at one end and is acted by three forces as shown in the figure. Find the tension in the rod at

1. section $a$-$a$, and
2. section $b$-$b$.

**Solution**

1. Let us cut the bar at section $a$-$a$ and consider the part of the bar to the right of the cut-section. The free-body diagram of this part of the bar is shown in Fig. 8.20. The scalar force balance in the horizontal direction gives

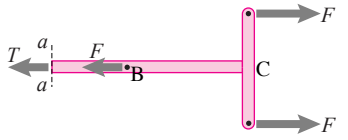



Figure 8.21:

Filename:sfig4-intern-axial1a

$$\begin{aligned} -T - F + 2F &= 0 \\ \Rightarrow \quad T &= F \\ &= 2\,\text{kN}. \end{aligned}$$

At section $a$-$a$: $T = 2\,\text{kN}$

2. Now, we cut the bar at section $b$-$b$ and again consider the section of the bar to the right of the cut-section. The free-body diagram of this part of the bar is shown in Fig. 8.22. Again, the force balance in the horizontal direction gives

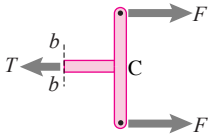



Figure 8.22:

Filename:sfig4-intern-axial1b

$$\begin{aligned} -T + 2F &= 0 \\ \Rightarrow \quad T &= 2F \\ &= 4\,\text{kN}. \end{aligned}$$

At section $b$-$b$: $T = 4\,\text{kN}$

**SAMPLE 8.7 Tension in a tapered bar due to self weight:** A tapered bar of height 1 m, base width 10 cm, top width 4 cm and uniform thickness 4 cm hangs upside down from a ceiling. If the density of the material is 7500 kg/ m$^3$, find the tension in the rod halfway from the top. You may take $g \approx 10\,\text{m/s}^2$.

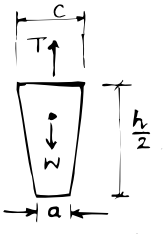



Figure 8.23:
Filename:sfig4-intern-axial2a

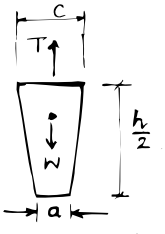



Figure 8.24:
Filename:sfig4-intern-axial2a

**Solution** Let us cut the bar at a section halfway from the top. The free-body diagram of the bar below the cut is shown in Fig. 8.24. From the scalar force balance in the vertical direction, we have

$$T = W$$

where $W$ is the weight of the lower part of bar below the cut section. Now, $W = \rho A t g$ where $A$ is the frontal area, $t$ is the thickness, and $\rho = 7500\,\text{kg/ m}^3$ is the density of the rod material. We need to compute $W$.

The width of the bar at the cut section is $c = (a + b)/2$ where $a = 4\,\text{cm}$ and $b = 10\,\text{cm}$. The frontal area of the bar-part is $A = (a + c)/2 \cdot (h/2)$ where $h = 1\,\text{m}$. Thus,

$$\begin{aligned} W &= \rho\left(\frac{a+c}{2}ht\right)g \\ &= 7500\,\text{kg/ m}^3\left(\frac{0.04\,\text{m}+0.07\,\text{m}}{2}\cdot\frac{1\,\text{m}}{2}\cdot(0.04\,\text{m})\right)10\,\text{m/s}^2 \\ &= 165\,\text{N}. \end{aligned}$$

Thus, $T = 165\,\text{N}$.

$$\boxed{T = 165\,\text{N}}$$

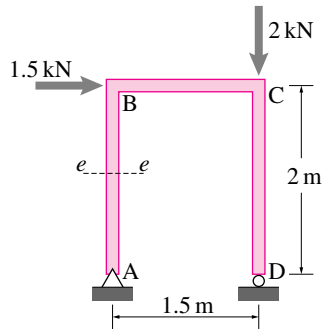



Figure 8.25:
Filename:sfig4-intern-frame

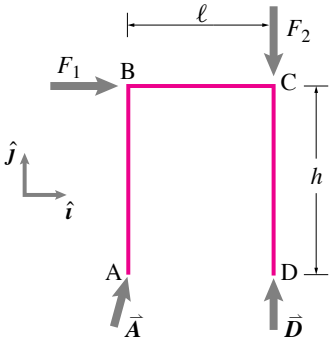



Figure 8.26:
Filename:sfig4-intern-frame-a

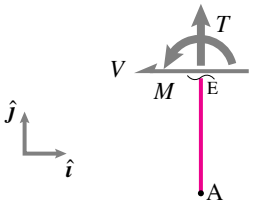



Figure 8.27:
Filename:sfig4-intern-frame-b

**SAMPLE 8.8 A simple frame:** A 2 m high and 1.5 m wide rectangular frame ABCD is loaded with a 1.5 kN horizontal force at B and a 2 kN vertical force at C. Find the internal forces and moments at the mid-section e-e of the vertical leg AB.

**Solution** To find the internal forces and moments, we need to cut the frame at the specified section e-e and consider the free-body diagram of either AE or EBCD. No matter which of the two we select, we will need the support reactions at A or D to determine the internal forces. Therefore, let us first find the support reactions at A and D by considering the free-body diagram of the whole frame (Fig. 8.26). The moment balance about point A, $\sum \vec{M}_{\rm A} = \vec{0}$, gives

$$\begin{aligned}
\vec{r}_{\rm B} \times \vec{F}_1 + \vec{r}_{\rm C} \times \vec{F}_2 + \vec{r}_{\rm D} \times \vec{D} &= \vec{0} \\
h\hat{\jmath} \times F_1\hat{\imath} + (h\hat{\jmath} + \ell\hat{\imath}) \times (-F_2\hat{\jmath}) + \ell\hat{\imath} \times D\hat{\jmath} &= \vec{0} \\
-F_1 h\hat{k} - F_2\ell\hat{k} + D\ell\hat{k} &= \vec{0} \\
\Rightarrow \quad D &= F_1\frac{h}{\ell} + F_2 \\
&= 1.5\,\text{kN} \cdot \frac{2}{1.5} + 2\,\text{kN} \\
&= 4\,\text{kN}.
\end{aligned}$$

From force equilibrium, $\sum \vec{F} = \vec{0}$, we have

$$\begin{aligned}
\vec{A} &= -\vec{F}_1 - \vec{F}_2 - \vec{D} \\
&= -F_1\hat{\imath} + F_2\hat{\jmath} - D\hat{\jmath} \\
&= -1.5\,\text{kN}\hat{\imath} - 2\,\text{kN}\hat{\jmath}.
\end{aligned}$$

Now we draw the free-body diagram of AE to find the shear force $V$, axial (tensile) force $T$, and the bending moment $M$ at section e-e.

From the force equilibrium of part AE, we get

$$\begin{aligned}
\vec{A} - V\hat{\imath} + T\hat{\jmath} &= \vec{0} \\
(A_x - V)\hat{\imath} + (A_y + T)\hat{\jmath} &= \vec{0} \\
\Rightarrow \quad V = A_x &= -1.5\,\text{kN} \\
T = -A_y &= 2\,\text{kN}.
\end{aligned}$$

From the moment equilibrium about point A, $\sum \vec{M}_{\rm A} = \vec{0}$, we have

$$\begin{aligned}
M\hat{k} + \frac{h}{2}\hat{\jmath} \times (-V\hat{\imath}) &= \vec{0} \\
M\hat{k} + V\frac{h}{2}\hat{k} &= \vec{0} \\
\Rightarrow \quad M &= -V\frac{h}{2} \\
&= -(-1.5\,\text{kN}) \cdot \frac{2\,\text{m}}{2} \\
&= 1.5\text{k N·m}.
\end{aligned}$$

$$\boxed{V = 1.5\,\text{kN}, \quad T = 2\,\text{kN}, \quad M = 1.5\text{k N·m}}$$

**SAMPLE 8.9 Shear force and bending moment diagrams:** A simply supported beam of length $\ell = 2\,\text{m}$ carries a concentrated vertical load $F = 100\,\text{N}$ at a distance $a$ from its left end. Find and plot the shear force and the bending moment along the length of the beam for $a = \ell/4$.

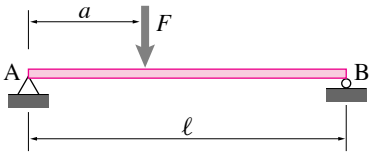



Figure 8.28:
Filename:sfig4-intern-ssvmx

**Solution** We first find the support reactions by considering the free-body diagram of the whole beam shown in Fig. 8.29. By now, we have developed enough intuition to know that the reaction at A will have no horizontal component since there is no external force in the horizontal direction. Therefore, we take the reactions at A and B to be only vertical. Now, from the moment equilibrium about point B, $\sum \vec{M}_{\text{B}} = \vec{0}$, we get

$$\begin{aligned} F(\ell - a)\hat{\boldsymbol{k}} - A_y \ell \hat{\boldsymbol{k}} &= \vec{0} \\ \Rightarrow \quad A_y &= \frac{F(\ell - a)}{\ell} \\ &= F\left(1 - \frac{a}{\ell}\right) \end{aligned}$$

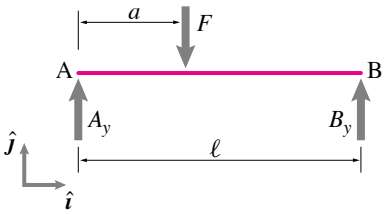



Figure 8.29:
Filename:sfig4-intern-ssvmx-a

and from the force equilibrium in the vertical direction, $(\sum \vec{F} = \vec{0}) \cdot \hat{\boldsymbol{j}}$, we get

$$B_y = F - A_y = F\frac{a}{\ell}.$$

Now we make a cut at an arbitrary (variable) distance $x$ from A where $x < a$ (see Fig. 8.30). Carrying out the force balance and the moment balance about point A, we get, for $0 \le x < a$,

$$V = A_y = F\left(1 - \frac{a}{\ell}\right) \tag{8.4}$$

$$M = Vx = F\left(1 - \frac{a}{\ell}\right)x \tag{8.5}$$

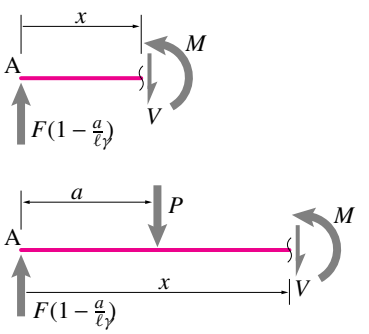



Figure 8.30:
Filename:sfig4-intern-ssvmx-b

Thus $V$ is constant for all $x < a$ but $M$ varies linearly with $x$.

Now we make a cut at an arbitrary $x$ to the right of load $F$, *i.e.*, $a < x \le \ell$. Again, from the force balance in the vertical direction, we get

$$V = -F + F\left(1 - \frac{a}{\ell}\right) = -F\frac{a}{\ell} \tag{8.6}$$

and from the moment balance about point A,

$$\begin{aligned} M &= Fa + Vx \\ &= Fa - F\frac{a}{\ell}x \\ &= Fa\left(1 - \frac{x}{\ell}\right). \end{aligned} \tag{8.7}$$

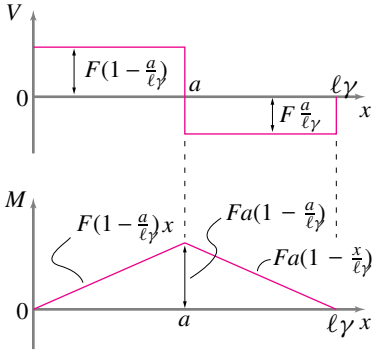



Figure 8.31:
Filename:sfig4-intern-ssvmx-c

Although eqn. (8.5) is strictly valid for $x < a$ and eqn. (8.7) is strictly valid for $x > a$, sustituting $x = a$ in these two equations gives the same value for $M(= Fa(1 - a/\ell))$ as it must because there is no reason to have a jump in the bending moment at any point along the length of the beam. The shear force $V$, however, does jump because of the concentrated load $F$ at $x = a$.

Now, we plug in $a = \ell/4 = 0.5\,\text{m}$, and $F = 100\,\text{N}$, in eqns. (8.4)–(8.7) and plot $V$ and $M$ along the length of the beam by varying $x$. The plots of $V(x)$ and $M(x)$ are shown in Fig. 8.31.

□

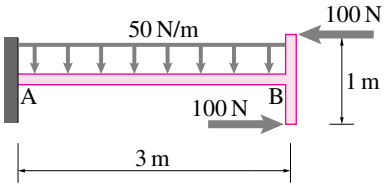



Figure 8.32:
Filename:sfig4-intern-cantvmx

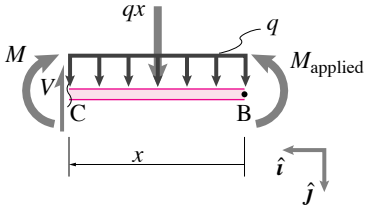



Figure 8.33:
Filename:sfig4-intern-cantvmx-a

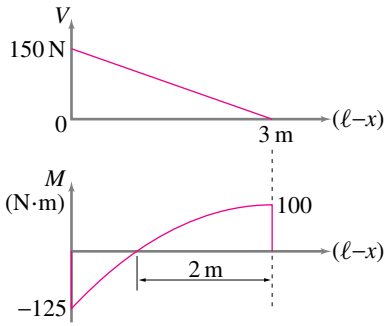



Figure 8.34:
Filename:sfig4-intern-cantvmx-b

**SAMPLE 8.10 Shear force and bending moment diagrams by superposition:** For the cantilever beam and the loading shown in the figure, draw the shear force and the bending moment diagrams by

1. considering all the loads together, and
2. considering each load (of one type) at a time and using superposition.

**Solution**

1. *$V(x)$ and $M(x)$ with all forces considered together:* The horizontal forces acting at the end of the cantilever are equal and opposite and, therefore, produce a couple. So, we first replace these forces by an equivalent couple $M_{\text{applied}} = 100\,\text{N} \cdot 1\,\text{m} = 100\,\text{N·m}$. Since we have a cantilever beam, we can consider the right hand side of the beam after making a cut anywhere for finding $V$ and $M$ without first finding the support reactions.

Let us cut the beam at an arbitrary distance $x$ from the right hand side. The free-body diagram of the right segment of the beam is shown in Fig. 8.33. From the force balance, $\sum \vec{F} = \vec{0}$, we find that

$$\begin{aligned} -V\hat{\boldsymbol{\jmath}} + qx\hat{\boldsymbol{\jmath}} &= \vec{0} \\ \Rightarrow \quad V &= qx \\ &= (50\,\text{N/m})x. \end{aligned} \tag{8.8}$$

Thus the shear force varies linearly along the length of the beam with

$$\begin{aligned} V(x=0) &= 0, \\ \text{and} \quad V(x=3\,\text{m}) &= 150\,\text{N}. \end{aligned}$$

The moment balance about point C, $\sum \vec{M}_{\text{C}} = \vec{0}$, gives

$$-M\hat{\boldsymbol{k}} - qx \cdot \frac{x}{2}\hat{\boldsymbol{k}} + M_{\text{applied}}\hat{\boldsymbol{k}} = \vec{0}$$

where the moment due to the distributed load is most easily computed by considering an equivalent concentrated load $qx$ acting at $x/2$ from the end B. Thus,

$$\Rightarrow \quad M = M_{\text{applied}} - q\frac{x^2}{2} \tag{8.9}$$

$$= 100\,\text{N·m} - 50\,\text{N/m} \cdot \frac{x^2}{2}. \tag{8.10}$$

Thus, the bending moment varies quadratically with $x$ along the length of the beam. In particular, the values at the ends are

$$\begin{aligned} M(x=0) &= 100\,\text{N·m} \\ \text{and} \quad M(x=3\,\text{m}) &= -125\,\text{N·m}. \end{aligned}$$

The shear force and the bending moment diagrams obtained from eqns. (8.8) and (8.9) are shown in Fig. 8.34. Note that $M = 0$ at $x = 2\,\text{m}$ as given by eqn. (8.9).

2. *$V(x)$ and $M(x)$ by superposition:* Now we consider the cantilever beam with only one type of load at a time. That is, we first consider the beam only with the uniformly distributed load and then only with the end couple. We draw the shear force and the bending moment diagrams for each case separately and then just *add them up*. That is superposition.

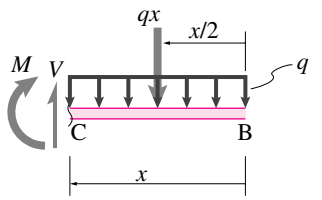



Figure 8.35:
Filename:sfig4-intern-cantvmx-c

So, first let us consider the beam with the uniformly distributed load. The free-body diagram of a segment CB, obtained by cutting the beam at a distance $x$ from the end B, is shown in Fig. 8.35. Once again, from force balance, we get

$$V = qx \qquad \text{for } 0 \le x \le \ell \tag{8.11}$$

and from the moment balance about point C, $\sum \vec{M}_C = \vec{0}$, we get

$$M = -qx \cdot \frac{x}{2} = -q\frac{x^2}{2} \qquad \text{for } 0 \le x \le \ell. \tag{8.12}$$

Figure 8.36 shows the plots of $V$ and $M$ obtained from eqns. (8.11) and (8.12), respectively, with the values computed from $x = 0$ to $x = 3\,\text{m}$ with $q = 50\,\text{N/m}$ as given.

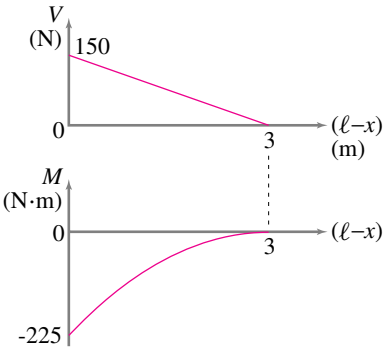



Figure 8.36:
Filename:sfig4-intern-cantvmx-d

Now we take the beam with only the end couple and repeat our analysis. A cut section of the beam is shown in Fig. 8.37. In this case, it should be obvious that from force balance and moment balance about any point, we get

$$\begin{aligned} V &= 0 \\ \text{and} \quad M &= M_{\text{applied}}. \end{aligned}$$

Thus, both the shear force and the bending moment are constant along the length of the beam as shown in Fig. 8.37.

Now superimposing (adding) the shear force diagrams from Figs. 8.36 and 8.37, and similarly, the bending moment diagrams from Figs. 8.36 and 8.37, we get the same diagrams as in Fig. 8.38.

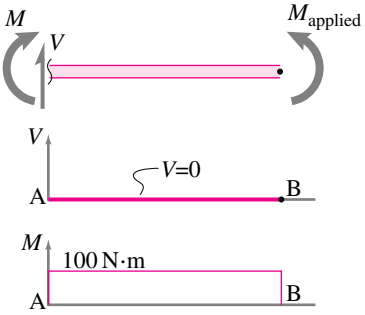



Figure 8.37:
Filename:sfig4-intern-cantvmx-e

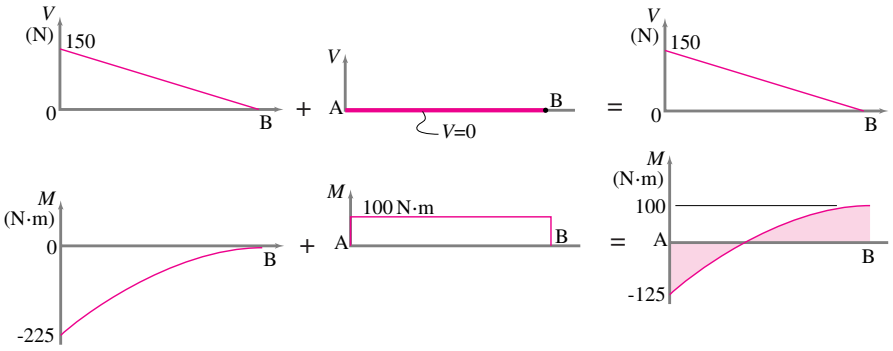



Figure 8.38:
Filename:sfig4-intern-cantvmx-f

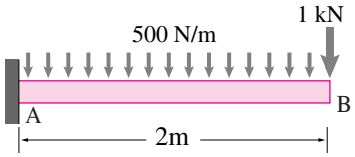
# Problems for Chapter 8

Tension, shear, and bending diagrams

## 8.1 Shear force, bending moment and tension diagrams

## Preparatory Problems

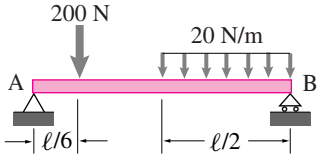
**8.1** A cantilever beam AB is loaded as shown in the figure. Find the support reactions on the beam at the left end A.




**problem 8.1:**

Filename:pfigure4-4-beamcl1

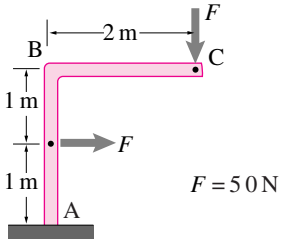
**8.2** A simply supported beam AB of length $\ell = 6\,\text{m}$ is partly loaded with a uniformly distributed load as shown in the figure. In addition, there is a concentrated load acting at $\ell/6$ from the left end A. Find the support reactions on the beam.




**problem 8.2:**

Filename:pfigure4-4-beamss2

**8.3** An (inverted) L-shaped frame is loaded with two equal concentrated forces of magnitude 50 N each as shown in the figure. Find the support reactions at A.

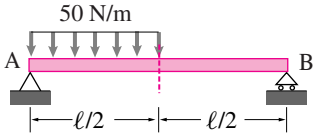



**problem 8.3:**

Filename:pfigure4-4-frame3

## More-Involved Problems

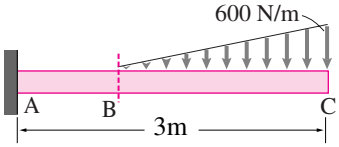
**8.4** Find the shear force and the bending moment at the mid section of the simply supported beam shown in the figure.




**problem 8.4:**

Filename:pfigure4-4-beamss4

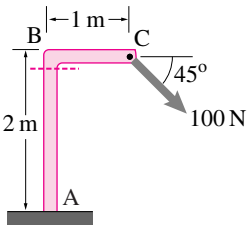
**8.5** A cantilever beam ABC is loaded with a linearly variable distributed load along two thirds of its span. The intensity of the load at the right end is $100\,\text{N}/m$. Find the shear force and the bending moment at section B of the beam.




**problem 8.5:**

Filename:pfigure4-4-beamcl5

**8.6** Analyze the frame shown in the figure and find the shear force and the bending moment at the end of the vertical section of the frame.

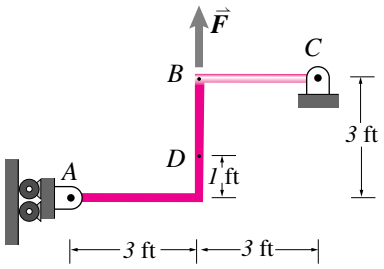



**problem 8.6:**

Filename:pfigure4-4-frame6

**8.7** A force $F = 100\,\text{lbf}$ is applied to the bent rod shown. Before doing any calculations, try to figure out the tension at D in your head.

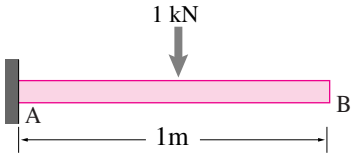
a) Find the reactions at A and C.
b) Find the tension, shear and bending moment at the section D. Check your answer against what you figured out in your head.




**problem 8.7:**

Filename:pfigure-bentrod

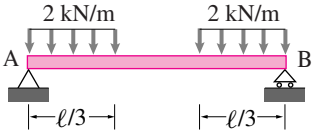
**8.8** Draw the shear force and the bending moment diagram for the cantilever beam shown in the figure.




**problem 8.8:**

Filename:pfigure4-4-beamcl7

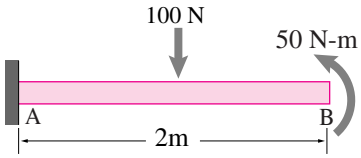
**8.9** A simply supported beam AB is loaded along one thirds of its span from both ends by a uniformly distributed load of intensity 20 N/m. Draw the shear force and the bending moment diagram of the beam.




**problem 8.9:**

Filename:pfigure4-4-beamss8

**8.10** The cantilever beam shown in the figure is loaded with a concentrated load and a concentrated moment as shown in the figure. Draw the shear force and the bending moment diagram of the beam.

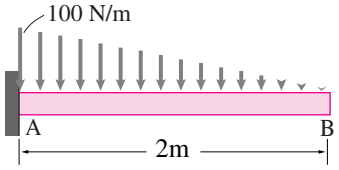



**problem 8.10:**

Filename:pfigure4-4-beamcl9

**8.11** A cantilever beam AB is loaded with a triangular shaped distributed load as

shown in the figure. Draw the shear force and the bending moment diagrams for the entire beam.

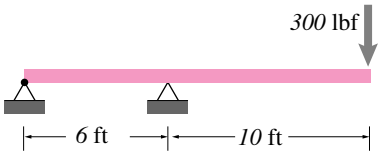



**problem 8.11:**

Filename:pfigure4-4-beamcl10

**8.12** A regulation 16 ft diving board is supported as shown.

a) Where is the bending moment the greatest and how big is it there?
b) Draw a bending moment diagram for this board.

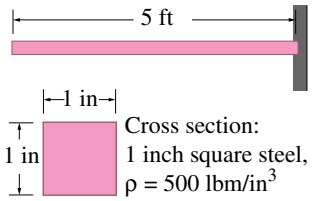



**problem 8.12:**

Filename:pfigure-divingboard

**8.13** The cantilever steel beam is loaded by its own weight.

a) Find the bending moment and shear force at the free and at the clamped end.
b) Draw a shear force diagram
c) Draw a bending moment diagram
d) The tension stress $\sigma$ in the beam at the top edge where it is biggest is given by $\sigma = 12M/h^3$ where $h = 1$ in for this beam. The strength (the maximum tension stress the material can bear) of soft steel is about $\sigma_{\max} = 30,000\,\text{lbf/in}^2$. What is the longest a beam with this cross section be made and still not fail?

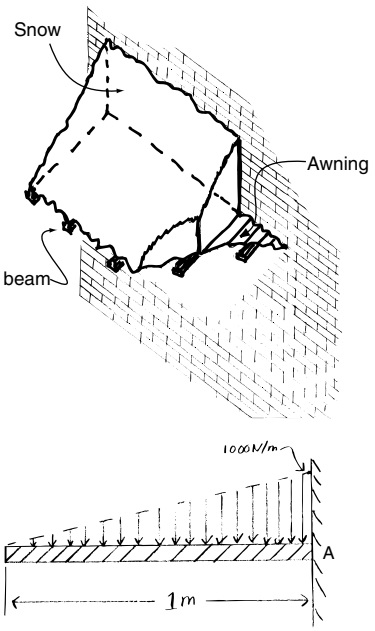



**problem 8.13:**

Filename:pfigure-bendingownweight

**8.14** A snow loaded bus-stop awning (shown partially cut away) on the side of a building is supported by horizontal, cantilevered, beams. The loading that is carried by one beam is as shown below.

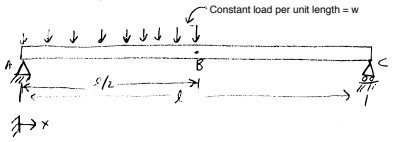
a) Find the reaction force and couple at the wall at A (the force and moment acting on one beam from the wall).
b) Draw shear force and bending moment diagrams for the beam.




**problem 8.14:**

Filename:F02final4-awning

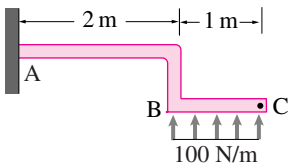
**8.15** Draw shear and bending moment diagrams of the beam shown. Clearly label the values of the heights of the curves at jumps, kinks and local maxima (if and where they exist).




**problem 8.15:**

Filename:F01p2-1-beam

**8.16** A frame ABC is much like a cantilever beam with a short bent section of length 0.5 m. The frame is loaded as shown in the figure. Draw the shear force and the bending moment diagrams of the entire frame indicating how it differs from an ordinary cantilever beam.

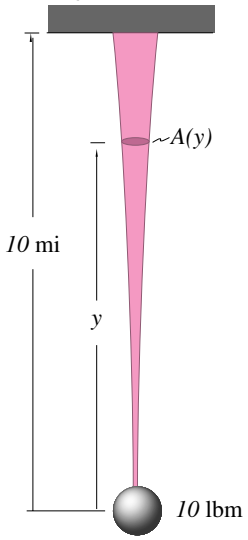



**problem 8.16:**

Filename:pfigure4-4-frame11

**8.17** A 10 pound ball is suspended by a long steel wire. The wire has a density of about $500\,\text{lbm/ft}^3$. The strength of the wire (the maximum force per unit area it can carry) is about $\sigma_{\max} = 60,000\,\text{lbf/in}^2$.

a) First, neglecting the weight of the wire in the calculation of stress, what is the weight of wire needed to hold the weight?
b) Taking account the weight of the wire in the load calculation, what is the weight of wire needed to hold the weight?




**problem 8.17:**

Filename:pfigure-hangingrod

# Part III: Dynamics

CHAPTER 9

# Dynamics in 1D

*The scalar equation $F = ma$ introduces the concepts of motion and time derivatives to mechanics. We explore various concepts and applications including power, work, kinetic and potential energies, oscillations, collisions and multi-particle systems.*

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We now progress from statics to dynamics. As the names imply, statics generally concerns things that don't move, or at least don't accelerate much, whereas dynamics concerns things whose motion is of central interest. In statics we neglected inertial (terms involving acceleration of mass); in statics the linear and angular momentum balance equations were reduced to Force and Moment balance. In dynamics the inertial terms in the momentum balance equations are important. In statics all the forces and moments cancel each other. In dynamics the forces and moments add up to cause the acceleration of mass.

Once you have mastered free-body diagrams and statics, the hard part of dynamics is learning to keep track of motion. The keeping track of motion, without yet worrying about the forces involved, is called *kinematics*. When we include both forces and pay attention to motion we are doing dynamics, in the mechanics sense of the word, and this is called *kinetics*. We will develop our understanding of dynamics (kinetics) by considering progressively more complex geometry of motion (kinematics).

This first dynamics chapter is limited to the unconstrained dynamics of a particle. What is a particle?

A *particle* is a system idealized as being totally characterized by its position (as a function of time) and its (fixed) mass (read more on page 160).

In this chapter we limit our attention to one spatial dimension (1D); each particle moves along a straight line and not on a planar or spatial curve.

**Unconstrained motion.** Finally, in this chapter we only consider cases where the applied forces are either given as a function of time or can be determined from the positions and velocities of the particles. The time-varying thrust from an engine might be thought of as a force given as a function of time. Gravity and springs cause forces which are functions of position. And the drag on a particle as it moves through air or water can be modeled as a force depending on velocity. The forces we do not consider until the next chapter are forces caused by geometric constraints, for example the forces between particles connected by strings or rods. These forces need to be solved-for using dynamics, and cannot be found *apriori* from position, velocity or time.

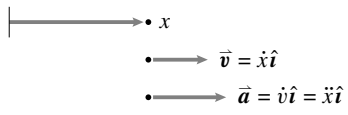



Figure 9.1: One-dimensional position, velocity, and acceleration. Positive values for $x$, $\dot{x}$ and $\ddot{x}$ describe positions, velocities and accelerations in the in the positive $x$ direction.

Filename:tfigure3-4-1

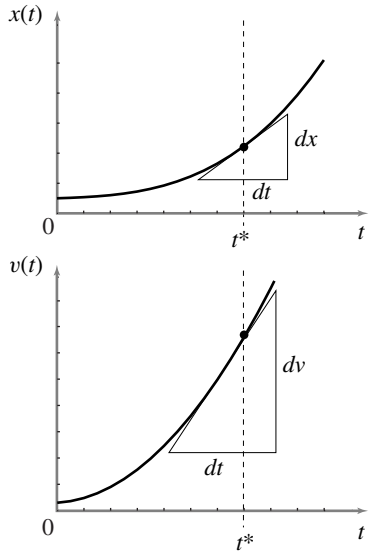



Figure 9.2: Graphs of $x(t)$ and $v(t) = \frac{dx}{dt}$ versus time. The slope of the position curve $dx/dt$ at $t^*$ is $v(t^*)$. And the slope of the velocity curve $dv/dt$ at $t^*$ is $a(t^*)$.

Filename:tfigure3-4-1a

**Kinematics and acceleration.** As mentioned, the main new concept here, which stays with us until the end of the book, is that things change with time. We keep track of that change using calculus. In particular, the equation $F = ma$ is a differential equation because

$$a \equiv \frac{d^2}{dt^2}x = \ddot{x} \ .$$

That is, any equation containing $a$ is an equation containing a term with a second derivative in time. And any equation that has terms which are derivatives of functions is a differential equation.

**Vectors are optional in 1D.** Although we emphasize the importance of vectors in most of this book, in this one chapter we keep things simpler with scalars only. We do this to better untangle the vector concepts from the new (or at least newly reviewed) calculus concepts. In later chapters time-derivatives of vectors will be of central interest.

### The organization of this chapter

The first three sections are a review and deepening of material covered in freshman physics: $F = ma$, energy methods, and the harmonic oscillator. The last three sections concern multi-particle systems, collisions and more advanced vibration analysis.

**Before going on** we suggest you get the lay of the land by reviewing the summary of mechanics on the inside cover and the general introduction to mechanics in Chapter 1.

## 9.1 Force and motion in 1D

Now we focus on a special case of particle motion: one particle moves on a given straight line. For this class of problems, with motion in only one direction, the kinematics is particularly simple. It's essentially a rehash of freshman calculus. Even in 1D, vectors can be useful because of their help with signs. But vectors are not really needed and we will not be zealous in their use (in this one chapter). As mentioned, we postpone until Chapter 11 issues about what forces might be required to keep the particle on that line.

### Position, velocity, and acceleration in one dimension

If, say, we call the direction of motion the $\hat{\imath}$ direction, then we can call $x$ the position of the particle (see figure 9.1). Even though we are neglecting the spatial extent of the particle, to be precise we can define $x$ to be the $x$ coordinate of the particle's center-of-mass. We can write the position $\vec{r}$, velocity $\vec{v}$ and acceleration $\vec{a}$ as

$$\vec{r} = x\hat{\imath}, \quad \vec{v} = v\hat{\imath} = \frac{dx}{dt}\hat{\imath} = \dot{x}\hat{\imath} \qquad \text{and} \qquad \vec{a} = a\hat{\imath} = \frac{dv}{dt}\hat{\imath} = \frac{d^2x}{dt^2}\hat{\imath} = \ddot{x}\hat{\imath}.$$

Figure 9.2 shows example graphs of $x(t)$ and $v(t)$ versus time.

**Signs.** Without vectors we need to be careful with signs; when in doubt, we will take $v$ and $a$ to be positive if they have the same direction as increasing $x$ (or $y$ or whatever coordinate describes position). Even though we pedantically declare that 'velocity is a vector' and 'acceleration is a vector', we will loosely use the words 'velocity' and 'acceleration' to stand for the coefficients of $\hat{\boldsymbol{\imath}}$ in the vector expressions above.

Example: **Position, velocity, and acceleration in one dimension**
If position is given as

$$x(t) = 3e^{4t/\,\mathrm{s}}\,\mathrm{m}$$

then $v(t) = dx/dt = 12e^{4t/\,\mathrm{s}}\,\mathrm{m/s}$ and $a(t) = dv/dt = 48e^{4t/\,\mathrm{s}}\,\mathrm{m/s^2}$.
So at, say, time $t = 2\,\mathrm{s}$ the acceleration is

$$a|_{t=2\,\mathrm{s}} = 48e^{4\cdot 2\,\mathrm{s}/\,\mathrm{s}}(\mathrm{m/s^2}) = 48\cdot e^8\,\mathrm{m/s^2} \approx 1.43\cdot 10^5\,\mathrm{m/s^2}.$$

**Units.** Note that, in the example above, the unit inverse-seconds 1/s is part of the argument of the exponential function. Thus when the exponential $e^{4t/\,\mathrm{s}}$ is differentiated with respect to time $t$ the 1/s is carried along with the 4; the coefficient of $t$ in the exponential is 4/s, so that same factor comes out front in the differentiation. If you treat units as quantities manipulated like all others, as we have done in the example above, the units come out right. Note how the units cancel in the last line when the dimensional quantity (2 s) is substituted in for the variable $t$. For more on units see appendix A.

**1D kinematics ⇔ calculus** One-dimensional kinematics problems can include almost all of the skills in elementary calculus. For example in kinematics you are often given position, velocity or acceleration as function of time and you have to differentiate it or integrate to find one of the other quantities. For example, if you are given the velocity $v(t)$ as a function of time and are asked to find the acceleration $a(t)$, you have to differentiate. If instead you were asked to find the position $x(t)$, you would be asked to calculate an integral (see figure 9.3). Using the fundamental theorem of calculus, we get the integral versions of the relations between position, velocity, and acceleration (see Fig. 9.3).

$$x(t) = x_0 + \int_{t_o}^{t} v(\tau)\,d\tau \qquad \text{with} \qquad x_0 = x(t_0),\ \text{and}$$

$$v(t) = v_0 + \int_{t_o}^{t} a(\tau)\,d\tau \qquad \text{with} \qquad v_0 = v(t_0).$$

With more indefinite notation, these equations can also be written as:

$$x = \int v\,dt$$
$$v = \int a\,dt.$$

If acceleration is given as a function of time, then position is found by integrating twice.

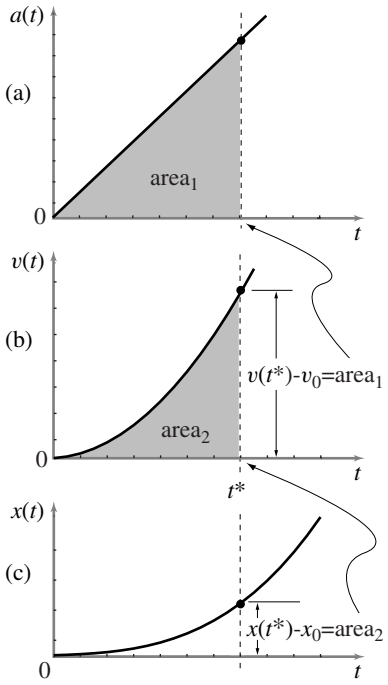



Figure 9.3: One-dimensional kinematics of a particle: (a) is a graph of the acceleration of a particle $a(t)$; (b) is a graph of the particle velocity $v(t)$ and the integral of $a(t)$ from $t_0 = 0$ to $t^*$, the shaded area under the acceleration curve; (c) is the position of the particle $x(t)$ and the integral of $v(t)$ from $t_0 = 0$ to $t^*$, the shaded area under the velocity curve.
Filename:tfigure3-4-1b

**1D kinematics, bicycles and calculus.** To put it another way, almost every calculus question could be phrased as a question about your bicycle speedometer*. With your bicycle speedometer (which includes a distance-measuring odometer) you can read your speed and distance travelled as functions of time. On the other hand, given one of those two functions you could find the other using calculus. As of this writing only a few bicycle speedometers also have accelerometers. Acceleration is also of interest whether or not you can explicitly measure it on your bicycle.

* To cover the range of calculus problems you need to be a very good rider, however, and be able to ride frontwards, backwards, at zero speed and infinitely fast.

## Differential equations

A *differential equation* is an equation that involves derivatives. Thus the equation relating position to velocity is

$$\frac{dx}{dt} = v \qquad \text{or, more explicitly} \qquad \frac{dx(t)}{dt} = v(t),$$

is a differential equation. An *ordinary differential equation* (ODE) is an equation that contains some terms that are ordinary derivatives (as opposed to partial derivatives and partial differential equations which we don't use in this book).

Example: **Calculating a derivative solves an ODE**
Given that the height of an elevator as a function of time on its 5 seconds long 3 meter trip from the first to second floor is

$$y(t) = (3\,\text{m})\frac{\left(1-\cos\left(\frac{\pi t}{5\,\text{s}}\right)\right)}{2}$$

we can solve the differential equation $v = \frac{dy}{dt}$ by differentiating to get

$$v = \frac{dy}{dt} = \frac{d}{dt}\left[(3\,\text{m})\frac{\left(1-\cos\left(\frac{\pi t}{5\,\text{s}}\right)\right)}{2}\right] = \frac{3\pi}{10}\sin\left(\frac{\pi t}{5\,\text{s}}\right)\,\text{m/s}$$

(Note: this would be a harsh elevator because of the jump in the acceleration (not calculated above) at the start and stop.)

A little less trivial is the case when you want to find a function when you are given the derivative.

Example: **Integration solves a simple ODE**
Assume that you start at home ($x = 0$) and, over about 30 seconds, you accelerate towards a steady-state speed of 4 m/s according to (see Fig. 9.4)

$$v(t) = 4(1 - e^{-t/(30\,\text{s})})\,\text{m/s}.$$

Your ride lasts 1000 seconds. We can find how far yougo by solving

$$\dot{x} = v(t) \quad \text{with the initial condition} \quad x(0) = 0.$$

This is simply solved by integration. Say, after 1000 seconds

$$\begin{aligned}
x(t = 1000\,\text{s}) &= \int_0^{1000\,\text{s}} v(t)\,dt = \int_0^{1000\,\text{s}} 4(1 - e^{-t/(30\,\text{s})})(\text{m/s})\,dt \\
&= \left(4t + (120\,\text{s})e^{-t/(30\,\text{s})}\right)\Big|_0^{1000\,\text{s}}\,\text{m/s} \\
&= \left((4\cdot 1000\,\text{s} + (120\,\text{s})e^{-100/3}) - (0 + (120\,\text{s})e^{0}\right)\,\text{m/s} \\
&= \left(4000 - 120 + 120e^{-100/3}\right)\,\text{m} \\
&\approx 3880\,\text{m} \qquad \text{(to within an angstrom or so)}
\end{aligned}$$

This is only 120 m less than if the whole trip was travelled at a steady 4 m/s (then $x = 4\,\text{m/s} \times 1000\,\text{s} = 4000\,\text{m}$).

Unlike the integral above, many integrals cannot be evaluated by hand (analytically).

Example: **An ODE that leads to an intractable integral**
Assume now that

$$v(t) = \frac{4t}{t + e^{-t/(30\,\text{s})}}\,\text{m}.$$

Again we have a bike trip where you start at zero speed and approach a steady speed of 4 m/s. So your position as a function of time should be similar. Following the last example, we have

$$\dot{x} = v(t) \quad \text{with the initial condition} \quad x(0) = 0$$

with the given $v(t)$. The integral for position is then

$$x(t = 1000\,\text{s}) = \int_0^{1000\,\text{s}} v(t)\,dt = \int_0^{1000\,\text{s}} \frac{4t}{t+e^{-t/(30\,\text{s})}}\,\text{m}\,dt \tag{9.1}$$

$$= \ldots$$

which is the kind of thing you have nightmares about seeing on an exam. You couldn't solve this integral if your life depended on it. No-one could. There is no formula for $x(t)$ that solves the differential equation, unless you regard eqn. (9.1) as a formula. In days of old they would say 'the problem has been reduced to quadrature' meaning that the remaining work was evaluating an integral*, even if they didn't know how to evaluate it exactly.

Just because a differential equation can't be solved analytically with pencil and paper doesn't mean it can't be solved. Most often the setup for numerical solution is not that difficult. Note that for numerical solution you either need dimensionless calculations, or at least need all variables in consistent units.

One of many ways to evaluate the integral of the above example numerically is by the following pseudo code.

```
ODE = { xdot = 4 t / (t+e^(-t/30)) }
IC  = { x(0) = 0  }
solve ODE with IC and evaluate at t=1000
```

The result is $x \approx 3988$ m which is also, as expected because of the similarity with the previous example, only slightly shy of the steady-speed approximation of 4000 m.*

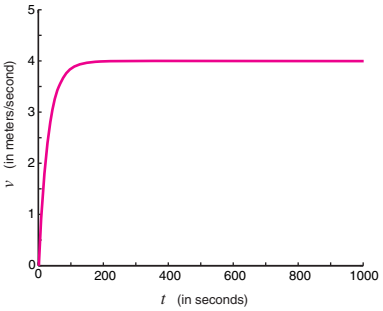



Figure 9.4: Velocity (speed) vs time for a 1000 m bike trip. The rider increases speed to exponentially approach 4 m/s.
Filename:tfigure-bikeplot

*Literally *quadrature* means finding a square (a 'quad') with area equal to the area under a given curve. The phrase is used more generally to mean integration, even if the integration is of several variables.

***Numerical error vs real difference.** When you notice such small differences (12m out of 4000m) based on computer calculation you need to question whether the difference is something real in the problem or, rather, is due to numerical errors. Two ways to check are with so-called convergence tests and by using your canned package's error estimate. We checked that the numerical integration is accurate to about $10^{-3}$m, less than the 1m resolution that we printed (no need to type lots of digits with little information). Thus the $\approx$ 12m difference between the constant $v$ solution and the solution where $v$ approaches a constant, is a real difference, the lag because the startup was slower in the second example, and not a numerical artifact.

## More differential equations.

As mentioned, because dynamics equations contain derivatives they are all differential equations. A catalogue of the simplest differential equations and their solutions is given in box 9.1 on page 392.

## The equations of dynamics

We want to understand kinetics (mechanics, dynamics), not just kinematics. The subject of mechanics is held up by the three pillars of material properties, geometry, and 'Newton's laws' (see page 4). Here we begin to flesh out the 'Newton's laws' pillar beyond statics (the first 8 chapters of this book), using kinematics (we just started with that above) to the the third pillar, dynamics.

## Linear momentum balance

For a particle moving in the $x$ direction the velocity and acceleration are $\vec{\boldsymbol{v}} = v\hat{\boldsymbol{\imath}}$ and $\vec{\boldsymbol{a}} = a\hat{\boldsymbol{\imath}}$. Thus the linear momentum and its rate of change are

$$\begin{aligned} \vec{\boldsymbol{L}} &\equiv \sum m_i \vec{\boldsymbol{v}}_i &&= m\vec{\boldsymbol{v}} = mv\hat{\boldsymbol{\imath}}, \text{ and} \\ \dot{\vec{\boldsymbol{L}}} &\equiv \sum m_i \vec{\boldsymbol{a}}_i &&= m\vec{\boldsymbol{a}} = \text{m}a\hat{\boldsymbol{\imath}}. \end{aligned}$$

Using any of the free body diagrams in Fig. 9.5, where $\vec{\boldsymbol{F}} = F\hat{\boldsymbol{\imath}}$, the equation of linear momentum balance, *eqn. I from the front inside cover, or equation **??** reduces to:

$$F\hat{\boldsymbol{\imath}} = ma\hat{\boldsymbol{\imath}} \tag{9.2}$$

which in scalar form is the central subject of this section.

$$F = ma$$

In scalar form, $F$ is the net force to the right and $a$ is the acceleration to the right. For the equation $F = ma$ to have content each of the terms must have some meaning in other contexts. And, at least intuitively, each does (see box 9.1 on page 385).

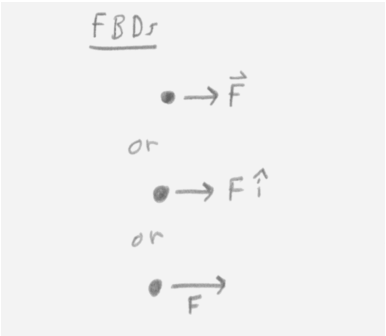



Figure 9.5: Two possible free body diagrams for 1D particle motion. It is assumed that $\vec{\boldsymbol{F}}$ represents the sum of all forces acting on the particle. Note, even though the inertial term $ma$ is of special interest, it does not show on the FBD. See the free body diagram Chapter 3 for the prescription about this. See box 9.2 on page 390 for a further warning about D'Alambert 'forces'.

Filename:tfigure-1dparticlefbd

*__Angular momentum?__ We skip thinking about angular momentum balance in this section. Why? If we pick an origin on the line of travel, all terms on both sides of all angular momentum balance equations are zero, and the equation $0 = 0$ tells us nothing new.

**Force.** The force $F$ could come from a spring, or a fluid or from your hand pushing the thing to the right or left, or any combination of these things. The most general case we want to consider here is that the force is determined by the position and velocity of the particle as well as the present time. Thus

$$F = f(x, v, t). \tag{9.3}$$

What do we mean 'determined by'? We mean that we have an independent way of knowing the force from its position, velocity and time, even without having yet thinking about the linear momentum balance equation $F = ma$. Special cases would be, say,

$$\begin{array}{lcccl} F & = & f(t) & = F_0 \sin(\beta t) & \text{for an oscillating load,} \\ F & = & mg & & \text{for the force of earth's gravity,} \\ F & = & f(v) & = -cv & \text{for a linear viscous drag,} \\ F & = & f(x) & = -kx & \text{for a linear spring, and} \\ F & = & f(x, v, t) & = -kx - cv + F_0 \sin(\beta t) & \text{for a combination of forces.} \end{array}$$

So all elementary 1D particle mechanics problems can be reduced to the solution of this pair of coupled first order differential equations,

$$\begin{aligned} \tfrac{dv}{dt} &= \underbrace{f(x, v, t)/m}_{a(t)} \quad (a) \\ \tfrac{dx}{dt} &= v(t) \quad (b) \end{aligned} \tag{9.4}$$

where the function $f(x, v, t)$ is given and $x(t)$ and $v(t)$ are to be found.

**Know a solution when you see one.** How can you tell if candidate functions solve a differential equation? First you can tell that the initial conditions are satisfied by evaluating the expressions at $t = 0$. To check that the differential equations are satisfied, you plug the candidate solutions into the equation and see that an identity results. Differential equations are satisfied when the unknown functions therein are replaced with specific functions that make the equations correct.

Example: **viscous drag**

If the only applied force is a viscous drag, $F = -cv$ (see Fig. 9.6), then linear momentum balance ($F = ma$) would be $-cv = ma$ and Eqns. 9.4 are

$$\frac{dv}{dt} = -cv/m$$

$$\frac{dx}{dt} = v$$

where $c$ and $m$ are constants and $x(t)$ and $v(t)$ are yet to be determined functions of time. Because the force only slows the particle there is be no motion unless the particle has some initial velocity. In general, you need to specify the initial position and velocity to find a solution. So we complete the problem statement with the initial conditions

$$x(0) = x_0 \quad \text{and} \quad v(0) = v_0$$

where $x_0$ and $v_0$ are given constants. Before worrying about how to solve such equations, you should know how to recognize a solution. The following two functions, assume for now that they fell from the sky,

$$v(t) = v_0 e^{-ct/m}, \quad \text{and}$$
$$x(t) = x_0 + m v_0 (1 - e^{-ct/m})/c$$

## 9.1 *THEORY*

### *What do the terms in $F = ma$ mean?*

For the equation $F = ma$ to have useful content, we need some independent ways of talking about each of the terms. Otherwise the equation is just defining, say, mass in terms of force or the other way around. The '$F = ma$ is just a definition of force' approach may or may not be legitimate, but is not useful if we care about force for other reasons, which we do.

**Mass.** Now that we know about atoms (these centuries) and what they are made of (these decades) we can approximately (about one percent accuracy) define the mass of a system by counting up (in principle) the total number of protons and neutrons and multiplying by $1.67 \cdot 10^{-27}$ kg. That is, mass is a measure of the extent of matter. Given that we think of mass as the amount of matter, we could more accurately and more easily use a reference volume of a pure chemical substance as a reference. Here we can get an accuracy of parts per million with some trouble. Actually mass is measured in comparison to a piece of metal in a box some basement. And that calibrated kilogram is accurate to about a part in 20,000,000. We can find the mass of a more complicated thing using that reference and a balance.

**Acceleration.** Because this is a course in mechanics and not in philosophy of science, we will just accept the concepts of space and time as given and measurable (using rulers and clocks) so acceleration is operationally well defined with no use of $F = ma$. At least to about one part per billion for most engineering purposes.

**Force.** Here is where people argue most. We find it easiest to think of force as defined in terms of deformation of solids. When one thing pushes on another, think of your little finger as caught in between. How much your finger is squeezed, as measured by how loud you yell, is a measure of force. More technically, we could look at the small amounts of deformation occurring where the bodies contact, and use the deformation as a measure of force. Or, more practically, we could interpose a calibrated material and measure its deformation. Such a chunk of material with deformation-measuring electronics is called a *load cell*. Load cell's are sold by the millions (say, in bathroom scales). A load cell uses nothing about $F = ma$ to operate accurately.

One reason it is nice to think of force as having a life away from $F = ma$ is that the whole coherent and useful subject of statics has no use for $F = ma$, yet functions well as a useful subject for designing bridges and such. Alternatively, still without thinking about $F = ma$, one could define force in terms of the net effect of earth's gravitational pull on a calibrated mass at some permanently-ordained location.

However you like to define force, the great result is that with any one definition, lots of equations come out about right using that concept. For example, the same concept $F$ works in all three of these contexts,

$$F = mg \text{ and } F = kx \text{ and } F = ma.$$

Pick your favorite as the one you think of as most fundamental.

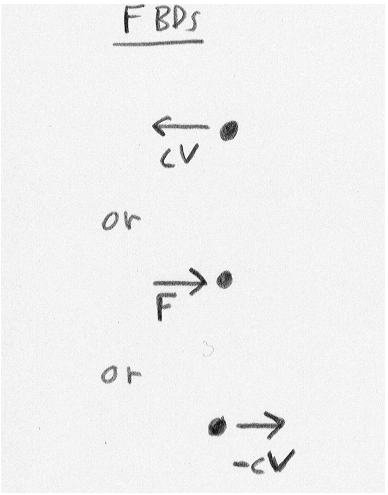



Figure 9.6: Three acceptable free body diagrams showing the viscous drag on a particle. The second one depends on being told on the side, say here, that $F = -cv$.

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solve the differential equations. Plugging the presumed solution for $v(t)$ into $\dot{v} = -cv/m$ gives, and this is what we want, $0 = 0$. And similarly, when the presumed solution for $x(t)$ when is plugged in to $\dot{x} = v$ you also get the 'satisfying' result that $0 = 0$.

Replacing the unknown functions $v(t)$ and $x(t)$ with the given formulas gives an identity. Thus the given formulas satisfy (or solve) the differential equations. Just like the case of integration (or equivalently the solution for $x$ of the ODE $\dot{x} = v(t)$), one often cannot find formulas for the solutions of differential equations.

Example: **A dynamics problem with no pencil and paper solution**
Consider the following case which models a particle in a sinusoidal force field with a second applied force that oscillates in time. Using the dimensional constants $c$, $d$, $F_0$, $\beta$, and $m$,

$$\frac{dv}{dt} = \left(c\sin(x/d) + F_0\sin(\beta t)\right)/m$$

$$\frac{dx}{dt} = v$$

with initial conditions $x(0) = 0$ and $v(0) = 0$.

There is no known formula for $x(t)$ that solves this ODE.

Just writing the ordinary differential equations and initial conditions is analogous to setting up an integral in freshman calculus. The solution is reduced to quadrature. Because numerical solution of sets of ordinary differential equations is a standard part of all modern computation packages you are in some sense done when you get to this point. You just ask a computer to finish up.

## The units of force

The simplest way to measure force is with the "metric"/SI convention and to use Newtons (N) where

$$1N \equiv \frac{1\,\mathrm{kg}}{\mathrm{m\,s^2}}.$$

Unfortunately, to everyone's confusion, there are other units of force.

**Kilogram force.** One confusing force unit is the kilogram of force, 1 kgf = g$\times$ 1 kg $\approx$ 9.81N. One kgf is the force of gravity at the earth's surface. Another name for kilogram force is kilopond, kp. How much do you weigh? In Europe most often people give their weight (weight is a force) in "kilograms" which means kgf. Basically we advise against using kgf for any purposes. But you should know that some people use it. To keep things accurate and simple, in Europe people should go on diets to lose mass.

**Pound force.** Analogous to the kilogram force is the pound force lbf. One lbf is the force of gravity (at the earth's surface) on one pound mass. Thus $1\,\mathrm{lbf} = g \times 1\,\mathrm{lbm} \approx 32.2\,\mathrm{lbm\,ft/\,s^2}$.

What is the force required to accelerate 10 lbm an amount of 5 ft/s$^2$?

$$F = ma = (10\,\text{lbm})(5\,\text{ft/s}^2) = (10\,\text{lbm})(5\,\text{ft/s}^2)\cdot\underbrace{\left(\frac{1\,\text{lbf}}{1\,\text{lbm}\cdot g}\right)}_{1}\underbrace{\left(\frac{g}{32\,\text{ft/s}^2}\right)}_{1}.$$

All of the units in the above expression cancel (appear an equal number of times on the top as the bottom of fractions) but for lbf. So

$$F \approx (10\cdot 5/32)\,\text{lbf} \approx 1.6\,\text{lbf}.$$

The surest way to know whether to multiply or divide by $g$ is by systematic multiplication by 1, as in this example. If you pick the wrong version of the number 1 you get the right answer, but in a strange mixture of units.

**Poundal.** If the English system imitated the metric system it would have a unit for the force needed to accelerate one lbm one ft/s$^2$. It does. Its called the poundal. 1 pdl = 1 lbm. Poundals should be a sensible to the English system as Newtons are to the English, but they are rarely used. Because the poundal is unfamiliar, and strange things are judged confusing, the poundal is generally catalogued as confusing. In principle the poundal is as simple as the Newton.

**The search for the number 1.** In the metric system the standard unit of force (N) is 1 (one) times the standards for mass, distance and inverse-time squared: 1 N = 1 kg m/s$^2$. One is a nice number. An attempt to get mass and force related by the number one, an attempt that has failed in the market place of engineering practice, uses the poundal. A second failed attempt defines a new unit of mass, the slug: 1 lbf = 1 slug ft/s$^2$. Slugs are also rarely used. But in principle a slug is just as simply related to a lbf as is a kg to a N.

**Europe is dynamic the USA static.** Thus the standard units used in Europe are easy if mostly you are studying dynamics. A unit of mass is accelerated a unit amount with a unit force. The standard units in the USA are easiest for gravitational loads. The unit of mass has a gravitational force on it of a unit of force.

Most people studying mechanics try to avoid all this confusion by sticking with SI, that's the real SI that has no such thing as kgf. In the USA and some other places, still in the 21st century, we have to learn to live with pound force and pound mass. At least we can be thankful that most of us can avoid dealing with the kilogram force (or kilopond), the poundal and the slug. Read more about such issues in the appendix on units.

## Some special cases in 1D mechanics

There are various special cases of eqn. (9.4) that occur in simple problems and which have simple solutions.

**Zero net force.** This is the simplest dynamics problem.

$$\begin{array}{lcll} F = ma \text{ with } F = 0 & \Rightarrow & 0 = ma & \\ \text{definition of} a & \Rightarrow & \dot{v} = 0 & \\ \text{integrating} & \Rightarrow & v = v_0 & (= \text{any constant}) \\ \text{integrating again} & \Rightarrow & x = x_0 + v_0 t & \end{array}$$

In a sense we have thus derived Newton's first law, an object in motion tends to stay in motion unless acted upon by a force, from his second law.

**Constant force.** Another simple case is constant force $F$ which leads to constant acceleration $a = F/m$. Using calculus you should know well by know, you get the following formulas:

$$\begin{aligned} a = const \Rightarrow \quad x &= x_0 + v_0 t + at^2/2 \\ a = const \Rightarrow \quad v &= v_0 + at \\ a = const \Rightarrow \quad v &= \pm\sqrt{v_0^2 + 2ax}. \end{aligned}$$

These are much seen in high school physics because, by permuting what is given and what is unknown, one can make up 100 homework problems that can be solved with these formulas and without calculus.

**Force given as a function of time.** Say $F$ is given as $F = F(t)$. This general case shows up when some kind of motor force is controlled by a human or computer to vary in time is some predetermined manner.

$$\begin{array}{lcll} F = ma \text{ with } F = F(t) & \Rightarrow & F(t) = ma & \\ \text{definition of } a & \Rightarrow & \dot{v} = F(t)/m & \\ \text{integrating} & \Rightarrow & v = v_0 + \frac{1}{m}\int_0^t F(\tau)\, d\tau & \end{array}$$

And we have to integrate once again to get position.

Example: **Ramping up the acceleration at the start**
If you get a car going by gradually depressing the 'accelerator' so that its acceleration increases linearly with time, we have

$$\begin{array}{rcccl} a & = & ct & & (\text{take } t = 0 \text{ at the start}) \\ \Rightarrow \quad v(t) & = & \int_0^t a d\tau + v_0 & = & \int_0^t c\tau d\tau = ct^2/2 \\ & & & & (\text{since } v_0 = 0) \\ \Rightarrow \quad x(t) & = & \int_0^t v d\tau + x_0 & = & \int_0^t (c\tau^2/2) d\tau = ct^3/6 \\ & & & & (\text{since } x_0 = 0). \end{array}$$

The distance the car travels is proportional to the cube of the time that has passed from dead stop.

The overall subject of 'vibrations' is in some sense about what happens when something is shaken. We can think of 'shaking' as applying a force which varies sinusoidally in time.

Example: **Force varies sinusoidally in time.**
Assume a 1 kg mass starts from rest and has a force of $F = 2\cos(2\pi t/\text{s})\,\text{N}$ applied. That's a force that oscillates once per second with an amplitude of 2 N. What is the position at $t = 10\,\text{s}$?

$$
\begin{array}{lcrcl}
F = ma \text{ with } F = F(t) & \Rightarrow & 2\cos(2\pi t/\text{s})\,\text{N} & = & m\dot{v} \\
\text{integrating} & \Rightarrow & v & = & v_0 + \frac{1}{m}\int_0^t 2\cos(2\pi t/\text{s})\,\text{N}\,d\tau \\
\text{using freshman calculus} & \Rightarrow & v & = & v_0 + \frac{1}{\pi m}\sin(2\pi t/\text{s})\,\text{N s} \\
\text{IC: } v(0) = 0 \Rightarrow v_0 = 0 & \Rightarrow & v & = & \frac{1}{\pi m}\sin(2\pi t/\text{s})\,\text{N s} \\
\text{integrate again, using } \dot{x} = v & \Rightarrow & x & = & x_0 - \frac{1}{2\pi^2 m}\cos(2\pi t/\text{s})\,\text{N s}^2 \\
\text{IC: } x(0) = 0 \Rightarrow x_0 = \frac{1}{2\pi^2 m} & \Rightarrow & x & = & \frac{1}{2\pi^2 m}(1 - \cos(2\pi t/\text{s}))\ \text{N s}^2
\end{array}
$$

Now we can substitute in $m = 1\,\text{kg}$ and $t = 10\,\text{s}$ to get $x = 0.0\,\text{m}$. The algebraic cancellation of units came about naturally from substituting in the definition of a Newton $1\,\text{N} = 1\,\text{kg m/ s}^2$. We carried the units through even though the final answer was 0.

## 9.2 D'Alembert's mechanics: beginners beware

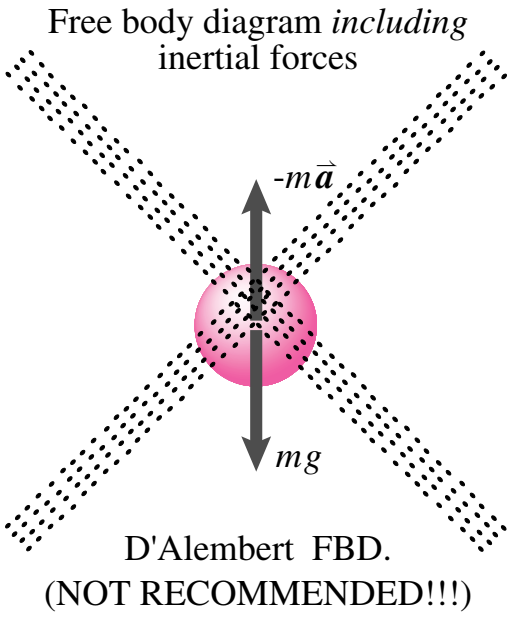
This box does not include any information needed for this course. As warned on page **??**, its worse than that. This box contains material that we think harms more than helps. But some people are curious about such things.

The D'Alembert's approach to mechanics, an alternative to the momentum balance approach, cannot be well absorbed by beginners. Students attempting to use D'Alembert methods make frequent mistakes. We advise against the use of D'Alembert mechanics for first-time dynamics students. We don't even allow its use in homework and exams.

On the other hand, the D'Alembert approach has an intuitive appeal to experts. And the D'Alembert equations are the first step in deriving the more advanced (*e.g.*, Lagrangian, Hamiltonian, 'method of virtual speed', and 'Kane') approaches to dynamics.

For completeness, to demystify the taboo, we briefly describe the approach.

First, label the free body diagram: *'free body diagram including inertial forces.'* Then, in addition to the applied forces draw pseudo-forces equal to $-m\vec{a}$ for every mass particle $m$. These pseudo-forces shown in the FBD of a falling ball using D'Alembert's approach to mechanics are sometimes called 'inertial' forces.




Instead of momentum balance equations you write 'pseudo-statics' equations of 'force' balance and 'moment' balance

pseudo-force balance

$$\underbrace{\sum \vec{F}}_{\text{including inertial forces}} = \vec{0}$$

pseudo-moment balance

$$\underbrace{\sum \vec{M}_C}_{\text{including torques from inertial forces}} = \vec{0}$$

These equations include the actual forces as well as the 'inertial' forces shown on the free body diagram.

By this means, the dynamics equations have been reduced to statics equations. Linear momentum balance is replaced by pseudo-statics force balance. Angular momentum balance is replaced by pseudo-statics moment balance.

The moving of the inertial terms from the right side of the equation to the left leads to both conceptual simplicity *and* puts the equations of dynamics in a form that is closer to most people's intuitions. The simplification is not so great as it may seem at first sight. Accelerations still need to be calculated and the sums involved in calculation of rate of change of linear and angular momentum still need to be calculated, only now they are sums of pseudo *inertial* forces.

Consider the example of sitting in a car as the car rounds a corner to the left. In the momentum balance approach, we write

$$\vec{F} = \underbrace{m\vec{a}}_{\dot{\vec{L}}}$$

and say the force from the car on you to the left is equal to the rate of change of your linear momentum as you accelerate to the left. In the D'Alambert approach, we write

$$\vec{F} - \underbrace{m\vec{a}}_{\text{inertia force}} = \vec{0}$$

and think the inertia force to the right is balanced by the interaction force of the car on your body to the left.

It is a puzzle of human consciousness why such a trivial algebraic manipulation, namely,

$$\vec{F} = \mathrm{m}\vec{a} \;\Rightarrow\; \vec{F} - m\vec{a} = \vec{0}$$

should lead to such a great conceptual confusion. But, it is an empirical fact that most of us are susceptible to this confusion.

That is, if you follow your likely first intuition and think of $m\vec{a}$ as a force you will probably join the ranks of many other talented students and make many sign errors.

Every teacher of mechanics has encountered the confusion in their students about whether $-m\vec{a}$ is or is not a force (and most likely in themselves as well.) To avoid such confusion, many teachers or texts take a firm stand and say

- '$m\vec{a}$ *is not* a force!'; but, as if believing in a different god, others will say with equal conviction
- '-$m\vec{a}$ *is* a force!'.

In this book, we take the former approach. We take the equation

$$\vec{F} = \mathrm{m}\vec{a}$$

to mean:

forces from interactions =$m\cdot$ (acceleration of mass).

If you insist on working with the D'Alambert approach instead, you must do so confidently and clearly. To repeat,

- instead of labeling your free body diagram '*FBD*', label it '*FBD including inertial forces*',
- instead of using '*Linear Momentum Balance*', use '*Pseudo-Force Balance*', and
- instead of using '*Angular Momentum Balance*' use '*Pseudo-Moment Balance*'

We do *not* recommend D'Alembert mechanics to beginners, but if you insist, good luck to you and don't blame us for your (almost inevitable) sign errors!

**Force depends on velocity.** This case is encountered when, say, an object moves through a fluid and other forces, say gravity, are negligible. Here we have

$$F = ma \quad \Rightarrow \quad F(v) = m\dot{v}.$$

This is solved by multiplying both sides by $dt$ and dividing both sides by $F(v)$ and integrating to get

$$\int dt = m \int \frac{dv}{F(v)} \quad \Rightarrow \quad t = m \int_{v_0}^{v} \frac{dv'}{F(v')}$$

If we want to know position vs time we have to integrate once again.

Example: **The slowing of a bullet.**
The main force on a bullet after it leaves the gun and before it hits its mark is from air drag. This drag is roughly proportional to the speed squared, thus

$$F = ma \quad \Rightarrow \quad -cv^2 = m\dot{v} \quad \Rightarrow \quad -c \int dt = m \int_{v_0}^{v} \frac{dv'}{v'^2}.$$

Carrying out the integrals ($\int dt = t$ and $\int v^{-2}\, dv = -v^{-1}$) we get

$$-ct = m\left(\frac{1}{v} - \frac{1}{v_0}\right) \quad \Rightarrow \quad v = \frac{v_0}{cv_0 t/m + 1}$$

To get position we would integrate again to get:

$$x = \int_0^t v(t')\, dt' = \int_0^t \frac{v_0}{cv_0 t'/m + 1}\, dt' = (m/c)\ln(1 + cv_0 t/m)$$

Interestingly, according to this equation (which becomes less and less accurate as the bullet slows and gravity and eventually viscous forces become important) the bullet goes an infinite distance before stopping.

**Force varies with position.** This case, where $F = F(x)$ will be treated in some detail in the next section on energy.

## The simplest ODEs

The simplest and most common ODEs in dynamics and the rest of science and engineering are

**Linear:** e.g., no functions squared.

**First or second order:** Have only first or second derivatives, respectively, and

**Constant coefficient:** All multiples of the derivatives are constants, not functions of time.

As mentioned previously, some special cases of these ODEs are listed and discussed in box 9.1 page 392.

## 9.3 The simplest ODEs, their solutions, and heuristic explanations.

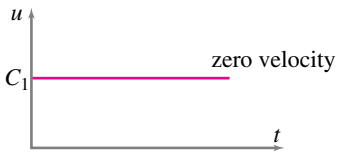
This box is *not* an aside. Rather it is a summary of material that each student should know well.

Here are some of the simplest useful ordinary differential equations (ODEs) and their general solutions. Think of $u$ as the distance an object has moved to the right of its 'home' at $u = 0$ in time $t$. The velocity and acceleration to the right are $du/dt = \dot{u}$ and $d^2/dt^2 = \ddot{u}$. If $\dot{u} < 0$ the particle is moving to the left. If $\ddot{u} < 0$ the particle is accelerating to the left. In all cases $A$ and $B$ are constants and $\lambda$ is a positive constant. $C_1$, $C_2$, $C_3$, and $C_4$ are arbitrary constants in the solutions that become unarbitrary when (are determined by) initial conditions are given.

**a)** $\dot{u} = 0 \;\Rightarrow\; u = C_1.$

$\dot{u} = 0$ means that the velocity is zero. This equation would arise in dynamics if a particle has no initial velocity and no force is applied to it. The particle doesn't move. Its position must be constant. But it could be anywhere, say at position $C_1$. Hence the general solution $u = C_1$, as can be found by direct integration.

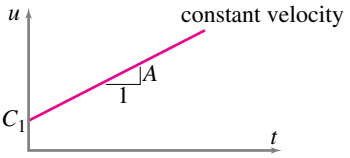
a)




**b)** $\dot{u} = A \;\Rightarrow\; u = At + C_1.$

$\dot{u} = A$ means the object has constant speed. This equation describes the motion of a particle that starts with speed $v_0 = A$ and because it has no force acting on it continues to move at constant speed. How far does it go in time $t$? It goes $v_0 t$. Where was it at time $t = 0$? It could have been anywhere then, say $C_1$. So where is it at time $t$? It's at its original position plus how far it has moved, $u = v_0 t + C_1$, as can also be found by direct integration.

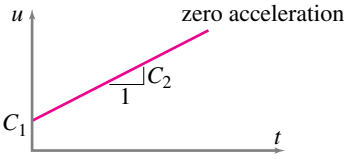
b)




**c)** $\ddot{u} = 0 \;\Rightarrow\; u = C_1 t + C_2.$

$\ddot{u} = 0$ means the acceleration is zero. That is, the rate of change of velocity is zero. This constant-velocity motion is the general equation for a particle with no force acting on it. The velocity, if not changing, must be constant. What constant? It could be anything, say $C_1$. Now we have the same situation as in case (b). So the position as a function of time is anything consistent with an object moving at constant velocity: $u = C_1 t + C_2$, where the constants $C_1$ and $C_2$ depend on the initial velocity and initial position. If you know that the position at $t = 0$ is $u_0$ and the velocity at $t = 0$ is $v_0$, then the position is $u = u_0 + v_0 t$.

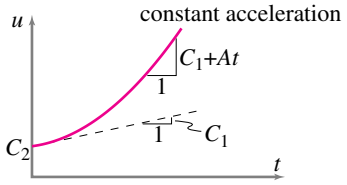
c)




**d)** $\ddot{u} = A$
$\Rightarrow\; u = At^2/2 + C_1 t + C_2.$

This constant acceleration $A$, constant rate of change of velocity, is the classic (all-too-often studied) case. This situation arises for vertical motion of an object in a constant gravitational field as well as in problems of constant acceleration or deceleration of vehicles. The velocity increases in proportion to the time that passes. The change in velocity in a given time is thus $At$ and the velocity is $v = \dot{u} = v_0 + At$ (given that the velocity was $v_0$ at $t = 0$). Because the velocity is increasing constantly over time, the average velocity in a trip of length $t$ occurs at $t/2$ and is $v_0 + At/2$. The distance traveled is the average velocity times the time of travel so the distance of travel is $t \cdot (v_0 + At/2) = v_0 t + At^2/2$. The position is the position at $t = 0$, $u_0$, plus the distance traveled since time zero. So $u = u_0 + v_0 t + At^2/2 = C_2 + C_1 t + At^2/2$. This solution can also be found by direct integration.

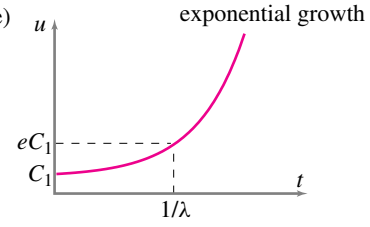
d)




**e)** $\dot{u} = \lambda u \;\Rightarrow\; u = C_1 e^{\lambda t}.$

The displacement $u$ grows in proportion to its present size. This equation describes the initial falling of an inverted pendulum in a thick viscous fluid. The bigger the $u$, the faster it moves. Such situations are called exponential growth (as in population growth or monetary inflation) for a good mathematical reason. The solution $u$ is an exponential function of time: $u(t) = C_1 e^{\lambda t}$, as can be found by separating variables or guessing.

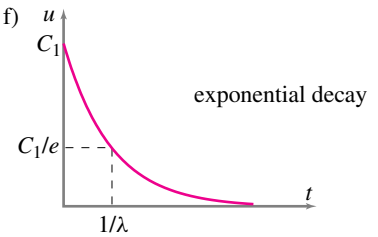
e)




**f)** $\dot{u} = -\lambda u \;\Rightarrow\; u = C_1 e^{-\lambda t}.$

The smaller $u$ is, the more slowly it gets smaller. $u$ gradually tapers towards nothing: $u$ decays exponentially. The solution to the equation is: $u(t) = C_1 e^{-\lambda t}$. This expression is essentially the same equation as in (e) above.

f)

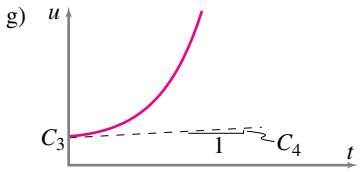


**g)** $\ddot{u} = \lambda^2 u$

$$\Rightarrow u = C_1 e^{\lambda t} + C_2 e^{-\lambda t}$$

or $$\Rightarrow u = C_3 \cosh(\lambda t) + C_4 \sinh(\lambda t).$$

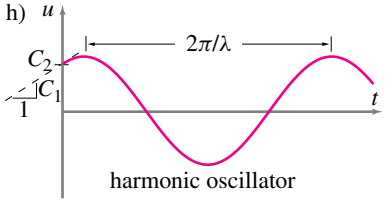
Note, sinh and cosh are just combinations of exponentials. For $\ddot{u} = \lambda^2 u$, the point accelerates more and more away from the origin in proportion to the distance from the origin. This equation describes the falling of a nearly vertical inverted pendulum when there is no friction. Most often, the solution of this equation gives roughly exponential growth. The pendulum accelerates away from being upright. The reason there is also an exponentially decaying solution to this equation is a little more subtle to understand intuitively: if a not quite upright pendulum is given just the right initial velocity it will slowly approach becoming just upright with an exponentially decaying displacement. This decaying solution is not easy to see experimentally because without the perfect initial condition the exponentially growing part of the solution eventually dominates and the pendulum accelerates away from being just upright.




**h)** $\ddot{u} = -\lambda^2 u$ or $\ddot{u} + \lambda^2 u = 0$

$$\Rightarrow \quad u = C_1 \sin(\lambda t) + C_2 \cos(\lambda t).$$

This equation describes a mass that is restrained by a spring which is relaxed when the mass is at $u = 0$. When $u$ is positive, $\ddot{u}$ is negative. That is, if the particle is on the right side of the origin it accelerates to the left. Similarly, if the particle is on the left it accelerates to the right. In the middle, where $u = 0$, it has no acceleration, so it neither speeds up nor slows down in its motion whether it is moving to the left or the right. So the particle goes back and forth: its position oscillates. A function that correctly describes this oscillation is $u = \sin(\lambda t)$, that is, sinusoidal oscillations. The oscillations are faster if $\lambda$ is bigger. Another solution is $u = \cos(\lambda t)$. The general solution is $u = C_1 \sin(\lambda t) + C_2 \cos(\lambda t)$. A plot of this function reveals a sine wave shape for any value of $C_1$ or $C_2$, although the phase depends on the relative values of $C_1$ and $C_2$. The equation $\ddot{u} = -\lambda^2 u$ or $\ddot{u} + \lambda^2 u = 0$ is called the 'harmonic oscillator' equation and is important in almost all branches of science. The solution may be found by guessing or other means (which are usually guessing in disguise). In the context of this equation, $\lambda$ is called the (angular) frequency of oscillation.




**More ODEs** Besides the ODEs listed here there are a few others that are often solved by hand rather than with numerical simulation. Most famously there is the forced, damped oscillator equation $A\ddot{u} + B\dot{u} + Cu = F\sin(Dt)$ which gets its own section.

With the exception of the damped or forced oscillator, most engineers now-a-days will use numerical integration if they want to solve an ODE not in this box.

Conversely,

> Anyone competent at dynamics knows all the equations and solutions on these two pages outside, inside, and inside-out.

Whether you have or have not learned these in a calculus course you should learn them every different way that you can.

**SAMPLE 9.1 Time derivatives:** The position of a particle varies with time as $\vec{r}(t) = (C_1 t + C_2 t^2)\hat{\imath}$, where $C_1 = 4\,\text{m/s}$ and $C_2 = 2\,\text{m/s}^2$.

1. Find the velocity and acceleration of the particle as functions of time.
2. Sketch the position, velocity, and acceleration of the particle against time from $t = 0$ to $t = 5\,\text{s}$.
3. Find the position, velocity, and acceleration of the particle at $t = 2\,\text{s}$.

**Solution**

1. We are given the position of the particle as a function of time. We need to find the velocity (time derivative of position) and the acceleration (time derivative of velocity).

$$\vec{r} = (C_1 t + C_2 t^2)\hat{\imath} = (4\,\text{m/s}\,t + 2\,\text{m/s}^2\,t^2)\hat{\imath} \tag{9.5}$$

$$\vec{v} \equiv \frac{d\vec{r}}{dt} = \frac{d}{dt}(C_1 t + C_2 t^2)\hat{\imath}$$

$$= (C_1 + 2C_2 t)\hat{\imath} = (4\,\text{m/s} + 4\,\text{m/s}^2\,t)\hat{\imath} \tag{9.6}$$

$$\vec{a} \equiv \frac{d\vec{v}}{dt} = \frac{d}{dt}(C_1 + 2C_2 t)\hat{\imath}$$

$$= 2C_2\hat{\imath} = (4\,\text{m/s}^2)\hat{\imath} \tag{9.7}$$

$$\boxed{\vec{v} = (4\,\text{m/s} + 4\,\text{m/s}^2\,t)\hat{\imath},\ \vec{a} = (4\,\text{m/s}^2)\hat{\imath}.}$$

Thus, we find that the velocity is a linear function of time and the acceleration is time-independent (a constant).

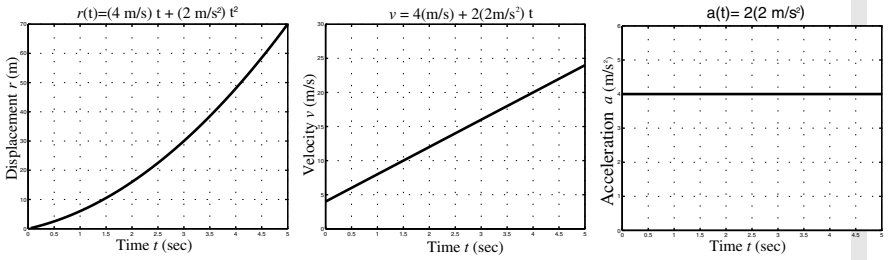



Figure 9.7:

Filename:sfig5-1-vraplot

2. We plot eqns. (9.5, 9.6, and 9.7) against time by taking 100 points between $t = 0$ and $t = 5\,\text{s}$, and evaluating $\vec{r}$, $\vec{v}$ and $\vec{a}$ at those points. The plots are shown in Fig. 9.7.
3. We can find the position, velocity, and acceleration at $t = 2\,\text{s}$ by evaluating their expressions at the given time instant:

$$\vec{r}|_{t=2\,\text{s}} = [(4\,\text{m/s}) \cdot (2\,\text{s}) + (2\,\text{m/s}^2) \cdot (2\,\text{s})^2]\hat{\imath}$$

$$= (16\,\text{m})\hat{\imath}$$

$$\vec{v}|_{t=2\,\text{s}} = [(4\,\text{m/s}) + (2\,\text{m/s}^2) \cdot (2\,\text{s})]\hat{\imath}$$

$$= (8\,\text{m/s})\hat{\imath}$$

$$\vec{a}|_{t=2\,\text{s}} = (2\,\text{m/s}^2)\hat{\imath} = \vec{a}(\text{for all t})$$

$$\boxed{\text{At } t = 2\,\text{s},\ \vec{r} = (16\,\text{m})\hat{\imath},\ \vec{v} = (8\,\text{m/s})\hat{\imath},\ \vec{a} = (2\,\text{m/s}^2)\hat{\imath}.}$$

**SAMPLE 9.2 Math review: Solving simple differential equations.** For the following differential equations, find the solution for the given initial conditions.

1. $\frac{dv}{dt} = a,\ v(t=0) = v_0$, where $a$ is a constant.
2. $\frac{d^2x}{dt^2} = a,\ x(t=0) = x_0,\ \dot{x}(t=0) = \dot{x}_0$, where $a$ is a constant.

**Solution**

1.

$$\begin{aligned} \frac{dv}{dt} &= a \quad \Rightarrow \quad dv = a\,dt \\ \text{or} \qquad \int dv &= \int a\,dt = a\int dt \\ \text{or} \qquad v &= at + C, \quad \text{where C is a constant of integration} \end{aligned}$$

Now, substituting the initial condition into the solution,

$$v(t=0) \equiv v_0 = a\cdot 0 + C \quad \Rightarrow \quad C = v_0.$$

Therefore,

$$v = at + v_0.$$

$$\boxed{v = v_0 + at}$$

**Alternatively,** we can use definite integrals:

$$\int_{v_0}^{v} dv = \int_0^t a\,dt \quad \Rightarrow \quad v - v_0 = at \quad \Rightarrow \quad v = v_0 + at.$$

2. This is a second order differential equation in $x$. We can solve this equation by first writing it as a first order differential equation in $v \equiv dx/dt$, solving for $v$ by integration, and then solving again for $x$ in the same manner.

$$\begin{aligned} \frac{d^2x}{dt^2} &= a \quad \text{or} \quad \frac{dv}{dt} = a \\ \text{or} \qquad \int dv &= \int a\,dt \\ \Rightarrow \quad v \equiv \dot{x} &= at + C_1 \qquad (9.8) \\ \text{but, } v \equiv \frac{dx}{dt}, \quad \Rightarrow \quad \int dx &= \int at\,dt + \int C_1\,dt \\ \text{or} \qquad x &= \frac{1}{2}at^2 + C_1 t + C_2, \qquad (9.9) \end{aligned}$$

where $C_1$ and $C_2$ are constants of integration. Substituting the initial condition for $\dot{x}$ in Eqn. (9.8), we get

$$\dot{x}(t=0) = \dot{x}_0 = a\cdot 0 + C_1 \quad \Rightarrow \quad C_1 = \dot{x}_0.$$

Similarly, substituting the initial condition for $x$ in Eqn. (9.9), we get

$$x(t=0) = x_0 = \frac{1}{2}a\cdot 0 + \dot{x}_0\cdot 0 + C_2 \quad \Rightarrow \quad C_2 = x_0.$$

Therefore,

$$x(t) = x_0 + \dot{x}_0 t + \frac{1}{2}at^2.$$

$$\boxed{x(t) = x_0 + \dot{x}_0 t + \tfrac{1}{2}at^2}$$

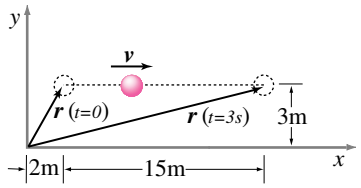



Figure 9.8:
Filename:sfig5-1-new1

**SAMPLE 9.3 Constant speed motion:** A ship cruises at a constant speed of 15 knots per hour due Northeast. It passes a lighthouse at 8:30 am. The next lighthouse is approximately 35 knots straight ahead. At what time does the ship pass the next lighthouse?

**Solution** We are given the distance $s$ and the speed of travel $v$. We need to find how long it takes to travel the given distance.

$$\begin{aligned} s &= vt \\ \Rightarrow \quad t &= \frac{s}{v} = \frac{35 \text{ knots}}{15 \text{ knots/hour}} = 2.33 \text{ hrs.} \end{aligned}$$

Now, the time at $t = 0$ is 8:30 am. Therefore, the time after 2.33 hrs (2 hours 20 minutes) will be 10:50 am.

$$\boxed{10:50 \text{ am}}$$

---

**SAMPLE 9.4 Constant velocity motion:** A particle travels with constant velocity $\vec{v} = 5\,\text{m/s}\hat{\imath}$. The initial position of the particle is $\vec{r}_0 = 2\,\text{m}\hat{\imath} + 3\,\text{m}\hat{\jmath}$. Find the position of the particle at $t = 3\,\text{s}$.

**Solution** Here, we are given the velocity, *i.e.*, the time derivative of position:

$$\vec{v} \equiv \frac{d\vec{r}}{dt} = v_0\hat{\imath}, \qquad \text{where } v_0 = 5\,\text{m/s}.$$

We need to find $\vec{r}$ at $t = 3\,\text{s}$, given that $\vec{r}$ at $t = 0$ is $\vec{r}_0$.

$$\begin{aligned} d\vec{r} &= v_0\hat{\imath}dt \\ \Rightarrow \quad \int_{\vec{r}_0}^{\vec{r}(t)} d\vec{r} &= \int_0^t v_0\hat{\imath}dt = v_0\hat{\imath}\int_0^t dt \\ \vec{r}(t) - \vec{r}_0 &= v_0\hat{\imath}t \\ \vec{r}(t) &= \vec{r}_0 + v_0t\hat{\imath} \\ \vec{r}(3\,\text{s}) &= (2\,\text{m}\hat{\imath} + 3\,\text{m}\hat{\jmath}) + (5\,\text{m/s})\cdot(3\,\text{s})\hat{\imath} \\ &= 17\,\text{m}\hat{\imath} + 3\,\text{m}\hat{\jmath}. \end{aligned}$$

$$\boxed{\vec{r} = 17\,\text{m}\hat{\imath} + 3\,\text{m}\hat{\jmath}}$$

**Comments:** We could solve this problem more compactly by working with scalars or components. It is given that the velocity is constant and is only in the $x$-direction. Therefore, the $y$-component of particle position will remain the same, *i.e.*, $r_y = r_{0y} = 3\,\text{m}$, and $r_x = r_{0x} + v_xt = 2\,\text{m} + (5\,\text{m/s})\cdot(3\,\text{s}) = 17\,\text{m}$. Thus, $\vec{r}(3\,\text{s}) = r_x\hat{\imath} + r_y\hat{\jmath} = 17\,\text{m}\hat{\imath} + 3\,\text{m}\hat{\jmath}$.

**SAMPLE 9.5 Constant acceleration:** A 0.5 kg mass starts from rest and attains a speed of 20 m/s$\hat{\boldsymbol{\imath}}$ in 4 s. Assuming that the mass accelerates at a constant rate, find the force acting on the mass.

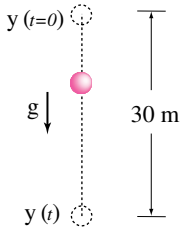



Figure 9.9:
Filename:sfig5-2-new3

**Solution** Here, we are given the initial velocity $\vec{\boldsymbol{v}}(0) = \vec{\boldsymbol{0}}$ and the final velocity $\vec{\boldsymbol{v}}$ after $t = 4$ s. We have to find the force acting on the mass. The net force on a particle is given by $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$. Thus, we need to find the acceleration $\vec{\boldsymbol{a}}$ of the mass to calculate the force acting on it. Now, the velocity of a particle under constant acceleration is given by

$$\vec{\boldsymbol{v}}(t) = \vec{\boldsymbol{v}}_0 + \vec{\boldsymbol{a}}t.$$

Therefore, we can find the acceleration $\vec{\boldsymbol{a}}$ as

$$\begin{aligned} \vec{\boldsymbol{a}} &= \frac{\vec{\boldsymbol{v}}(t) - \vec{\boldsymbol{v}}(0)}{t} \\ &= \frac{20\,\text{m/s}\hat{\boldsymbol{\imath}} - \vec{\boldsymbol{0}}}{4\,\text{s}} \\ &= 5\,\text{m/s}^2\hat{\boldsymbol{\imath}}. \end{aligned}$$

The force on the particle is

$$\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}} = (0.5\,\text{kg}) \cdot (5\,\text{m/s}^2\hat{\boldsymbol{\imath}}) = 2.5\,\text{N}\hat{\boldsymbol{\imath}}.$$

$$\boxed{\vec{\boldsymbol{F}} = 2.5\,\text{N}\hat{\boldsymbol{\imath}}}$$

---

**SAMPLE 9.6 Time of travel for a given distance:** A ball of mass 200 gm falls freely under gravity from a height of 50 m. Find the time taken to fall through a distance of 30 m, given that the acceleration due to gravity $g = 10\,\text{m/s}^2$.

**Solution** The entire motion is in one dimension — the vertical direction. We can, therefore, use scalar equations for distance, velocity, and acceleration. Let $y$ denote the distance travelled by the ball. Let us measure $y$ vertically downwards, starting from the height at which the ball starts falling (see Fig. 9.9). Under constant acceleration $g$, we can write the distance travelled as

$$y(t) = y_0 + v_0 t + \frac{1}{2}gt^2.$$

Note that at $t = 0$, $y_0 = 0$ and $v_0 = 0$. We are given that at some instant $t$ (that we need to find) $y = 30$ m. Thus,

$$\begin{aligned} y &= \frac{1}{2}gt^2 \\ t &= \sqrt{\frac{2y}{g}} = \sqrt{\frac{2 \times 30\,\text{m}}{10\,\text{m/s}^2}} = 2.45\,\text{s}. \end{aligned}$$

$$\boxed{t = 2.45\,\text{s}}$$

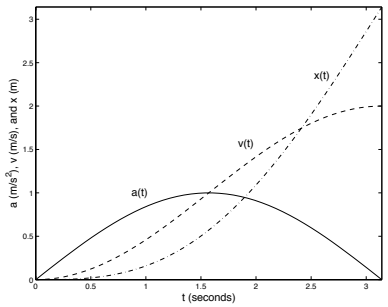



Figure 9.10: Plots of acceleration $a(t)$, speed $v(t)$ and displacement $x(t)$ as functions of time: $a(t) = a_0 \sin\lambda t$, $v(t) = \frac{a_0}{\lambda}(1-\cos\lambda t)$, and $x(t) = \frac{a_0}{\lambda}(t - \frac{1}{\lambda}\sin\lambda t)$ where $a_0 = 1\,\text{m/s}^2$ and $\lambda = 1/\,\text{s}$.

Filename:sfig9-1-matlab1

**SAMPLE 9.7 Time varying acceleration:** A force $F(t) = F_0 \sin\lambda t$ acts on an initially still cart of mass $m$ in a particular direction. Find the speed and the distance travelled by the cart as functions of time. Plot the acceleration, the speed and the displacement of the cart against time for $0 \le t \le \pi$ s, assuming $\lambda = 1/\,\text{s}$. What are the speed and the displacement of the cart at $t = \pi$ s if $F_0 = 1\,\text{N}$ and $m = 1\,\text{kg}$?

**Solution** We are given the applied force and the mass of the cart. Therefore, we know the acceleration ($a = F/m$). Thus,

$$\begin{aligned} a \equiv \frac{dv}{dt} &= \frac{F_0}{m}\sin\lambda t \\ \Rightarrow \quad dv &= a_0 \sin\lambda t dt \end{aligned}$$

where $a_0 = F_0/m$. Hence,

$$\begin{aligned} \int_0^{v(t)} dv &= \int_0^t a_0 \sin\lambda\tau d\tau \\ \Rightarrow \quad v(t) &= -\frac{a_0}{\lambda}(\cos\lambda t - 1) \\ &= \frac{a_0}{\lambda}(1-\cos\lambda t). \end{aligned}$$

Since the speed $v = \frac{dx}{dt}$, we have,

$$\begin{aligned} dx &= \frac{a_0}{\lambda}(1-\cos\lambda t)dt \\ \int_0^{x(t)} dx &= \int_0^t \frac{a_0}{\lambda}(1-\cos\lambda\tau)d\tau \\ \Rightarrow \quad x(t) &= \frac{a_0}{\lambda}\left(t - \frac{1}{\lambda}\sin\lambda t\right). \end{aligned}$$

$$\boxed{v(t) = \tfrac{F_0/m}{\lambda}(1-\cos\lambda t), \quad x(t) = \tfrac{F_0/m}{\lambda}\left(t - \tfrac{1}{\lambda}\sin\lambda t\right)}$$

Substituting $a_0 = F_0/m = 1\,\text{N}/1\,\text{kg} = 1\,\text{m/s}^2$, $\lambda = 1/\,\text{s}$ and $t = \pi$ s in the expressions for $v$ and $x$ above, we find the speed and the displacement (distance travelled by the cart) at $t = \pi$ seconds as follows.

$$\begin{aligned} v(t = \pi\,\text{s}) &= \frac{1\,\text{m/s}^2}{1/\,\text{s}}(1-\cos\pi) \\ &= 2\,\text{m/s}, \\ x(t = \pi\,\text{s}) &= 1\,\text{m/s}\left(\pi\,\text{s} - \frac{1}{1/\,\text{s}}\sin(\frac{1}{\text{s}}\cdot\pi\,\text{s})\right) \\ &= \pi\,\text{m}. \end{aligned}$$

$$\boxed{\text{At } t = \pi\,\text{s}, \quad v = 2\,\text{m/s}, \quad x = \pi\,\text{m}}$$

The graph of $a(t)$, $v(t)$, and $x(t)$ are shown in Fig. 9.10 for $0 \le t \le \pi$ s assuming the given values of $m$, $\lambda$, and $F_0$. Note the behavior of $v(t)$ and $x(t)$ close to $t = 0$. Since the cart starts from rest, the speed builds up slowly, and the displacement builds up even more slowly because the speed is very low in the beginning.

**SAMPLE 9.8 Numerical integration of ODE's:**

1. Write the second order linear nonhomogeneous differential equation, $\ddot{x} + c\dot{x} + kx = a_0 \sin \omega t$, as a set of first order equations that can be used for numerical integration.
2. Write the second order nonlinear homogeneous differential equation, $\ddot{x} + c\dot{x}^2 + kx^3 = 0$, as a set of first order equations that can be used for numerical integration.
3. Solve the nonlinear equation given in (b) by numerical integration taking $c = 0.05$, $k = 1$, $x(0) = 0$, and $\dot{x}(0) = 0.1$. Compare this solution with that of the linear equation in (a) by setting $a_0 = 0$ and taking other values to be the same as for (b).

**Solution**

1.

$$\begin{aligned} \text{If we let} \quad & \dot{x} &&= y, \\ \text{then} \quad & \dot{y} &&= \ddot{x} = -c\dot{x} - kx + a_0 \sin \omega t \\ & &&= -cy - kx + a_0 \sin \omega t \end{aligned}$$

$$\text{or} \qquad \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{Bmatrix} 0 \\ a_0 \sin \omega t \end{Bmatrix}. \tag{9.10}$$

Equation (9.10) is written in matrix form to show that it is a set of *linear* first-order ODE's. In this case linearity means that the dependent variables only appear linearly, not as powers *etc*.

2.

$$\begin{aligned} \text{If} \quad & \dot{x} &&= y \\ \text{then} \quad & \dot{y} &&= \ddot{x} = c\dot{x} - kx^3 = -cy - kx^3 \end{aligned}$$

$$\text{or} \qquad \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{Bmatrix} y \\ -cy - k^3 \end{Bmatrix}. \tag{9.11}$$

Equation (9.11) is a set of *nonlinear* first order ODE's. It cannot be arranged as Eqn. 9.10 because of the nonlinearity in $x$ and $y$. It is, however, in an appropriate form for numerical integration.

3. Now we solve the set of first order equations obtained in (b) using a numerical ODE solver with the following pseudocode.

```
ODEs = {xdot = y,  ydot = -c y - k x^3}
IC   = {x(0) = 0, y(0) = 0.1}
Set  k=1, c=0.05
Solve ODEs with IC for t=0 to t=200
Plot x(t) and y(t)
```

The plot obtained from numerical integration using a Runge-Kutta based integrator is shown in Fig. 9.11. A similar program used for the equation in (a) with $a_0 = 0$ gives the plot shown in Fig. 9.12. The two plots show how a simple nonlinearity changes the response drastically.

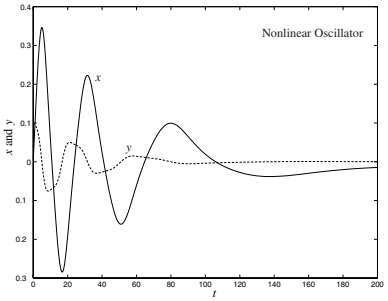



Figure 9.11: Numerical solution of the nonlinear ODE $\ddot{x} + c\dot{x}^2 + kx^3 = 0$ with initial conditions $x(0) = 0$ and $\dot{x}(0) = 0.1$.

Filename:sfig5-1-nonlin

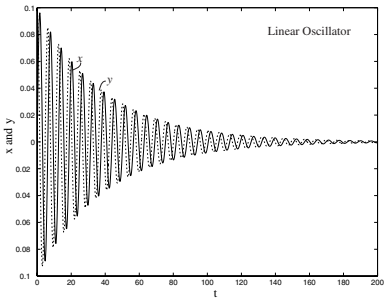



Figure 9.12: Numerical solution of the linear ODE $\ddot{x} + c\dot{x}^2 + kx = 0$ with initial conditions $x(0) = 0$ and $\dot{x}(0) = 0.1$.

Filename:sfig5-1-lin

## 9.2 Energy methods in 1D

Energy is an important concept in science and engineering and is even a kind of currency in human trade. To start with, an energy equation is primarily a short-cut for solving some mechanics problems. Later we accept energy as a concept that is somewhat bigger than can be defined in classical mechanics, and we can look at, say, the chemical energy cost of various mechanical tasks.

For a student learning mechanics energy is first a method, or trick, for solving some simple problems of the type assigned in elementary courses like this one. As problems become more difficult (have more degrees of freedom or include, say, more-than-just-constant friction) energy becomes less useful as a problem solving technique. However, in more advanced mechanics problems energy gets a central role again. Energy is the central concept in various theories including advanced ways to write equations of motion and methods of understanding stability.

*

In fourteen hundred and ninety-two,
Columbus sailed the ocean blue,
And that's twice the watts
in a horsepower too.

$$\begin{array}{r} 746 \\ \times 2 \\ \hline 1492 \end{array}$$

### Power, work, kinetic energy and potential energy

Before we get to the facts and theorems, we start with some definitions. Here are four words. We will use these definitions, or generalizations of them, throughout dynamics.

**Power.** The power of a force $F$ is its product with the velocity $v$ of the point on which it is acting,

$$\text{Power} = P = Fv.$$

This is the 1D version of the more general $P = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}$ which we will use once we go on to 2D and 3D dynamics. In full generality, power is a scalar (not a vector). The common units for power are watts ($1\,\text{W} = 1\,\text{N m/s} = 1\,\text{J/s}$), kilowatts ($1\text{KW} = 10^3$ W), $\text{lbf ft/s}$ (no special name) and horsepower ($1\,\text{hp} \equiv 550\,\text{lbf ft/s} \approx 745.7 \approx 746$* watts).

Example:
A 5 N force acting on a particle moving 3 m/s has a power of
$P = Fv = (5\,\text{N})(3\,\text{m/s}) = 15\,\text{N m/s} = 15\,\text{W} \approx 0.02\,\text{hp}$.

**Work.** The work $W$ of a force is most easily defined incrementally ($\Delta W$) for small motions $\Delta x$ of a particle; motions so small that variations in force can be neglected and the force viewed as constant,

$$\text{increment of work} = \Delta W = F\Delta x.$$

This is a special 1D reduction of the more general $\Delta W = \vec{\boldsymbol{F}} \cdot \Delta\vec{\boldsymbol{r}}$. Even in 2D and 3D work is a scalar. Often we want to know the work for larger (non-infinitesimal) displacements. We do this by adding up the increments. Using sloppy calculus (implicitly taking the the limit of a Reimann sum):

$W = \sum \Delta W = \int dW = \int F\,dx$, which we can write more definitely as

$$\text{Work} = W = \int_{x_0}^{x} F(x')\,dx'.$$

which is the 1D version of the more general $W = \int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$. Common units of work are Joules ($1\,\text{J} = 1\,\text{N m} = 1\,\text{kg m}^2/\text{s}^2$), foot-pounds ($= 1\,\text{ft lbf}$) and kilo-watt hours ($= 3.6 \cdot 10^6\,\text{J}$).

Example:
The force $F = F_0 \sin(cx)$ pushes a mass from $x_0 = 0\,\text{m}$ to $x_1 = \pi\,\text{m}$ where $F_0 = 7\,\text{N}$ and $c = 1/\,\text{m}$. Then

$$W = \int_{x_0}^{x_1} F\,dx = \int_0^{\pi\,\text{m}} F_0 \sin(cx)\,dx = -(F_0/c)\cos(cx)\big|_0^{\pi\,\text{m}} = 14\,\text{N}$$

**Kinetic energy.** The kinetic energy quantifies the motion of a little differently than momentum does. In kinetic energy high speed gets extra credit ($v^2$ instead of just $v$). Further, for kinetic energy we don't worry about which way a particle moves. The kinetic energy $E_\text{K}$ of a particle in 1D is

$$\text{kinetic energy} = E_\text{K} = \frac{1}{2}mv^2$$

In two and three dimensions the formula above applies for one particle (taking $v = |\vec{\boldsymbol{v}}|$). For a collection of particles $E_\text{K}$ is defined as a sum of $E_\text{K}$ for each particle separately. In full generality work is a scalar. The units of work (force×distance) and of kinetic energy (mass×speed$^2$) are the same (mass×distance$^2$/ time$^2$) and so are the common measures, namely Joules, foot-pounds and kilo-watt hours.

Example:
A 3 kg mass moving at a speed of 4 m/s has a kinetic energy of $E_\text{K} = mv^2/2 = (3\,\text{kg})(4\,\text{m/s})^2/2 = 24\,\text{kg m}^2/\,\text{s}^2 = 24\,\text{J}$.

**Potential energy.** This is the most abstract of the definitions. The potential energy $E_\text{P}$ associated with a force $F$ is defined as that function of $x$ with these properties

$$E_\text{P}(x) = -\int_{x_0}^{x} F(x')\,dx' \qquad \text{and} \qquad F(x) = -\frac{d}{dx}E_\text{P}(x)$$

which people write more indefinitely as $E_\text{P} = -\int F\,dx$ and $F = E_\text{P}'$. In two and three dimensions the concept of potential energy is more subtle still, being defined by a path integral which may or may not be sensible. But it is still a scalar.

Example:
The force $F = c/x^2$ is associated with the potential energy $E_\text{P} = -\int F\,dx = c/x + C_0$.

**The datum for potential energy.** The potential energy always has an undetermined, and generally irrelevant, integration constant. The integration constant is rrelevant because usually we care about *changes* in energy. So in the example above we could set $C_0 = 0$ and write $E_{\rm P} = -\int F\,dx = c/x$. In general we define the *datum* for potential energy as that position where we set the potential energy to zero.

- For near-earth gravity the datum is usually set at the height of the ground (so that $E_{\rm P} = mgh$), a launch point, or of a conspicuous physical point (say the hinge of a pendulum).
- For inverse-square gravity the datum is usually set at $\infty$ so that formulas are most simple.
- For springs that datum is usually set at the position where the spring is 'relaxed' (unstretched and at its rest-length), again simplifying the terms in energy equations.

**Potential energy is a shortcut for calculating work.** From the definition of potential energy we can calculate work of a force in moving a particle from one place to another as:

$$\text{work} = \int_{x_1}^{x_2} F(x')\,dx' = -\left(E_{\rm P2} - E_{\rm P1}\right).$$

Of course you need to know, or find, $E_{\rm P}(x)$ first in order to use this shortcut.

Example:
The work of $F = c/x^2$ in moving a mass from $x_1$ to $x_2$ is

$$\text{work} = \int_{x_1}^{x_2} F(x')\,dx' = -\left(E_{\rm P2} - E_{\rm P1}\right) = c/x_1 - c/x_2$$

Where we used that $E_{\rm P} = c/x$ has the needed property that $F = -\dfrac{d}{dx}E_{\rm P}$.

**Why all this new language?** All of the words above are defined in terms of position, velocity and force. So anything we say about power, work and kinetic and potential energies we could say already using $x$, $v$ and $F$. More particularly, we already have two ways of quantifying the motion of a particle, $v$ and Ł $= mv$, Why do we need a third, $E_{\rm K} = mv^2/2$? The answer is for simpler thinking and simpler formulas. Various facts and theorems are simpler if commonly appearing groups of terms are given names. And all of the definitions above are common groups. Then, luckily, some of them turn out to be more general than just 1D particle mechanics.

The new vocabulary makes thinking easier. Various so-called 'one degree of freedom' problems can be solved by noting that energy is conserved. And features of solutions of more-complex problems can be extracted or checked by making sure that energy balance comes out right.

## Power and work

The simplest relation between the quantities we have defined above is that between Power and work:

$$W = \int F\,dx = \int Fv\frac{dx}{v} = \int Fv\,dt$$

or more definitely

$$W = \int_{x_0}^{x_1} F\,dx = \int_{t_0}^{t_1} Fv\,dt = \int_{t_0}^{t_1} P\,dt$$

Example:
The power of a some force acting on some particle is $P = P_0(ct^2)$ where $P_0 = 10\,\mathrm{W}$ and $c = 3/\,\mathrm{s}^2$ then over 3 seconds the work done by the force is:

$$W = \int_{t_0}^{t_1} P\,dt == \int_{t_0}^{t_1} P_0(ct^2)\,dt = P_0ct^3/3\Big|_{t_0}^{t_1} = (10\,\mathrm{W})(3/\,\mathrm{s}^2)t^3/3\Big|_0^{3\,\mathrm{s}} = 270\,\mathrm{W\,s} = 270\,\mathrm{J}$$

### Power and rate-of-change of kinetic energy

On the inside cover the third basic law of mechanics is energy balance. Energy balance takes a number of different forms, depending on context. The power balance equation from the front cover and simplified for a particle is

$$P = \dot{E}_\mathrm{K},$$

where, recall, $P = Fv$ is the power of the applied force $F$. The derivation of this result from $F = ma$ for a particle is simple enough, and is good to know. First note the following result from using the chain of differentiation:

$$\frac{d}{dt}\left(v^2\right) = 2v\frac{dv}{dt} = 2v\dot{v} = 2va.$$

When we need to call on this simple kinematics (calculus) result it usually comes to us the other way around. So what you should remember is this formula, one of the basic tricks of the trade:

$$va = \frac{d}{dt}\left(\frac{v^2}{2}\right).$$

Multiplying both sides by $m$ and substituting in $F = ma$ we get our 1D power balance equation:

$$Fv = \frac{d}{dt}\left(\frac{mv^2}{2}\right).$$

The power of a given force depends on the speed of the object to which it is applied. When a finite force is applied to a stationary object the power of the force is zero and so is the rate of change of kinetic energy. The object has an acceleration, its speed is increasing, but until it has finite speed $\dot{E}_K = 0$.

Example:
A constant force $F$ is applied to an initially stationary mass $m$ starting at $t = 0$. Then $v = Ft/m$, $E_K = mv^2/2 = F^2t^2/(2m)$ and $P = Fv = F^2t/m$. Note that $\dot{E}_K = P$ and both are zero at $t = 0$.

## Work is change of kinetic energy

Integrating the power balance equation in time we get

$$\int P\,dt = \int \dot{E}_K\,dt = \Delta E_K \tag{9.12}$$

More definitely, and also using the work integral, we have that the work of the net force on a particle is the change of its kinetic energy:

$$\int_{t_1}^{t_2} \underbrace{Fv}_{P}\,dt = \int_{x_1}^{x_2} F\,dx = E_{K2} - E_{K1}$$

Once we remember that

work is change in kinetic energy,

we can use it without deriving it every time from $F = ma$ or from more general energy balance equations.

Example:
A force applied to a particle $m$ varies sinusoidally with position according to $F = F_0\cos(cx)$. At $x = 0$ the particle has speed $v = v_0$. Then

$$W = \Delta E_K \quad \Rightarrow \quad \int_0^x F(x')\,dx' = \Delta\left(mv^2/2\right) \quad \Rightarrow \quad F_0\sin(cx)/c = mv^2/2 - mv_0^2/2$$

$$\text{so} \qquad v = m\sqrt{v_0^2 + 2F_0\sin(cx)/(mc)}$$

The above example illustrates three points you should remember:

- The work-energy equations always leave the *sign* of the velocity unknown. You can see this because the derivation involves $v^2$. You can also see it in formulas you get for velocity. They involve a square root, and thus, implicitly a $\pm$. Whether one, the other or both roots are relevant depends on reasoning that lies outside the energy equation itself.
- The work-energy equations can generate formulas that, in certain situations, are nonsense: If the initial speed $v_0$ is not high enough the particle will not get very far. In particular if $v_0^2 < 2F_0/(mc)$ the inside of the square root will be negative for some $x$ and the "answer" will be imaginary. These are values of $x$ that the particle will never reach.

- Here we have apparently solved for something about the motion of a particle. And we have, partially. But to find the $x(t)$ we would have to integrate again. And that next integral is hard. That is, energy balance lets us solve for some aspects of the motion, namely speed vs position, without ever needing to know in detail how position varies with time.

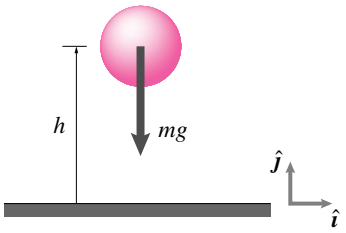



Figure 9.13: Free body diagram of a falling ball, assuming gravity is the only significant external force acting on the ball.

Filename:tfigure1-falling-ball

## Conservation of energy

Most people leave high-school physics loving conservation of energy. It makes certain special homework problems easy. In the real world the principle is also useful for building intuition, and sometimes also for problem solving.

Most generally we cannot think of energy conservation as necessarily applicable nor, if applicable, as derivable from the equations of mechanics. But in 1D particle mechanics energy conservation is a theorem.

Recall that if a particle is acted on by a force that varies with position, $F = F(x)$, then we can define a potential energy $E_P = -\int F\,dx$ and that the work done by the force when the particle moves from $x_1$ to $x_2$ is

$$-(E_{P2} - E_{P1}) = -\Delta E_P.$$

That is, the decrease in $E_P$ is the amount of work that the force does. Or, in other words, $E_P$ represents a potential to do work. Because work causes an increase in kinetic energy, $E_P$ is called the *potential energy* of the force field. Now we can compare this result with the work-energy equation 9.12 to find that

$$-\Delta E_P = \Delta E_K \qquad \Rightarrow \qquad 0 = \Delta \underbrace{(E_P + E_K)}_{E_T}.$$

The *total energy* $E_T$ doesn't change ($\Delta E_T = 0$) and thus is a constant. In other words,

> as a particle moves in the presence of a force field with a potential energy, the total energy $E_T = E_K + E_P$ is constant.

This fact goes by the name of *conservation of energy*.

Example: **Falling ball**
Consider the ball in the free body diagram 9.13. If we define gravitational potential energy as minus the work gravity does on a ball while it is lifted from the ground, then

$$E_P = -\int_0^y (-mg)\,dy' = mgy = mgh.$$

For vertical motion

$$E_K = \frac{1}{2}m\dot{y}^2.$$

So conservation of energy says that in free fall:

$$\text{Constant} = E_P + E_K = mgy + m\dot{y}^2$$

which you could also derive directly from $m\ddot{y} = -mg$.

**Using conservation of energy to find equations of motion.** On the one hand conservation of energy sometimes gives us a (partial) solution to a mechanics problem. On the other, we can use conservation of energy to find the "equations of motion". The basic strategy is to take the derivative of the conservation of energy equation.

Example: **Falling ball eqns. from energy.**

$$\begin{aligned} E_{\mathrm{T}} = \text{constant} \quad \Rightarrow \quad 0 &= \tfrac{d}{dt} E_{\mathrm{T}} \\ &= \tfrac{d}{dt}(E_{\mathrm{P}} + E_{\mathrm{K}}) \\ &= \tfrac{d}{dt}(mgy + m\dot{y}^2/2) \\ &= (mg\dot{y} + m\dot{y}\ddot{y}) \\ \Rightarrow \quad m\ddot{y} &= -mg. \end{aligned}$$

We had to assume (and this is just a technical point) that $\dot{y} \neq 0$ in one of the cancellations. We have used energy balance to derive linear-momentum balance.

One can also find equations of motion starting with power balance.

$$P = \dot{E}_{\mathrm{K}}$$

as derived here in detail here for the case of gravity acting on a particle.

$$\begin{aligned} P &= \tfrac{d}{dt}(E_{\mathrm{K}}) && \text{(Power balance)} \\ \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}} &= \tfrac{d}{dt}(E_{\mathrm{K}}) && \text{(Power of external force)} \\ (-mg\hat{\boldsymbol{j}}) \cdot (\dot{y}\hat{\boldsymbol{j}}) &= \tfrac{d}{dt}\left[\tfrac{1}{2}mv^2\right] && \text{(expanding terms)} \\ -mg\dot{y} &= \tfrac{1}{2}m\tfrac{d}{dt}\left(\dot{y}^2\right) && \text{(carrying out dot product, substitute for } v\text{)} \\ -mg\dot{y} &= \tfrac{1}{2}m\,(2\dot{y}\cdot\ddot{y}) && \text{(the chain rule)} \\ \ddot{y} &= -g && \text{(cancel terms, switch sides),} \end{aligned} \tag{9.13}$$

**The potential energy of a spring is $k(\Delta\ell)^2/2$.** Besides near-earth gravity, which we already covered ($E_{\mathrm{P}} = mgh$), the main elementary use of potential energy is for the stretch of a linear spring. Integrating $dW = Fdx$ for a linear spring with $F = kx$, where $x$ is the spring stretch, from the rest length, we get

$$E_{\mathrm{P}} = -\int_0^x F(x')\,dx' = -\int_0^x -kx'\,dx' = \frac{1}{2}kx^2. \tag{9.14}$$

In the above example we measured $x$ from the rest position of one end of the spring. But often the natural $x$ coordinate will not be so nicely set up. It is safer to remember the spring's potential energy in terms of its stretch:

$$E_{\mathrm{P}} = \frac{k\,\Delta\ell^2}{2}, \tag{9.15}$$

where we measure $\Delta\ell = \ell - \ell_0$ where $\ell_0$ is the spring's rest length ($\ell_0 =$ length when the tension is zero).

Thus for a spring and mass oscillator, the subject of the next section, conservation of energy tells us that $mv^2/2 + kx^2/2 = constant$.

## Is energy balance a principle or a calculation trick?

For one dimensional particle motion, momentum balance, power balance, and energy balance can each be derived from either of the others. If we take $F = ma$ as primary, energy calculations are just a convenience of notation or, in the case of the work-energy relation, a useful calculation technique (trick).

Historically, conservation of energy was first noted in particle mechanics problems. But because the position-dependent forces of springs and gravity seemed so fundamental, that they had a description as the derivative of a potential gave the energy relations the smell of something more fundamental. And so it has turned out that energy is an important topic for chemistry, thermodynamics, electrodynamics and sub-atomic physics. Its not just an analogy, its the same energy. Thus energy is the primary currency of exchange between, say, the superficially disparate chemical and mechanical systems.

The exchange of energy between these forms, in the context of particle mechanical models, can give the sense that we are doing the same 1D momentum based mechanics calculations when actually we are using more general energy balance equations, equations that cannot be derived from $F = ma$.

## Terrestrial locomotion: Trains, cars, bicycles and animals

A free body diagram of an accelerating car, treated as a 1D particle system, is shown in Fig. 9.5 on page 384. The point represents the car, the force is the propulsion force from the wheel-ground interaction, and for now, we have neglected air friction. Without worrying about details we could say then that the power of the propulsion force is equal to the rate of change of kinetic energy.

Example: **Accelerating car**
An aggressive 1 ton car has an acceleration of $0.5g$ while going 60 mph and passing another car. Neglecting friction and air resistance, the power of the propulsion force is

$$\begin{aligned} P &= Fv = mav = (1\,\text{ton})(0.5g)(50\,\text{mi/hr}) \\ &= (1\,\text{ton})(0.5g)(60\,\text{mi/hr})\underbrace{\left(\frac{2000\,\text{lbm}}{\text{ton}}\right)}_{1}\underbrace{\left(\frac{5280\,\text{ft}}{\text{mi}}\right)}_{1}\underbrace{\left(\frac{1\,\text{hr}}{3600\,\text{s}}\right)}_{1}\underbrace{\left(\frac{1\,\text{lbf}}{g\cdot\,\text{lbm}}\right)}_{1}\underbrace{\left(\frac{1\,\text{hp}}{550\,\text{ft}\cdot\,\text{lbf/s}}\right)}_{1} \\ &= (0.5\cdot 60\cdot 2000\cdot 5280)/(3600\cdot 500)\,\text{hp} \approx 160\,\text{hp} \end{aligned}$$

The car engine needs to supply this 160 hp plus any internal transmission dissipation. Note the judicious multiple multiplication by 1 so that all units cancel but for horse-power; ton cancels ton, hr cancels hr, $g$ cancels $g$ and so on.

**??** Such calculations are deceptively simple. Some apparent paradoxes:

- the propulsive force on the car comes from the interaction of the ground with the car. Are we saying that the (dead-as-a-doormat) ground supplies a power of, say, 160 hp to an accelerating car?
- The point of application of the force on the car is at the bottom of the tire. That point has no velocity. So the actual power of the ground force

on the car (tire) is zero. How is that reconciled with, say, the 160 hp that we get from particle mechanics.

These are legitimate concerns. None-the-less, the calculation turns out, perhaps by the demands of dimensional consistency, to be useful and correct. These issues are discussed further in box 9.4 on page 410.

**Drag power.** The drag force of air on moving things has an effect on the energy balance. Air drag is important for cars, bicycles and animals that are moving quickly (say, running people). The air drag is proportional to

$$F_d = \frac{1}{2}\rho C A v^2$$

What are the proportionalities in the drag formula?

- The cross-sectional area (the area visible from directly in front) $A$. The bigger the area the more air has to be pushed out of the way. For a car $A \approx 2\,\mathrm{m}^2$
- The density of air $\rho$. The more mass has to be pushed out of the way, the bigger the force. For rough calculations one can remember that the density of air is about one thousandth that of water $\rho_{air} \approx 1\,\mathrm{kg/\,m}^3$. But the density varies in human environments from about $1.1\,\mathrm{kg/\,m}^3$ in high-altitude (low pressure), high-temperature (gas expands when hot), humid (water vapor is lighter than air) environments up to about $1.4\,\mathrm{kg/\,m}^3$ in low-lying cold dry places.
- The relative speed squared $v^2$. The faster you are moving the more air per unit time you must displace, and each bit of air gets displaced with a bigger speed. Typical highway speeds are about $v = 30\,\mathrm{m/\,s}$ ($\approx 67$ mph) and a typical human walking speed is about $v = 1\,\mathrm{m/s}$ (about 10% over $\approx 2$ mph).
- A shape coefficient, sometimes called a drag coefficient $C_d$. Different shapes of the same size, can displace the air more or less as the vehicle passes through. Streamlined shapes have small $C_d$
- One half (1/2). Convention has a factor of 1/2. This simplifies the power interpretation.

The drag power is

$$P = F_d v = \frac{1}{2}\rho C A v^3$$

which is a key result: increasing the speed 1% increases the power demand by 3% and doubling the speed multiplies the power demand by a factor of 8. This huge dependence of power on speed motivates smug energy-conservers to drive annoyingly slowly on highways.

Another way of writing the drag equation is

$$P = C_d \times (\text{ The relative kinetic energy swept by the vehicle per unit time})$$

How's that? The volume "swept" by the vehicle per unit time is its area times speed $vA$. The air mass swept per unit time is thus $\rho v A$. The kinetic energy

of the air, measured as moving relative to the vehicle, is $v^2/2$ per unit mass. Putting this together we get the swept kinetic energy of the air per unit time is $\rho A v^3/2$.

**How big is the drag coefficient $C_d$?** When in doubt take dimensionless constants as 1 and you are usually not too far off. At one extreme, a flat plate has $C_d \approx 1.25$ and good airfoils have $C_d \approx 0.05$. People, animals, and bicyclists all have $C_d$ close to 1.

**Drag on cars.** For the worst cars $C_d$ is actually almost 1. For typical cars on the street $C_d \approx 0.35$. For the best high-efficiency cars on the market in 2007, $C_d \approx 0.25$. Real marketed cars may one day get drag as low as $C_d \approx 0.2$. And concept cars that are shaped like trout can have drag as low as $C_d \approx 0.1$.

The drag power of a $2\,\mathrm{m}^2$ car going $30\,\mathrm{m/s}$ ($\approx 67\,\mathrm{mph} = 108\,\mathrm{km/hr}$ ) is about

$$\begin{aligned} P_{drag} &= \frac{1}{2} C\rho A v^3 \approx \frac{1}{2} \cdot 0.35 \cdot (1\,\mathrm{kg/m^3}) \cdot (2\,\mathrm{m^2}) \cdot (30\,\mathrm{m/s})^3 \\ &= 9450\,\mathrm{kg\,m^2/s^3} = 9.45\,\mathrm{KW} \approx 13\,\mathrm{hp} \end{aligned}$$

That is, comparing with the example above, a car that needs 160 hp extra to make a zippy pass needs only 13 hp to move steadily along at a typical highway speed. For the units conversion we used $1\,\mathrm{N} = 1\,\mathrm{kg\,m/s^2}$, $1\,\mathrm{J} = 1\,\mathrm{N\,m}$, $1\,\mathrm{W} = 1\,\mathrm{J/s}$, and $1\,\mathrm{KW} \approx 1.34\,\mathrm{hp}$.

**Caveat on the drag "law".** While there is some physics in the reasoning behind the drag law $F_d = \rho C A v^2/2$ the emphasis should be on the word "some", the whole chaotic nature of turbulent flow is not captured. The quadratic drag law is an empirical fit. For a given shape the $C_d$ actually depends on the surface texture. And for a given shape and texture the $C_d$ depends on $v$, the $v^2$ doesn't capture all of the velocity dependence. Nonethe-less, the drag law is a reasonable approximation for most engineering purposes where drag is important.

## Summary

There are two basic types of energy problems

- Problems where force or acceleration is given as a function of position ($a = a(x)$ or $F = F(x)$) and energy methods are basically a trick for finding $v(x)$.
- Problems where work, energy or power is of interest for its own sake because of, say, interest in engine power, dissipated energy, etc.

Of course the two problem types can overlap also.

## 9.4 *THEORY*

*Energetics of locomotion: using particle equations for non-particle systems*

On page **??** we showed a naive locomotion power example in which we used

$$P = Fv$$

where $v$ was the car velocity, $F$ the thrust on the car, and $P$ was 'the power' of the locomotion force. We pointed out two issues.

- How does it make sense for the passive ground, the source of the propulsive force, to supply power?
- The point of application of the ground force on the car is at the bottom of a wheel, a point that is not moving ($v = 0$). So how can $Fv$ be other than zero?

The basic issue is that a car is not a particle, it has many moving parts and also some chemistry, so particle equations need to be interpreted with some care.

## Particle equations are exact for non-particle systems

It turns out that the most general form for linear momentum balance as applied to a complicated system moving and deforming in complicated ways, reduces to equation $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$. That is, so long as we interpret $\vec{\boldsymbol{F}}$ to be the total force on the sytem, $\vec{\boldsymbol{a}}$ to be the acceleration of the center of mass, and $m$ to be the total mass of the system.

The power and energy equations in this chapter have been based on $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ (or their 1D scalar version $F = ma$) so apply to *any* system. But the terms $P$ and $E_{\mathrm{K}}$ have meanings that go beyond particle mechanics. So while it is correct that (we derived it from $F = ma$),

$$Fv = \frac{d}{dt}\left(\frac{1}{2}mv^2\right)$$

for non-particle systems it is *not correct* that $Fv$ is the actual power of the force applied nor that $mv^2/2$ is the kinetic energy of the system.

To understand the situation depends on understanding multi-body systems where we will see that the power of a force is $\vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}_P$ where $\vec{\boldsymbol{v}}_P$ is the velocity of the material point to which the force is applied; and the kinetic energy is larger than $\frac{1}{2}mv^2$ because of motion relative to the average motion. Remember to reconsider these issues when you know more.

## More general energy balance equations

Without worrying about what we can derive from what, there is no doubt that for any closed system we can write the energy balance equation from the front inside cover of the book, the first law of thermodynamics, as:

$$\dot{Q} + P = \dot{E}_{\mathrm{K}} + \dot{E}_{\mathrm{P}} + \dot{E}_{\mathrm{int}}.$$

About the ever-shifting sign conventions, here we use $\dot{Q}$ as the heat flow *in* to the system, $P$ is the power of external forces *on* the system, $\dot{E}_{\mathrm{K}}$ and $\dot{E}_{\mathrm{P}}$ are the rate of *increase* of the kinetic and potential energies of the system, and $\dot{E}_{\mathrm{int}}$ is the rate of *increase* of internal energy. We can consider an accelerating car using this energy equation. For simplicity assume that no external forces do work on the car (the ground certainly does no work, and let's neglect air friction for now). We can also look at a car on level ground so there are no changes in gravitational potential energy. Finally, even though a car has many moving parts, the bulk of the material goes at the speed of a typical point on the body of the car. Thus the particle formula for kinetic energy is reasonably accurate. Putting this altogether we have

$$\begin{aligned} \dot{Q} + \underbrace{P}_{0} &= \frac{d}{dt}\left(\frac{1}{2}mv^2\right) + \underbrace{E_{\mathrm{P}}}_{0} + \dot{E}_{\mathrm{int}} \\ \Rightarrow \quad \frac{d}{dt}\left(\frac{1}{2}mv^2\right) &= -\dot{E}_{\mathrm{int}} + \dot{Q} \end{aligned}$$

The rate of loss of chemical potential energy $-\dot{E}_{\mathrm{int}}$ less the heat flow out $-\dot{Q}$ is what we call the power of the engine. Say chemical energy is being lost (used up) at a rate of $-\dot{E}_{\mathrm{int}} \doteq 40\,\mathrm{KW}$. Say the heat flow out the exhaust is $-\dot{Q} = 30\,\mathrm{KW}$. Then, with that 40 KW of fuel use and that 25% efficient engine, we would have

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = -\dot{E}_{\mathrm{int}} + \dot{Q} = 40\,\mathrm{KW} - 30\,\mathrm{KW} = 10\,\mathrm{KW} \approx 13\,\mathrm{hp}.$$

But even this is not quite right because it does not take account the flow of gases in and out of the car. Things can be messy if you look carefully.

**What's the bottom line?** In the end, with some sloppiness of thought but not much inaccuracy, we are not far off thinking that the change of kinetic energy of the car has to come from some place. And that place is the work of the engine as supplied by the decrease in chemical potential energy of the fuel. When we write $P = \dot{E}_{\mathrm{K}}$ for a car, the $P$ in that equation is the force applied to the car times the velocity of the car. But that $P$ is not the power supplied by the outside agents on the car (e.g., the passive ground). Rather it is the power of forces inside the car. Never mind that we're modeling a car as a particle with no internal structure, at least for momentum-balance purposes.

This whole situation can only be properly clarified when we look at the power of internal and external forces in multi-body systems.

**SAMPLE 9.9** How much time does it take for a car of mass 800 kg to go from 0 mph to 60 mph, if we assume that the engine delivers a constant power P of 40 horsepower during this period. (1 horsepower = 745.7 W)

**Solution**

$$\begin{aligned} P &= \dot{W} \equiv \frac{dW}{dt} \\ dW &= P\,dt \\ W_{12} &= \int_{t_0}^{t_1} P\,dt = P(t_1 - t_0) = P\Delta t \\ \Delta t &= \frac{W_{12}}{P}. \end{aligned}$$

Now, from IIIa in the inside front cover,

$$\begin{aligned} W_{12} &= (E_{\mathrm{K}})_2 - (E_{\mathrm{K}})_1 \\ &= \frac{1}{2}m(v_2^2 - v_1^2) \\ &= \frac{800\,\mathrm{kg}[(60\,\mathrm{mph})^2 - 0]}{2} \\ &= \frac{1}{2}\cdot 800\,\mathrm{kg}\left(60\frac{\mathrm{mi}}{\mathrm{hr}}\cdot\frac{1.61\times10^3\,\mathrm{m}}{1\,\mathrm{mi}}\cdot\frac{1\,\mathrm{hr}}{3600\,\mathrm{s}}\right)^2 \\ &= 288.01\times10^3\ \mathrm{kg}\cdot\mathrm{m}\cdot\mathrm{m/s}^2 \\ &= 288\ \mathrm{KJoule}. \end{aligned}$$

Therefore,

$$\Delta t = \frac{288\times10^3\,\mathrm{J}}{40\times745.7\,\mathrm{W}} = 9.66\,\mathrm{s}.$$

Thus it takes about 10 s to accelerate from a standstill to 60 mph.

$$\boxed{\Delta t = 9.66\,\mathrm{s}}$$

Note 1: This model gives a roughly realistic answer *but* it is not a realistic model, at least at the start, at time $t_0$. In the model here, the acceleration is infinite at the start (the power jumps from zero to a finite value at the start, when the velocity is zero), something the finite-friction tires would not allow.

Note 2: We have been a little sloppy in quoting the energy equation. Since there are no external forces doing work on the car, somewhat more properly we should perhaps have written

$$0 = \dot{E}_{\mathrm{K}} + \dot{E}_{\mathrm{int}} + \dot{E}_{\mathrm{P}}$$

and set $-(\dot{E}_{\mathrm{int}} + \dot{E}_{\mathrm{P}}) =$ 'the engine power' where the engine power is from the decrease in gasoline potential energy ($-\dot{E}_{\mathrm{P}}$ is positive) less the increase in 'heat' ($\dot{E}_{\mathrm{int}}$) from engine inefficiencies.

**SAMPLE 9.10 Which is the best bicycle helmet?** Assume a bicyclist moves with speed 25 mph when her head hits a brick wall. Assume her head is rigid and that it has constant deceleration as it travels through the 2 inches of the bicycle helmet. What is the deceleration? What force is required? (Neglect force from the neck on the head.)

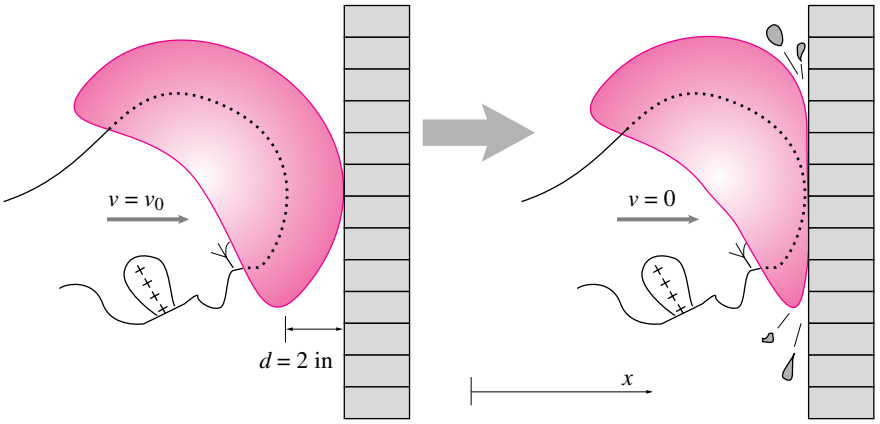



Figure 9.14:

Filename:sfig3-2-ouchie

**Solution**

**Solution 1 – Kinematics method 1:** We are given the initial speed of $V_0$, a final speed of 0, and a constant acceleration $a$ (which is negative) over a given distance of travel $d$. If we call $t_c$ the time when the helmet is fully crushed,

$$
\begin{aligned}
v(t) &= v_0 + \int_0^{t_c} a(t')dt' = v_0 + at_c \\
0 = v(t_c) &= v_0 + at_c \quad \Rightarrow \quad t_c = -v_0/a \qquad (9.16)\\
x(t) &= x_0 + \int_0^{t_c} v(t')dt' = 0 + \int_0^{t_c} (v_0 + at)dt \\
d = x(t_c) &= 0 + v_0 t_c + a t_c^2/2 \\
d &= v_0\left(\frac{-v_0}{a}\right) + a\left(\frac{v_0}{a}\right)^2/2 \quad \Rightarrow \quad d = \frac{-v_0^2}{2a} \qquad \text{(using (9.16))} \\
\Rightarrow \quad a &= \frac{-v_0^2}{2d} \\
&= \frac{-(25\,\text{mph})^2}{2\cdot(2\,\text{in})} \\
&= \frac{-25^2}{4}\cdot\frac{\text{mi}^2}{\text{hr}^2\cdot\text{in}}\cdot\underbrace{\left(\frac{5280\,\text{ft}}{\text{mi}}\right)^2}_{1}\cdot\underbrace{\left(\frac{1\,\text{hr}}{3600\,\text{s}}\right)^2}_{1}\cdot\underbrace{\left(\frac{12\,\text{in}}{\text{ft}}\right)}_{1}\cdot\underbrace{\left(\frac{1g}{32.2\,\text{ft/s}^2}\right)}_{1} \\
&= \frac{-25}{4}\cdot\frac{5280^2}{3600^2}\cdot 12\cdot\frac{1}{32.2}g \\
a &= -125g
\end{aligned}
$$

To stop from 25 mph in 2 inches requires an acceleration that is 125 times that of gravity.

**Solution 2 – Kinematics method 2:**

$$\begin{aligned} \frac{dv}{dt} &= a \quad \Rightarrow \quad dv = a\,dt \\ \Rightarrow \quad v\,dv &= a\,v\,dt \quad \Rightarrow \quad v\,dv = a\frac{dx}{dt}dt \\ \Rightarrow \quad v\,dv &= a\,dx \\ \Rightarrow \quad \int v\,dv &= \int a\,dx \\ \Rightarrow \quad \Delta\frac{v^2}{2} &= ax \qquad \text{(since } a = \text{constant)} \\ \Rightarrow \quad 0 - \frac{v_0^2}{2} &= ad \quad \Rightarrow \quad a = \frac{-v_0^2}{2d} \qquad \text{(as before)} \end{aligned}$$

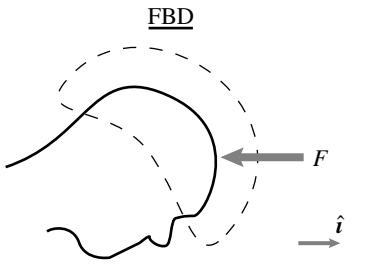



Figure 9.15: $F$ is the force of the helmet on the moving head.

Filename:sfig3-2-ouchie-fbd

**Solution 3 – Quote formulas:**

$$\begin{aligned} \text{“}v &= \sqrt{2ad}\text{”} \\ \Rightarrow \quad a &= \frac{v^2}{2d} \qquad \text{which is right if you know how to interpret it!} \end{aligned}$$

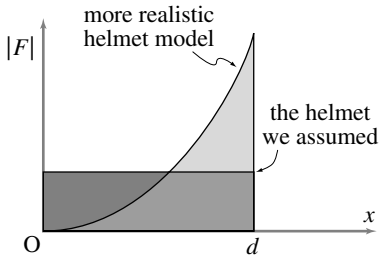



Figure 9.16:

Filename:sfig3-2-ouchie-graph

**Solution 4 – Work-Energy:**

Constant acceleration $\Rightarrow$ constant force

$$\begin{aligned} \text{Work in} &= \Delta E_{\text{K}} \\ -Fd &= 0 - \frac{mv_0^2}{2} \quad \Rightarrow \quad F = \frac{mv_0^2}{2d} \\ \text{But } \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}} \quad &\Rightarrow \quad -F\hat{\boldsymbol{\imath}} = -ma\hat{\boldsymbol{\imath}} \Rightarrow a = \frac{-F}{m} \\ \text{So } a &= \frac{-v_0^2}{2d} \qquad \text{(again)} \end{aligned}$$

Assuming a head mass of 8 lbm, the force on the head during impact is

$$|F| = \frac{mv_0^2}{2d} \quad = \quad ma \quad = \quad 8\,\text{lbm} \cdot 125g.$$

$$\boxed{|F| = 1000\,\text{lbf}}$$

During a collision in which an 8 lbm head decelerates from 25 mph to 0 in 2 inches, the force applied to the head is 1000 lbf.

Note 1: The way to minimize the peak acceleration when stopping from a given speed over a given distance is to have constant acceleration. The ‘best’ possible helmet, the one we assumed, causes constant deceleration. There is no helmet of any possible material with 2 in thickness that could make the deceleration for this collision less than $125g$ or the peak force less than 1000 lbf.

Note 2: Collisions with head decelerations of $250g$ or greater are often fatal. Even $125g$ usually causes brain injury. So, the best possible helmet does not insure against injury for fast riders hitting solid objects.

Note 3: Epidemiological evidence suggests that, on average, chances of serious brain injury are decreased by about a factor of 5 by wearing a helmet.

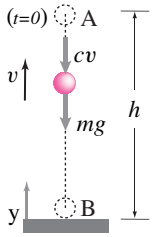



Figure 9.17: Free body diagram of the falling ball. Note that the drag force $F_s = cv$ is shown acting downwards. This is because we have assumed $v$ to be positive upwards and the drag force always acts in the opposite direction of the velocity.

Filename:sfig9-2-fallingball

**SAMPLE 9.11 Dissipated energy in viscous drag:** A ball of mass $m = 1\,\text{kg}$ is dropped from rest from a height $h = 100\,\text{m}$ under gravity. The air resistance on the ball is modeled as viscous drag $F_s = cv$ where $v$ is the speed of the ball and $c = 0.25\,\text{kg/s}$ is the drag coefficient. Find the energy dissipated in overcoming the air resistance during the entire flight of the ball.

**Solution** There are various ways in which we could calculate the energy dissipated in viscous drag. The most straightforward way is to compute the work done by the drag force on the body, $\int F_s dx$ during the entire flight. This calculation will be very easy if we knew the drag force as a function of position, that is, if we have $F_s(x)$. Unfortunately, we have $F_s \equiv F_s(v) = cv$ and we do not know $v$ as a function of position. However, we can find the speed $v$ as a function of time by solving the equation of motion $F = ma$ and determine the speed just before the ball hits the ground. Now, we can find the energy of the ball in two positions — just when it starts falling and just before it hits the ground. The difference between the two energies is what is lost or dissipated in the overcoming the air resistance. Let 'A' denote position-1 from where the ball is dropped, *i.e.*, $y_A = h$, and 'B' denote position-2, a hair above the ground, *i.e.*, $y_B = 0$. Taking the ground as the datum for potential energy, we have,

$$\begin{aligned} E_A &= (E_\text{K} + E_\text{P})_A = \frac{1}{2}m\overbrace{v_A^2}^{0} + mg\overbrace{y_A}^{h} \\ &= mgh \\ E_B &= (E_\text{K} + E_\text{P})_B = \frac{1}{2}m\overbrace{v_B^2}^{?} + mg\overbrace{y_B}^{0} \\ &= \frac{1}{2}mv_B^2. \end{aligned}$$

Therefore, the energy dissipated in air resistance is

$$E_\text{drag} = \Delta E = E_A - E_B = mgh - \frac{1}{2}mv_B^2 \tag{9.17}$$

Now, we just need to find $v_B$. From the free-body diagram shown in Fig. 9.17, we have,

$$\begin{aligned} m\overbrace{\ddot{y}}^{\dot{v}} &= -mg - cv \\ \text{or} \quad \frac{dv}{dt} &= -g - \frac{c}{m}v \\ \Rightarrow \quad \int_0^{v(t)} \frac{dv}{\tilde{c}v + g} &= -\int_0^t d\tau \end{aligned}$$

where $\tilde{c} = \frac{c}{m}$. Thus,

$$\begin{aligned} \frac{1}{\tilde{c}}\ln(\tilde{c}v + g)\Big|_0^{v(t)} &= -t \\ \Rightarrow \quad \ln\frac{\tilde{c}v(t) + g}{g} &= -\tilde{c}t \\ \Rightarrow \quad v(t) &= \frac{g}{\tilde{c}}(e^{-\tilde{c}t} - 1). \end{aligned} \tag{9.18}$$

So, we have solved for $v(t)$. Unfortunately, we cannot find $v_B$ from this expression because we do not know what $t$ when the ball reaches the ground. Thus we need to first find $t_B$ and

then substitute it in eqn. (9.18) to find $v_B$. From eqn. (9.18), we have,

$$\begin{aligned} \frac{dy}{dt} &= \frac{g}{\tilde{c}}(e^{-\tilde{c}t} - 1) \\ \Rightarrow \quad \int_h^0 dy &= \frac{g}{\tilde{c}} \int_0^{t_B} (e^{-\tilde{c}t} - 1)dt \\ \Rightarrow \quad -h &= \frac{g}{\tilde{c}} \left( \frac{e^{-\tilde{c}t}}{-\tilde{c}} - t \right) \Bigg|_0^{t_f} \\ &= \frac{g}{\tilde{c}} \left( -\frac{e^{-\tilde{c}t_f}}{\tilde{c}} - t_f + \frac{1}{\tilde{c}} \right) \\ \text{or} \quad \frac{\tilde{c}h}{g} &= \frac{1}{\tilde{c}}(e^{-\tilde{c}t_f} - 1) + t_f. \end{aligned} \tag{9.19}$$

This turns out to be a transcendental equation* with no simple solution for $t_f$. We can, however, solve it numerically (either using a computer program, or by trial and error). For the given values of $m$, $c$, and $h$, we solve eqn. (9.19) by trial and error (to locate zero crossing), and find that $t_f = 5.5495\,\text{s}$ (see Fig. 9.18). Substituting $t = t_f$ in eqn. (9.18), we get

$$\begin{aligned} v_B &= \frac{mg}{c}(e^{-\frac{c}{m}t_f} - 1) \\ &= \frac{1\,\text{kg} \cdot 9.81\,\text{m/s}^2}{0.25\,\text{kg/s}}(e^{-\frac{0.25\,\text{kg/s}}{1\,\text{kg}} \cdot 5.5495\,\text{s}} - 1) \\ &= -29.44\,\text{m/s}. \end{aligned}$$

Note that $v$ comes out to be negative, which is expected because we assumed $v$ to be positive upwards. The velocity is clearly directed downwards once the ball starts falling. Now, substituting the values of $m$, $g$, $h$, and $v_B$ in eqn. (9.17), we get

$$\begin{aligned} E_{\text{drag}} &= mgh - \frac{1}{2}mv_B^2 \\ &= 1\,\text{kg} \cdot 9.81\,\text{m/s}^2 \cdot 100\,\text{m} - \frac{1}{2} \cdot 1\,\text{kg} \cdot (29.44\,\text{m/s})^2 \\ &= 547.64\,\text{N·m} \end{aligned}$$

Thus more than half of the initial energy is dissipated in air friction. If there was no viscous drag on the ball, its speed just before hitting the ground would be

$$v_B = \sqrt{2gh} = 44.29\,\text{m/s}.$$

$$\boxed{E_{\text{drag}} = 547.64\,\text{N·m}}$$

* A transcendental equation in $t$ is one where $t$ appears both as an argument of a trigonometric or exponential function and elsewhere. Such equations can almost never be solved by hand in closed form.

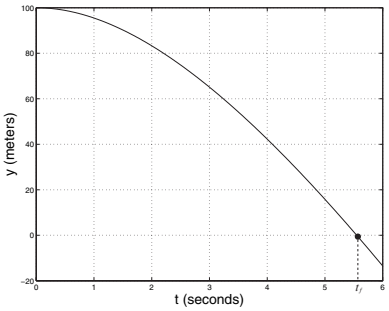



Figure 9.18: A graph of $y(t) = h - \frac{mg}{c}\left(\frac{m}{c}e^{-\frac{c}{m}t} + t - \frac{m}{c}\right)$ for determining $t_f$ when $y = 0$. This is the same as solving eqn. (9.19) for $t_f$.

Filename:sfig9-2-dragfall

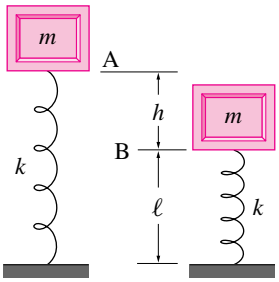



Figure 9.19:

Filename:sfig2-6-1

**SAMPLE 9.12 Energy of a mass-spring system.** A mass $m = 2\,\text{kg}$ is attached to a spring with spring constant $k = 2\,\text{kN/m}$. The relaxed (unstretched) length of the spring is $\ell = 40\,\text{cm}$. The mass is pulled up and released from rest at position A shown in Fig. 9.19. The mass falls by a distance $h = 10\,\text{cm}$ before reaching position B, which is the relaxed position of the spring. Find the speed at point B.

**Solution** The total energy of the mass-spring system at any instant or position consists of the energy stored in the spring and the sum of potential and kinetic energies of the mass. For potential energy of the mass, we need to select a datum where the potential energy is zero. We can select any horizontal plane to be the datum. Let the ground support level of the spring be the datum. Then, at position A,

$$\begin{aligned}\text{Energy in the spring} &= \frac{1}{2}k\,(\text{stretch})^2 = \frac{1}{2}kh^2 \quad \text{(see eqn. (9.15), page 406)}\\ \text{Energy of the mass} &= E_\text{K} + E_\text{P} = \frac{1}{2}m\underbrace{v_A^2}_{0} + mg(\ell + h) = mg(\ell + h).\end{aligned}$$

Therefore, the total energy at position A

$$E_A = \frac{1}{2}kh^2 + mg(\ell + h).$$

Let the speed of the mass at position B be $v_B$. When the mass is at B, the spring is relaxed, i.e., there is no stretch in the spring. Therefore, at position B,

$$\begin{aligned}\text{Energy in the spring} &= \frac{1}{2}k\,(\text{stretch})^2 = 0\\ \text{Energy of the mass} &= E_\text{K} + E_\text{P} = \frac{1}{2}mv_B^2 + mg\ell,\end{aligned}$$

and the total energy

$$E_B = \frac{1}{2}mv_B^2 + mg\ell.$$

Because the net change in the total energy of the system from position A to position B is

$$\begin{aligned}0 &= \Delta E\\ &= E_A - E_B = \frac{1}{2}kh^2 + mg(\ell + h) - \frac{1}{2}mv_B^2 - mg\ell\\ &= \frac{1}{2}(kh^2 - mv_B^2) + mgh\\ \Rightarrow \quad v_B^2 &= kh^2/m + 2gh\\ \Rightarrow \quad |v_B| &= \left(kh^2/m + 2gh\right)^{1/2}\\ &= \left((2000\,\text{N/m}\cdot(0.1\,\text{m})^2/2\,\text{kg}) + 2\cdot 9.81\,\text{m/s}^2\cdot 0.1\,\text{m}\right)^{1/2}\\ &= 3.46\,\text{m/s}.\end{aligned}$$

$$\boxed{|v_B| = 3.46\,\text{m/s}}$$

# 9.3 Vibrations: mass, spring and dashpot

When the mass in the spring-mass system of Fig. 9.20 is to the right ($x > 0$) of the rest position ($x = 0$) it accelerates to the left; when the mass is to the left ($x < 0$) it accelerates to the right. The resulting motion is an oscillation. The harmonic oscillator has no friction (no inelastic deformation) so mechanical energy is conserved; so vibrations, once started, persist forever even with no pushing, pumping, or energy supply of any kind. The oscillations are sinusoidal in time. Air pressure which varies sinusoidal in time is perceived by the ear as a pure note. Hence the spring mass system set into motion is called a *harmonic oscillator* even if it oscillates too slowly to be heard. With varying degrees of approximation, car suspensions, violin strings, buildings responding to earthquakes, earthquake faults themselves, and vibrating machines are modeled as mass-spring systems. Almost all of the concepts in vibration theory are based on concepts associated with the behavior of the harmonic oscillator.

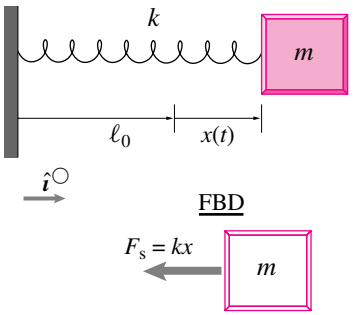



Figure 9.20: A spring mass system. For simplicity, because we are only doing a 1D analysis, not shown on the FBD are forces that have no component in the relevant direction such as from gravity or the (assumed massless and frictionless) support wheels. It would be more correct to show those forces, but it would add clutter to the picture.

Filename:tfigure3-MS

The unforced oscillations of a spring and mass is the basic model for all vibrating systems.

Most engineering materials are elastic (spring-like) under working conditions. And all real things have mass. This elasticity and mass make vibration possible.

The two key ingredients for oscillations (vibrations) are a spring-like 'restoring' force pulling the mass to the center and inertia, so that the mass continues to move through the center position once pulled there.

Even stiff structures will vibrate if encouraged to do so by the shaking of an unbalanced motor, the rumbling of a truck, a party upstairs, or the ground motion of an earthquake. The vibrations of one thing can excite oscillations of another. For example a vibrating bridge can excite oscillations in the air flowing by, which in turn can excite the bridge oscillate more; this mutual excitement of fluids and solids causes vibrations in a clarinet reed*, and caused the wild oscillations leading up to the infamous Tacoma Narrows bridge collapse. Mechanical vibrations are the source of most music and of most annoying sounds. They are the main function of a vibrating massager, and the main defect of a squeaking hinge. Mechanical vibrations in pendula or quartz crystals are used to measure time, but vibrations can cause a machine to go out of control (e.g., bicycle shimmy), or a building to collapse. So the study of vibrations, good vibrations and bad vibrations, is a common application of dynamics.

Because the motions associated with vibrations have features which are

* If you squeeze a blade of grass between your parallel thumbs and then blow in the slot between the left and right thumb joints, at the grass, you can make a squeal sound with mechanics that is similar to that of a clarinet — self-excited oscillations. With practice you can adjust the pitch (the elastic restoring force is related to the tension in the grass which you can control by straightening and bending your thumbs). Music, kind of.

common over all structures and machines, a special vocabulary and special methods of approach have been developed. For example, you will be able to usefully discuss *natural frequency*, *damping*, *resonance*, *normal modes*, and *frequency response*, concepts which we will introduce in this and the following sections, without writing any equations. But to make sense of these words, you should know about the equations first.

In this section we cover the governing equations for the harmonic oscillator and their solution. You will also see how the system is changed if there is some linear friction (damping).

## The harmonic oscillator

The mother of all vibrating machines is the simple harmonic oscillator of Fig. 9.20. The mass slides on a frictionless surface. The spring is relaxed at $x = 0$. The spring is thus stretched from $\ell_0$ to $\ell_0 + \Delta\ell$, a stretch of $\Delta\ell = x$. A free body diagram of the mass, cut 'free' from the spring in its extended state, is shown in the lower part of figure 9.20. Linear momentum balance in the $x$ direction ($\{\sum \vec{F} = \dot{\vec{L}}\} \cdot \hat{\imath}$) gives:

$$\begin{aligned} \sum F_x &= \dot{L}_x \\ -kx &= m\ddot{x}. \end{aligned}$$

Rearranging, we get one of the most famous and useful differential equations of all time*:

$$\ddot{x} + \frac{k}{m}x = 0. \tag{9.20}$$

Eqn. (9.20) appears in many contexts both in and out of dynamics. In non-mechanical contexts the variable $x$ and the parameter combination $k/m$ are replaced by other physical quantities ($> 0$). In an electrical circuit, for example, $x$ might represent a voltage and the term corresponding to $k/m$ might be $1/LC$, where $C$ is a capacitance and $L$ an inductance. But even in dynamics the equation appears with other physical quantities besides $k/m$ multiplying the $x$, and $x$ itself could represent rotation, say, instead of displacement. In order to avoid being specific about the physical system being modeled, the harmonic oscillator equation is often written as

$$\ddot{x} + \lambda^2 x = 0. \tag{9.21}$$

The constant in front of the $x$ is called $\lambda^2$ instead of just, say, $\lambda$ ('lambda')sidenote Most books use $p^2$ or $\omega^2$ in the place we have put $\lambda^2$. Using $\omega$ ('omega') can lead to confusion because we will later use $\omega$ for angular velocity. If one is studying vibrations of a rotating shaft then there would be two very different $\omega$'s in the problem. One, the coefficient of a differential

*Caution: If you make a sign error in your setup you can get $\ddot{x} - (k/m)x = 0$ or $\ddot{x} = (k/m)x$. This is a very different equation with a very different solution. See box 9.1 on page 392.

equation and, the other, the angular velocity. To add to the confusion, simple harmonic oscillations and circular motion have a deep connection, so the coincidence of notation is not accidental. Deep connection or not, the $\omega$ in the harmonic oscillator equation is not the same thing as the $\omega$ describing angular motion of a physical object. We avoid this confusion by using $\lambda$ instead of $\omega$. (This $\lambda$ is unrelated to the magnitude of the unit vector $\hat{\boldsymbol{\lambda}}$). , for two reasons:

1. This convention shows that $\lambda^2$ is positive,
2. In the solution we need the square root of this coefficient, so it is convenient to have $\sqrt{\lambda^2} = \lambda$.

For the spring-block system, $\lambda^2$ is $k/m$ and in other problems $\lambda^2$ is some other combination of physical quantities.

## Solution of the harmonic oscillator differential equation

You can learn how to find solutions to the harmonic oscillator differential equation 9.21 from first principles in a math class*. Here we content ourselves with remembering its general solution, using the intuition from the first sentences of this section. Until you remember it once and for all (which should be soon) you can look up the solution in box 9.1 on page 392, namely

$$\begin{aligned} x(t) &= A\,\cos(\lambda\, t) + B\,\sin(\lambda\, t), \\ \text{or}\quad x(t) &= C_1\cos(\lambda\, t) + C_2\sin(\lambda\, t). \end{aligned} \tag{9.22}$$

This sum of two sine waves* and is a solution of differential equation 9.21 for any values of the constants $A$ (or $C_1$) and $B$ (or $C_2$).

What does it means to say "$u = C_1\sin(\lambda\, t) + C_2\cos(\lambda\, t)$ solves (or satisfies) the equation: $\ddot{u} = -\lambda^2 u$?" Differential equations want to digest their food completely. You can satisfy a differential equation by feeding it a function that fully eliminates. If you plug a candidate solution into a differential equation and get $0 = 0$ you have satisfied the equation.

For the harmonic oscillator equation a solution is a function $u$ that has the property that its second derivative is the same as minus the original function multiplied by the constant $\lambda^2$. That is, the function $u(t) = C_1\sin(\lambda\, t) + C_2\cos(\lambda\, t)$ has the property that its second derivative is the original function multiplied by $-\lambda^2$. You need not take this property on faith.

* You want to derive the solution to the harmonic oscillor ODE? We don't especially advise it, but here's how. The energy equation gives (using $C^2$ to show that the energy is positive)

$$\begin{aligned} \dot{x}^2/\lambda^2 + x^2 &= C^2 \\ \text{so}\quad dx/\sqrt{C^2 - x^2} &= \lambda dt. \end{aligned}$$

If you're good at such things, then integrate (how? substitute $x = C\sin\theta$ or guess), or else look it up on your symbolic calculator or a symbolic math program and get

$$\begin{aligned} \cos^{-1}(x/C) &= \lambda t - c_2 \\ \text{so}\quad x &= C\cos(\lambda t - c_2) \end{aligned}$$

where $C$ and $c_2$ are arbitrary constants. We picked the signs of arbitrary constants to please us. That's one form of the general solution of the harmonic oscillator equation. You can plug it back into eqn. (9.21) and see that you get $0 = 0$ for all values of $C$, $c_2$ and $t$.

* A cosine function is also a sine wave.

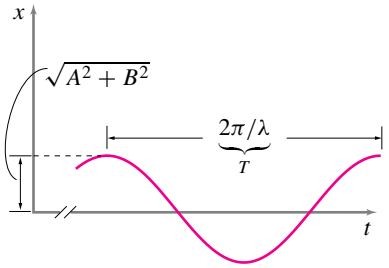



Figure 9.21: **Position versus time for an undamped, unforced harmonic oscillator.** $x$ is the position of the mass, $t$ is time.

Filename:tfigure12-sinewave

* A plausibility argument for uniqueness goes like this. If you release a mass from a given position $x_0$ at a given speed $v_0$ it will move in a definite way and no other way. This is a special case of what is called "determinism". But all solutions have some position and speed at $t = 0$ and we can find a $C_1$ and $C_2$ in eqn. (9.22) to match each such. Thus we have found *the* motion for every possible situation, and there can be no others.

## Checking the solution in detail

To check if a function is a solution, plug it into the differential equation and see if an identity is obtained.

Is this equality correct for the proposed $u(t)$?

$$\frac{d^2}{dt^2}u = -\lambda^2 u$$

$$\frac{d^2}{dt^2}\underbrace{[C_1\sin(\lambda\, t) + C_2\cos(\lambda\, t)]}_{u(t)} \stackrel{?}{=} -\lambda^2\underbrace{[C_1\sin(\lambda\, t) + C_2\cos(\lambda\, t)]}_{u(t)}$$

$$\frac{d}{dt}\left(\frac{d}{dt}[C_1\sin(\lambda\, t) + C_2\cos(\lambda\, t)]\right) \stackrel{?}{=} -\lambda^2[C_1\sin(\lambda\, t) + C_2\cos(\lambda\, t)]$$

$$\frac{d}{dt}[C_1\lambda\cos(\lambda\, t) - C_2\lambda\sin(\lambda\, t)] \stackrel{?}{=} -\lambda^2[C_1\sin(\lambda\, t) + C_2\cos(\lambda\, t)]$$

$$\underbrace{-C_1\lambda^2\sin(\lambda t) - C_2\lambda^2\cos(\lambda\, t)}_{\ddot{u}} \stackrel{?}{=} -\lambda^2\underbrace{[C_1\sin(\lambda\, t) + C_2\cos(\lambda\, t)]}_{u(t)}$$

The equation $\ddot{u} = -\lambda^2 u$ *does* hold with the given $u(t)$. Right and left sides match. This shows at a glance if we write one more line.

$$0 \stackrel{:)}{=} 0 \qquad \text{(Satisfying.)}$$

Whatever the constants $C_1$ and $C_2$, the proposed solution eqn. (9.22) satisfies the differential equation eqn. (9.21).

**Uniqueness.** * We have not proved 'uniqueness', that there are not other solutions to this differential equation than eqn. (**??**). There are not, as the mathematics-for-its-own-sake inclined can learn to prove elsewhere

## Interpreting the solution of the harmonic oscillator equation

We could use a coiled metal spring and a block on low friction rollers to make a machine schematically like the system shown in figure 9.20. We could then watch it move. We would see (approximately) that

$$x(t) = A\cos(\lambda\, t) + B\sin(\lambda\, t),$$

as shown in the graph in figure 9.21.

### Angular frequency, period, and frequency

Three related measures of the rate of oscillation are angular frequency, period, and frequency. The simplest of these is *angular frequency* $\lambda$ =

$\sqrt{(k/m)}$, sometimes called *circular frequency*. The period $T$ is the amount of time that it takes to complete one oscillation. One oscillation of both the *sine* function and the *cosine* function occurs when the argument of the function advances by $2\pi$, that is when

$$\lambda T = 2\pi, \qquad \text{so} \qquad T = \frac{2\pi}{\lambda} = \frac{2\pi}{\sqrt{(k/m)}},$$

Some people memorize these formulas in high school. The *natural frequency* $f$ is the reciprocal of the period

$$f = \frac{1}{T} = \frac{\lambda}{2\pi} = \frac{\sqrt{(k/m)}}{2\pi}.$$

Typically, natural frequency $f$ is measured in cycles per second or *Hertz* and the angular frequency $\lambda$ in radians per second. A computer or watch quartz timing crystal has mechanical vibrations at a frequency of millions of cycles per second, some molecules about a million times faster than that. On the other extreme, the free vibrations of the whole earth have frequencies of thousandths of a cycle per second (i.e. thousands of seconds per cycle). The slowest vibration mode of the earth has a period of about 54 minutes*.

**Amplitude.** The amplitude of the sine wave that results from the addition of the *sine* function and the *cosine* function is given by the square root of the sum of the squares of the two amplitudes. That is, the amplitude of the resulting sine wave is $\sqrt{A^2 + B^2}$. Another way of describing this sum is through the trigonometric identity:

$$A\cos(\lambda\, t) + B\sin(\lambda\, t) = R\cos(\lambda\, t - \phi), \tag{9.23}$$

where $R = \sqrt{A^2 + B^2}$ and $\tan\phi = B/A$ (see box 9.5 on page 422. So,

the only possible motion of a spring and mass is a sinusoidal oscillation which can be thought of either as the sum of a *cosine* function and a *sine* function or as a single *cosine* function with *phase shift* $\phi$.

## Initial conditions determine the constants $A$ and $B$

The general motion of the harmonic oscillator, equation 9.22, has the constants $A$ and $B$ which could have any value. Or, equivalently, the amplitude $R$ and phase $\phi$ in equation 9.23 could be anything. They are determined by the way motion is started, the *initial conditions*. Two special initial conditions are worth getting a feel for: release from rest and initial velocity with no spring stretch.

* Why does the earth oscillate? First because it can. It has both mass and an 'elastic restoring force' The elastic restoring force comes from a combination two things: 1) the elasticity of rock and 2) the self-gravitation of the earth trying to bundle itself into a ball. What gets the earth started oscillating? Mostly big earthquakes.

## Release from rest

The simplest motion to consider is when the spring is stretched a given amount and the mass is *released from rest*, meaning the initial velocity of the mass is zero. we find the motion by looking at the general solution

$$x(t) = A\cos(\sqrt{(k/m)}\,t) + B\sin(\sqrt{(k/m)}\,t).$$

At $t = 0$, this general solution has to agree with the initial condition that the displacement is $x(0) = x_0$ and the initial velocity is $v(0) = v_0 = 0$. In this case

$$x(0) = x_0 \text{ and } v(0) = 0 \quad \Rightarrow \quad A = x_0 \text{ and } B = 0.$$

The next example shows the details.

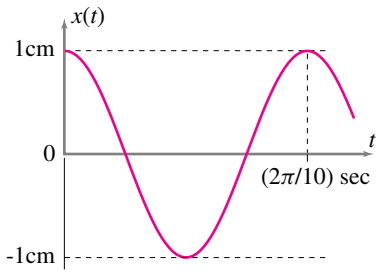



Figure 9.22: The position of a mass as a function of time if $k = 50\,\text{N/m}$, $m = 0.5\,\text{kg}$, $x(0) = 1\,\text{cm}$ and $v(0) = 0$.

Filename:tfigure12-cosine

Example:
The mass in figure 9.20 is 0.5 kg, the spring constant is $k = 50\,\text{N/m}$, and the initial displacement is 2 cm,1 cm,then

$$x(0) = A\underbrace{\cos(0)}_{1} + B\underbrace{\sin(0)}_{0} = A \qquad \Rightarrow \qquad A = 2\,\text{cm}.$$

The initial velocity must also match, so first we find the velocity by differentiating the position to get

$$v(t) = \dot{x}(t) = -A\sqrt{(k/m)}\sin(\sqrt{(k/m)}\,t) + B\sqrt{(k/m)}\cos(\sqrt{(k/m)}\,t).$$

## 9.5 *THEORY*

### *Derivation and visualization of the formula* $A\cos(\lambda t) + B\sin(\lambda t) = R\cos(\lambda t - \phi)$

Here is a demonstration that the sum of a *cosine* function and a *sine* function is a new sine wave. By sine wave we mean a function whose shape is the same as the sine function, though it may be displaced along the time axis. For example $\cos t$ and $\cos(t - const)$ are both sinewaves.

**The trig identity approach.** The quickest approach is to start with the function $f(t) = R\cos(\lambda t - \phi)$ and use the trig addition identity for cosines

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi.$$

Thus:

$$\begin{aligned} R\cos(\lambda t - \phi) &= R\cos\lambda t\cos\phi + R\sin\lambda t\theta\sin\phi \\ &= A\sin\lambda t + B\cos\lambda t. \end{aligned}$$

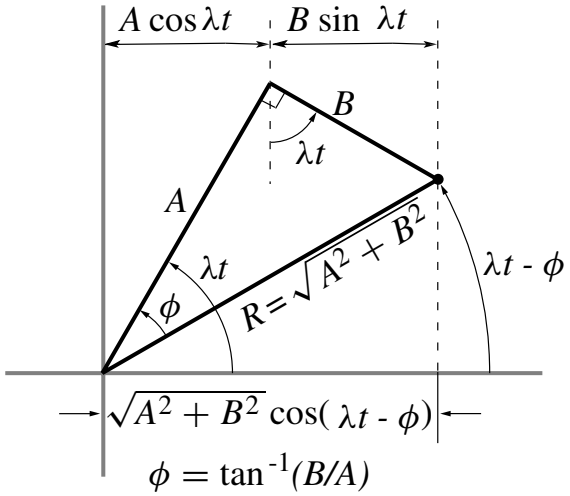
We can run the reasoning from right to left and set $A = R\cos\phi$ and $B = R\sin\phi$ and then solve for $R$ and $\phi$ in terms of $A$ and $B$. Thus demonstrating the formula titling this box. If you have trouble remembering the trig identity, one derivation uses the picture to the right. Trigonometry is full of such circular reasoning.

**The geometric approach.** Consider the line segment $A$ spinning in circles about the origin at rate $\lambda$; that is, the angle the segment makes with the positive $x$ axis is $\lambda t$. The projection of that segment onto the $x$ axis is $A\cos(\lambda t)$, a sine wave. Now consider the segment labeled $B$ in the figure, glued at a right angle to $A$. The length of its projection on the $x$-axis is $B\sin(\lambda t)$. So, the sum of these two projections is $A\cos(\lambda t) + B\sin(\lambda t)$. The two segments $A$ and $B$ make up a right triangle with diagonal $R = \sqrt{A^2 + B^2}$.

The projection or 'shadow' of $R$ on the $x$ axis is the same as the sum of the shadows of $A$ and $B$. The angle it makes with the $x$ axis is $\lambda t - \phi$ where one can see from the triangle drawn that $\phi = \arctan(B/A)$. So, by adding the shadow lengths, we see

$$A\cos(\lambda t) + B\sin(\lambda t) = \sqrt{A^2 + B^2}\cos(\lambda t - \phi)\ .$$

The function $f(t) = R\cos(\lambda t - \phi)$ is a sine wave. In particular it is the cosine function with a maximum at $\lambda t - \phi$.



Now, we evaluate this expression at $t = 0$ and set it equal to the given initial velocity which in this case was zero:*

$$v(0) = -A\sqrt{(k/m)}\underbrace{\sin(0)}_{0} + B\sqrt{(k/m)}\underbrace{\cos(0)}_{1} = B\sqrt{(k/m)} \qquad \Rightarrow \qquad B = 0.$$

Substituting in the values for $k = 5\,\mathrm{N/m}$ and $m = 0.5\,\mathrm{kg}$, we get

$$x(t) = 2\cos\left(\sqrt{\underbrace{\left(\frac{0.5\,\mathrm{kg}}{50\,\mathrm{N/m}}\right)}_{0.01\,\mathrm{s}^{-1}}}\,t\right)\mathrm{cm} = 2\cos(0.1t/\,\mathrm{s})\,\mathrm{cm}$$

which is plotted in figure 9.22.

## Initial velocity with no spring stretch

Another simple case is when the spring has no initial stretch but the mass has some initial velocity. Such might be the case just after a resting mass is hit by a hammer.

Example:
Using the same 0.5 kg mass and $k = 50\,\mathrm{N/m}$ spring, we now consider an initial displacement of zero but an initial velocity of 10 cm/s. We can find the motion for this case from the general solution by the same procedure we just used. We get

$$x(t) = B\sin(\sqrt{(k/m)}\,t)$$

with $B\sqrt{(k/m)} = 10\,\mathrm{cm/s} \qquad \Rightarrow \qquad B = 1\,\mathrm{cm}.$

The resulting motion, $x(t) = (1\,\mathrm{cm}) \cdot \sin(\frac{0.1t}{s})$, is shown in figure 9.23.

# Work, energy, and the harmonic oscillator

In the previous section we showed that momentum balance implies conservation of energy for a harmonic oscillator. Similarly we showed that the harmonic oscillator equation follows from conservation of energy. Energy accounting gives an extra intuitive way to think about what happens in an oscillator.

## Conservation of energy

We have neglected all dissipation in the harmonic oscillator. So the total mechanical energy, the sum of the kinetic energy $E_\mathrm{K} = \frac{1}{2}mv^2$ and the potential energy (from eqn. (9.15)) $E_\mathrm{P} = \frac{1}{2}k(\Delta L)^2$, is constant in time.

$$E_\mathrm{T} = E_\mathrm{K} + E_\mathrm{P} = \text{constant}.$$

As the mass moves, energy is exchanged back and forth between kinetic and potential energy. At the extremes in the displacement, the spring is most stretched. At these extreme points the potential energy is at a maximum, and the kinetic energy is zero. When the mass passes through the center position the spring is relaxed. At this middle position the potential energy is at a minimum (zero), and the mass is at its peak speed, and the kinetic energy reaches its maximum value.

* It is tempting, but wrong, to evaluate $x(t)$ at $t = 0$ and then differentiate to get $v(0)$. This procedure is wrong because $x(0)$ is just a number, differentiating it would always give zero, even when the initial velocity is not zero.

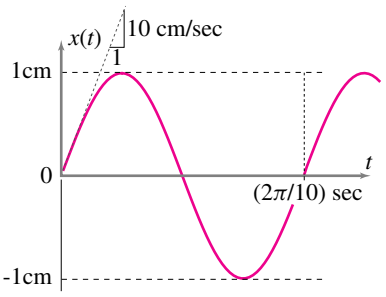
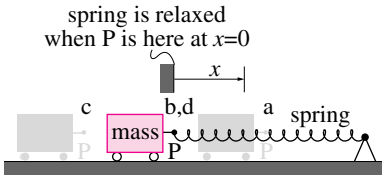
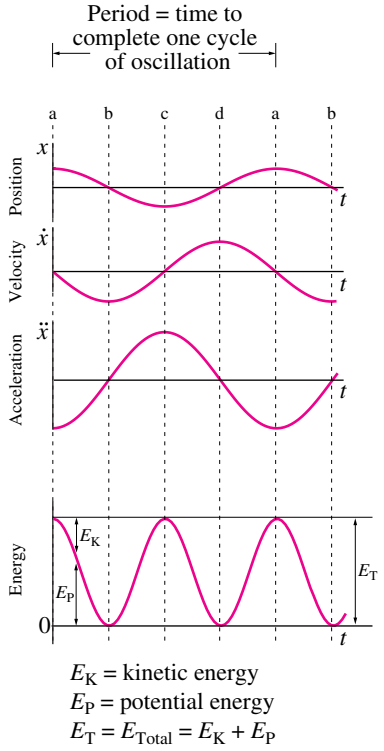


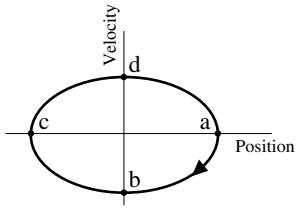

Figure 9.23: The position of a mass as a function of time if $k = 50\,\mathrm{N/m}$, $m = 0.5\,\mathrm{kg}$, $x(0) = 0$ and $v(0) = 10$ cm/s.

Filename:tfigure12-sine




$E_K$ = kinetic energy
$E_P$ = potential energy
$E_T = E_{Total} = E_K + E_P$




Cross plot or phase plane portrait

Figure 9.24: **Harmonic oscillator.** At $t = 0$ the mass is released from rest at a. The spring is relaxed at $x = 0$ (pts b and d). Some things to note: The acceleration curve is proportional to the negative of the displacement curve; The displacement is at a maximum or minimum when the velocity is zero; The velocity is at a maximum or minimum when the displacement is zero; The kinetic and potential energy fluctuate at twice the frequency as the position; The motion is an ellipse in the cross plot of velocity *vs.* position.

Filename:tfigure12-oscplots

Although energy conservation is a basic principle, this is a case where it can be derived, or more easily, checked. Using the special case where the motion starts from rest (i.e., $x(t) = A\cos(\sqrt{k/m}\,t)$), we can make sure that the total energy really is constant.

$$\begin{aligned} E_T &= E_P + E_K \\ &= \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \\ &= \frac{1}{2}k(\underbrace{A\cos(\sqrt{k/m}t)}_{x})^2 + \frac{1}{2}m(\underbrace{A\sqrt{k/m}\sin(\sqrt{k/m}t)}_{v})^2 \\ &= \frac{1}{2}kA^2\underbrace{\{\cos^2(\sqrt{k/m}t) + \sin^2(\sqrt{k/m}t)\}}_{1} \\ &= \frac{1}{2}kA^2 = \quad \text{initial energy in spring} \end{aligned}$$

which does not change with time.

## Using energy to derive the oscillator equation

As mentioned above and in the previous section, rather than just checking the energy balance, we could use the energy balance to help us find the equations of motion. As for all one-degree-of-freedom systems, the equations of motion can be derived by taking the time derivative of the energy balance equation. Starting from $E_T = \text{constant}$, we get

$$\begin{aligned} 0 &= \frac{d}{dt}E_T \\ &= \frac{d}{dt}(E_P + E_K) \\ &= \frac{d}{dt}(\frac{1}{2}kx^2 + \frac{1}{2}mv^2) \\ &= kx\underbrace{\dot{x}}_{v} + mv\underbrace{\dot{v}}_{a} \\ &= kx\cancel{\dot{x}} + m\cancel{\dot{x}}\underbrace{a}_{\ddot{x}} \\ 0 &= kx + m\ddot{x} \end{aligned}$$

which is the harmonic oscillator equation*.

Similarly, power balance also leads to the harmonic oscillator equation. Referring to the FBD in figure 9.20, the equation of power balance for the

block during its motion after release is:

$$\underbrace{P}_{\text{Power in}} = \underbrace{\dot{E}_K}_{\text{Rate of change of kinetic energy}}$$

$$\begin{aligned} \vec{F}_{spring} \cdot \vec{v}_A &= \frac{d}{dt}\left(\frac{1}{2}mv_A^2\right) \\ -kx_A\hat{\imath} \cdot \dot{x}_A\hat{\imath} &= \frac{d}{dt}\left(\frac{1}{2}m\dot{x}_A^2\right) \\ -kx_A\dot{x}_A &= m\dot{x}_A\ddot{x}_A \,. \end{aligned}$$

Dividing both sides by $\dot{x}_A$ (assuming it is not zero), we again get

$$-kx_A = m\ddot{x}_A \qquad \text{or} \qquad m\ddot{x}_A + kx_A = 0,$$

which is by now familiar enough to be called our friend.

**Energy oscillations.** Let's assume the block is released from rest at $x = x_A > 0$. The mass begins to move to the left and the spring does positive work on the mass since the motion and the force are in the same direction. After the block passes through the rest point $x = O$, it does work on the spring until it comes to rest at its left extreme. The spring then commences to do work on the block again as the block gains kinetic energy in its rightward motion. The block then passes through the rest position and does work on the spring until its kinetic energy is all used up and it is back in its rest position. Note that the potential and kinetic energy each have a two local maxima and minima for each oscillation of the mass, thus their plots are sine-waves with twice the frequency of the basic oscillation.

## A spring-mass system with gravity

When a mass is attached to a spring but gravity also acts one has to take some care to get things right (see fig. 9.26). Once a good free body diagram is drawn using well defined coordinates, all else follows easily.

Note that there are three natural choices for measuring the position of the mass in Fig. 9.26. $y$ measures position from the fixed end of the spring, $x$ measures from the position of the mass when the spring is relaxed, and $z$ measures from the position of the mass when it is in static equilibrium (with the gravity force balancing the spring compression).

# Damping

Dashpots are used to absorb energy. One is shown schematically in fig. 9.29. Often springs and dashpots are light in comparison to the machinery to which they are attached so their mass and weight are neglected. Often they are attached with pin joints, ball and socket joints, or other kinds of flexible connections so only forces are transmitted. Because they only have forces at

* A technical defect of the derivation of the oscillator equation from conservation of energy is that the derivation does not apply at the instant when $v = 0$ ($0 \cdot x = 0 \cdot y$ does not imply that $x = y$). You can ignore this technical defect because it has essentially no physical consequence

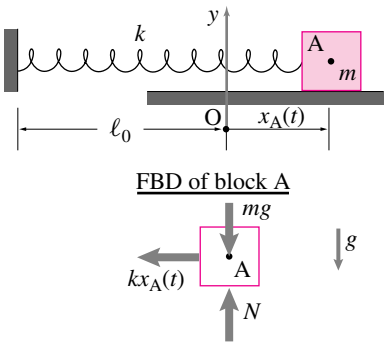






Figure 9.25:
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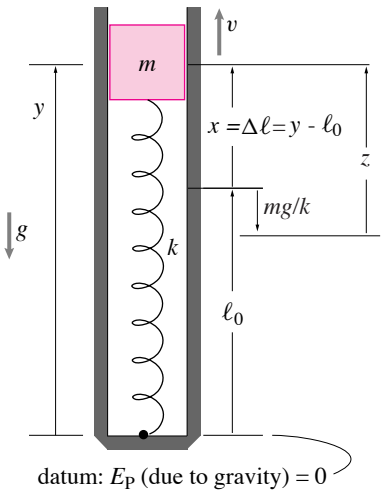



Figure 9.26: Spring and mass with gravity.
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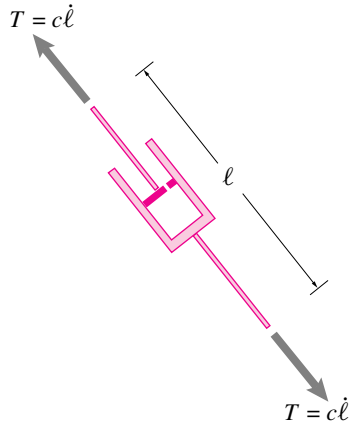
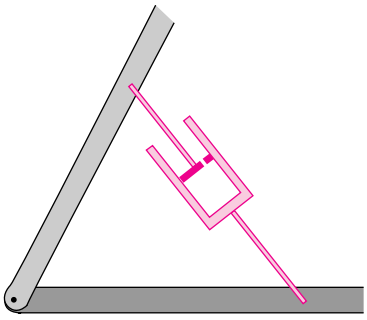



Figure 9.27: **A damper or dashpot.** The symbol shown represents a device which resists the relative motion of its endpoints. The schematic is supposed to suggest a plunger in a cylinder. For the plunger to move, fluid must leak around the cylinder. This leakage happens for either direction of motion. Thus the damper resists relative motion in either direction; i. e., for $\dot{L} > 0$ and $\dot{L} < 0$.

Filename:tfigure12-dashpot



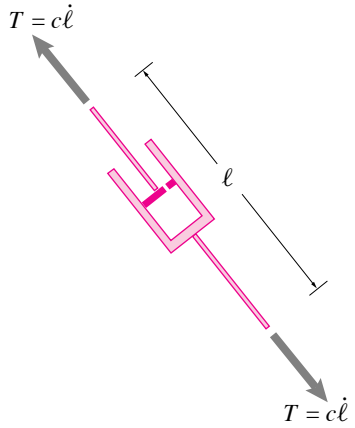



Figure 9.29: **A dashpot.** A dashpot is shown here connecting two parts of a mechanism. The tension in the dashpot is proportional to the rate at which it lengthens.

Filename:tfigure2-dashpot

their ends they are 'two-force' bodies and, by the reasoning of section 4.2, the forces at their ends are equal, opposite, and along the line of connection. The most familiar example is in the shock absorbers of a car. The symbol for a dashpot shown in figure 9.29 is meant to suggest the mechanism.

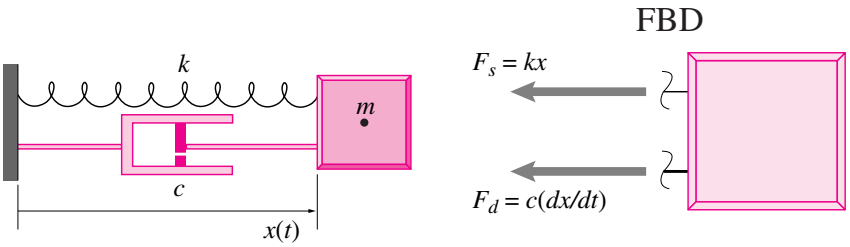



Figure 9.28: A mass spring dashpot system, or damped harmonic oscillator. Also shown is a free body diagram of the mass.

Filename:tfigure12-MSD

The dashpot provides resistance to motion by drawing air or oil in and out of the cylinder through a small opening. Due to the viscosity of the air or oil, a pressure drop is created across the opening that is related to the speed of the fluid flowing through. Ideally, this viscous resistance produces linear damping, meaning that the force is exactly proportional to the velocity. In a physical dashpot nonlinearities are introduced from the fluid flow and from friction between the piston and the cylinder. Also, dashpots that use air as a working fluid may have compressibility that introduces extra springiness to the system.

The tension in the dashpot is usually assumed to be proportional to the rate at which it lengthens, although this approximation is not especially accurate for most dampers one can buy. The relation is assumed to hold for negative lengthening as well. So the compression (negative tension) is proportional to the rate at which the dashpot shortens (negative lengthens).

The defining equation for a linear dashpot is:

$$T = C\,\dot{\ell}$$

where C is the dashpot constant.

## Damped oscillations

We now add a dashpot in parallel with the spring of a mass-spring system creates a *mass-spring-dashpot* system, or *damped harmonic oscillator*. The system is shown in figure 9.28. Also in figure 9.28 is a free body diagram of the mass. It has two forces acting on it, neglecting gravity:

$F_s = kx$ is the spring force, assuming a linear spring, and

$F_d = c\,dx/dt = c\dot{x}$ is the dashpot force assuming a linear dashpot.

The system is a one degree of freedom system because a single coordinate $x$ is sufficient to describe the complete motion of the system. The equation of motion for this system is

$$m\ddot{x} = -F_d - F_s \quad \text{where} \quad \ddot{x} = d^2x/dt^2. \tag{9.24}$$

Assuming a linear spring and a linear dashpot this expression becomes

$$m\ddot{x} + c\dot{x} + kx = 0. \tag{9.25}$$

We have taken care with the signs of the various terms. Make sure you can confidently derive equation 9.25 without introducing sign errorsThe analytical solution of the damped-oscillator equation is in box 9.6. Some qualitative features of the damped solutions are shown in Fig. 9.30

## Summary of equations for the unforced harmonic oscillator

- $\ddot{x} + \frac{k}{m}x = 0$, mass-spring equation
- $\ddot{x} + \lambda^2 x = 0$, harmonic oscillator equation
- $x(t) = A\cos(\lambda t) + B\sin(\lambda t)$, general solution to harmonic oscillator equation
- $x(t) = R\cos(\lambda t - \phi)$, amplitude-phase version of solution to harmonic oscillator solution, $R = \sqrt{A^2 + B^2}, \phi = \tan^{-1}(\frac{B}{A})$ (See box on page 422).
- $\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$, mass-spring-dashpot equation (see equations 9.26-9.28 for solutions)
- $D = \ln\left(\frac{x_n}{x_{n+1}}\right)$, logarithmic decrement. $c = \frac{2mD}{T}$. (see box 9.6 on page 428)

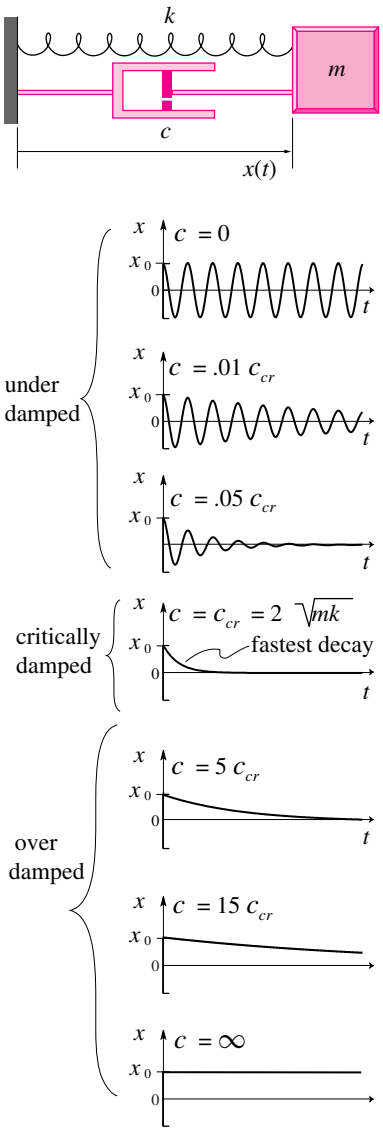



Figure 9.30: The effect of varying the damping with a fixed mass and spring. In all the plots the mass is released from rest at $x = x_0$. In the case of under-damping, oscillations persist for a long time, forever if there is no damping. In the case of over-damping, the dashpot doesn't relax for a long time; it stays locked up forever in the limit of $c \to \infty$. The fastest relaxation occurs for critical damping.

Filename:tfigure12-damping

## 9.6 *THEORY*
### *Solution of the damped-oscillator equations*

These solutions are important for those aiming at a more-mathematical understanding. They are of much lower status than those in box 9.1 on page 392. Don't attempt to memorize these solutions in detail.

The governing equation 9.25 has a solution which depends on the values of the constants. There are cases where one wants to consider negative springs or negative dashpots, but for the purposes of understanding classical vibration theory we can assume that $m$, $c$, and $k$ are all positive. Even with this restriction the solution depends on the relative values of $m$, $c$, and $k$. You can learn all about these solutions in any book that introduces ordinary differential equations; most freshman calculus books have such a discussion.

The three solutions are categorized as follows:

- *Under-damped*: $c^2 < 4mk$. In this case the damping is small and oscillations persist forever, though their amplitude diminishes exponentially in time. The general solution for this case is:

$$x(t) = e^{(-\frac{c}{2m})t}[A\cos(\lambda_d t) + B\sin(\lambda_d t)], \qquad (9.26)$$

  where $\lambda_d$ is the damped natural frequency and is given by $\lambda_d = \sqrt{\left(\frac{k}{m} - \frac{c}{2m}\right)^2}$.

- *Critically damped*: $c^2 = 4mk$. In this case the damping is at a critical level that separates the cases of under-damped oscillations from the simply decaying motion of the over-damped case. The general solution is:

$$x(t) = Ae^{(-\frac{c}{2m})t} + Bte^{(-\frac{c}{2m})t}. \qquad (9.27)$$

- *Over-damped*: $c^2 > 4mk$. Here there are no oscillations, just a simple return to equilibrium with at most one crossing through the equilibrium position on the way to equilibrium. The general solution in the over-damped case is:

$$x(t) = Ae^{(-\frac{c}{2m}+\sqrt{(\frac{c}{2m})^2-\frac{k}{m}})t} + Be^{(-\frac{c}{2m}-\sqrt{(\frac{c}{2m})^2-\frac{k}{m}})t}. \qquad (9.28)$$

  The solution 9.28 actually includes equations 9.27 and 9.26 as special cases. To interpret equation refoverdampe as the general solution you need to know the relation between complex exponentials and trigonometric functions for the cases when the argument of the square root term is negative.

For a given mass and spring we can imagine the damping as a variable to adjust. A system which has small damping (small $c$) is *under-damped* and does not come to equilibrium quickly because oscillations last for a long time. A system which has a lot of damping (big $c$) is *over-damped* does not come to equilibrium quickly because the dashpot doesn't leak fast enough. A system which is inbetween, *critically-damped* comes to equilibrium most quickly. The purpose of damping is often to purge motion after a disturbance. If the only design variable available for adjustment is the damping, then the quickest purge is accomplished with critical damping, $c = \sqrt{(4km)}$. In practice, a damping value close to critical is often used.

## Measurement of damping: logarithmic decrement method

In the under-damped case, the viscous damping constant $c$ may be determined experimentally by measuring the rate of decay of unforced oscillations. This decay can be quantified using the logarithmic decrement. The logarithmic decrement is the natural logarithm of the ratio of any two successive amplitudes. The larger the damping, the greater will be the rate of decay of oscillations and the bigger the logarithmic decrement:

$$\text{logarithmic decrement} \equiv D = \ln(\frac{x_n}{x_{n+1}}) \qquad (9.29)$$

where $x_n$ and $x_{n+1}$ are the heights of two successive peaks in the decaying oscillation pictured in figure **??**. Because of the exponential envelope that this curve has, $x_n = (\text{const.})e^{-(\frac{c}{2m})t_n}$ and $x_{n+1} = (\text{const.})e^{-(\frac{c}{2m})t_n+T}$.

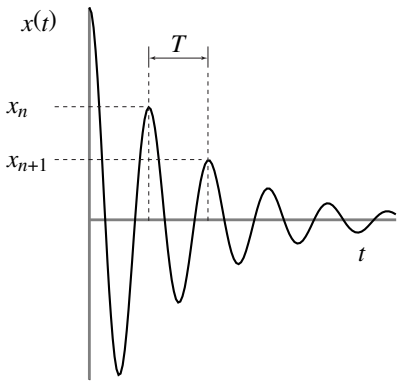
$$D = \ln[(e^{-(\frac{c}{2m})t_n})/(e^{-(\frac{c}{2m})t_n+T})]$$

Simplifying this expression, we get that

$$D = \frac{cT}{2m}$$

where $T$ is the period of oscillation. Thus, the damping constant $c$ can be measured by measuring the logarithmic decrement $D$ and the period of oscillation $T$ as

$$c = \frac{2mD}{T}.$$




**The logarithmic decrement method.** $D = \ln(x_n/x_{n+1})$

**SAMPLE 9.13** A block of mass $m = 20\,\text{kg}$ is attached to two identical springs each with spring constant $k = 1\,\text{kN/m}$. The block slides on a horizontal surface without any friction.

1. Find the equation of motion of the block.
2. What is the oscillation frequency of the block?
3. How much time does the block take to go back and forth 10 times?

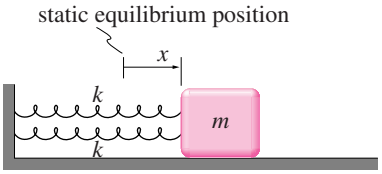



Figure 9.31:
Filename:sfig10-1-1-1

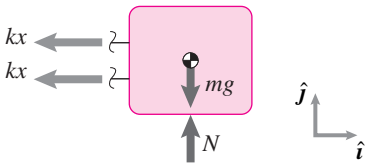



Figure 9.32:
Filename:sfig10-1-1-1a

**Solution**

1. The free body diagram of the block is shown in Figure 9.32. The linear momentum balance, $\sum \vec{F} = m\vec{a}$, for the block gives

$$-2kx\hat{\imath} + (N - mg)\hat{\jmath} = m\vec{a}$$

Dotting both sides with $\hat{\imath}$ we have,

$$\begin{aligned} -2kx = ma_x &= m\ddot{x} && (9.30)\\ \text{or}\quad m\ddot{x} + 2kx &= 0 && (9.31)\\ \text{or}\quad \ddot{x} + \frac{2k}{m}x &= 0. && (9.32)\end{aligned}$$

$$\boxed{\ddot{x} + \tfrac{2k}{m}x = 0}$$

2. Comparing Eqn. (9.32) with the standard harmonic oscillator equation, $\ddot{x} + \lambda^2 x = 0$, where $\lambda$ is the oscillation frequency, we get

$$\begin{aligned} \lambda^2 &= \frac{2k}{m}\\ \Rightarrow\quad \lambda &= \sqrt{\frac{2k}{m}}\\ &= \sqrt{\frac{2\cdot(1\,\text{kN/m})}{20\,\text{kg}}}\\ &= 10\,\text{rad/s}.\end{aligned}$$

$$\boxed{\lambda = 10\,\text{rad/s}}$$

3. Time period of oscillation $T = \frac{2\pi}{\lambda} = \frac{2\pi}{10\,\text{rad/s}} = \frac{\pi}{5}\,\text{s}$. Since the time period represents the time the mass takes to go back and forth just once, the time it takes to go back and forth 10 times (*i.e.*, to complete 10 cycles of motion) is

$$t = 10T = 10\cdot\frac{\pi}{5}\,\text{s} = 2\pi\,\text{s}.$$

$$\boxed{t = 2\pi\,\text{s}}$$

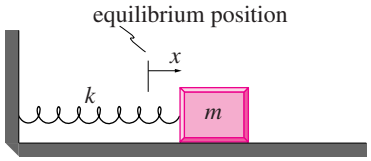



Figure 9.33:
Filename:sfig10-1-3

**SAMPLE 9.14** A spring-mass system executes simple harmonic motion: $x(t) = A\cos(\lambda t - \phi)$. The system starts with initial conditions $x(0) = 25$ mm and $\dot{x}(0) = 160$ mm/ s and oscillates at the rate of 2 cycles/sec.

1. Find the time period of oscillation and the oscillation frequency $\lambda$.
2. Find the amplitude of oscillation $A$ and the phase angle $\phi$.
3. Find the displacement, velocity, and acceleration of the mass at $t = 1.5$ s.
4. Find the maximum speed and acceleration of the system.
5. Draw an accurate plot of displacement *vs*. time of the system and label all relevant quantities. What does $\phi$ signify in this plot?

**Solution**

1. We are given $f = 2$ Hz. Therefore, the time period of oscillation is

$$T = \frac{1}{f} = \frac{1}{2\text{ Hz}} = 0.5\text{ s},$$

and the oscillation frequency $\lambda = 2\pi f = 4\pi$ rad/s.

$$\boxed{T = 0.5\text{ s}, \quad \lambda = 4\pi\text{ rad/s}.}$$

2. The displacement $x(t)$ of the mass is given by

$$x(t) = A\cos(\lambda t - \phi).$$

Therefore the velocity (actually the speed) is

$$\dot{x}(t) = -A\lambda\sin(\lambda t - \phi)$$

At $t = 0$, we have

$$x(0) = A\cos(-\phi) = A\cos\phi \tag{9.33}$$
$$\dot{x}(0) = -A\lambda\sin(-\phi) = A\lambda\sin\phi \tag{9.34}$$

By squaring Eqn (9.33) and adding it to the square of [Eqn (9.34) divided by $\lambda$], we get

$$A^2\cos^2\phi + \frac{A^2\lambda^2\sin^2\phi}{\lambda^2} = A^2 = x^2(0) + \frac{\dot{x}^2(0)}{\lambda^2}$$
$$\Rightarrow \quad A = \sqrt{(25\text{ mm})^2 + \frac{(160\text{ mm/ s})^2}{(4\pi\text{ rad/s})^2}}$$
$$= 28.06\text{ mm}.$$

Substituting the value of $A$ in Eqn (9.33), we get

$$\phi = \cos^{-1}\frac{x(0)}{A}$$
$$= \cos^{-1}\frac{25\text{ mm}}{28.06\text{ mm}}$$
$$= 0.471\text{ rad} \approx 27^\circ.$$

$$\boxed{A = 28.06\text{ mm}. \quad \phi = 0.471\text{ rad}.}$$

3. The displacement, velocity, and acceleration of the mass at any time $t$ can now be calculated as follows

$$\begin{aligned} & x(t) &=& A\cos(\lambda t-\phi) \\ \Rightarrow\quad & x(1.5\,\text{s}) &=& 28.06\,\text{mm}\cdot\cos(6\pi-0.471) \\ & &=& 25\,\text{mm}. \end{aligned}$$

$$\begin{aligned} & \dot{x}(t) &=& -A\lambda\sin(\lambda t-\phi) \\ \Rightarrow\quad & \dot{x}(1.5\,\text{s}) &=& 28.06\,\text{mm}\cdot(4\pi\ \text{rad/s})\cdot\sin(6\pi-0.471) \\ & &=& 160\,\text{mm/ s}. \end{aligned}$$

$$\begin{aligned} & \ddot{x}(t) &=& -A\lambda^2\cos(\lambda t-\phi) \\ \Rightarrow\quad & \ddot{x}(1.5\,\text{s}) &=& 28.06\,\text{mm}\cdot(4\pi\ \text{rad/s})^2\cdot\cos(6\pi-0.471) \\ & &=& -3.95\times10^3\,\text{mm/ s}^2 \\ & &=& -3.95\,\text{m/ s}^2. \end{aligned}$$

*

$$\boxed{x(1.5\,\text{s})=25\,\text{mm}.\quad \dot{x}(1.5\,\text{s})=160\,\text{mm/ s}.\quad \ddot{x}(1.5\,\text{s})=-3.93\,\text{m/ s}^2.}$$

* We can find the displacement and velocity at $t = 1.5$ s without any differentiation. Note that the system completes 2 cycles in 1 second, implying that it will complete 3 cycles in 1.5 seconds. Therefore, at $t = 1.5$ s, it has the same displacement and velocity as it had at $t = 0$ s.

4. Maximum speed:

$$|\dot{x}_{\max}| \;=\; A\lambda = (28.06\,\text{mm})\cdot(4\pi\ \text{rad/s}) = 0.35\,\text{m/s}.$$

Maximum acceleration:

$$|\ddot{x}_{\max}| \;=\; A\lambda^2 = (28.06\,\text{mm})\cdot(4\pi\ \text{rad/s})^2 = 4.43\,\text{m/s}^2.$$

$$\boxed{|\dot{x}_{\max}| = 0.35\,\text{m/s},\quad |\ddot{x}_{\max}| = 4.43\,\text{m/s}^2.}$$

5. The plot of $x(t)$ versus $t$ is shown in Fig. 9.34. The phase angle $\phi$ represents the shift in $\cos(\lambda t)$ to the right by an amount $\frac{\phi}{\lambda}$.

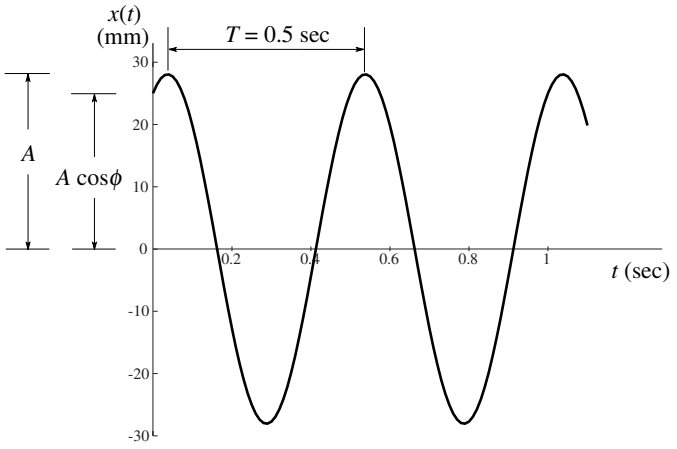



Figure 9.34:

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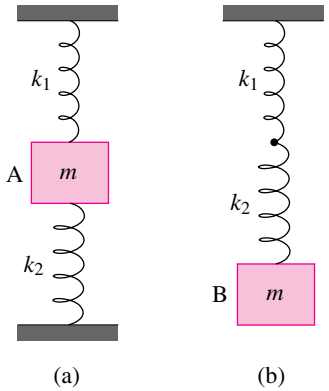



Figure 9.35:

Filename:sfig3-4-2

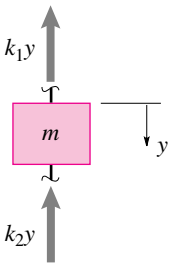



Figure 9.36: Free body diagram of the mass.

Filename:sfig3-4-2a

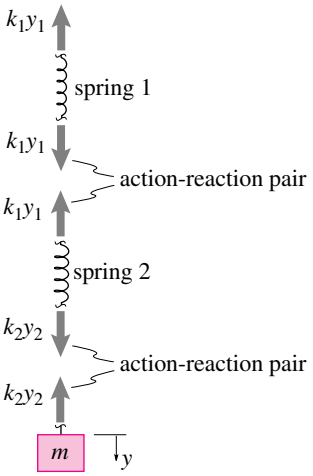



Figure 9.37: Free body diagrams

Filename:sfig3-4-2b

**SAMPLE 9.15 Springs in series versus springs in parallel:** Two massless springs with spring constants $k_1$ and $k_2$ are attached to mass A *in parallel* (although they look superficially as if they are in series) as shown in Fig. 9.35. An identical pair of springs is attached to mass B *in series*. Taking $m_A = m_B = m$, find and compare the natural frequencies of the two systems. Ignore gravity.

**Solution** Let us pull each mass downwards by a small vertical distance $y$ and then release. Measuring $y$ to be positive downwards, we can derive the equations of motion for each mass by writing the balance of linear momentum for each as follows.

- **Mass A:** The free body diagram of mass A is shown in Fig. 9.36. As the mass is displaced downwards by $y$, spring 1 gets stretched by $y$ whereas spring 2 gets compressed by $y$. Therefore, the forces applied by the two springs, $k_1 y$ and $k_2 y$, are in the same direction. The linear momentum balance of mass A in the vertical direction gives:

$$\begin{aligned} & \sum F = ma_y \\ \text{or} \quad & -k_1 y - k_2 y = m\ddot{y} \\ \text{or} \quad & \ddot{y} + \left(\frac{k_1 + k_2}{m}\right) y = 0. \end{aligned}$$

Let the natural frequency of this system be $\omega_p$. Comparing with the standard simple harmonic equation $\ddot{x} + \lambda^2 x = 0$ (see box 9.1 on page 392), we get the natural frequency ($\lambda$) of the system:

$$\omega_p = \sqrt{\frac{k_1 + k_2}{m}} \tag{9.35}$$

$$\boxed{\omega_p = \sqrt{\frac{k_1+k_2}{m}}}$$

- **Mass B:** The free body diagram of mass B and the two springs is shown in Fig. 9.37. In this case both springs stretch as the mass is displaced downwards. Let the net stretch in spring 1 be $y_1$ and in spring 2 be $y_2$. $y_1$ and $y_2$ are unknown, of course, but we know that

$$y_1 + y_2 = y \tag{9.36}$$

Now, using the free body diagram of spring 2 and then writing linear momentum balance we get,

$$\begin{aligned} k_2 y_2 - k_1 y_1 &= \underbrace{m}_{0}\, a = 0 \\ y_1 &= \frac{k_2}{k_1} y_2 \end{aligned} \tag{9.37}$$

Solving (9.36) and (9.37) we get

$$y_2 = \frac{k_1}{k_1 + k_2} y.$$

Now, linear momentum balance of mass B in the vertical direction gives:

$$\begin{aligned} -k_2 y_2 = ma_y &= m\ddot{y} \\ \text{or} \quad m\ddot{y} + k_2 \overbrace{\frac{k_1}{k_1 + k_2} y}^{y_2} &= 0 \\ \text{or} \quad \ddot{y} + \frac{k_1 k_2}{m(k_1 + k_2)} y &= 0. \end{aligned} \tag{9.38}$$

Let the natural frequency of this system be denoted by $\omega_s$. Then, comparing with the standard simple harmonic equation as in the previous case, we get

$$\omega_s = \sqrt{\frac{k_1 k_2}{m(k_1 + k_2)}}. \tag{9.39}$$

$$\boxed{\omega_s = \sqrt{\frac{k_1 k_2}{m(k_1+k_2)}}}$$

From (9.35) and (9.39)

$$\frac{\omega_p}{\omega_s} = \frac{k_1 + k_2}{\sqrt{k_1 k_2}}.$$

Let $k_1 = k_2 = k$. Then, $\omega_p/\omega_s = 2$, *i.e.*, the natural frequency of the system with two identical springs in parallel is twice as much as that of the system with the same springs in series. Intuitively, the restoring force applied by two springs in parallel will be more than the force applied by identical springs in series. In one case the forces add and in the other they don't and each spring is stretched less. Therefore, we do expect mass A to oscillate at a faster rate (higher natural frequency) than mass B.

**Comments:**

1. Although the springs attached to mass A do not visually seem to be in parallel, from mechanics point of view they are parallel. You can easily check this result by putting the two springs visually in parallel and then deriving the equation of mass A. You will get the same equations. For springs in parallel, each spring has the *same displacement* but *different forces*. For springs in series, each has *different displacements* but the *same force*.
2. When many springs are connected to a mass in series or in parallel, sometimes we talk about their *effective* spring constant, *i.e.*, the spring constant of a single imaginary spring which could be used to replace all the springs attached in parallel or in series. Let the effective spring constant for springs in parallel and in series be represented by $k_{pe}$ and $k_{se}$ respectively. By comparing eqns. (9.35) and (9.39) with the expression for natural frequency of a simple spring mass system, we see that

   $$k_{pe} = k_1 + k_2 \quad \text{and} \quad \frac{1}{k_{se}} = \frac{1}{k_1} + \frac{1}{k_2}.$$

   These expressions can be easily extended for any arbitrary number of springs, say, $N$ springs:

   $$k_{pe} = k_1 + k_2 + \ldots + k_N \quad \text{and} \quad \frac{1}{k_{se}} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots + \frac{1}{k_N}.$$

**SAMPLE 9.16** Figure 9.38 shows two responses obtained from experiments on two spring-mass systems. For each system

1. Find the natural frequency.
2. Find the initial conditions.

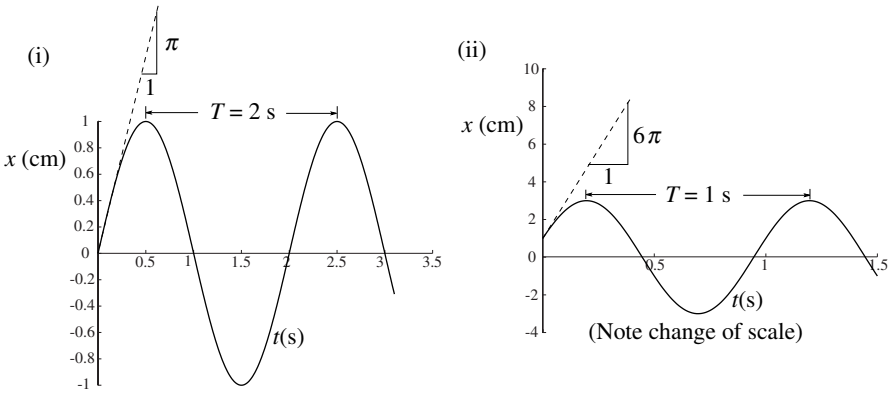



Figure 9.38:

Filename:sfig10-1-4

**Solution**

1. **Natural frequency:** By definition, the natural frequency $f$ is the number of cycles the system completes in one second. From the given responses we see that:
   **Case(i):** the system completes $\frac{1}{2}$ a cycle in 1 s.
   $$\Rightarrow \qquad f = \frac{1}{2}\ \text{Hz}.$$
   **Case(ii):** the system completes 1 cycle in 1 s.
   $$\Rightarrow \qquad f = 1\ \text{Hz}.$$
   It is usually hard to measure the fraction of cycle occurring in a short time. It is easier to first find the time period, *i.e.*, the time taken to complete 1 cycle. * Then the natural frequency can be found by the formula $f = \frac{1}{T}$. From the given responses, we find the time period by estimating the time between two successive peaks (or troughs): From Figure 9.38 we find that for
   **Case (i):**
   $$f = \frac{1}{T} = \frac{1}{2\,\text{s}} = \frac{1}{2}\ \text{Hz},$$
   **Case (ii):**
   $$f = \frac{1}{T} = \frac{1}{1\,\text{s}} = 1\ \text{Hz}$$
   $$\boxed{\text{case (i) } f = \tfrac{1}{2}\ \text{Hz}. \quad \text{case (ii) } f = 1\ \text{Hz}.}$$
2. **Initial conditions:** Now we are to find the displacement and velocity at $t = 0\,\text{s}$ for each case. Displacement is easy because we are given the displacement plot, so we just read the value at $t = 0$ from the plots:
   **Case (i):**
   $$x(0) = 0.$$
   **Case (ii):**
   $$x(0) = 1\,\text{cm}.$$
   The velocity (actually the speed) is the time-derivative of the displacement. Therefore, we get the initial velocity from the slope of the displacement curve at $t = 0$.

* To estimate the frequency of some repeated motion in an experiment, it is best to measure the time for a large number of cycles, say 5, 10 or 20, and then divide that time by the total number of cycles to get an average value for the time period of oscillation.

**Case (i):**

$$\dot{x}(0) = \frac{dx}{dt}(t=0) = \frac{\pi \text{ cm}}{1 \text{ s}} = 3.14 \text{ cm/ s}.$$

**Case (ii):**

$$\dot{x}(0) = \frac{dx}{dt}(t=0) = \frac{6\pi \text{ cm}}{1 \text{ s}} = 18.85 \text{ cm/ s}.$$

Thus the initial conditions are

| | | |
|---|---|---|
| Case (i): | $x(0) = 0$ | $\dot{x}(0) = 3.14 \text{ cm/ s}$ |
| Case (ii): | $x(0) = 1 \text{ cm}$ | $\dot{x}(0) = 18.85 \text{ cm/ s}$ |

**Comments:** Estimating the speed from the initial slope of the displacement curve at $t = 0$ is not a very good method because it is hard to draw an accurate tangent to the curve at $t = 0$. A slightly different line but still seemingly tangential to the curve at $t = 0$ can lead to significant error in the estimated value. A better method, perhaps, is to use the known values of displacement at different points and use the energy method to calculate the initial speed. We show sample calculations for the first system:

**Case(i):** We know that $x(0) = 0$. Therefore the entire energy at $t = 0$ is the kinetic energy $= \frac{1}{2}mv_0^2$. At $t = 0.5 \text{ s}$ we note that the displacement is maximum, *i.e.*, the speed is zero. Therefore, the entire energy is potential energy $= \frac{1}{2}kx^2$, where $x = x(t = 0.5 \text{ s}) = 1 \text{ cm}$.

Now, from the conservation of energy:

$$\begin{aligned}
\frac{1}{2}mv_0^2 &= \frac{1}{2}k(x_{t=0.5\text{ s}})^2 \\
\Rightarrow \quad v_0 &= \sqrt{\frac{k}{m}} \cdot (x_{t=0.5\text{ s}}) \\
&= \underbrace{\sqrt{\frac{k}{m}}}_{\lambda} \cdot (1 \text{ cm}) \\
&= 2\pi f \cdot (1 \text{ cm}) \\
&= 2\pi \cdot \frac{1}{2} \text{ Hz} \cdot 1 \text{ cm} \\
&= 3.14 \text{ cm/ s}.
\end{aligned}$$

Similar calculations can be done for the second system.

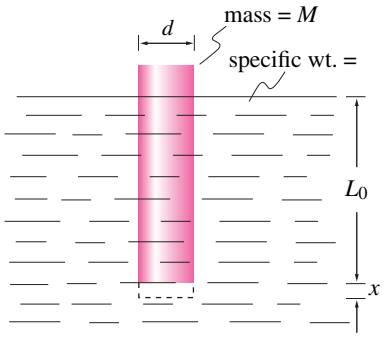



Figure 9.39:
Filename:sfig3-4-1

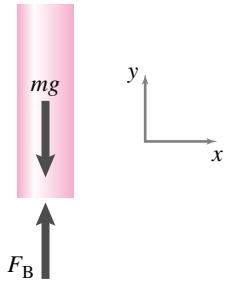



Figure 9.40:
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**SAMPLE 9.17 Simple harmonic motion of a buoy.** A cylinder of cross sectional area $A$ and mass $M$ is in static equilibrium inside a fluid of specific weight $\gamma$ when $L_o$ length of the cylinder is submerged in the fluid. From this position, the cylinder is pushed down vertically by a small amount $x$ and let go. Assume that the only forces acting on the cylinder are gravity and the buoyant force and assume that the buoy's motion is purely vertical. Derive the equation of motion of the cylinder using Linear Momentum Balance. What is the period of oscillation of the cylinder?

**Solution** The free body diagram of the cylinder is shown in Fig. 9.40 where $F_B$ represents the buoyant force. Before the cylinder is pushed down by $x$, the linear momentum balance of the cylinder gives

$$F_B - Mg = M \underbrace{a}_{0} = 0 \qquad \Rightarrow \qquad F_B = Mg$$

Now $F_B$ = (volume of the displaced fluid)· (its specific weight) $= AL_o\gamma$. Thus,

$$AL_o\gamma = Mg. \tag{9.40}$$

Now, when the cylinder is pushed down by an amount $x$,

$$F_B' = \text{ new buoyant force } = (L_o + x)A\gamma.$$

Therefore, from LMB we get

$$\begin{aligned} & F_B' - Mg &&= -M\ddot{x} \\ \text{or} \quad & (L_o + x)A\gamma - Mg &&= -M\ddot{x} \\ & & &= \overbrace{-AL_o\gamma + Mg}^{=0 \text{ from (9.40).}} \\ \text{or} \quad & M\ddot{x} + A\gamma x &&= 0 \\ \text{or} \quad & M\ddot{x} + A\gamma x &&= 0 \\ \text{or} \quad & \ddot{x} + \frac{A\gamma}{M}x &&= 0. \end{aligned}$$

$$\boxed{\ddot{x} + \tfrac{A\gamma}{M}x = 0}$$

Comparing this equation with the standard simple harmonic equation (*e.g.*, eqn.(g), in the box on ODE's on page 392).

$$\text{The circular frequency } \lambda = \sqrt{\frac{A\gamma}{M}},$$

$$\text{Therefore,} \qquad \text{the period of oscillation } T = \frac{2\pi}{\lambda} = 2\pi\sqrt{\frac{M}{A\gamma}}$$

.

$$\boxed{T = 2\pi\sqrt{\tfrac{M}{A\gamma}}}$$

**Comments:** Note this calculation neglects the fluid mechanics. The common way of making a correction is to use 'added mass' to account for fluid that moves more-or-less with the cylinder. The added mass is usually something like one-half the mass of the fluid with volume equal to that of the cylinder. Another way to see the error is to realize that the pressure used in this calculation assumes fluid statics when in fact the fluid is moving.

**SAMPLE 9.18** A block of mass 10 kg is attached to a spring and a dashpot as shown in Figure 9.41. The spring constant $k = 1000\,\text{N/m}$ and a damping rate $c = 50\,\text{N}\cdot\text{s/m}$. When the block is at a distance $d_0$ from the left wall the spring is relaxed. The block is pulled to the right by 0.5 m and released. Assuming no initial velocity, find

1. the equation of motion of the block.
2. the position of the block at $t = 2\,\text{s}$.

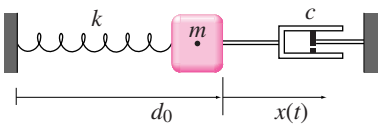



Figure 9.41: Spring-mass dashpot.

Filename:sfig10-2-1

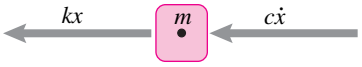



Figure 9.42: Free body diagram of the mass at an instant $t$.

Filename:sfig10-2-1a

**Solution**

1. Let $x$ be the position of the block, measured positive to the right of the static equilibrium position, at some time $t$. Let $\dot{x}$ be the corresponding speed. The free body diagram of the block at the instant $t$ is shown in Figure 9.42.

   Since the motion is only horizontal, we can write the linear momentum balance in the $x$-direction ($\sum F_x = m\,a_x$):

$$\underbrace{-kx - c\dot{x}}_{\sum F_x} = m\,\underbrace{\ddot{x}}_{a_x}$$

$$\text{or} \quad \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0 \tag{9.41}$$

   which is the desired equation of motion of the block.

$$\boxed{\ddot{x} + \tfrac{c}{m}\dot{x} + \tfrac{k}{m}x = 0.}$$

2. To find the position and velocity of the block at any time $t$ we need to solve Eqn (9.41). Since the solution depends on the relative values of $m$, $k$, and $c$, we first compute $c^2$ and compare with the *critical value* $4mk$.

$$\begin{aligned} & c^2 &=& \ 2500(\text{N}\cdot\text{s/m})^2 \\ \text{and} \quad & 4mk &=& \ 4\cdot 10\,\text{kg}\cdot 1000\,\text{N/m} = 4000(\text{N}\cdot\text{s/m})^2. \\ \Rightarrow \quad & c^2 &<& \ 4mk. \end{aligned}$$

   Therefore, the system is underdamped and we may write the general solution as (see box 9.6 on page 428)

$$x(t) = e^{-\frac{c}{2m}t}\,[A\cos\lambda_D t + B\sin\lambda_D t] \tag{9.42}$$

   where

$$\lambda_D = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} = 9.682\,\text{rad/s}.$$

   Substituting the initial conditions $x(0) = 0.5\,\text{m}$ and $\dot{x}(0) = 0\ \text{m/s}$ in Eqn (9.42) (we need to differentiate Eqn (9.42) first to substitute $\dot{x}(0)$), we get

$$\begin{aligned} & x(0) &=& \ 0.5\,\text{m} = A. \\ & \dot{x}(0) &=& \ 0 = -\frac{c}{2m}\cdot A + \lambda_D \cdot B \\ \Rightarrow \quad & B &=& \ \frac{A\,c}{2m\lambda_D} = \frac{(0.5\,\text{m})\cdot(50\,\text{N}\cdot\text{s/m})}{2\cdot(10\,\text{kg})\cdot(9.682\,\text{rad/s})} = 0.13\,\text{m}. \end{aligned}$$

   Thus, the solution is

$$x(t) = e^{(-2.5\frac{1}{\text{s}})t}\,[0.50\cos(9.68\,\text{rad/s}\ t) + 0.13\sin(9.68\,\text{rad/s}\ t)]\ \text{m}.$$

   Substituting $t = 2\,\text{s}$ in the above expression we get $x(2\,\text{s}) = 0.003\,\text{m}$.

$$\boxed{x(2\,\text{s}) = 0.003\,\text{m}.}$$

**SAMPLE 9.19** A structure, modeled as a single degree of freedom system, exhibits characteristics of an underdamped system under free oscillations. The response of the structure to some initial condition is determined to be $x(t) = Ae^{-\xi\lambda t}\sin(\lambda_D t)$ where $A = 0.3\,\text{m}$, $\xi \equiv$ damping ratio $= 0.02$, $\lambda \equiv$ undamped circular frequency $= 1\,\text{rad/s}$, and $\lambda_D \equiv$ damped circular frequency $= \lambda\sqrt{1-\xi^2} \approx \lambda$.

1. Find an expression for the ratio of energies of the system at the $(n+1)$th displacement peak and the $n$th displacement peak.
2. What percent of energy available at the first peak is lost after 5 cycles?

**Solution**

1. We are given that
$$x(t) = Ae^{-\xi\lambda t}\sin(\lambda_D t).$$
The structure attains its first displacement peak when $\sin\lambda_D t$ is maximum, *i.e.*,
$$\lambda_D t = \frac{\pi}{2} \quad \Rightarrow \quad t = \frac{\pi}{2\lambda_D}.$$
At this instant,
$$\begin{aligned} x(t) &= Ae^{-\xi\cdot\lambda\cdot\frac{\pi}{2\lambda_D}} \\ &= Ae^{-\frac{\pi}{2}\cdot\frac{\xi}{\sqrt{1-\xi^2}}} \\ &= (0.3\,\text{m})\cdot e^{-0.0314} \\ &= 0.29\,\text{m}. \end{aligned}$$
Let $x_n$ and $x_{n+1}$ be the values of the displacement at the $n$th and the $(n+1)$th peak, respectively. Since $x_n$ and $x_{n+1}$ are peak displacements, the respective velocities are zero at these points. Therefore, the energy of the system at these peaks is given by the potential energy stored in the spring. That is
$$E_n = \frac{1}{2}kx_n^2 \quad \text{and} \quad E_{n+1} = \frac{1}{2}kx_{n+1}^2. \tag{9.43}$$
Let $t_n$ be the time at which the $n$th peak displacement $x_n$ is attained, *i.e.*,
$$x_n = Ae^{-\xi\lambda t_n} \tag{9.44}$$
Since $x_{n+1}$ is the next peak displacement, it must occur at $t = t_n + T_D$ where $T_D$ is the time period of damped oscillations. Thus
$$x_{n+1} = Ae^{-\xi\lambda(t_n+T_D)} \tag{9.45}$$
From Eqns (9.43), (9.44), and (9.45)
$$\begin{aligned} \frac{E_{n+1}}{E_n} &= \frac{\frac{1}{2}k(Ae^{-\xi\lambda(t_n+T_D)})^2}{\frac{1}{2}k(Ae^{-\xi\lambda t_n})^2} \\ &= e^{-2\xi\lambda T_D}. \end{aligned}$$
$$\boxed{\frac{E_{n+1}}{E_n} = e^{-2\xi\lambda T_D}.}$$

2. Noting that $T_D = \frac{2\pi}{\lambda_D}$ and $\lambda_D = \lambda\sqrt{1-\xi^2}$, we get

$$\begin{aligned} E_{n+1} &= E_n\, e^{-2\xi\lambda\cdot\frac{2\pi}{\lambda\sqrt{1-\xi^2}}} \\ &= e^{-4\pi\frac{\xi}{\sqrt{1-\xi^2}}} \approx e^{-4\pi\xi} \\ \Rightarrow \qquad E_{n+1} &= e^{-4\pi\xi}\, E_n. \end{aligned}$$

Applying this equation recursively for $n = n-1, n-2, \ldots, 1, 0,$, we get

$$\begin{aligned} E_n &= e^{-4\pi\xi}\cdot E_{n-1} \\ &= e^{-4\pi\xi}\cdot(e^{-4\pi\xi}\cdot E_{n-2}) \\ &= (e^{-4\pi\xi})^3\cdot E_{n-3} \\ &\vdots \\ &= (e^{-4\pi\xi})^n\cdot E_0. \end{aligned}$$

Now we use this equation to find the percentage of energy of the first peak ($n = 0$) lost after 5 cycles ($n = 5$):

$$\begin{aligned} \Delta E_5 &= \frac{E_0 - E_5}{E_0}\times 100 \\ &= \left(1 - e^{-4\pi\xi\,\cdot\,5}\right)\times 100 \\ &= 71.5\%. \end{aligned}$$

$$\boxed{\Delta E_5 = 71.5\%.}$$

**SAMPLE 9.20 A SDOF spring-mass model from given data:** The following table is obtained for successive peaks of displacement from the simulation of free vibration of a mechanical system. Make a single degree of freedom mass-spring-dashpot model of the system choosing appropriate values for mass, spring stiffness, and damping rate.

**Data:**

| peak number $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| time ( s) | 0.0000 | 0.6279 | 1.2558 | 1.8837 | 2.5116 | 3.1395 | 3.7674 |
| peak disp. ( m) | 0.5006 | 0.4697 | 0.4411 | 0.4143 | 0.3892 | 0.3659 | 0.3443 |

**Solution** Since the data provided is for successive peak displacements, the time between any two successive peaks represents the period of oscillations. It is also clear that the system is underdamped because the successive peaks are decreasing. We can use the logarithmic decrement method to determine the damping in the system.

First, we find the time period $T_D$ from which we can determine the damped circular frequency $\lambda_D$. From the given data we find that

$$t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = \cdots = 0.6279\,\text{s}$$

Therefore,

$$\begin{aligned} T_D &= 0.6279\,\text{s}. \\ \Rightarrow \quad \lambda_D &= \frac{2\pi}{T_D} = 10\,\text{rad/s}. \end{aligned} \tag{9.46}$$

Now we make a table for the logarithmic decrement of the peak displacements:

| peak disp. $x_n$ ( m) | 0.5006 | 0.4697 | 0.4411 | 0.4143 | 0.3892 | 0.3659 | 0.3443 |
|---|---|---|---|---|---|---|---|
| $\frac{x_n}{x_{n+1}}$ | 1.0658 | 1.0648 | 1.0647 | 1.0645 | 1,0637 | 1.0627 | |
| $\ln\left(\frac{x_n}{x_{n+1}}\right)$ | 0.0637 | 0.0628 | 0.0627 | 0.0624 | 0.0618 | 0.0608 | |

Thus, we get several values of the logarithmic decrement $D = \ln\left(\frac{x_n}{x_{n+1}}\right)$ *.

We take the average value of $D$:

$$D = \bar{D} = 0.0624. \tag{9.47}$$

Let the equivalent single degree of freedom model have mass $m$, spring stiffness $k$, and damping rate $c$. Then

$$\lambda_D = \lambda\sqrt{1-\xi^2} \approx \lambda = \sqrt{\frac{k}{m}}.$$

Thus, from Eqn (9.46),

$$\frac{k}{m} = \lambda^2 = 100(\,\text{rad/s})^2, \tag{9.48}$$

* Theoretically, all of these values should be the same, but it is rarely the case in practice. When $x_n$'s are measured from an experimental setup, the values of $D$ may vary even more.

and, since $D = \frac{cT_D}{2m}$, from Eqn (9.47) we get

$$\begin{aligned} c &= \frac{2m\,D}{T_D} \\ &= \frac{2m(0.0624)}{0.6279\,\mathrm{s}} \\ &= (0.1988\frac{1}{\mathrm{s}})m. \end{aligned} \tag{9.49}$$

Equations (9.48) and (9.49) have three unknowns: $k$, $m$, and $c$. We cannot determine all three uniquely from the given information. So, let us pick an arbitrary mass $m = 5\,\mathrm{kg}$. Then

$$\begin{aligned} k &= (100\frac{1}{\mathrm{s}^2})\cdot(5\,\mathrm{kg}) \\ &= 500\,\mathrm{N/\,m}, \end{aligned}$$

and

$$\begin{aligned} c &= (0.1988\frac{1}{\mathrm{s}})\cdot(5\,\mathrm{kg}) \\ &= 0.99\,\mathrm{N\cdot s/\,m}. \end{aligned}$$

$$\boxed{\begin{aligned} m &= 5\,\mathrm{kg}, \\ k &= 500\,\mathrm{N/\,m}, \\ c &= 0.99\,\mathrm{N\cdot s/\,m}. \end{aligned}}$$

Of course, we could choose many other sets of values for $m$, $k$, and $c$ which would match the given response. In practice, there is usually a little more information available about the system, such as the mass of the system. In that case, we can determine $k$ and $c$ uniquely from the given response.

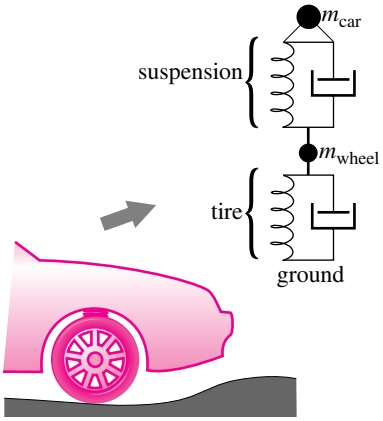



Figure 9.43: The vertical motions of a car and wheel might be considered as coupled motions in 1D.

Filename:tefig3-4-car-susp

# 9.4 Coupled motions in 1D

Many important engineering systems have parts that move independently. A one-particle model is not adequate.

Example: **Car suspension.**
A model of a car suspension treats the wheel as one particle and the car as another. The wheel is coupled to the ground by a tire and to the car by the suspension. In a first analysis the only motion to consider would be vertical for both the wheel and the car.

So here, still using one-dimensional mechanics, we consider systems that can be modelled as two or more particles. Such one-dimensional coupled motion analysis is common in engineering practice in situations where there are connected parts that all move in about the same direction, but not the same amount at the same time. Many of the ideas generalize (but not in this section) to systems where parts, each with one degree of freedom, are couple together, even when each degree of freedom is quite different from the others (the particles don't all move on a common line).

The goal is to develop two skills:

- To write correct equations of motion for a line of particles connected to each other with springs and dashpots, and
- To simulate the motions of such systems on a computer.
- (the third of the two things, it should be implicit) To look at the computer simulation and use it to find errors in the equations.

The simplest way of dealing with the coupled motion of two or more particles is to write $\vec{F} = m\vec{a}$ for each particle and use the forces on the free body diagrams to evaluate the forces. Because the most common models for the interaction forces are springs and dashpots (see chapter 3), one needs to account for the relative positions and velocities of the particles.

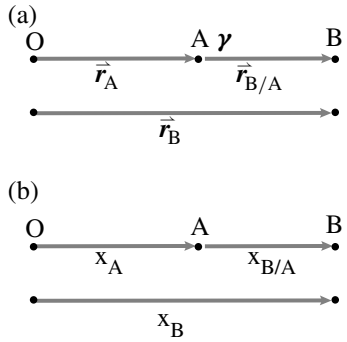



Figure 9.44: The relative position of points A and B in one dimension.

Filename:tfigure-relpos1D

## Relative motion in one dimension

If the position of A is $\vec{r}_A$, and B's position is $\vec{r}_B$, then B's position *relative* to A is

$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A.$$

Relative velocity and acceleration are similarly defined by subtraction, or by differentiating the above expression, as

$$\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A \text{ and } \vec{a}_{B/A} = \vec{a}_B - \vec{a}_A.$$

In one dimension, the relative position diagram of Fig. 2.5 on page 21 becomes Fig. 9.44. $\vec{r} = x\hat{\imath}$, $\vec{v} = v\hat{\imath}$, and $\vec{a} = a\hat{\imath}$. So, we can write,

$$\begin{aligned} x_{B/A} &\equiv x_B - x_A, \\ v_{B/A} &\equiv v_B - v_A, \text{ and} \\ a_{B/A} &\equiv a_B - a_A. \end{aligned}$$

Example: **Two masses connected by a spring.**
Consider the two masses on a frictionless support (Fig. 9.45). Assume the spring is unstretched when $x_1 = x_2 = 0$. After drawing free body diagrams of the two masses we can write $\vec{F} = m\vec{a}$ for each mass:

$$\begin{aligned} \textbf{mass 1:} \quad & \vec{F}_1 = m\vec{a}_1 \quad \Rightarrow \quad T\hat{\imath} = m_1\ddot{x}_1\hat{\imath} \\ \textbf{mass 2:} \quad & \vec{F}_2 = m\vec{a}_2 \quad \Rightarrow \quad -T\hat{\imath} = m_2\ddot{x}_2\hat{\imath} \end{aligned} \tag{9.50}$$

The stretch of the spring is

$$\begin{aligned} & \Delta\ell = x_2 - x_1 \\ \text{so} \quad & T = k\Delta\ell = k(x_2 - x_1). \end{aligned} \tag{9.51}$$

Combining (9.50) and (9.51) we get

$$\begin{aligned} \ddot{x}_1 &= \left(\tfrac{1}{m_1}\right) k(x_2 - x_1) \\ \ddot{x}_2 &= \left(\tfrac{1}{m_2}\right) (-k(x_2 - x_1)) \end{aligned} \tag{9.52}$$

Note: Take care with signs when setting up this type of problem. You can check for example that if $x_2 > x_1$, mass 1 accelerates to the right ($\ddot{x}_1 > 0$) and mass 2 accelerates to the left($\ddot{x}_2 < 0$). You've been warned!

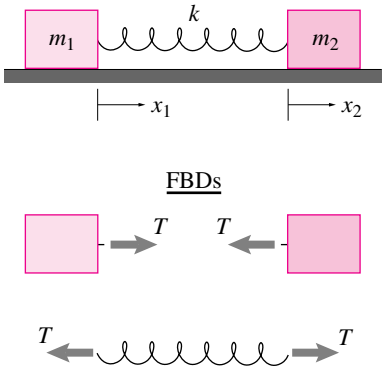



Figure 9.45:
Filename:tefig3-4-mass-springer

The differential equations that result from writing $\vec{F} = m\vec{a}$ for the separate particles are coupled second-order equations. They are often solved on a computer by writing them as a system of first-order equations.

Example: **Writing second-order ODEs as first-order ODEs.**
Refer again to Fig. 9.45 If we define $v_1 = \dot{x}_1$ and $v_2 = \dot{x}_2$ we can rewrite equation 9.52 as

$$\begin{aligned} \dot{x}_1 &= v_1 \\ \dot{v}_1 &= \left(\frac{1}{m_1}\right) k(x_2 - x_1) \\ \dot{x}_2 &= v_2 \\ \dot{v}_2 &= \left(\frac{1}{m_2}\right) (-k)\,(x_2 - x_1) \end{aligned}$$

or, defining $z_1 = x_1, z_2 = v_1, z_3 = x_2, z_4 = v_2$, we get

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -\left(\tfrac{k}{m}\right) z_1 + \left(\tfrac{k}{m_1}\right) z_3 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= \tfrac{k}{m_2} z_1 - \tfrac{k}{m_2} z_3 \,. \end{aligned}$$

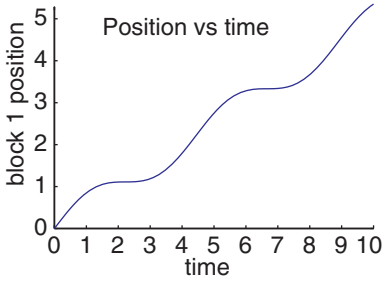



Figure 9.46: Plot of the position of the left mass *vs.* time.
Filename:tfig-coupledmasses

Most numerical solutions depend on specifying numerical values for the various constants and initial conditions.

Example: **computer solution**
If we take, in consistent units, $m_1 = 1, k = 1, m_2 = 1, x_1(0) = 0, x_2(0) = 0, v_1(0) = 1$, and $v_2(0) = 0$, we can set up a well defined computer problem (please see the preface for a discussion of the computer notation). This problem corresponds to finding the motion just after the left mass was hit on the left side with a hammer.:

```
ODEs = {z1dot =  z2
        z2dot = -z1 + z3
        z3dot =  z4
        z4dot =  z1 - z3}
ICs  = {z1(0) 0, z2(0)=1, z3(0)=0, z4(0)=0}
solve ODEs with ICs from t=0 to t=10
plot z1 vs t.
```

This yields the plot shown in Fig. 9.46.

As the samples show, the same methods work for problems involving connections with dashpots.

## Center of mass

For both theoretical and practical reasons it is often useful to pay attention to the motion of the average position of mass in the system. This average position is called the center-of-mass. For a collection of particles in one dimension the center-of-mass is

$$x_{\text{CM}} = \frac{\sum x_i m_i}{m_{\text{tot}}}, \tag{9.53}$$

where $m_{\text{tot}} = \sum m_i$ is the total mass of the system. The velocity and acceleration of the center-of-mass are found by differentiation to be

$$v_{\text{CM}} = \frac{\sum v_i m_i}{m_{\text{tot}}} \quad \text{and} \quad a_{\text{CM}} = \frac{\sum a_i m_i}{m_{\text{tot}}}. \tag{9.54}$$

If we imagine a system of interconnected masses and add the $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ equations from all the separate masses we can get on the left hand side only the forces from the outside; the interaction forces cancel because they come in equal and opposite (action and reaction) pairs. So we get:

$$\sum F_{\text{external}} = \sum a_i m_i = m_{\text{tot}} a_{\text{CM}}. \tag{9.55}$$

So the center-of-mass of a system (a system that may be deforming wildly) obeys the same simple governing equation as a single particle. Although our demonstration here was for particles in one dimension. The result holds for any bodies of any type in 1,2, or 3 dimensions.

**SAMPLE 9.21** For the given quantities and initial conditions, find $x_1(t)$. Assume the spring is unstretched when $x_1 = x_2$.

$$m_1 = 1\,\text{kg}, \quad m_2 = 2\,\text{kg}, \quad k = 3\,\text{N/m}, \quad c = 5\,\text{N/(m/s)}$$
$$x_1(0) = 1\,\text{m}, \quad \dot{x}_1(0) = 0, \quad x_2(0) = 2\,\text{m}, \quad \dot{x}_2(0) = 0.$$

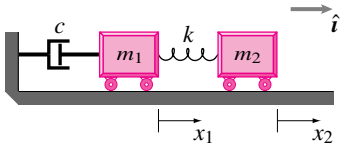



Figure 9.47:
Filename:sfig3-4-unstr-ini

**Solution** The free body diagrams of all components of the given system are shown below.

FBDs

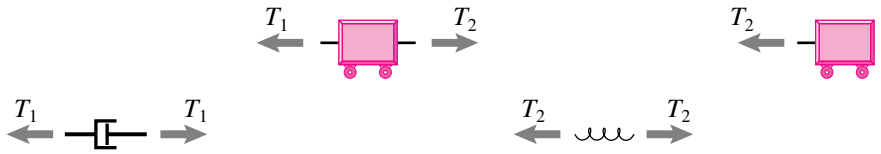



Figure 9.48:
Filename:sfig3-4-unstr-ini-fbd

The spring and dashpot laws give

$$T_1 = c\dot{x}_1 \qquad T_2 = k(x_2 - x_1). \tag{9.56}$$

The linear momentum balance for the two masses gives

$$\begin{aligned} &\textstyle\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}} \\ \text{mass 1:}\quad & -T_1\hat{\imath} + T_2\hat{\imath} = m_1\ddot{x}_1\hat{\imath} \\ \text{mass 2:}\quad & -T_2\hat{\imath} = m_2\ddot{x}_2\hat{\imath}. \end{aligned} \tag{9.57}$$

Applying the constitutive laws (9.56) to the momentum balance equations (9.57) gives

$$\begin{aligned} \ddot{x}_1 &= [k(x_2 - x_1) - c\dot{x}_1]/m_1 \\ \ddot{x}_2 &= [-k(x_2 - x_1)]/m_2. \end{aligned}$$

Defining $z_1 = x_1, z_2 = \dot{x}_1, z_3 = x_2, z_4 = \dot{x}_2$ gives

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= [k(z_3 - z_1) - cz_2]/m_1 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= [-k(z_3 - z_1)]/m_2. \end{aligned}$$

The initial conditions are

$$z_1(0) = 1\,\text{m}, \quad z_2(0) = 0, \quad z_3(0) = 2\,\text{m}, \quad z_4(0) = 0.$$

We are now set for numerical solution.

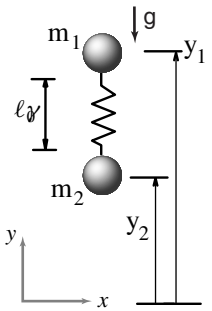



Figure 9.49:
Filename:sfig5-6-hopper

* You can think of it as a model of a hopping robot or a hopping animal where the foot mass is modeled with $m_2$ and the body mass with $m_1$. The spring models the elasticity of the system. If you restrict this model to one dimensional vertical motion under gravity, it will hop, once released from some initial height. The number of hops depends on the intial conditions. Theoretically, it can hop forever with suitable initial conditions.

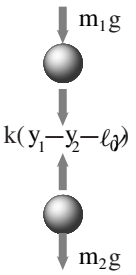



Figure 9.50: Free body diagram of the two masses $m_1$ and $m_2$
Filename:sfig5-6-hopper-a

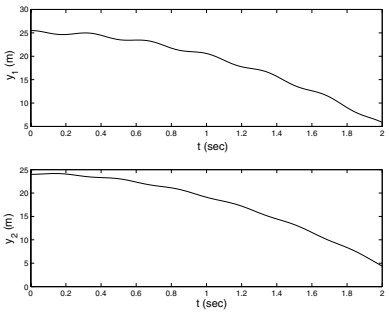



Figure 9.51: Numerically obtained solutions $y_1(t)$ and $y_2(t)$
Filename:sfig5-6-hopper-b

**SAMPLE 9.22 Flight of a toy hopper.** A hopper model * is made of two masses $m_1 = 0.4\,\text{kg}$ and $m_2 = 1\,\text{kg}$, and a spring with stiffness $k = 100\,\text{N/m}$ as shown in Fig. 9.49. The unstretched length of the spring is $\ell_0 = 1\,\text{m}$. The model is released from rest from the configuration shown in the figure with $y_1 = 25.5\,\text{m}$ and $y_2 = 24\,\text{m}$.

1. Find and plot $y_1(t)$ and $y_2(t)$ for $t = 0$ to $2\,\text{s}$.
2. Plot the motion of $m_1$ and $m_2$ with respect to the center-of-mass of the hopper during the same time interval.
3. Plot the motion of the center-of-mass of the hopper from the solution obtained for $y_1(t)$ and $y_2(t)$ and compare it with analytical values obtained by integrating the center-of-mass motion directly.

**Solution** The free-body diagrams of the two masses are shown in Fig. 9.50. From the linear momentum balance in the $y$ direction, we can write the equations of motion at once.

$$\begin{aligned} m_1\ddot{y}_1 &= -k(y_1 - y_2 - \ell_0) - m_1 g \\ \Rightarrow \quad \ddot{y}_1 &= -\frac{k}{m_1}(y_1 - y_2) + \frac{k\ell_0}{m_1} - g \qquad (9.58) \\ m_2\ddot{y}_2 &= k(y_1 - y_2 - \ell_0) - m_2 g \\ \Rightarrow \quad \ddot{y}_2 &= \frac{k}{m_2}(y_1 - y_2) - \frac{k\ell_0}{m_2} - g. \qquad (9.59) \end{aligned}$$

1. The equations of motion obtained above are coupled linear differential equations of second order. We can solve for $y_1(t)$ and $y_2(t)$ by numerical integration of these equations. As we have shown in previous examples, we first need to set up these equations as a set of first order equations.

   Letting $\dot{y}_1 = v_1$ and $\dot{y}_2 = v_2$, we get

$$\begin{aligned} \dot{y}_1 &= v_1 \\ \dot{v}_1 &= -\frac{k}{m_1}(y_1 - y_2) + \frac{k\ell_0}{m_1} - g \\ \dot{y}_2 &= v_2 \\ \dot{v}_2 &= \frac{k}{m_2}(y_1 - y_2) - \frac{k\ell_0}{m_2} - g. \end{aligned}$$

   Now we solve this set of equations numerically using some ODE solver and the following pseudocode.

```
ODEs = {y1dot = v1,
        v1dot = -k/m1*(y1-y2-l0) - g,
        y2dot = v2,
        v1dot = k/m1*(y1-y2-l0) - g}
IC   = {y1(0)=25.5, v1(0)=0, y2(0)=24, v2(0)=0}
Set  k=100, m1=0.4, m2=1, l0=1
Solve ODEs with IC for t=0 to t=2
Plot y1(t) and y2(t)
```

   The solution obtained thus is shown in Fig. 9.51.

2. We can find the motion of $m_1$ and $m_2$ with respect to the center-of-mass by subtraction the motion of the center-of-mass, $y_{\rm cm}$ from $y_1$ and $y_2$. Since,

$$y_{\rm cm} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} \tag{9.60}$$

we get,

$$\begin{aligned} y_{1/\rm cm} &= y_1 - y_{\rm cm} = \frac{m_2}{m_1 + m_2}(y_1 - y_2) \\ y_{2/\rm cm} &= y_2 - y_{\rm cm} = -\frac{m_1}{m_1 + m_2}(y_1 - y_2). \end{aligned}$$

The relative motions thus obtained are shown in Fig. 9.52. We note that the motions of $m_1$ and $m_2$, as seen by an observer sitting at the center-of-mass, are simple harmonic oscillations.

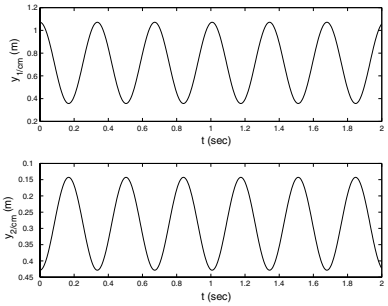



Figure 9.52: Numerically obtained solutions $y_{1/\rm cm}(t)$ and $y_{1/\rm cm}(t)$.
Filename:sfig5-6-hopper-c

3. We can find the center-of-mass motion $y_{\rm cm}(t)$ from $y_1$ and $y_2$ by using eqn. (9.60). The solution obtained thus is shown as a solid line in Fig. **??**. We can also solve for the center-of-mass motion analytically by first writing the equation of motion of the center-of-mass and then integrating it analytically.

   The free-body diagram of the hopper as a single system is shown in Fig. 9.53. The linear momentum balance for the system in the vertical direction gives

$$\begin{aligned} (m_1 + m_2)\ddot{y}_{\rm cm} &= -m_1 g - m_2 g \\ \Rightarrow \quad \ddot{y}_{\rm cm} &= -g. \end{aligned}$$

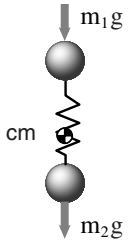



Figure 9.53: Free body diagram of the hopper as a single system. The spring force does not show up here since it becomes an internal force to the system
Filename:sfig5-6-hopper-e

We recognize this equation as the equation of motion of a freely falling body under gravity. We can integrate this equation twice to get

$$y_{\rm cm}(t) = y_{\rm cm}(0) + \dot{y}_{\rm cm}(0)t - \frac{1}{2}gt^2.$$

Noting that $y_{\rm cm}(0) = 24.43\,\text{m}$ (from eqn. (9.60)), and $\dot{y}_{\rm cm}(0) = 0$ (the system is released from rest), we get

$$y_{\rm cm}(t) = 24.43\,\text{m} - \frac{1}{2} \cdot 9.81\,\text{m/s}^2 \cdot t^2.$$

The values obtained for the center-of-mass position from the above expression are shown in Fig. 9.54 by small circles.

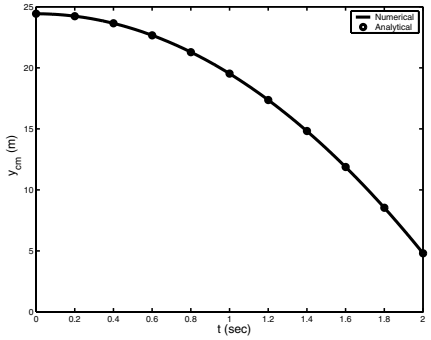



Figure 9.54: Numerically obtained solution for the position of the center-of-mass, $y_{\rm cm}(t)$.
Filename:sfig5-6-hopper-dd

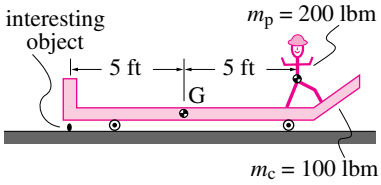



Figure 9.55: Mr. P spots an interesting object.

Filename:sfig2-6-5

**SAMPLE 9.23 Conservation of linear momentum.** Mr. P with mass $m_p$ = 200 lbm is standing on a cart with frictionless and massless wheels. The cart weighs half as much as Mr. P. Standing at one end of the cart, Mr. P spots an interesting object at the other end of the cart. Mr. P decides to walk to the other end of the cart to pick up the object. How far does he find himself from the object after he reaches the end of the cart?

**Solution** From your own experience in small boats perhaps, you know that when Mr. P walks to the left the cart moves to the right. Here, we want to find how far the cart moves.

Consider the cart and Mr. P together to be the system of interest. The free-body diagram of the system is shown in Fig. 9.56(a).

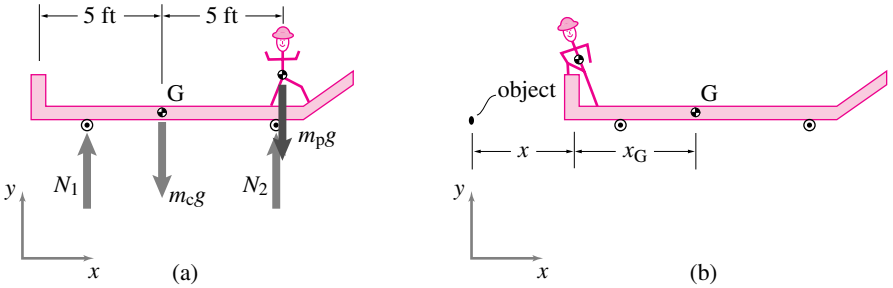



Figure 9.56: (a) Free-body diagram of Mr. P-and-the-cart system. (b) The cart has moved to the right by distance $x$ when Mr. P reaches the other end.

Filename:sfig2-6-5a

From the diagram it is clear that there are no external forces in the $x$-direction. Therefore,

$$\dot{L}_x = \sum F_x = 0 \quad \Rightarrow \quad L_x = \text{constant}$$

that is, the linear momentum of the system in the $x$-direction is 'conserved'. But the initial linear momentum of the system is zero. Therefore,

$$L_x = m_{tot}(v_{cm})_x = 0 \text{ all the time} \quad \Rightarrow \quad (v_{cm})_x = 0 \text{ all the time.}$$

Because the horizontal velocity of the center-of-mass is always zero, the center-of-mass does not change its horizontal position. Now let $x_{cm}$ and $x'_{cm}$ be the $x$-coordinates of the center-of-mass of the system at the beginning and at the end, respectively. Then,

$$x'_{cm} = x_{cm}.$$

Now, from the given dimensions and the stipulated position at the end in Fig. 9.56(b),

$$x_{cm} = \frac{m_c x_G + m_p x_p}{m_c + m_p} \quad \text{and} \quad x'_{cm} = \frac{m_c(x_G + x) + m_p x}{m_c + m_p}.$$

Equating the two distances we get,

$$\begin{aligned} m_c x_G + m_p x_p &= m_c(x_G + x) + m_p x \\ &= m_c x_G + x(m_c + m_p) \\ \Rightarrow \quad x &= \frac{m_p x_p}{m_c + m_p} \\ &= \frac{200\,\text{lbm} \cdot 10\,\text{ft}}{300\,\text{lbm}} = 6\frac{2}{3}\,\text{ft}. \end{aligned}$$

6.67 ft

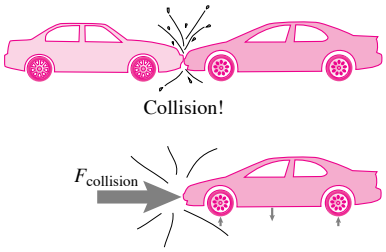
[Note: if Mr. P and the cart have the same mass, the cart moves to the right the same distance Mr. P moves to the left.]

# 9.5 Collisions in 1D

Sometimes things interact in a sudden manner, like two cars in a head-on crash or a dropped cell-phone hitting the floor. Some sudden interactions are intentional, for example in sports the banging of racquets, bats, clubs, sticks, hands and legs with balls, pucks and bodies. And in machines there are sometimes intentionally sudden interactions like the clicking of a ratchet and the flip of an electric light switch. More esoteric 'sudden' interactions include those between subatomic particles in an accelerator and near passes of satellites with planets.

When two solids bump into each other a nearly discontinuous change in their velocities and/or angular velocities is needed to keep the bodies from interpenetrating. This sudden change in velocity demands large interaction. In the case of subatomic particles near nuclei and satellites near planets there might be no contact, but none-the-less there are large forces when the interaction distances get small. Estimating the effects of these large yet short-lived forces is the central problem in collision mechanics.

Two objects are said to *collide* when some interaction force or moment between them becomes so large that other forces acting on the bodies become negligible. For example, in a car collision the force of interaction at the bumpers may be many times the weight of the car or the reaction forces acting on the wheels. And so short acting that, although velocities change, positions change negligibly during the collision.




The collision forces are assumed to be much bigger than all other forces on the FBD during the collision.

Figure 9.57: Here cars are shown colliding. A free body diagram of the right car shows the collision force and should not show other forces which are negligibly small. Here they are shown as negligibly small forces to give the idea that they may be much smaller than the collision force. The wheel reaction forces are neglected because of the spring compliance of the suspension and tires.

Filename:tfigure2-collisions

**Collisional free body diagrams** The analysis of collisions is a little different than the analysis of smooth motions, but still depends on free body diagrams (See figure 9.57). Knowing which forces to include and which to ignore in a collisional free-body-diagram is a subtle issue. Some rules of thumb:

- ignore forces from gravity, springs, and at places where contact is broken in the collision, and
- include forces at places where new contact is made, or where contact is maintained.

The elementary analysis of rigid body collisions is based on these ideas:

I. Collision forces are big, so non-collisional forces are neglected in collisional free body diagrams.
II. Collision forces are of short duration, so the position and orientation of the colliding bodies do not change during the collision.

## What happens during a collision

During a collision between what would generally be called "rigid" bodies *things get wild*. There are huge contact forces and stresses in the regions near the nominally* contacting points, there could be plastic deformation, fracture, and frictional slip. Elastic waves may travel all over the body, reflect and scatter this way and that. Altogether the contact interaction during the collision is the result of very complex deformations (see Fig. 9.58).

Deformations (the lack of rigidity) give rise to the forces between colliding bodies. So what could the phrase "rigid-object collisions" mean? It is an oxymoron. Trying to understand the collision forces in detail, and how they are related to deformations, is way beyond this book. Actually, there is no unified theory of collisions so you can't read about it in any book. Loosely one might imagine that during part of the collision material is being squeezed, this is called the *compression* phase and later on it expands back in a *restitution* phase. But the realities of collisions are not necessarily so simple; the forces and deformations can vary in complex ways.

Soon after the collision, however, the vibrations often die out, each object may have negligible permanent change in shape, and the object returns to motions that are well described by rigid-object kinematics. To find out the net effect of the collision forces we use this one key idea:

> III. The laws of mechanics apply *during* collisions even though rigid-object kinematics does not.

While the motions during a collision may be wildly complex, the general linear and angular momentum balance laws are still applicable. Rather than applying these laws to understand the details *during* a collision, we use them to summarize the overall result of the collision.

That is, in rigid-object collision analysis we do not pay attention to how the forces vary in time, or to the detailed trajectories, velocities or accelerations of any material points. Rather, we focus on the net change in the velocities of the colliding bodies that the collision forces cause. Thus, instead of using the differential-equation form of the linear momentum balance, angular-momentum balance and energy equations (Ia, IIa, and IIIa from the inside front cover) we use the time integrated forms (Ib, IIb, and IIIb). All that we note about a collisional force is its net impulse

$$\vec{\boldsymbol{P}}_{\text{coll}} = \int_{\text{collision time}} \vec{\boldsymbol{F}}_{\text{coll}}\, dt$$

in terms of which we have, for one object experiencing this impulse at point C

$$\vec{\boldsymbol{P}}_{\text{coll}} = \Delta\vec{\boldsymbol{L}}, \tag{9.61}$$

$$\vec{\boldsymbol{r}}_{C/0} \times \vec{\boldsymbol{P}}_{\text{coll}} = \Delta\vec{\boldsymbol{H}}_{O}, \quad \text{and} \tag{9.62}$$

$$\text{Collisional dissipation} = \Delta E_{K}. \tag{9.63}$$

* Nominally means "in name". That is, what one calls "contacting points" are not points at all, but regions of complex interaction.

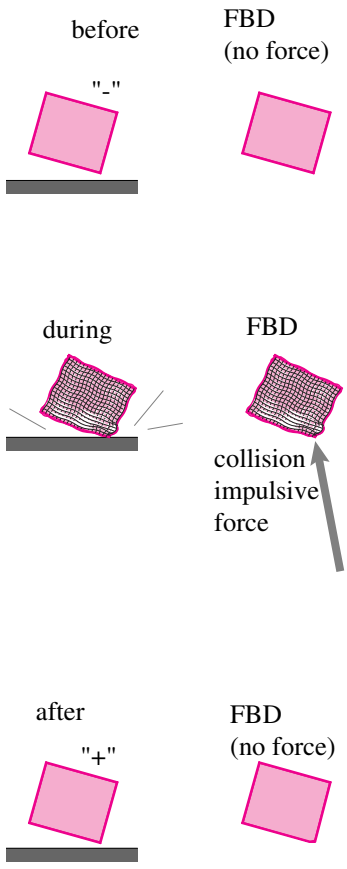



Figure 9.58: Just before a collision is called "-", and just after is called "+". The only forces that show on a collisional free body diagram are those that are large and part of the impact. Either a force or an impulse may be shown. This figure exaggerates the difference between the before (-) and after (+) states. In analysis we assume that there is no change in the body's position or orientation from just before to just after the collision. The only net changes caused by the collision are the body's velocity and rotation rate.

Filename:tfigure-collisiongeneral

Most often the first two of these, the impulse-momentum equations are used to find the motion after collision. The energy equation is just a check to make sure that the collisional dissipation is positive (otherwise the collision would be an energy source).

## Extra assumptions are needed

The momentum balance equations, with the assumptions already discussed, are never enough in themselves to determine the outcome of a collision. The extra assumptions come in various forms. To minimize the algebra we discuss the issues first with one-dimensional collisions.

## One dimensional collisions

Here we only consider collisions in the context of one-dimensional mechanics: all motion is constrained to one direction of motion by forces which we ignore. Only momentum and forces in, say, the $\hat{\boldsymbol{\imath}}$ direction are included.

Example: **1-D collisions**
Consider two masses which collide along their common line of motion. All velocities and momenta are positive if to the right and $P$ is the impulse *on* mass 2 from mass 1. The relevant impulse-momentum relations are

$$\begin{aligned} \text{For mass 1} \qquad -P &= m_1(v_1^+ - v_1^-), \\ \text{For mass 2} \qquad P &= m_2(v_2^+ - v_2^-), \text{ and} \\ \text{For the system} \qquad 0 &= (m_1 v_1^+ + m_2 v_2^+) - (m_1 v_1^- + m_2 v_2^-). \end{aligned}$$

The third equation comes from a free body diagram of the system (ie, conservation of momentum) or by adding the first two equations. In any case, given the masses and initial velocities we have only two independent equations and we have three unknowns: $v_1^+$, $v_2^+$ and $P$. Momentum balance is not enough to determine the outcome of a collision.

To "close" (make solvable) the set of equations one needs to make extra assumptions.

### Sticking collisions

The simplest assumption is that the masses *stick* together after the collision so

$$v_1^+ = v_2^+.$$

Such a collision is sometimes called a *perfectly plastic*, a *perfectly inelastic*, or a *dead* collision. Algebraic manipulations of the momentum equations and the "sticking" constitutive law give

$$\begin{aligned} v_1^+ = v_2^+ &= (m_1 v_1^- + m_2 v_2^-)/m_{\text{tot}} \qquad (\text{where } m_{\text{tot}} = m_1 + m_2) \quad \text{and} \\ P &= (v_1^- - v_2^-) m_{\text{coll}} \qquad (\text{where } m_{\text{coll}} = \frac{m_1 m_2}{m_1 + m_2}). \end{aligned}$$

The *collisional mass* or *contact mass* $m_{\text{coll}}$

$$m_{\text{coll}} = \frac{m_1 m_2}{m_1 + m_2} = \frac{1}{\frac{1}{m_1} + \frac{1}{m_2}}$$

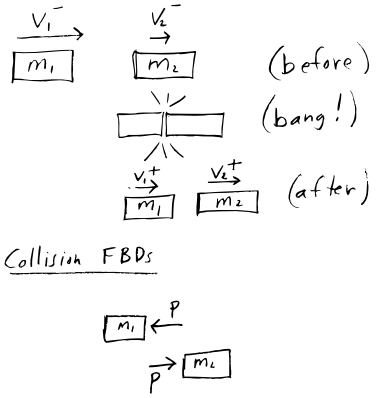



Figure 9.59: Before the collisions the masses have velocities to the right of $v_1^-$ and $v_2^-$. After the collision the velocities are $v_1^+$ and $v_2^+$. During the collision the impulse $P$ acts to the right on mass 2 and to the left on mass 1.
Filename:tfigure-1Dcollisions

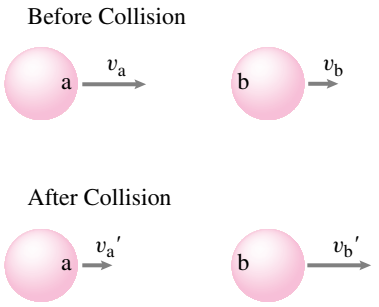



Figure 9.60: The rate of approach of two about-to-collide points is $v_a - v_b$. The rate of separation after collision is $v_b' - v_a'$. The simplest collision law says $v_b' - v_a' = e(v_a - v_b)$, where $e$ is the coefficient of restitution.

Filename:tfigure11-vba

is not the mass of anything. It is just a quantity that shows up repeatedly in collision calculations and theory. It is the reciprocal of the sum of the reciprocals of the two masses. If one mass is much bigger than the other, the contact mass is $m_{\text{coll}} \approx$ the smaller of the two masses. It is the proportionality constant relating the interaction force and the relative acceleration of the particles during the collision

$$m_{\text{coll}}(a_2 - a_1) = F \qquad \text{(with } F \text{ being the force of body 1 on body 2)}$$

and is thus related to the effective mass of box **??** on page **??**.

## More general 1-D collisions

The momentum equations can be re-arranged to better get at the essence of the situation which is that

- In the collision the system's center-of-mass velocity is unchanged, and
- The effect of the collision is to change the difference between the two mass velocities.

So we define the center-of-mass velocity $v_{\text{cm}}$ and the velocity difference $v_{\text{rel}}$ as

$$v_{\text{cm}} \equiv (m_1 v_1 + m_2 v_2)/m_{\text{tot}} \quad \text{and} \quad v_{\text{rel}} \equiv v_2 - v_1.$$

Note that before a collision the masses are approaching each other so $v_1^- > v_2^-$ and $v_{\text{rel}}^- < 0$. A little more algebra shows that for any $P$,

$$\begin{aligned} v_2^+ &= v_{\text{cm}} + \frac{m_1}{m_1 + m_2} v_{\text{rel}}^+, \\ v_1^+ &= v_{\text{cm}} - \frac{m_2}{m_1 + m_2} v_{\text{rel}}^+, \quad \text{and} \\ P &= (v_{\text{rel}}^+ - v_{\text{rel}}^-) m_{\text{coll}} \end{aligned}$$

That is, $P$ acts on $v_{\text{rel}}$ as if $v_{\text{rel}}$ were the velocity of an object with mass $m_{\text{coll}}$. If $P = 0$ the equations above are a long winded way of saying that nothing happened, $v_1^+ = v_1^-$ and $v_2^+ = v_2^-$, and the masses pass right through each other.

If $P = -v_{\text{rel}}^- m_{\text{coll}}$ there is a sticking collision.

## Elastic collisions

Application of the above formulas will show that if

$$P = -2 v_{\text{rel}}^- m_{\text{coll}}$$

then the kinetic energy of the system after the collision is the same as the kinetic energy before. That is

$$\begin{aligned} {E_{\text{K}}}^+ &= {E_{\text{K}}}^- \\ \frac{m_1 {v_1^+}^2 + m_2 {v_2^+}^2}{2} &= \frac{m_1 {v_1^-}^2 + m_2 {v_2^-}^2}{2}. \end{aligned}$$

Also, $v^+_{\rm rel} = -v^-_{\rm rel}$, the relative velocity maintains its magnitude and reverses its sign.

## The coefficient of restitution

* We have that as $P$ ranges from $-v^-_{\rm rel} m_{\rm coll}$ to $-2v^-_{\rm rel} m_{\rm coll}$, the collision ranges from sticking to an energy conserving reversal of relative velocities. The *coefficient of restitution* $e$ is introduced as a way of interpolating between these cases. The most commonly used collision law can be summarized with this simple equation,

The speed with which colliding points are separating after the collision.

The speed at which colliding points are approaching before collision.

$$\overbrace{(v'_b - v'_a)} = \underbrace{e}\ \overbrace{(v_a - v_b)}, \tag{9.64}$$

The coefficient of restitution, assumed to be a constant for given materials.

Or, more simply expressed, the collision law can be defined by either of the following two equations

$$\begin{aligned} v^+_{\rm rel} &= -e v^-_{\rm rel} \quad \text{or} \\ P &= -(1+e) v^-_{\rm rel} m_{\rm coll}. \end{aligned}$$

If $e = 0$ we have a sticking collision. If $e = 1$ we have an energy conserving elastic collision. If $e$ is between 0 and 1 the collision is somewhere between as dead and as alive as can be. which can be summarized as, *the rate of separation is proportional to the rate of approach.* The coefficient $e$ is called Newton's (see box 9.5) or Poisson's coefficient of restitution. Somewhat of a miracle is that a given pair of objects seems to have a coefficient of restitution that is roughly independent of the velocities. This is the result of a conspiracy by all kinds of deformation mechanisms that we don't really understand. But that $e$ is a constant for a given pair of bodies is only an approximation that has roughly the same status (accuracy) as, say, the friction coefficient. Much lower status than the momentum balance equations.

**What saith Newton about collisions?** On page 25 of Newton's Principia (Motte's translation revised, by Florian Cajori, Univ. of CA press, 1947) he discusses collisions of spheres as measured in pendulum experiments. He takes account of air friction. He has already discussed momentum conservation.

"In bodies imperfectly elastic the velocity of the return is to be diminished together with the elastic force; because that force (except when the parts of

* A common mistake is to take $e$ as a material property. It is not. $e$ generally depends on the shapes and sizes of the contacting objects also (see box 9.5 on 456).

bodies are bruised by their impact, or suffer some such extension as happens under the strokes of a hammer) is (as far as I can perceive) certain and determined, and makes bodies to return one from the other with a relative velocity, which is in a given ratio to that relative velocity with which they met. This I tried in balls of wool, made up tightly, and strongly compressed. For, first, by letting go the pendula's bodies, and measuring their reflection, I determined the quantity of their elastic force; and then, according to this force, estimated the reflections that ought to happen in other cases of impact. And with this computation other experiments made afterwards did accordingly agree; the balls always receding one from the other with a relative velocity, which was to the relative velocity to which they met, as about 5 to 9. Balls of steel returned with almost the same velocity; those of cork with a velocity something less; but in balls of glass the proportion was as about 15 to 16. ”

**SAMPLE 9.24 Collision without energy loss:** A block of mass $m_1 = 2\,\text{kg}$ moves with speed $v_1 = 0.5\,\text{m/s}$ along the $x$-axis on a frictionless level ground behind another block of mass $m_2 = 10\,\text{kg}$ moving at a speed $v_2 = 0.2\,\text{m/s}$ in the same direction. The first block collides with the second block. Given that there is no loss of energy in this collision, find the speeds of the two blocks immediately after the collision.

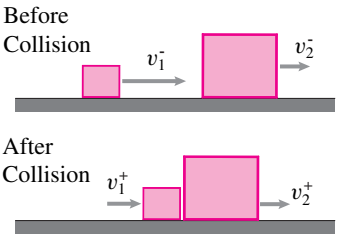



Figure 9.61:
Filename:sfig9-5-coll1

**Solution** We are given the speeds of two blocks (of known masses) just before the collision. It is also given that there is no loss of energy in the collision. We have to find the speed of the two masses immediately after collision.

We know that the linear momentum of the system consisting of the two blocks is conserved during the collision. Thus, if $v_1^-$ and $v_2^-$ are the speeds of the two masses just before the collision and $v_1^+$ and $v_2^+$ are their respective speeds immediately after the collision, then we have

$$m_1 v_1^+ + m_2 v_2^+ = m_1 v_1^- + m_2 v_2^- \tag{9.65}$$

Since there is no loss of energy in the collision, the energy of the system is conserved. Thus, $E^- = E^+$, or

$$\frac{1}{2}m_1(v_1^+)^2 + \frac{1}{2}m_2(v_2^+)^2 = \frac{1}{2}m_1(v_1^-)^2 + \frac{1}{2}m_2(v_2^-)^2. \tag{9.66}$$

Thus, we have two equations (eqn. (9.65) and eqn. (9.66)) in two unknowns, $v_1^+$ and $v_2^+$, and hence we can solve for them. It is now only a question in algebra. From eqn. (9.66), we have

$$\begin{aligned} m_1\left[(v_1^+)^2 - (v_1^-)^2\right] &= m_2\left[(v_2^-)^2 - (v_2^+)^2\right] \\ \Rightarrow \quad m_1(v_1^+ + v_1^-)(v_1^+ - v_1^-) &= m_2(v_2^- + v_2^+)(v_2^- - v_2^+) \end{aligned} \tag{9.67}$$

But, from eqn. (9.65), $m_1(v_1^+ - v_1^-) = m_2(v_2^- - v_2^+)$. Hence,eqn. (9.67) simplifies to

$$\begin{aligned} v_1^+ + v_1^- &= v_2^+ + v_2^- \\ \Rightarrow \quad v_1^+ - v_2^+ &= v_2^- - v_1^-. \end{aligned} \tag{9.68}$$

Multiplying the above equation by $m_1$ and subtracting from eqn. (9.65), we get

$$\begin{aligned} (m_1 + m_2)v_2^+ &= 2m_1 v_1^- + v_2^-(m_2 - m_1) \\ \Rightarrow \quad v_2^+ &= \frac{2m_1}{m_1 + m_2}v_1^- + \frac{m_2 - m_1}{m_1 + m_2}v_2^-. \end{aligned}$$

Now substituting the given values, $m_1 = 2\,\text{kg}$, $m_2 = 10\,\text{kg}$, $v_1^- = 0.5\,\text{m/s}$ and $v_2^- = 0.2\,\text{m/s}$ above, we get $v_2^+ = 0.3\,\text{m/s}$. Further, substituting the values of $v_2^+$ in eqn. (9.68), we get $v_1^+ = 0$, *i.e.*, the first mass comes to a halt!

$$\boxed{v_1^+ = 0 \text{ and } v_2^+ = 0.3\,\text{m/s}}$$

**Comments:** : Note that rather than using energy conservation equation directly as we did above, we could have used the given energy information to set $e = 1$ (perfectly elastic collision) in eqn. (**??**) to get $v_2^+ - v_1^+ = -v_2^- + v_1^-$ (rather than deriving it as we did above). We can then solve this equation along with eqn. (9.65) to solve for $v_1^+$ and $v_2^+$.

## 9.7 *THEORY*

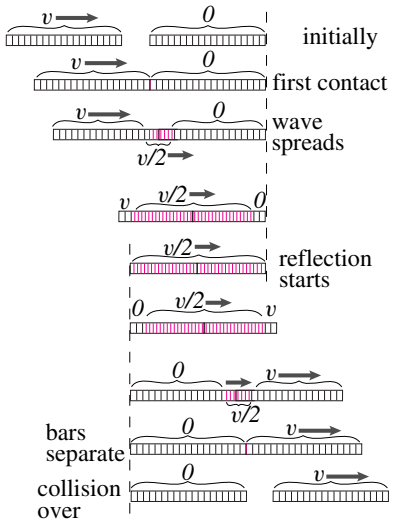
### *The axial collision of elastic rods: the unusual disappearance of vibrations*

This box is not related to the skills covered in this book. It is an aside for those wondering how things work.

One approach to understanding collisions is to look at the stresses and deformations *during* the collision. This leads to the solution of partial differential equations. The material behavior needed to define those equations is usually not that well understood. So, hard as it is to solve such equations, even on a computer, the solution can be far from reality.

But to get a sense of things one can study an ideal system. The simple system we look at here was somewhat controversial amongst the great 19th century scientists Cauchy, Poisson and Saint-Venant (so said E.J. Routh in 1905).

**Two identical linear elastic rods.** Imagine two identical uniform linear elastic rods with length $\ell$. The right one is stationary and the left one approaches it with speed $v$.




No matter how the rods shake and vibrate, their elastic potential energy plus kinetic energy is constant.

Using reasoning beyond this book (see the paragraph for experts at the end of this box) one can explain this collision in detail, as illustrated in the sketches above. The pictures exaggerate the compression in the bar (For most materials the compression wouldn't be visible).

First the left rod moves like a rigid body towards the still rod at the right. Then contact is made and a compressional sound wave starts off spreading to the left and right. Behind the wave fronts is compressed material moving at speed $v/2$ to the right. To the right of the right wave front the material is still. To the left of the left-moving wave front the material is still moves at $v$. When the wave fronts meet the ends of their respective bars, the bars are compressed and all material is going to the right at $v/2$. Then both wave-fronts reflect off the ends of the bars and head back towards the contact point. To the left of the right-moving wave front (on the left bar) the material is still and uncompressed. To the right of the left-moving wave front (on the right bar) the material is uncompressed but moving to the right at speed $v$ . Finally, the waves meet in the center and the bars separate. The right bar is now uniformly moving to the right at speed $v$ and the left bar is still.

The result of this collision is that all of the momentum of the left bar is transferred to the right bar. The separation velocity is equal in magnitude to the approach velocity. The coefficient of restitution $e$ is 1, and the kinetic energy of the system is the same after the collision as it was before.

Note that the collision itself was quick. The wave-fronts move at the speed of sound, about 1000 m/s for metals. So for 1 meter metal rods the collision takes a few thousandths of a second. But during that few thousandths of a second, the initial energy was partitioned into elastic strain energy and kinetic energy in different time-changing regions of the bar.

Despite all the complicated details, the elastic bars lead to the prediction of an 'elastic' collision. Maybe this is not surprising.

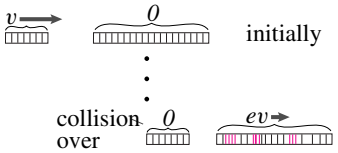
**An elastic rod hits a rigid wall** If you drop a 3 foot wooden dowel straight down on a thick concrete or stone floor it bounces quite well. Why? A wave analysis like that described above shows that a wave travelling from the first contact at the floor travels up the top and reflecting back to the bottom, leaving the rod moving uniformly up after the collisions just as fast as it was moving down before. Of course a wooden dowel is not perfectly described by the simple wave theory. And the ground is not perfectly rigid. So a real dowel's collision is not perfectly elastic.

But again we find that if we assume an elastic material that we predict an elastic collision. Again, no surprise. But

> the previous two examples are completely misleading!

Actually these are maybe *the only* examples where a detailed elastic theory predicts an elastic collision. More commonly the details are more like the next example.

**Rods of different length** If the rods have length $\ell_1$ and $\ell_2 > \ell_1$ then the collision works out differently.




When the reflection from the left end of the left rod comes back to the contact point, the rods separate. The left rod is stationary but the right rod has waves moving up and back. The average speed of the right rod is $(\ell_1/\ell_2)v$ so the effective coefficient of restitution is $e = \ell_1/\ell_2 < 1$. Later, after the vibrations have died out, the energy of the system will be less than initially. Or, even if the waves don't die out, the kinetic energy that can be accounted for in rigid-body mechanics is lost to remnant vibrations. Thus a totally elastic system leads to *inelastic* collisions. It is wrong to think that the restitution constant $e$ depends on material; it also depends on the shapes and sizes of the objects. The amount of vibrational energy left after contact ends depends on shape and size.

• • •

**For experts only: the wave equation** In one-dimensional linear elasticity the displacement $u$ to the right, of a point at location $x$ on one or the other rod follows this partial differential equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2}.$$

That is, the collision mechanics in detail is the finding of $u(x,t)$ that solves the wave equation above with the given initial conditions (one bar is moving the other isn't) and the boundary conditions (the ends of the bars have no stresses but when where they are in contact where they can have equal compressive stresses). The solution is most easily found by constructing right and left going waves that add to meet the initial conditions and boundary conditions (Routh).

**SAMPLE 9.25 Estimating peak force in a collision:** A metal ball of mass $m = 0.5\,\text{kg}$ strikes a stationary surface $S_1$ with velocity $\vec{v} = 10\,\text{m/s}\hat{\imath}$ and rebounds with velocity $\vec{v} = -9\,\text{m/s}\hat{\imath}$. The same ball strikes another stationary surface $S_2$ with the same velocity and has the same rebound velocity. The contact time during the two collisions is, however, found to be 0.1 sand 0.001 srespectively. Assuming that the collisional force between the ball and the two surfaces can be modeled as $F(t) = \frac{F_0}{2}(1 + \cos\frac{2\pi t}{T})$ (see Fig. 9.62) where $-T/2 \le t \le T/2$ and $T$ is the contact time, find the peak force $F_0$ in each case.

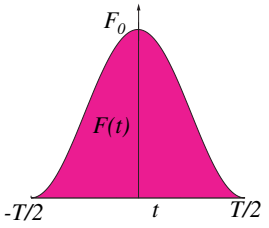



Figure 9.62:
Filename:sfig9-5-impulse

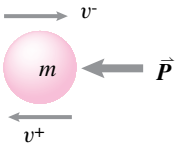



Figure 9.63: Free body diagram of the ball during collision with collisional impulse $\vec{P}$ acting on it.
Filename:sfig9-5-ball

**Solution** Let the collisional impulse acting on the ball be $\vec{P}$ (see Fig. 9.63) given by

$$\vec{P} = \int_{-T/2}^{T/2} \vec{F}(t)dt.$$

From impulse-momentum relationship, we have

$$\vec{P} = \Delta\vec{L} = m\Delta\vec{v}.$$

Since in the case of each surface, $\Delta\vec{v}$ is the same ($\vec{v}^+ - \vec{v}^- = -19\,\text{m/s}\hat{\imath}$), the change in linear momentum $\Delta L = m\Delta\vec{v}$ is also the same. Hence, the impulse acting on the ball in each case has to be the same. Now, let $\vec{P}_1$ and $\vec{P}_2$ be the impulses acting on the ball during the collision with surface $S_1$ and $S_2$ respectively. Then,

$$\begin{aligned}
\vec{P}_1 &= -\int_{T_1/2}^{T_1/2} \vec{F}_1(t)dt\,\hat{\imath} \\
&= -\int_{-T_1/2}^{T_1/2} \frac{(F_0)_1}{2}\left(1 + \cos\frac{2\pi t}{T_1}\right)dt\hat{\imath} \\
&= -\frac{(F_0)_1 T_1}{2}\,\hat{\imath}
\end{aligned}$$

Similarly,

$$\vec{P}_2 = -\frac{(F_0)_2 T_2}{2}\,\hat{\imath}.$$

Now, setting $\vec{P}_1 = \Delta L$, we get

$$\begin{aligned}
-\frac{(F_0)_1 T_1}{2}\hat{\imath} &= -m\Delta v\,\hat{\imath} \\
\Rightarrow \quad (F_0)_1 &= \frac{2m\Delta v}{T_1} \\
&= \frac{2\cdot 0.5\,\text{kg}19\,\text{m/s}}{0.1\,\text{s}} \\
&= 1.9\,\text{N}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
(F_0)_2 &= \frac{2m\Delta v}{T_2} \\
&= \frac{2\cdot 0.5\,\text{kg}19\,\text{m/s}}{0.001\,\text{s}} \\
&= 190\,\text{N}.
\end{aligned}$$

Clearly, the peak force is inversely proportional to the collision time. In fact, it is easy to see that for the given model of the impulsive force, the peak force $F_0 = \frac{2m\Delta v}{T}$. Thus if the change in momentum is constant, then the peak force varies as $1/T$.

$$\boxed{(F_0)_1 = 1.9\,\text{N} \quad \text{and} \quad (F_0)_2 = 190\,\text{N}}$$

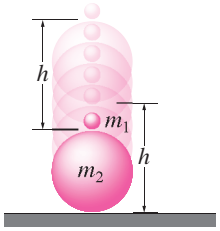
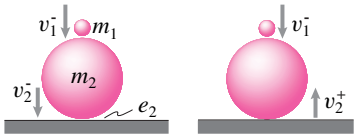



Figure 9.64:

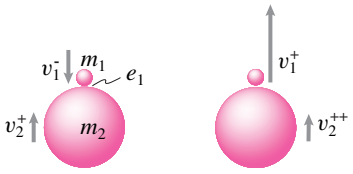
Filename:sfig9-5-twoballs




Before ground Collision | After ground Collision

Figure 9.65: Collision of the basketball with the ground.

Filename:sfig9-5-twoballs-a




Before ball-to-ball Collision | After ball-to-ball Collision

Figure 9.66: Collision of the tennis ball with the basketball.

Filename:sfig9-5-twoballs-b

**SAMPLE 9.26 A two-ball multiple collision experiment:** A tennis ball of approximate mass $m_1 = 60\,\text{gm}$ and a basketball of approximate mass $m_2 = 600\,\text{gm}$ are used in a fun collision experiment. The two balls are held in air, one on top of the other with a tiny gap between them, at a height $h$ from the ground as shown in the figure. The two balls are released simultaneously from rest. The coefficient of restitution between the tennis ball and the basketball is $e_1 = 0.6$ and that between the basketball and the floor is $e_2 = 0.9$. Assume that the collision between the two balls takes place immediately after the basketball rebounds from the floor. Find the height of the tennis ball flight in terms of $h$ as a result of the collision.

**Solution** We need to track two separate collisions here — one between the basketball and the floor, and second, between the tennis ball and the basketball. We can find the relevant vertical velocities before and after the collisions to determine the velocity of the tennis ball's flight which we can use to find the height of the flight. We will assume upward velocities to be positive.

**Collision-1:** Just before the basketball hits the floor, let its vertical velocity be $v_2^-$ and let the tennis ball's speed at the same instant be $v_1^-$. Since both balls undergo free fall from height $h$ before attaining these speeds, we have

$$v_1^- = v_2^- = -\sqrt{2gh}.$$

Now let $v_2^+$ be the speed of the basketball immediately after the collision with the ground (see Fig. 9.65). Then,

$$v_2^+ = -e_2 v_2^- = e_2\sqrt{2gh}.$$

**Collision-2:** We assume that the second collision, the collision between the tennis ball and the basketball, takes place immediately after the first collision. Hence, the velocity of the tennis ball just before the collision with the basketball can be assumed to be $v_1^- = \sqrt{2gh}$. The second collision is shown in Fig. 9.66. The after impact velocities of the two balls are $v_1^+$ and $v_2^{++}$. Now, from collision law, we have

$$v_1^+ - v_2^{++} = -e_1(v_1^- - v_2^+) = -e_1(-\sqrt{2gh} - e_2\sqrt{2gh}) = \sqrt{2gh}e_1(1+e_2). \qquad (9.69)$$

The conservation of linear momentum for the two-ball system gives

$$\begin{aligned} m_1 v_1^+ + m_2 v_2^{++} &= m_1 v_1^- + m_2 v_2^+) \\ \Rightarrow \quad v_1^+ + \frac{m_2}{m_1} v_2^{++} &= v_1^- + \frac{m_2}{m_1} v_2^+ \end{aligned}$$

Taking $M = m_2/m_1$, and substituting the values of $v_1^-$ and $v_2^+$, we get

$$v_1^+ + M v_2^{++} = \sqrt{2gh}(1 + Me_2). \qquad (9.70)$$

Now solving eqn. (9.69) and eqn. (9.70) simultaneously, we get

$$v_1^+ = \frac{\sqrt{2gh}}{1+M}\left[M(e_1 + e_2 + e_1e_2) - 1\right].$$

This is the velocity with which the tennis ball takes off on its vertical flight. Let the height of this flight be $h_f$. Then, from constant acceleration motion formula, we get $(v_1^+)^2 = 2gh_f$, or $h_f = (v_1^+)^2/2g$. Thus, from the derived expression for $v_1^+$ above, we get

$$h_f = \frac{h}{(1+M)^2}\left[M(e_1 + e_2 + e_1e_2) - 1\right]^2.$$

Substituting $M = m_2/m_1 = 10$, $e_1 = 0.6$, and $e_2 = 0.9$ above, we get $h_f = 3.11h$. Thus the tennis ball flies off to three times its original height.

$$\boxed{h_f = 3.11h}$$

**Note:** From the expression obtained for $v_1^+$, we see that if $M$ is very large then $v_1^+ = \sqrt{2gh}(e_1 + e_2 + e_1e_2)$ and $h_f = (e_1 + e_2 + e_1e_2)^2 h$.

# 9.6 Advanced vibrations: forcing and resonance

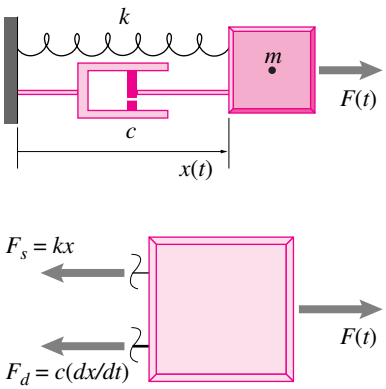



Figure 9.67: A forced mass-spring-dashpot is just a mass held in place by a spring and dashpot but pushed by a force $F(t)$ from some external source.

Filename:tfigure12-MSDforced

If the world of oscillators was as we have described them so far, there wouldn't be much to talk about. The undamped oscillators would be oscillating away and the damped oscillators (all the real ones) would be all damped out. The reason vibrations exist is because they are some how excited. This excitement is also called *forcing* whether or not it is due to a literal mechanical force.

The simplest example of a 'forced' harmonic oscillator is the mass-spring-dashpot system with an additional mechanical force applied to the mass. A picture of such a system is shown in figure 9.67. The governing equation for a forced harmonic oscillator is:

$$m\ddot{x} + c\dot{x} + kx = F(t). \tag{9.71}$$

When $F(t) = 0$ there is no forcing and the governing equation reduces to that of the un-forced damped harmonic oscillator, eqn. (9.25). There are two special forcings of common interest:

- Constant force, and
- Sinusoidal forcing.

Constant force idealizes situations where the force doesn't vary much as due say, to gravity, a steady wind, or sliding friction. Sinusoidally varying forces are used to approximate oscillating forces as caused, say, by vibrating machine parts or earthquakes. Sums of sine waves can accurately approximate any force that varies with time*.

## Forcing with a constant force

The case of constant forcing is both common and easy to analyze, so easy that it is often ignored. If $F = constant$, then the general solution of equation 9.71 for $x(t)$ is the same as the unforced case but with a constant added. The constant is $F/k$. The usual way of accommodating this case is to describe a new equilibrium point at $x = F/k$ and to pick a new deflection variable that is zero at that point. If we pick a new variable $w$ and define it as $w = x - F/k$, the amount of motion away from equilibrium, then, substituting into equation 9.71 the forced oscillator equation becomes

$$m\ddot{w} + c\dot{w} + kw = 0, \tag{9.72}$$

which is the unforced oscillator equation. The case of constant forcing reduces to the case of no forcing if one merely changes what one calls the equilibrium point to be the place where the mass is in equilibrium, taking account of the constant applied force.

$$x(t) - F/k = e^{(-\frac{c}{2m})t}\left(A\cos(\lambda t) + B\sin(\lambda t)\right)$$

* The best approximation of a function with a sum of sine waves is a Fourier series, a topic for other courses.

where $\lambda_d = \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$.

An alternative approach is to use *superposition*. Here we say $x(t) = x_h(t) + x_p(t)$ where $x_h(t)$ satisfies $m\ddot{x} + c\dot{x} + kx = 0$ and $x_p(t)$ is any solution of $m\ddot{x} + c\dot{x} + kx = F$. Such a solution is $x_p = F/k$ if $F(t)$ is constant. So the net solution is $F/k$ plus a solution to the 'homogeneous' equation 9.72.

$$x(t) = \underbrace{e^{(-\frac{c}{2m})t}\left(A\cos(\lambda t) + B\sin(\lambda t)\right)}_{x_h} + \underbrace{F/k}_{x_p}$$

## Forcing with a sinusoidally varying force

The motion resulting from sinusoidal forcing is of central interest in vibration analysis. In this case we imagine that $F(t) = F_0\cos(pt)$ where $F_0$ is the amplitude of forcing and $p$ is the angular frequency of the forcing.

The general solution of equation 9.71 is given by the sum of two parts. One is the general solution of equation 9.25, $x_h(t)$, and the other is *any* solution of equation 9.71, $x_p(t)$. The solution $x_h(t)$ of the damped oscillator equation 9.25 is called the 'homogeneous' or 'complementary' solution. Any solution $x_p(t)$ of the forced oscillator equation 9.71 is called a 'particular' solution.

We already know the solution $x_h(t)$ of the undamped governing differential equation 9.25. This solution is equation 9.26, 9.27, or 9.28, depending on the values of the mass, spring and damping constants. So the new problem is to find any solution to the forced equation 9.71. The easiest way to solve this (or any other) differential equation is to make a fortuitous guess (you may learn other methods in your math classes). In this case if $F(t) = F_0\cos(pt)$ we make the guess that

$$x_p(t) = A\cos(pt) + B\sin(pt). \tag{9.73}$$

If we plug this guess into the forced oscillator equation (9.71), we find, after much tedious algebra, that we do in fact have a solution if

$$A = \frac{\frac{F_0}{k}\left(1 - \frac{p^2}{(\frac{k}{m})}\right)}{\left(\frac{c^2}{km}\right)\left(\frac{p^2}{\frac{k}{m}}\right) + \left(1 - \frac{p^2}{(\frac{k}{m})}\right)^2},$$

$$\text{and} \qquad B = \frac{\frac{F_0}{k}\left(\frac{cp}{k}\right)}{\left(\frac{c^2}{km}\right)\left(\frac{p^2}{\frac{k}{m}}\right) + \left(1 - \frac{p^2}{(\frac{k}{m})}\right)^2} \quad .$$

So the response to the cosine-wave forcing is the sum of a sine wave and a

cosine wave.

$$x_p(t) = \overbrace{\left(\frac{\frac{F_0}{k}\left(1-\frac{p^2}{\left(\frac{k}{m}\right)}\right)}{\left(\frac{c^2}{km}\right)\left(\frac{p^2}{\frac{k}{m}}\right)+\left(1-\frac{p^2}{\left(\frac{k}{m}\right)}\right)^2}\right)}^{A}\cos(pt) + \underbrace{\left(\frac{\frac{F_0}{k}\left(\frac{cp}{k}\right)}{\left(\frac{c^2}{km}\right)\left(\frac{p^2}{\frac{k}{m}}\right)+\left(1-\frac{p^2}{\left(\frac{k}{m}\right)}\right)^2}\right)}_{B}\sin(pt)$$

Alternatively sum of sine waves can be written as a cosine wave that has been shifted in phase as

$$x_p(t) = C\cos(pt-\phi),$$

$$\text{where} \quad C = \sqrt{(A^2+B^2)} = \frac{\frac{F_0}{k}}{\sqrt{\left(\frac{c^2}{km}\right)\left(\frac{p^2}{\frac{k}{m}}\right)+\left(1-\frac{p^2}{\left(\frac{k}{m}\right)}\right)^2}}, \tag{9.74}$$

$$\text{and} \quad \phi = \tan^{-1}\left(\frac{B}{A}\right) = \tan^{-1}\left(\frac{\left(\frac{c^2}{km}\right)\left(\frac{p^2}{\frac{k}{m}}\right)}{\left(1-\frac{p^2}{\left(\frac{k}{m}\right)}\right)}\right). \tag{9.75}$$

The general solution, therefore, is

$$x(t) = x_h(t) + x_p(t). \tag{9.76}$$

## Uses of resonance

Though resonance is often a problem, it is also often of engineering use. Nuclear Magnetic Resonance imaging is used for medical diagnosis. The resonance of quartz crystals is used to time most watches now-a-days. In the old days, the resonant excitation of a clock pendulum was used to keep time. Self excited resonance is what makes musical instruments have such clear pitches.

## Frequency response

One way to characterize a structures sensitivity to oscillatory loads is by a *frequency response* curve. The frequency response curve might be found by a physical experiment or from a calculation based on a simplified model of the structure. The curve somewhat describes the answer to the following question about a structure:

*How does the size of the motion of a structure depend on the frequency and amplitude of an applied sinusoidal forcing?*

Here is how the method works. First, you must apply a sinusoidal force, say $F = F_0 \cos(pt)$, to the structure at a physical point of interest. Then you measure the motion of a part of the structure of interest. You might instead measure a strain or rotation, but for definiteness let's assume you measure the displacement of some point on the structure $\delta$.

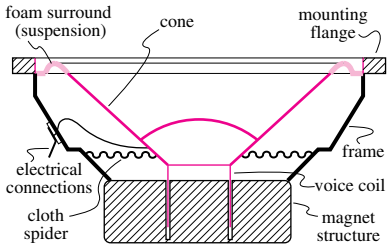
If the structure is linear and has some damping, the eventual motion of the structure will be a sinusoidal oscillation. In particular, you will measure that

$$\delta = C \cdot \cos(pt - \phi). \tag{9.78}$$

where $C$ and $\phi$ have been defined previously in equation 9.74. If you had applied half as big a force, you would have measured half the displacement, still assuming the structure is linear, so the ratio of the displacement to the force $C/F_0$ is independent of the size of the force $F_0$. Let's define:

$$R = \frac{C}{F_0}$$

### 9.8 A Loudspeaker cone is a forced oscillator.




**Cross-sectional view of a speaker.**

A speaker, similar to the ones used in many home and auto speaker systems, is one of many devices which may be conveniently modeled as a one-degree-of-freedom mass-spring-dashpot system. A typical speaker has a paper or plastic cone, supported at the edges by a roll of plastic foam (the surround), and guided at the center by a cloth bellows (the spider). It has a large magnet structure, and (not visible from outside) a coil of wire attached to the point of the cone, which can slide up and down inside the magnet. (The device described above is, strictly speaking, the speaker driver. A complete speaker system includes an enclosure, one or more drivers, and various electronic components.) When you turn on your stereo, it forces a current through the coil in time with the music, causing the coil to alternately attract and repel the magnet. This rapid oscillation of attraction and repulsion results in the vibration of the cone which you hear as sound.

In the speaker, the primary mass is comprised of the coil and cone, though the air near the cone also contributes as 'added mass.' The 'spring' and 'dashpot' effects in the system are due to the foam and cloth supporting the cone, and perhaps to various magnetic effects. Speaker system design is greatly complicated by the fact that the air surrounding the speaker must also be taken into account. Changing the shape of the speaker enclosure can change the effective values of all three mass-spring-dashpot parameters. (You may be able to observe this dependence by cupping your hands over a speaker (gently, without touching the moving parts), and observing amplitude or tone changes.) Nevertheless, knowledge of the basic characteristics of a speaker (e.g., resonance frequency), is invaluable in speaker system design.

Our approximate equation of motion for the speaker is identical to that of the ideal mass-spring-dashpot above, even though the forcing is from an electromagnetic force, rather a than a direct mechanical force:

$$m\ddot{x} + c\dot{x} + kx = F(t) \text{ with } F(t) = \alpha i(t) \tag{9.77}$$

where $i(t)$ is the electrical current flow through the coil in amps, and $\alpha$ is the electro-mechanical coupling coefficient, in force per unit current.

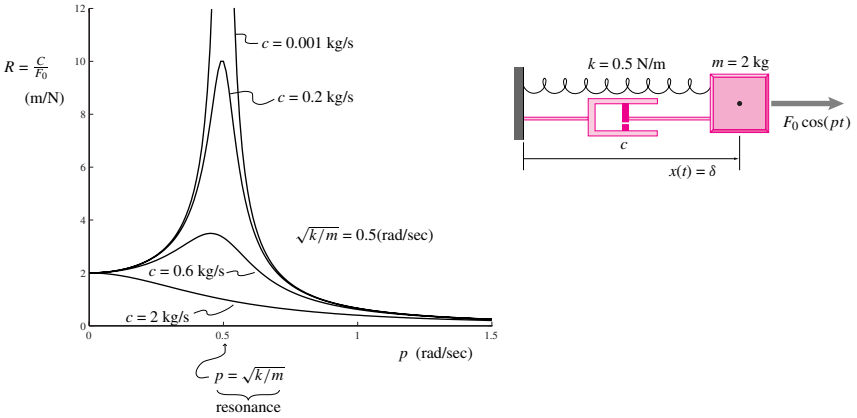



Figure 9.68:

Filename:tfigure12-ampl-vs-freq

That is, the response variable $R$ is the ratio of the amplitude of the displacement sine wave to the amplitude of the forcing sine wave.

Now, this experiment can be repeated for different values of the angular forcing frequency $p$. The ratio of the vibrating displacement $\delta$ to that of the applied forcing $F_0$ will depend on $p$. The structure has different sensitivities to forcing at different frequencies. So the response ratio amplitude $R$ depends on $p$. The function $R = R(p)$ is called the frequency response. A plot of the amplitude ratio $R$ versus the driving frequency $p$ is shown in figure 9.68 for various values of the damping coefficient $c$. Numerical values are shown for definiteness although the plot could be shown as dimensionless.

## Experimental measurement

To measure the frequency response function experimentally, one can apply forcing at a whole range of forcing frequencies. Another approach is to apply a sudden, 'impulsive', force and look at the response. This second method is equivalent, it turns out, as you may learn in the context of Laplace transforms or Fourier analysis.

Why does on want to know the frequency response? The answer is because it is one way to think about structural response. A car suspension may never be tested on a sinusoidal road. But knowing how the suspension would respond to sine wave shaped roads of all possible wave lengths somehow characterizes the car's response to roads with any kind of bumpiness.

Example: **Resonance of a building**
A mildly damped structure has a natural frequency of 17 hz and is forced at 17 hz. Because the frequency response function has a peak at 17 hz, resonance, the structures motions will be very large.

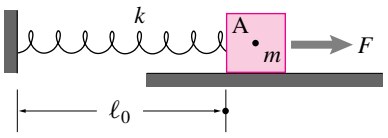



Figure 9.69:
Filename:sfig5-5-forcedosc

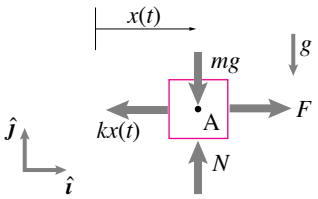



Figure 9.70: Free-body diagram of the mass.
Filename:sfig5-5-forcedosc-a

**SAMPLE 9.27 Response to a constant force:** A constant force $F = 50\,\text{N}$ acts on a mass-spring system as shown in the figure. Let $m = 5\,\text{kg}$ and $k = 10\,\text{kN/}m$.

- Write the equation of motion of the system.
- If the system starts from the initial displacement $x_0 = 0.01\,\text{m}$ with zero velocity, find the displacement of the mass as a function of time.
- Plot the response (displacement) of the system against time and describe how it is different from the unforced response of the system.

**Solution**

1. The free-body diagram of the mass is shown in Fig. 9.70 at a displacement $x$ (assumed positive to the right). Applying linear momentum balance in the $x$-direction, *i.e.*, $(\sum \vec{F} = m\vec{a}) \cdot \hat{\imath}$, we get

$$\begin{aligned} F - kx &= m\ddot{x} \\ \Rightarrow \quad m\ddot{x} + kx &= F \end{aligned} \tag{9.79}$$

   which is the equation of motion of the system.

2. The equation of motion has a non-zero right hand side. Thus, it is a nonhomogeneous differential equation. A general solution of this equation is made up of two parts — the homogeneous solution $x_h$ which is the solution of the unforced system (eqn. (9.79) with $F = 0$), and a particular solution $x_p$ that satisfies the nonhomogeneous equation. Thus,

$$x(t) = x_h(t) + x_p(t). \tag{9.80}$$

   Now, let us find $x_h(t)$ and $x_p(t)$.

   **Homogeneous solution:** $x_h(t)$ has to satisfy $m\ddot{x} + kx = 0$. Let $\lambda = \sqrt{k/m}$. Then, from the solution of unforced harmonic oscillator, we know that

$$x_h(t) = A\sin(\lambda t) + B\cos(\lambda t)$$

   where $A$ and $B$ are constants to be determined later from initial conditions.

   **Particular solution:** $x_p$ must satisfy eqn. (9.79). Since the nonhomogeneous part of the equation is a constant ($F$), we guess that $x_p$ must be a constant too (of the same form as $F$). Let $x_p = C$. Now we substitute $x_p = C, \quad \Rightarrow \quad \ddot{x}_p = \ddot{(C)} = 0$ in eqn. (9.79) to determine C.

$$kC = F \quad \Rightarrow \quad C = F/k \quad \text{or} \quad x_p = F/k.$$

   Substituting $x_h$ and $x_p$ in eqn. (9.80), we get

$$x(t) = A\sin(\lambda t) + B\cos(\lambda t) + F/k. \tag{9.81}$$

   Now we use the given initial conditions to determine $A$ and $B$.

$$\begin{aligned} x(t=0) &= B + F/k = x_0 \text{ (given)} \quad \Rightarrow \quad B = x_0 - F/k \\ \dot{x}(t) &= A\lambda\cos(\lambda t) - B\lambda\sin(\lambda t) \\ \Rightarrow \quad \dot{x}(t=0) &= A = 0 \text{ (given)} \quad \Rightarrow \quad A = 0. \end{aligned}$$

Thus,

$$x(t) = (x_0 - F/k)\cos(\lambda t) + F/k.$$

3. Let us plug the given numerical values, $k = 10\,\text{kN/}m, m = 5\,\text{kg}, \quad \Rightarrow \quad \lambda = \sqrt{k/m} = 44.72\,\text{rad/s}$, $F = 50\,\text{N}$ and $x0 = 0.01\,\text{m}$ in eqn. (9.82). The displacement is now given as

$$x(t) = -(.04\,\text{m})\cos(44.72 \cdot t) + .05\,\text{m}.$$

This response is plotted in Fig. 9.71 against time. Note that the oscillations of the mass are about a non-zero mean value, $x_{eq} = 0.04\,\text{m}$. A little thought should reveal that this is what we should expect. When a mass hangs from a spring under gravity, the spring elongates a little, by $mg/k$ to be precise, to balance the mass. Thus, the new static equilibrium position is not at the relaxed length $\ell_0$ of the spring but at $\ell_0 + mg/k$. Any oscillations of the mass will be about this new equilibrium.

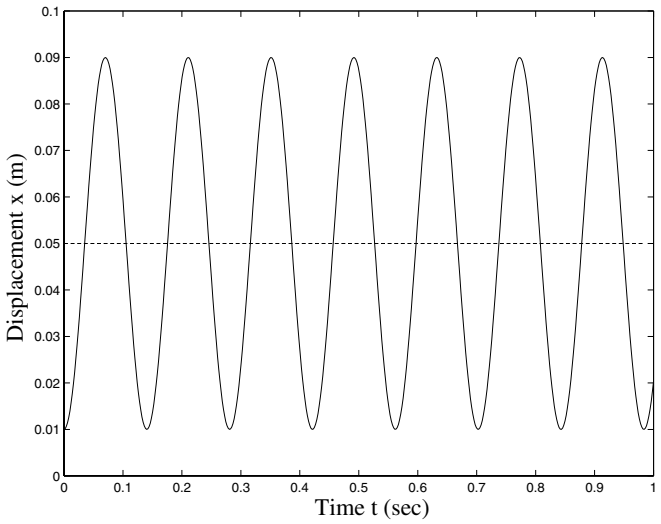



Figure 9.71: Displacement of the mass as a function of time. Note that the mass oscillates about a nonzero value of $x$.

Filename:sfig5-5-forcedosc-b

This problem is exactly like a mass hanging from a spring under gravity, a constant force, but just rotated by 90°. The new static equilibrium is at $x_eq = F/k$ and any oscillations of the mass have to be around this new equilibrium.

We can rewrite the response of the system by measuring the displacement of the mass from the new equilibrium. Let $\tilde{x} = x - F/k$. Then, eqn. (9.82) becomes

$$\tilde{x} = \tilde{x}_0 \cos(\lambda t)$$

where $\tilde{x}_0 = x_0 - F/k$ is the initial displacement. Clearly, this is the response of an unforced harmonic oscillator. Thus the effect of a constant force on a spring-mass system is just a shift in its static equilibrium position.

□

**SAMPLE 9.28 Particular solution:** Find a particular solution of the forced oscillator equation $\ddot{x} + \lambda^2 x = F(t)$ where

1. $F(t) = mg$ (a constant),
2. $F(t) = At$,
3. $F(t) = C\sin(pt)$.

**Solution** The given differential equation is a second order linear ordinary differential equation with a non-zero right hand side. A particular solution of this equation must satisfy the entire equation. For such equations, we guess a particular solution to have the *same functional form* as the right hand side (the forcing function) and plug it into the equation to see if our guess works. We can usually determine the values of any unknown, assumed constants so that the assumed solution satisfies the equation. Let us see how it works here.

1. The forcing function is a constant, $mg$. So, let us assume the particular solution to be a constant, *i.e.*, let $x_p = C$. Plugging it into the equation, we have

$$\underbrace{\ddot{C}}_{0} + \lambda^2 C = mg \quad \Rightarrow \quad C = mg/\lambda^2 \quad \Rightarrow \quad x_p = mg/\lambda^2$$

$$\boxed{x_p = mg/\lambda^2}$$

2. The forcing function is linear in $t$. So, let us assume a linear function as a particular solution, $x_p = \alpha t$ where $\alpha$ is a constant. Now, noting that $\dot{x}_p = \alpha \quad \Rightarrow \quad \ddot{x_p} = 0$, and plugging back into the differential equation, we get

$$\lambda^2 \alpha t = At \quad \Rightarrow \quad \alpha = A/\lambda^2 \quad \Rightarrow \quad x_p(t) = (A/\lambda^2)t.$$

$$\boxed{x_p(t) = (A/\lambda^2)t}$$

3. The forcing function is a harmonic function. So, let $x_p = \beta\sin(pt)$ where $\beta$ is a constant to be determined later. Now, plugging $x_p$ into the differential equation and noting that $\ddot{x}_p = -\beta\omega^2\sin(pt)$, we get

$$(-\beta p^2 + \beta\lambda^2)\sin(pt) = C\sin(pt) \quad \Rightarrow \quad \beta = \frac{C}{\lambda^2 - p^2}.$$

Thus the particular solution in this case is

$$x_p(t) = \frac{C}{\lambda^2 - p^2}\sin(pt).$$

$$\boxed{x_p(t) = \frac{C}{\lambda^2 - p^2}\sin(pt)}$$

**SAMPLE 9.29 Damping and forced response:** When a single-degree-of-freedom damped oscillator (mass-spring-dashpot system) is subjected to a periodic forcing $F(t) = F_0 \sin(pt)$, then the response of the system is given by

$$x(t) = C\cos(pt - \phi)$$

where $C = \frac{F_0/k}{\sqrt{(2\zeta r)^2+(1-r^2)^2}}$, $\phi = \tan^{-1}\frac{2\zeta r}{1-r^2}$, $r = \frac{p}{\lambda}$, $\lambda = \sqrt{k/m}$ and $\zeta$ is the damping ratio.

1. For $r \ll 1$, *i.e.*, the forcing frequency $p$ much smaller than the natural frequency $\lambda$, how does the damping ratio $\zeta$ affect the response amplitude $C$ and the phase $\phi$?
2. For $r \gg 1$, *i.e.*, the forcing frequency $p$ much larger than the natural frequency $\lambda$, how does the damping ratio $\zeta$ affect the response amplitude $C$ and the phase $\phi$?

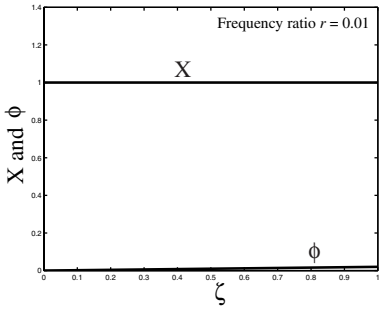



Figure 9.72:
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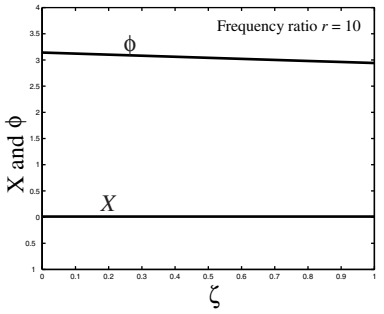



Figure 9.73:
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**Solution**

1. If the frequency ratio $r \ll 1$, then $r^2$ will be even smaller; so we can ignore $r^2$ terms with respect to 1 in the expressions for $C$ and $\phi$. Thus, for $r \ll 1$,

$$\begin{aligned} C &= \frac{F_0/k}{\sqrt{(2\zeta r)^2 + (1-r^2)^2}} \approx \frac{F_0/k}{1} = \frac{F_0}{k} \\ \phi &= \tan^{-1}(2\zeta r) \approx \tan^{-1} 0 = 0 \end{aligned}$$

that is, the response amplitude does not vary with the damping ratio $\zeta$, and the phase also remains constant at zero. As an example, we use the full expressions for $C$ and $\phi$ for plotting them against $\zeta$ for $r = 0.01$ in Fig. 9.72

$$\boxed{\text{For } r \ll 1,\ C \approx F_0/k,\ \text{and}\ \ \phi \approx 0}$$

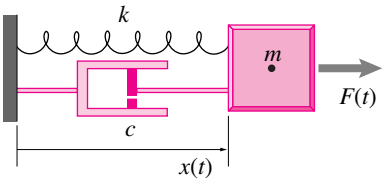
2. If $r \gg 1$, then the denominator in the expression for $C$, $4\zeta^2 r^2 + (1-r^2)^2 \approx r^4$ (because we can ignore all other terms with respect to $r^4$. Similarly, we can ignore 1 with respect to $r^2$ in the expression for $\phi$. Thus, for $r \gg 1$,

$$\begin{aligned} C &= \frac{F_0/k}{\sqrt{(2\zeta r)^2 + (1-r^2)^2}} C \approx \frac{F_0/k}{r^2} = 0 \\ \phi &= \tan^{-1}\frac{2\zeta r}{-r^2} \approx \tan^{-1}\frac{2\zeta}{-r} \approx \tan^{-1}(-0) = \pi. \end{aligned}$$

Once again, we see that the response amplitude and phase do not vary with $\zeta$. This is also evident from Fig. 9.73 where we plot $C$ and $\phi$ using their full expressions for $r = 10$. The slight variation in $\phi$ around $\pi$ goes away as we take higher values of $r$.

$$\boxed{\text{For } r \gg 1,\ C \approx 0,\ \text{and}\ \ \phi \approx \pi}$$

Thus, we see that the damping in a system does not affect the response of the system much if the forcing frequency is far away from the natural frequency.



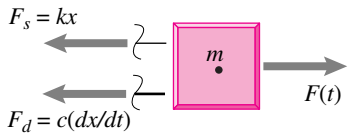




Figure 9.74:

Filename:sfig5-5-reswork

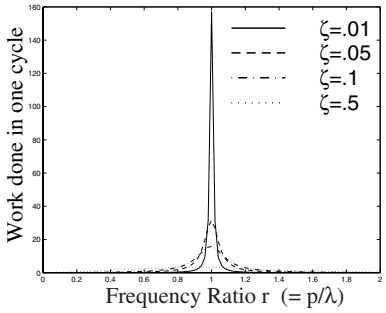



Figure 9.75:

Filename:sfig5-5-reswork-plot

**SAMPLE 9.30 Energetics of resonance:** Consider the response of a damped harmonic oscillator to a periodic forcing. Find the work done on the system by the periodic force during a single cycle of the force and show how this work varies with the forcing frequency and the damping ratio.

**Solution** Let us consider the damped harmonic oscillator shown in Fig. 9.74 with $F(t) = F_0 \sin(pt)$. The equation of motion of the system is $m\ddot{x} + c\dot{x} + kx = F_0 \sin(pt)$ and the response of the system may be expressed as $X \sin(pt - \phi)$ where $X = (F_0/k)/\sqrt{(2\zeta r)^2 + (1 - r^2)^2}$ and $\phi = \tan^{-1}(2\zeta r/(1 - r^2))$, with $r = p/\lambda$, $\lambda = \sqrt{k/m}$ and $\zeta = c/\sqrt{2km}$.

We can compute the work done by the applied force on the system in one cycle by evaluating the integral

$$W = \int_{\text{onecycle}} F\, dx$$

But, $x = X \sin(pt - \phi) \quad \Rightarrow \quad dx = Xp \cos(pt - \phi) dt$. Therefore,

$$\begin{aligned}
W &= \int_0^{2\pi/\lambda} F_0 \sin(pt) \cdot Xp \cos(pt - \phi)\, dt \\
&= F_0 Xp \int_0^{2\pi/\lambda} \sin(pt) \cos(pt - \phi)\, dt \\
&= F_0 Xp \int_0^{2\pi/\lambda} \sin(pt)(\cos(pt)\cos\phi + \sin(pt)\sin\phi)\, dt \\
&= F_0 Xp \left[ \cos\phi \cdot \frac{1}{2} \int_0^{2\pi/\lambda} \sin(2pt)\, dt + \sin\phi \cdot \frac{1}{2} \int_0^{2\pi/\lambda} (1 - \cos(2pt))\, dt \right] \\
&= \frac{F_0 Xp}{2} \left[ \cos\phi \left( -\frac{\cos(2pt)}{2p} \right) \Big|_0^{2\pi/\lambda} + \sin\phi \left( t - \frac{\sin(2pt)}{2p} \right) \Big|_0^{2\pi/\lambda} \right] \\
&= \frac{F_0 Xp}{2} \left[ \frac{\cos\phi}{2p}(-1 + 1) + \frac{2\pi}{p} \sin\phi + 0 \right] \\
&= \frac{F_0 Xp}{2} \cdot \frac{2\pi}{p} \sin\phi \\
&= F_0 \pi X \sin\phi
\end{aligned}$$

Although the expression obtained above for $W$ looks simple, we must substitute for $X$ and $\phi$ to see the dependence of $W$ on the damping ratio $\zeta$ and the frequency ratio $r$.

$$W = \frac{F_0^2 \pi}{\pi \sqrt{(2\zeta r)^2 + (1 - r^2)^2}} \cdot \sin\left( \tan^{-1} \frac{2\zeta r}{1 - r^2} \right) \tag{9.82}$$

Unfortunately, this expression is too complicated to see the dependence of $W$ on $\zeta$ and $r$. However, we know that for small $r(< 1)$, $\phi \approx 0$ and for large $r(> 1)$, $\phi = \approx \pi$, implying that $W$ is almost zero in both these cases. On the other hand, for $r$ close to one, that is, close to resonance, $\phi \approx \pi/2 \quad \Rightarrow \quad \sin\phi \approx 1$, but the response amplitude $X$ is large (for small $\zeta$), which makes $W$ to be big near the resonance. Figure 9.75 shows a plot of $W$ against $r$, using eqn. (9.82), for different values of $\zeta$. It is clear from the plot that the work done on the system in a single cycle is much larger close to the resonance for lightly damped systems. This explains why the response amplitude keeps on growing near resonance.

$$\boxed{W = F_0 \pi X \sin\phi}$$

# Problems for Chapter 9

Unconstrained 1D dynamics

## 9.1 Force and motion in 1D

### Preparatory Problems

**9.1** Give three examples of unconstrained 1D motion of real life objects where you can use the particle idealization for dynamic calculations.

**9.2** A car is going downhill on a constant slope straight road. You consider the car as a particle for finding out its speed at the end of the road. For specifying initial velocity, which point on the car would you consider?

**9.3** The acceleration of a particle is given as a function of time, $a(t)$. Is this information sufficient to find the speed of the particle at the end of, say, $T$ seconds?

**9.4** If a particle has constant acceleration, its linear momentum (a) remains constant, (b) changes linearly with time, or (c) changes quadratically with time. Which one is true?

**9.5** In a motorcycle race on a straight track, the speed of a motorcyclist at the 200 mmark is recorded. Given that the rider started from rest position, you can find the acceleration of the motorcyle from the given information, provided the acceleration (a) is constant, (b) varies linearly with time, or (c) is a sinusoidal function of time.

**9.6** The force acting on a particle is given as a function of time. If you plot the force function and find the area under the graph, you can determine (a) the net displacement of the particle, (b) average velocity of the particle, or (c) the change in linear momentum of the particle.

**9.7** If the linear momentum of a body remains constant in time, it must have (a) a constant force acting on it, (b) no net force acting on it, or (c) a sinusoidal force acting on it.

**9.8** The distance between two points in a bicycle race is 10 km. How many minutes does a bicyclist take to cover this distance if he/she maintains a constant speed of 15 mph.

**9.9** A 5 kNconstant force acts on an object of mass 1 kgfor 5 seconds. If the object was initially at rest, find the final speed of the object.

**9.10** Given that $\dot{x} = k_1 + k_2 t$, $k_1 = 1\,\text{ft/s}$, $k_2 = 1\,\text{ft/s}^2$, and $x(0) = 1\,\text{ft}$, what is the displacement at the end of 10 seconds?

### More-Involved Problems

**9.11** Find $x(3\,\text{s})$ given that

$$\dot{x} = x/(1\,\text{s}) \quad \text{and} \quad x(0) = 1\,\text{m}$$

or, expressed slightly differently,

$$\dot{x} = cx \quad \text{and} \quad x(0) = x_0,$$

where $c = 1\,\text{s}^{-1}$ and $x_0 = 1\,\text{m}$. Make a sketch of $x$ versus $t$.

**9.12** A ball of mass $m$ is dropped from rest at a height $h$ above the ground. Find the position and velocity as a function of time. Neglect air friction. When does the ball hit the ground? What is the velocity of the ball just before it hits?

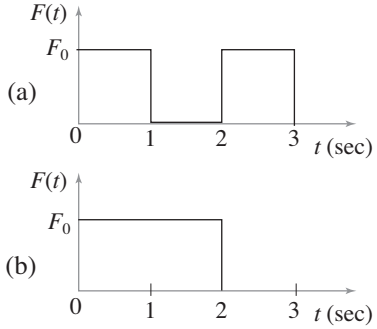
**9.13** The speed of a particle varies sinusoidally as $v = A\sin[(3\,\text{rad/s})t]$, where $A = 0.5\,\text{m/s}$. Let the initial position of the particle be $x(0) = 0$. Find the position of the particle at $t = \pi/2\,\text{s}$.

**9.14** The speed of a particle is directly proportional to its position and is given as $\dot{x} = x/\,\text{s}$. If the initial position, $x(0\,\text{s}) = 1\,\text{m}$, how far would the particle be from the origin in 5 seconds?

**9.15** Consider a force $F(t)$ acting on a cart for a short duration. In case (a), the force acts in two impulses of one second duration each as shown in Fig. 9.15. In case (b), the force acts continuously for two seconds. Given that the mass of the cart is 10 kg, $v(0) = 0$, and $F_0 = 10\,\text{N}$, for each force profile,

a) Find the speed of the cart at the end of 3 seconds, and
b) Find the distance travelled by the cart in 3 seconds.

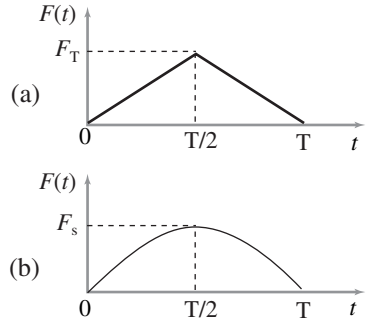
Comment on your answers for the two cases.




**problem 9.15:**

Filename:pfigure9-1-fcompare

**9.16** A car of mass $m$ is accelerated by applying a triangular force profile shown in Fig. 9.16(a). Find the speed of the car at $t = T$ seconds. If the same speed is to be achieved at t=T seconds with a sinusoidal force profile, $F(t) = F_s \sin\frac{\pi t}{T}$, find the required force magnitude $F_s$.




**problem 9.16:**

Filename:pfigure9-1-fcompare2

**9.17** A particle of mass $m = 1\,\text{kg}$ is acted upon by a short duration force given by

$$F(t) = \begin{cases} F_0 t & 0 \le t \le 1\,\text{s} \\ F_0(2-t) & 1\,\text{s} < t \le 2\,\text{s} \end{cases}$$

where $F_0 = 5\,\text{N}$. If the particle starts from rest, find the speed of the particle as a function time. Sketch the given force profile

as a function of time and draw the corresponding speed $v(t)$ as a function of time. What is the speed of the particle at $t = 2$ s?

**9.18** A ball of mass $m$ is dropped vertically from rest at a height $h$ above the ground. Air resistance causes a drag force on the ball directly proportional to the speed $v$ of the ball, $F_d = bv$. Find the velocity and position of the ball as a function of time. Find the velocity as a function of position. Gravity is non-negligible, of course.

**9.19** In quadratic drag problems, the deceleration is proportional to the square of velocity, *i.e.*, $a = \frac{dv}{dt} = -kv^2$. Assume that a particle with initial velocity $v(0) = v_0$. experiences quadratic drag.

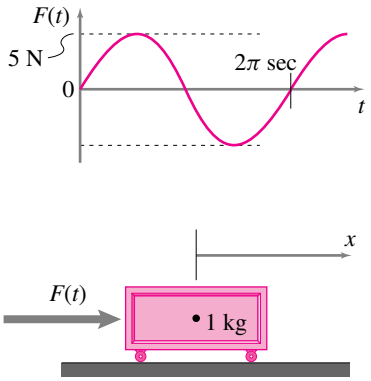
a) How long does it take for the particle to reduce its speed to half of its initial speed (*i.e.*, find $t$ such that $v(t) = \frac{1}{2}v_0$ )?
b) Find the position of the particle as a function of velocity. How far does the particle move from its initial position when its velocity drops to half its initial value?

**9.20** A sinusoidal force acts on a 1 kg mass as shown in the figure and graph below. The mass is initially still; i. e.,

$$x(0) = v(0) = 0$$

.

a) What is the velocity of the mass after $2\pi$ seconds?
b) What is the position of the mass after $2\pi$ seconds?
c) Plot position $x$ versus time $t$ for the motion.




**problem 9.20:**

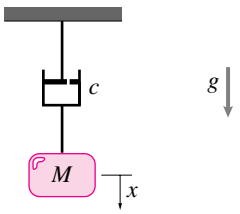
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**9.21** A motorcycle accelerates from 0 mph to 60 mph in 5 seconds. Find the average acceleration in m/s$^2$. How does this acceleration compare with $g$, the acceleration of an object falling near the earth's surface?

**9.22** A car moves on a straight road with an initial velocity $v_0 = 30$ m/s. Let its position $x = 0$ at $t = 0$. For the first 5 s it has no acceleration, and thereafter it brakes with a retarding force that gives it a constant acceleration $a_x = -10m/s^2$. Calculate the velocity and the $x$-coordinate of the car when $t = 8$ s and when $t = 12$ s, and find the maximum distance travelled by the car.

**9.23** A grain of sugar falling through honey has a negative acceleration proportional to the difference between its velocity and its 'terminal' velocity (which is a known constant $v_t$). Write this sentence as a differential equation, defining any constants you need. Solve the equation assuming some given initial velocity $v_0$.

**9.24** The mass-dashpot system shown below is released from rest at $x = 0$. Determine an equation of motion for the particle of mass $m$ that involves only $\dot{x}$ and $x$ (a first-order ordinary differential equation). The damping coefficient of the dashpot is $c$.




**problem 9.24:**

Filename:pfigure-blue-151-2

**9.25** Due to gravity, a particle falls in air with a drag force proportional to the speed squared.

1. Write $\sum \vec{F} = m\vec{a}$ in terms of variables you clearly define,
2. find a constant speed motion that satisfies your differential equation,
3. pick numerical values for your constants and for the initial height. Assume the initial speed is zero
   a) set up the equation for numerical solution,
   b) solve the equation on the computer,
   c) make a plot with your computer solution and show how that plot supports your answer to (2).

**9.26** A force pulls a particle of mass $m$ towards the origin according to the law (assume same equation works for $x > 0, x < 0$)

$$F = Ax + Bx^2 + C\dot{x}$$

Assume $\dot{x}(0) = 0$.
Using numerical solution, find values of $A, B, C, m$, and $x_0$ so that

1. the mass never crosses the origin,
2. the mass crosses the origin once,
3. the mass crosses the origin many times.

[Hint: Vary one parameter at a time and choose a different set of parameter values for each case.]

## 9.2 Energy methods in 1D

### Preparatory Problems

**9.27** A mass $m$ is moving at position $x$ moving at velocity $v$ and being acted upon by force $F$. For each of the quantities below:

i give the symbol used for the quantity
ii describe the quantity in words
iii give a formula to evaluate the quantity in terms of some or all of $m, x, v$ and $F$ and any other variables you may need.
iv Give the standard units for the quantity in the SI system.
v Give the standard units for the quantity in the English system.

a) Power
b) Kinetic energy
c) Work
d) Potential energy

**9.28** Write an equation relating the two words in each of these pairs. If any conditions or descriptions of the situation are needed, give them. If you know more than one equation (or form for a given equation), give all that you know. All should be given in the context of this section: 1D motion.

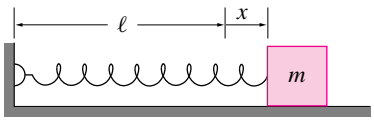
a) work and power
b) work and kinetic energy
c) power and kinetic energy
d) work and potential energy
e) potential energy and kinetic energy

**9.29** A force $F = F_0 \sin(ct)$ acts on a particle with mass $m$ which has position $x = 3\,\text{m}$, velocity $v = 5\,\text{m/s}$ at $t = 2\,\text{s}$. $F_0 = 4\,\text{N}$ and $c = 2/\,\text{s}$. At $t = 2\,\text{s}$ evaluate (give numbers and units):
a) $a$,
b) $E_\text{K}$,
c) $P$,
d) $\dot{E}_\text{K}$,
e) the rate at which the force is doing work.

**9.30** A force only depends on position according to $F = C_0 + C_1 x$ where $C_0$ and $C_1$ are constants. What is the work done by this force when the point to which it is applied moves from $x_1$ to $x_2$. Answer in terms of some or all of $C_0, C_1, x_1$ and $x_2$.

**9.31** Find the potential $E_\text{P}$ associated with each of these force fields.
a) $F = 0$.
b) $F = F_0$ (=constant).
c) $F = kx$.
d) $F = A\sin(x/x_0)$.
e) $F = c/x^2$.

**9.32** Consider a spring-mass system with $m = 2\,\text{kg}$ and $k = 5\,\text{N/m}$. The mass is pulled to the right a distance $x = x_0 = 0.5\,\text{m}$ from the unstretched position and released from rest. No external forces act on the mass.
a) What are the initial potential and kinetic energy of the system?
b) What is the potential and kinetic energy of the system as the mass passes through the static equilibrium (unstretched spring) position?
c) What is the speed of the mass when it passes through the static equilibrium position?




**problem 9.32:**

Filename:ch2-10-a

**9.33** A mass $m$ is held in place by a spring whose restoring force is $T(x) = kx$. Derive the equation of motion of the system (that is, find the acceleration $a$ in terms of $x$).

**9.34** The peak propulsion force on a 4-wheel-drive car is about $\mu mg$ where $\mu \approx 1$ for rubber on road (a bit more for fancy racing tires). Assume a car starts from rest at position zero. Answer the following questions with symbols and with numbers (using $\mu = 1$, $m = 1000\,\text{kg}$, and $g = 10\,\text{m/s}^2$).
a) What is the minimum distance required to reach $v_1 = 60\,\text{mph}$?
b) What is the extra distance required to get from $v_1 = 60\,\text{mph}$ up to $v_2 = 70\,\text{mph}$?
c) What is the peak power used by the engine in getting up to $v_1 = 60\,\text{mph}$?

**9.35** A car (mass $m = 1000\,\text{kg}$) traveling at speed $v_0 30\,\text{m/s}$ crashes into a brick wall and comes to a stop as the front end of the car compresses a distance $d = 1\,\text{m}$. Answer with symbols and numbers. Assume constant deceleration during the crash.
a) What is the total energy dissipated in the crash?
b) What is the force of the car on the wall?
c) What is the force of the wall on the car?
d) What is the deceleration of the car passengers (assuming they are strapped in and move with the bulk of the car). Answer in $g$'s?
e) Assuming an $m_p = 50\,\text{kg}$ person, what is the force of the seat belts on the person (answer in body weight).
f) If a parent was holding a 15 kg child on his lap, what force would he need to hold on to the child through the crash (answer in N and in number of child body weights).

## More-Involved Problems

**9.36** A 10 year old ($m = 90\,\text{lbf}$) jumps off an $h = 10\,\text{ft}$ wall and accelerates down with $g = 32\,\text{ft/s}^2$. She bends her legs a distance $d = 1\,\text{ft}$ to brake her fall and bring her body to a stop. Neglect the mass of her legs. Assume constant deceleration as she brakes the fall.
a) What is the total distance her body falls?
b) What is the potential energy lost?
c) How much work must be absorbed by her legs?
d) What is the force of her legs on her body (answer in symbols, numbers and numbers of body weight)?

**9.37** In traditional archery, when pulling an arrow back the force increases approximately linearly up to the peak 'draw force' $F_{draw}$ that varies from about $F_{draw} = 25\,\text{lbf}$ for a bow made for a small person to about $F_{draw} = 75\,\text{lbf}$ for a bow made for a big strong person. The distance the arrow is pulled back, the draw length $\ell_{draw}$, varies from about $\ell_{draw} = 2\,\text{ft}$ for a small adult to about 30 inch for a big adult. An arrow has mass of about 300 grain (1 grain $\approx 54.8\,\text{gm}$, so an arrow has mass of about $19.44 \approx 20\,\text{gm} \approx 3/4$ ounce). Give all answers in symbols and numbers.
a) What is the range of speeds you can expect an arrow to fly?
b) What is the range of heights an arrow might go if shot straight up?

**9.38** A big person ($m = 100\,\text{kg}$) jumps on a trampoline which we model as a linear spring with stiffness $k$. You know that the trampoline deflects $d_0 = 20\,\text{cm}$ under the stationary weight $mg$ of the person (use $g = 10\,\text{m/s}^2$). Assume there is no dissipation and the person is jumping repeatedly a height $h = 1\,\text{m}$ above the unloaded surface of the trampoline. Give all answers with symbols and numbers.
a) What is the stiffness $k$ of the spring (answer in terms of some or all of $m, g$ and $d_0$).
b) What is the maximum deflection of the trampoline during these jumps?
c) What is the peak force of the trampoline on the jumper? (answer in symbols, Newtons, and numbers of body weights).

**9.39** For the car of problem 9.34 what is the average power required to reach speed $v_1$? There are two plausible ways to calculate this power:

$$\bar{P}_1 = \int_0^x P(x')\,dx' \text{ and}$$

$$\bar{P}_2 = \int_0^t P(t')\,dt' .$$

Use both. Do the two methods give the same answer? If so, why, if not, why not?

**9.40** For problem 9.35 which answers would change in which way if the deceleration was not exactly constant during the crash? That is, for which quantities would be bigger, which smaller, which the same, for which would the answer depend on the nature of the non-constant acceleration?

**9.41** The earth's gravitational pull on a mass $m$ is $F = -\frac{mgR^2}{r^2}$, where $mg$ is the pull at the surface of the earth and $R$ is the radius of the earth. Assume a ballistic rocket is shot straight with a launch velocity of $v_0$ (measured in a 'fixed' not-rotating-with-the-earth frame). Assume the rocket goes in a straight radial line as the earth turns underneath it (relative to the surface of the earth this rocket would be launched somewhat to the West to cancel the earths rotation). Assume the period of active thrust is negligibly short (hence the word *ballistic*: "relating to or characteristic of the motion of objects moving under their own momentum and the force of gravity").

a) Solve for $v$ as a function of $r$ (and some or all of $m$, $g$, $R$ and $v_0$.
b) Find the maximum height the rocket reaches.
c) Find the 'escape velocity' $v_{escape}$, the minimum launch speed needed for the rocket to never return.
d) On one graph plot height ($r$ or $r - R$) vs $t$ for a $v_0$ just below $v_{escape}$ and for $v_0$ just greater than $v_{escape}$. If you use numerical methods to make this plot use $g = 10\,\mathrm{m/s^2}$, $R = 6400\,\mathrm{km}$, and $m = 1\,\mathrm{kg}$. Make sure your axis are such that you can see a clear qualitative difference between the two cases.

**9.42** The power available to a very strong accelerating cyclist over short periods of time (up to, say, about 1 minute) is about 1 horsepower. Assume a rider starts from rest and uses this constant power. Assume a mass (bike + rider) of 150 lbm, a realistic drag force of $.006\,\mathrm{lbf/(ft/s)^2}v^2$. Neglect other drag forces.

1. What is the peak speed of the cyclist?
2. Using analytic or numerical methods make a plot of speed vs. time.
3. What is the acceleration as $t \to \infty$ in this solution?
4. What is the acceleration as $t \to 0$ in your solution?

• • •

Also see several problems in the harmonic oscillator section.

## 9.3 Elementary vibration analysis

## Preparatory Problems

**9.43** The basic model.

a) Draw a spring ($k$) mass ($m$) system in a configuration where the spring is stretched.
b) On the drawing indicate the variable $x$.
c) Draw a free body diagram of the mass.
d) Write the equation of linear momentum balance for the mass.
e) Rearrange the momentum balance equation to get the Harmonic-oscillator equation in standard form.
f) Write the general solution to the harmonic oscillator equation in 2 different ways (one as a sum of a sine and cosine function and one as a phase shifted sine or cosine function).
g) What is the natural frequency of this system?
h) What is the period?
i) What is the frequency (or circular frequency)?
j) Find the solution for the special case that the mass is released from rest at $x(0) = x_0$.
   - give the analytic expression.
   - plot the position vs time for at least one whole cycle of motion.
   - with the same time scale, plot velocity vs time (what is the peak velocity).
   - with the same time scale, plot both the potential and kinetic energies vs time.
k) Find the solution for the special case that the mass is launched at $v_0$ from the rest position (just the analytic form, no need to repeat all the parts just above).

**9.44** Does the function $x = C_1 e^{\lambda t} + C_2 e^{-\lambda t}$ satisfy the harmonic oscillator equation $\ddot{x} + \lambda^2 x = 0$ for any, possibly special, values of $C_1$ and $C_2$? Show that it does or does not.

**9.45** Given that $\ddot{x} = -(1/\mathrm{s}^2)x$, $x(0) = 1\,\mathrm{m}$, and $\dot{x}(0) = 0$ find:

a) $x(\pi\ \mathrm{s}) = ?$
b) $\dot{x}(\pi\ \mathrm{s}) = ?$

**9.46** Given that $\ddot{x} + x = 0$, $x(0) = 1$, and $\dot{x}(0) = 0$, find the value of $x$ at $t = \pi/2$ s.

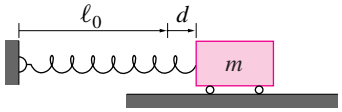
**9.47** Given that $\ddot{x} + \lambda^2 x = C_0$, $x(0) = x_0$, and $\dot{x}(0) = 0$, find the value of $x$ at $t = \pi/\lambda$ s.

• • •

**9.48** A mass $m$ is connected to a spring $k$ and released from rest with the spring stretched a distance $d$ from its static equilibrium position. It then oscillates back and forth repeatedly crossing the equilibrium. How much time passes from release until the mass moves through the equilibrium position for the second time? Neglect gravity and friction. Answer in terms of some or all of $m$, $k$, and $d$.

**9.49** A spring with rest length $\ell_0$ is attached to a mass $m$ which slides frictionlessly on a horizontal ground as shown. At time $t = 0$ the mass is released with no initial speed with the spring stretched a distance $d$. *[Remember to define any coordinates or base vectors you use.]*

a) What is the acceleration of the mass just after release?
b) Find a differential equation which describes the horizontal motion of the mass.
c) What is the position of the mass at an arbitrary time $t$?
d) What is the speed of the mass when it passes through the position where the spring is relaxed?



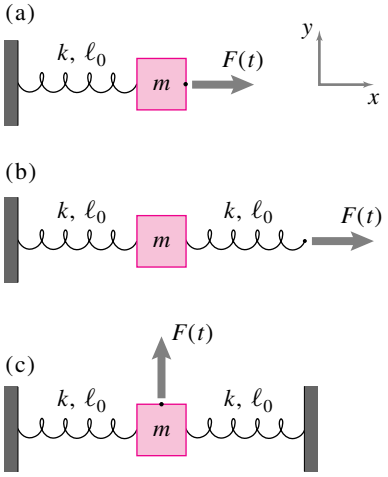

**problem 9.49:**

Filename:s97f1

**9.50** Reconsider the spring-mass system from problem 9.49.

a) Find the potential and kinetic energy of the spring mass system as functions of time.
b) Assigning numerical values to the various variables, use a computer to make a plot of the potential and kinetic energy as a function of time for several periods of oscillation. Are the potential and kinetic energy ever equal at the same time? If so, at what position $x(t)$?
c) Make a plot of kinetic energy versus potential energy. What is the phase relationship between the kinetic and potential energy?

**9.51** For the three spring-mass systems shown in the figure, find the equation of motion of the mass in each case. All springs are massless and are shown in their relaxed states. Ignore gravity. (In problem (c) assume vertical motion.)



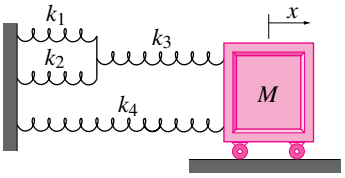

**problem 9.51:**

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## More-Involved Problems

**9.52** A spring and mass system is shown in the figure.

a) First, as a review, let $k_1$, $k_2$, and $k_3$ equal zero and $k_4$ be nonzero. What is the natural frequency of this system?
b) Now, let all the springs have nonzero stiffness. What is the stiffness of a single spring equivalent to the combination of $k_1, k_2, k_3, k_4$? What is the frequency of oscillation of mass $M$?
c) What is the equivalent stiffness, $k_{eq}$, of all of the springs together. That is, if you replace all of the springs with one spring, what would its stiffness have to be such that the system has the same natural frequency of vibration?

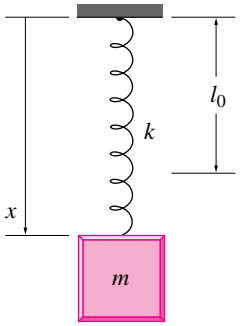



**problem 9.52:**

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**9.53 Mass hanging from a spring.** A mass $m$ is hanging from a spring with constant $k$ which has the length $l_0$ when it is relaxed (i. e., when no mass is attached). It only moves vertically.

a) Draw a Free Body Diagram of the mass.
b) Write the equation of linear momentum balance.
c) Reduce this equation to a standard differential equation in $x$, the position of the mass.
d) Verify that one solution is that $x(t)$ is constant at $x = l_0 + mg/k$.
e) What is the meaning of that solution? (That is, describe in words what is going on.)
f) Define a new variable $\hat{x} = x - (l_0 + mg/k)$. Substitute $x = \hat{x} + (l_0 + mg/k)$ into your differential equation and note that the equation is simpler in terms of the variable $\hat{x}$.
g) Assume that the mass is released from an an initial position of $x = D$. What is the motion of the mass?
h) What is the period of oscillation of this oscillating mass?
i) Why might this solution not make physical sense for a long, soft spring if $D > \ell_0 + 2mg/k$)?

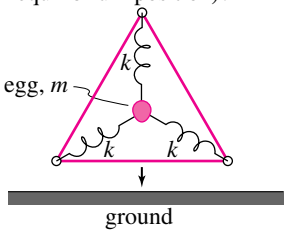



**problem 9.53:**

Filename:pg141-1

**9.54** One of the winners in an egg-drop contest was a structure in which rubber bands held the egg at the center of it. Here is a model. Consider the egg to be a particle of mass $m$ and the springs to be linear with spring constants $k$. Consider only a two-dimensional version of the winning design as shown in the figure. Assume the frame hits the ground on one of the straight sections. Assume small motions (deflection $\ll$ side-length) and that the springs do not buckle.

a) what will be the frequency of vibration of the egg after impact?
b) What is the maximum vertical deflection of the egg (relative to its equilibrium position)?

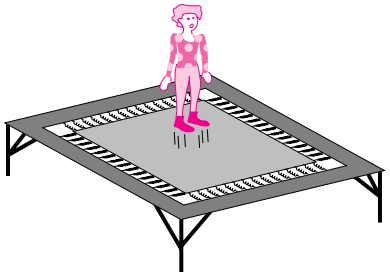



**problem 9.54:**

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**9.55 A person jumps on a trampoline.** The trampoline is modeled as having an effective vertical undamped linear spring with stiffness $k = 200\,\text{lbf/ft}$. The person is modeled as a rigid mass $m = 150\,\text{lbm}$. $g = 32.2\,\text{ft/s}^2$.

a) What is the period of motion if the person's motion is so small that her feet never leave the trampoline?
b) What is the maximum amplitude of motion for which her feet never leave the trampoline?
c) (harder) If she repeatedly jumps so that her feet clear the trampoline by a height $h = 5\,\text{ft}$, what is the period of this motion?



**problem 9.55:** A person jumps on a trampoline.

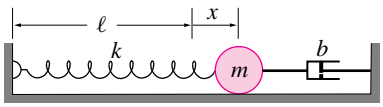
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**9.56** A mass moves on a frictionless surface. It is connected to a dashpot with damping coefficient $b$ to its right and a spring with constant $k$ and rest length $\ell$ to its left. At the instant of interest, the mass is moving to the right and the spring is stretched a distance $x$ from its position where the spring is unstretched. There is gravity.

a) Draw a free body diagram of the mass at the instant of interest.

b) Derive the equation of motion of the mass.

.




**problem 9.56:**

Filename:ch2-11

**9.57** The equation of motion of an unforced mass-spring-dashpot system is, $m\ddot{x} + c\dot{x} + kx = 0$, as discussed in the text. For a system with $m = 0.4\,\text{kg}$, $c = 10\,\text{kg/s}$, and $k = 5\,\text{N/m}$,

a) Find whether the system is underdamped, critically damped, or overdamped.

b) Sketch a typical solution of the system.

c) Make an accurate plot of the response of the system (displacement vs time) for the initial conditions $x(0) = 0.1\,\text{m}$ and $\dot{x}(0) = 0$.

.

**9.58** Experiments conducted on free oscillations of a damped oscillator reveal that the amplitude of oscillations drops to 25% of its peak value in just 3 periods of oscillations. The period os oscillation is measured to be 0.6 s and the mass of the system is known to be 1.2 kg. Find the damping coefficient and the spring stiffness of the system.

**9.59** You are required to design a mass-spring-dashpot system that, if disturbed, returns to its equilibrium position the quickest. You are given a mass, $m = 1\,\text{kg}$, and a damper with $c = 10\,\text{kg/s}$. What should be the stiffness of the spring? Your solution needs to include your definition of "'quickest".

## 9.4 Coupled motion in 1D

The primary emphasis of this section is setting up correct differential equations (without sign errors) and solving these equations on the computer. Experts note: normal modes are not covered here.

## Preparatory Problems

**9.60** Write the following set of coupled second order ODE's as a system of first order ODE's.

$$\begin{aligned} \ddot{x}_1 &= k_2(x_2 - x_1) - k_1 x_1 \\ \ddot{x}_2 &= k_3 x_2 - k_2(x_2 - x_1) \end{aligned}$$

**9.61** See also problem 9.62. The solution of a set of a second order differential equations is:

$$\begin{aligned} \xi(t) &= A\sin\omega t + B\cos\omega t + \xi^* \\ \dot{\xi}(t) &= A\omega\cos\omega t - B\omega\sin\omega t, \end{aligned}$$

where $A$ and $B$ are constants to be determined from initial conditions. Assume $A$ and $B$ are the only unknowns and write the equations in matrix form to solve for $A$ and $B$ in terms of $\xi(0)$ and $\dot{\xi}(0)$.

**9.62** Solve for the constants $A$ and $B$ in Problem 9.61 using the matrix form, if $\xi(0) = 0$, $\dot{\xi}(0) = 0.5$, $\omega = 0.5\,\text{rad/s}$ and $\xi^* = 0.2$.

**9.63** A set of first order linear differential equations is given:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 + kx_1 + cx_2 &= 0 \end{aligned}$$

Write these equations in the form $\dot{\vec{x}} = [\mathbf{A}]\vec{x}$, where $\vec{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$.

**9.64** Write the following pair of coupled ODE's as a set of first order ODE's.

$$\begin{aligned} \ddot{x}_1 + x_1 &= \dot{x}_2\sin t \\ \ddot{x}_2 + x_2 &= \dot{x}_1\cos t \end{aligned}$$

**9.65** The following set of differential equations can not only be written in first order form but in matrix form $\dot{\vec{x}} = [\mathbf{A}]\vec{x} + \vec{c}$. In general things are not so simple, but this *linear* case is prevalent in the analytic study of dynamical systems.

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 + 5\Omega^2 x_1 - 4\Omega^2 x_2 &= 2\Omega^2 v_1^* \\ \dot{x}_4 - 4\Omega^2 x_1 + 5\Omega^2 x_2 &= -\Omega^2 v_1^* \end{aligned}$$

**9.66** Write each of the following equations as a system of first order ODE's.

a) $\ddot{\theta} + \lambda^2\theta = \cos t$,

b) $\ddot{x} + 2p\dot{x} + kx = 0$,

c) $\ddot{x} + 2c\dot{x} + k\sin x = 0$.

**9.67** A train is moving at constant absolute velocity $v\hat{\imath}$. A passenger, idealized as a point mass, is walking at an absolute absolute velocity $u\hat{\imath}$, where $u > v$. What is the velocity of the passenger relative to the train?

• • •

**9.68** Two equal masses, each denoted by the letter $m$, are on an air track. One mass is connected by a spring to the end of the track. The other mass is connected by a spring to the first mass. The two spring constants are equal and represented by the letter $k$. In the rest (springs are relaxed) configuration, the masses are a distance $\ell$ apart. Motion of the two masses $x_1$ and $x_2$ is measured relative to this configuration.

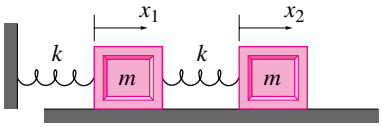
a) Draw a free body diagram for each mass.

b) Write the equation of linear momentum balance for each mass.

c) Write the equations as a system of first order ODEs.

d) Pick parameter values and initial conditions of your choice and simulate a motion of this system. Make a plot of the motion of, say, one of the masses vs time,

e) Explain how your plot does or does not make sense in terms of your understanding of this system. Is the initial motion in the right direction? Are the solutions periodic? Bounded? etc.




**problem 9.68:**

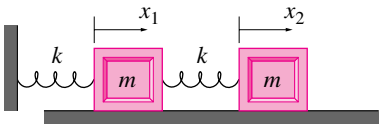
Filename:pfigure-s94f1p4

**9.69** Two equal masses, each denoted by the letter $m$, are on an air track. One mass is connected by a spring to the end of the track. The other mass is connected by a spring to the first mass. The two spring constants are equal and represented by the letter $k$. In the rest configuration (springs are relaxed) the masses are a distance $\ell$ apart. Motion of the two masses $x_1$ and $x_2$ is measured relative to this configuration.

a) Write the potential energy of the system for arbitrary displacements $x_1$ and $x_2$ at some time $t$.

b) Write the kinetic energy of the system at the same time $t$ in terms of $\dot{x}_1$, $\dot{x}_2$, $m$, and $k$.

c) Write the total energy of the system.




**problem 9.69:**

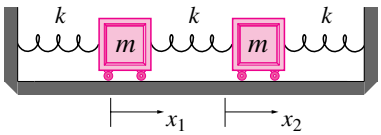
Filename:pfigure-twomassenergy

**9.70 Normal Modes.** Three equal springs ($k$) hold two equal masses ($m$) in place. There is no friction. $x_1$ and $x_2$ are the displacements of the masses from their equilibrium positions.

a) How many independent normal modes of vibration are there for this system?

b) Assume the system is in a normal mode of vibration and it is observed that $x_1 = A\sin(ct) + B\cos(ct)$ where $A$, $B$, and $c$ are constants. What is $x_2(t)$? (The answer is not unique. You may express your answer in terms of any of $A$, $B$, $c$, $m$ and $k$. )

c) Find all of the frequencies of normal-mode-vibration for this system in terms of $m$ and $k$.



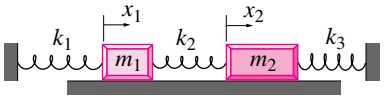

**problem 9.70:**

Filename:pfigure-f93f2

**9.71** A two degree of freedom mass-spring system, made up of two unequal masses $m_1$ and $m_2$ and three springs with unequal stiffnesses $k_1$, $k_2$ and $k_3$, is shown in the figure. All three springs are relaxed in the configuration shown. Neglect friction.

a) Derive the equations of motion for the two masses.

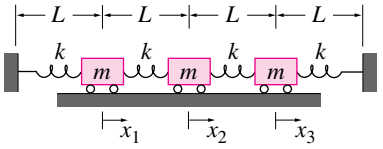
b) Does each mass undergo simple harmonic motion?




**problem 9.71:**

Filename:pfigure-s94h5p1

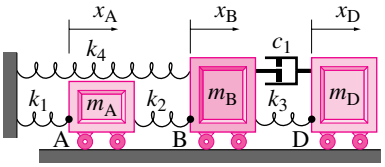
**9.72** For the three-mass system shown, draw a free body diagram of each mass. Write the spring forces in terms of the displacements $x_1$, $x_2$, and $x_3$.




**problem 9.72:**

Filename:s92f1p1

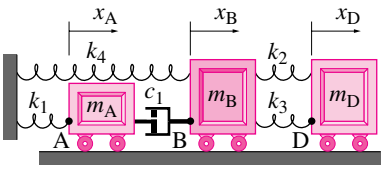
**9.73** The springs shown are relaxed when $x_A = x_B = x_D = 0$. In terms of some or all of $m_A, m_B, m_D, x_A, x_B, x_D, \dot{x}_A, \dot{x}_B, \dot{x}_C$, and $k_1, k_2, k_3, k_4, c_1$, and $F$, find the acceleration of block B.




**problem 9.73:**

Filename:pfigure-s95q3

**9.74** A system of three masses, four springs, and one damper are connected as shown. Assume that all the springs are relaxed when $x_A = x_B = x_D = 0$. Given $k_1$, $k_2$, $k_3$, $k_4$, $c_1$, $m_A$, $m_B$, $m_D$, $x_A$, $x_B$, $x_D$, $\dot{x}_A$, $\dot{x}_B$, and $\dot{x}_D$, find the acceleration of mass B, $\vec{a}_B = \ddot{x}_B\hat{\imath}$.




**problem 9.74:**

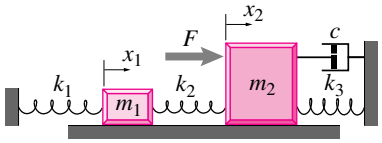
Filename:pfigure-s95f1

**9.75 Equations of motion.** Two masses are connected to fixed supports and each other with the three springs and dashpot shown. The force $F$ acts on mass 2. The displacements $x_1$ and $x_2$ are defined so that $x_1 = x_2 = 0$ when the springs are unstretched. The ground is frictionless. The governing equations for the system shown can be written in first order form if we define $v_1 \equiv \dot{x}_1$ and $v_2 \equiv \dot{x}_2$.

a) Write the governing equations in a neat first order form. Your equations should be in terms of any or all of the constants $m_1$, $m_2$, $k_1$, $k_2$,$k_3$, $C$, the constant force $F$, and $t$. Getting the signs right is important.

b) Write computer commands to find and plot $v_1(t)$ for 10 units of time. Make up appropriate initial conditions.

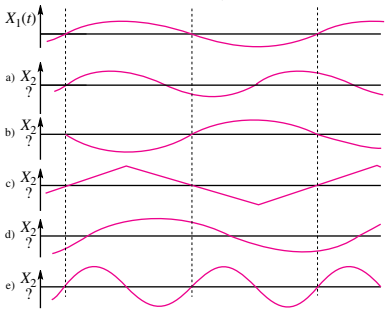
c) For constants and initial conditions of your choosing, plot $x_1$ vs $t$ for enough time so that decaying erratic oscillations can be observed.




**problem 9.75:**

Filename:p-f96-f-3

**9.76** $x_1(t)$ and $x_2(t)$ are measured positions on two points of a vibrating structure. $x_1(t)$ is shown. Some candidates for $x_2(t)$ are shown. Which of the $x_2(t)$ could possibly be associated with a normal mode vibration of the structure? Answer "could" or "could not" next to each choice. (If a curve looks like it is meant to be a sine/cosine curve, it is.)



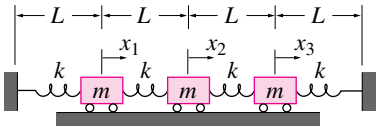

**problem 9.76:**

Filename:pfigure-blue-144-1

**9.77** For the three-mass system shown, one of the normal modes is described with the eigenvector (1, 0, -1). Assume $x_1 = x_2 = x_3 = 0$ when all the springs are fully relaxed.

a) What is the angular frequency $\omega$ for this mode? Answer in terms of $L, m, k$, and $g$. (Hint: Note that in this mode of vibration the middle mass does not move.)

b) Make a neat plot of $x_2$ versus $x_1$ for one cycle of vibration with this mode.




**problem 9.77:**

Filename:pfigure-blue-160-2

**9.78** The three beads of masses $m$, $2m$, and $m$ connected by massless linear springs of constant $k$ slide freely on a straight rod. Let $x_i$ denote the displacement of the $i^{th}$ bead from its equilibrium position at rest.

a) Write expressions for the total kinetic and potential energies.

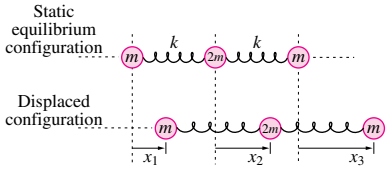
b) Write an expression for the total linear momentum.

c) Draw free body diagrams for the beads and use Newton's second law to derive the equations for motion for the system.

d) Verify that total energy and linear momentum are both conserved.

e) Show that the center of mass must either remain at rest or move at constant velocity.

f) What can you say about vibratory (sinusoidal) motions of the system?




**problem 9.78:**

Filename:pfigure-blue-161-1

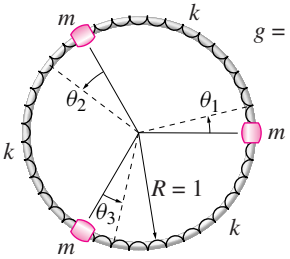
**9.79** The system shown below comprises three identical beads of mass $m$ that can slide frictionlessly on the rigid, immobile, circular hoop. The beads are connected by three identical linear springs of stiffness $k$, wound around the hoop as shown and equally spaced when the springs are unstretched (the strings are unstretched when $\theta_1 = \theta_2 = \theta_3 = 0$.)

a) Determine the natural frequencies and associated mode shapes for the system. (Hint: you should be able to deduce a 'rigid-body' mode by inspection.)

b) If your calculations in (a) are correct, then you should have also obtained the mode shape $(0, 1, -1)^T$. Write down the most general set of initial conditions so that the ensuing motion of the system is simple harmonic in that mode shape.

c) Since $(0, 1, -1)^T$ is a mode shape, then by "symmetry", $(-1, 0, 1)^T$ and $(1, -1, 0)^T$ are also mode shapes (draw a picture). Explain how we can have three mode shapes associated with the same frequency.

d) Without doing any calculations, compare the frequencies of the constrained system to those of the unconstrained system, obtained in (a).

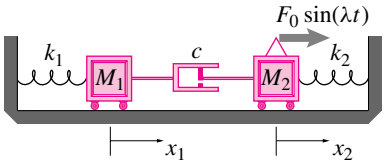



**problem 9.79:**

Filename:pfigure-blue-158-1

**9.80 Equations of motion.** Two masses are connected to fixed supports and each other with the two springs and dashpot shown. The displacements $x_1$ and $x_2$ are defined so that $x_1 = x_2 = 0$ when both springs are unstretched.

For the special case that $C = 0$ and $F_0 = 0$ clearly define two different set of initial conditions that lead to normal mode vibrations of this system.



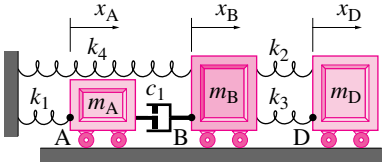

**problem 9.80:**

Filename:p-s96-p3-1

**9.81** As in problem 9.74, a system of three masses, four springs, and one damper are connected as shown. Assume that all the springs are relaxed when $x_A = x_B = x_D = 0$.

a) In the special case when $k_1 = k_2 = k_3 = k_4 = k$, $c_1 = 0$, and $m_A = m_B = m_D = m$, find a normal mode of vibration. Define it in any clear way and explain or show why it is a normal mode in any clear way.

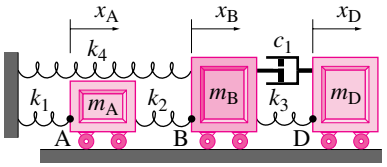
b) In the same special case as in (a) above, find another normal mode of vibration.




**problem 9.81:**

Filename:pfigure-s95f1a

**9.82** As in problem **??**, a system of three masses, four springs, and one damper are connected as shown. In the special case when $c_1 = 0$, find the normal modes of vibration.




**problem 9.82:**

Filename:pfigure-s95q3a

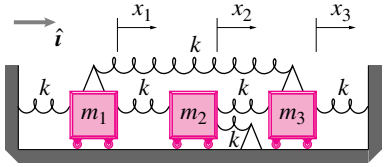
**9.83 Normal modes.** All three masses have $m = 1\,\text{kg}$ and all 6 springs are $k = 1\,\text{N/m}$. The system is at rest when $x_1 = x_2 = x_3 = 0$.

a) Find as many different initial conditions as you can for which normal mode vibrations result. In each case, find the associated natural frequency. (we will call two initial conditions $[v]$ and $[w]$ different if there is no constant $c$ so that $[v_1\ v_2\ v_3] = c[w_1\ w_2\ w_3]$. Assume the initial velocities are zero.)

b) For the initial condition

$$[x_0] = [\,0.1\,\text{m}\ \ 0\ \ 0\,],$$
$$[\dot{x}_0] = [\,0\ \ 2\,\text{m/s}\ \ 0\,]$$

what is the initial (immediately after the start) acceleration of mass 2?



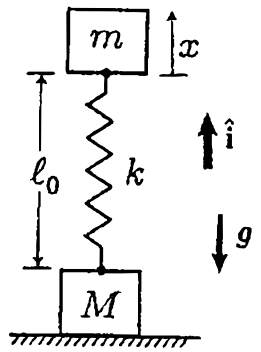

**problem 9.83:**

Filename:pfigure3-f95p2p2

**9.84** Two blocks with masses $M$ and $m$ are connected by a spring with constant $k$ and free length $\ell_0$ that can sustain compression. Mass $M$ is resting on the ground at the

start. There is gravity. The upwards vertical displacement of mass $m$ is $x$, which is zero when the spring is at its rest length and $M$ is on the ground.

a) For what value of $x$ is the system in static equilibrium?
b) Find a differential equation governing the motion of the $m$ assuming $M$ remains on the ground.
c) Draw a free body diagram of $M$.
d) For what value of $x$ is $M$ on the verge of lifting off the ground.
e) Defining $y$ as the height of the lower mass, write two coupled differential equations for the motion of $m$ and $M$ if both masses are in the air.
f) Find the value of $x < 0$ so that if the system is started from rest with that $x$ and $y = 0$ that the ground reaction force on $M$ just goes to zero.
g) Starting here, this problem is more of a project than a typical homework problem. Assume $x(t = 0)$ is less than the value computed above. Write a computer program that integrates the equations of motion until $M$ lifts off and then switches to integrating the equations for the two masses in the air.
h) modify your program so that if $M$ hits the ground again, it sticks until the ground reaction force goes to zero again.
i) By playing around, this way or that, see if you can find a special value for $x(t = 0)$ so that the bouncing continues indefinitely. (This is a perhaps surprising result, that a system with plastic collisions can continue to bounce indefinitely.)




**problem 9.84:**

Filename:pfigure-2blocksandspring

**9.85** A 3 kg mass is suspended by a spring ($k = 10\,\mathrm{N/m}$) and forced by a 5 N sinusoidally oscillating force with a period of 1 s. What is the amplitude of the steady-state oscillations (ignore the "homogeneous" solution)

**9.86** Given that $\ddot{\theta} + k^2\theta = \beta \sin \omega t$, $\theta(0) = 0$, and $\dot{\theta}(0) = \dot{\theta}_0$, find $\theta(t)$ .

**9.87** A machine produces a steady-state vibration due to a forcing function described by $Q(t) = Q_0 \sin \omega t$, where $Q_0 = 5000N$. The machine rests on a circular concrete foundation. The foundation rests on an isotropic, elastic half-space. The equivalent spring constant of the half-space is $k = 2,000,000\,\mathrm{N{\cdot}m}$ and has a damping ratio $d = c/c_c = 0.125$. The machine operates at a frequency of $\omega = 4$ Hz.

1. What is the natural frequency of the system?
2. If the system were undamped, what would the steady-state displacement be?
3. What is the steady-state displacement given that $d = 0.125$?
4. How much additional thickness of concrete should be added to the footing to reduce the damped steady-state amplitude by 50%? (The diameter must be held constant.)

## 9.5 Advanced vibration

APPENDIX A

# Units and dimensions

*A brief essay about some issues related to units and dimensions.*

**Contents**

Many engineering texts have, somewhere near the start, a tedious and pedantic section about units and dimensions. This book is different. That section is here at the end; not to diminish the importance of the topic but because students are immune to preaching. The only way a student will get good at managing units is by imitation, or in time of panic or idle curiosity. As for imitation, we have tried to set a good example in the whole of the book. As for panic and curiosity, this section is here. The central messageis this:

*balance your units and carry your units.*

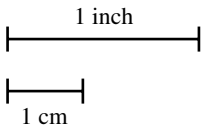



Figure A.1: Relative size of an inch and a centimeter.

Filename:tfigure1-a

## Balance your units

Every line of every calculation should be dimensionally sensible. That is, the dimensions on the left of the equal sign should be consistent with the dimensions on the right the same way numbers have to balance. Otherwise the equations are not equations. For example, if two bicycles tied in a race you could say they were in some way equal. But even if you noticed that the weight difference between them was 10% over 2 pounds you would not write

$$8\,\mathrm{kg} = 9\,\mathrm{kg}.$$

The equivalence between the two bikes does not make eight kilograms equal to nine kilograms. In this same way it would be wrong to write

$$1\,\mathrm{in} = 1\,\mathrm{s}.$$

if you noticed that it takes a bug about a minute (60 seconds) to walk the length of your body (say about 60 inches). That the passing of a second corresponds to the passing of an inch, so for some purposes an inch is equivalent to a second, and $1 = 1$, does not mean that an inch is a second. An inch has dimensions of length which cannot be equal to a second with dimensions of time. Length can equal time no more than 8 can equal 9.

Of course it is correct to write that

$$5.08\,\mathrm{cm} = 2\,\mathrm{in}$$

whether or not you have noticed anything. Both centimeters and inches have dimensions of length and one inch is equivalent to 2.54 centimeters always

(figure A.1). An equation where the units on both sides of the equation are the same physical quantities (length in the example above) is *balanced* with regard to units.

## Carry your units

When you go from one line of a calculation to the next you should carry (keep written track of) the units with as much care as any other numerical or algebraic quantities. This written presentation of your units will help you as well as the people to whom you show your work. The rest of this section is, more or less, a discussion of how and why to 'carry your units.'

Most physical quantities are dimensional and are represented by a number multiplying a unit: 7 m means 7 times (one meter). Thus, the 'm' and the 7 are of equal status in any equations in which they are used. When you do arithmetic and don't forget any terms you have 'carried' the numbers. Similarly, *carrying* the units just means not forgetting them *in* your calculations (not just next to your calculations).

## Dimensions, units and changing units

Distance has dimensions of length $[L]$ that can be measured with various units — centimeters ( cm), yards ( yd), or furlongs (an obsolete unit equal to $1/8$ mile). A meter is the standard unit of length in the SI system. In answer to the question 'What is the length of a bicycle crank $\ell$?' we say '$\ell$ is seven inches' and write $\ell = 7$ in or say '$\ell$ is seventeen point seven centimeters' and write $\ell = 17.7$ cm. In each case, a number multiplies a dimensional unit.

Force has dimension of mass times acceleration $[m \cdot a]$. Because acceleration itself has dimensions of length over time squared $[L/T^2]$, force also has dimensions of mass times length divided by time squared $[M \cdot L/T^2]$. Because force has such a central role in mechanics, it is often convenient to think of force as having its own units. Force then has dimensions of, simply, force $[F]$. The most common units for force are Newton ( N) and the pound ( lbf). The 'f' in the notation for the pound lbf is to distinguish a pound force lbf from the pound mass lbm, $1\,\text{lbf} = \text{lbm} \cdot g \approx 32.2\,\text{lbm} \cdot \text{ft/s}^2$. Some people use lb to mean pound force or pound mass, depending on context. We use lbm for pound mass and lbf for pound force to avoid confusion.

## Changing units

We can say 'The typical force of a seated racing bicyclist on a bicycle pedal is thirty pounds,' and write any of the following:

$$\begin{aligned} F &= 30\ \text{lbf} \\ F &= 30\ \text{lbf} \cdot (1) \\ F &= 30\ \text{lbf} \cdot \underbrace{\left(\frac{4.45\ \text{N}}{1\ \text{lbf}}\right)}_{1} \\ F &= 133.5\ \text{N}. \end{aligned}$$

Here we have shown one way to change units. Multiply the expression of interest by one (1) and then make an appropriate substitution for one. Any table of units will tell us that 1 lbf is approximately 4.45 N. So we can write $1 = (4.45\,\text{Newtons}/1\,\text{lbf})$ and multiply any part of an equation by it without affecting the equation's validity. See figure A.2 to get a sense of the relation between a pound force, a Newton, and the less used force units, the poundal and the kilogram-force.

What if we had made a mistake and instead multiplied the right hand side by $1 = (1\,\text{lbf}/4.45\,\text{Newton})$? No problem. We would then have

$$F = 30\ \text{lbf} = 30\ \text{lbf} \cdot \frac{1\ \text{lbf}}{4.45\ \text{Newton}} = \frac{30}{4.45}\ \text{lbf}^2/\,\text{N}.$$

This expression is admittedly weird, but it is correct. If you should end up with such a weird but correct solution you can compensate by multiplying by one again and again until the units cancel in a way that you find pleasing. In this case we could get an answer in a more conventional form by multiplying the right hand side by $1^2$ using $1 = (4.45\,\text{N}/\,\text{lbf})$:

$$F = \frac{30}{4.45}\frac{\text{lbf}^2}{\text{N}} \cdot 1^2 = \frac{3.0}{4.45}\frac{\text{lbf}^2}{\text{N}} \cdot \left(\frac{4.45\,\text{N}}{1\,\text{lbf}}\right)^2 = 133.5\,\text{N} \qquad \text{(as expected)}.$$

A trivial but surprisingly useful observation is that $F = F$. A quantity is equal to itself no matter how it is represented. That is, $30\,\text{lbf} = 133.5\,\text{N}$ even though $30 \neq 133.5$. To summarize:

> *Units are manipulated in any and all calculations as if they were numbers or algebraic symbols. For example, canceling equal units from the top and bottom of a fraction is the same as canceling numbers or algebraic symbols.*

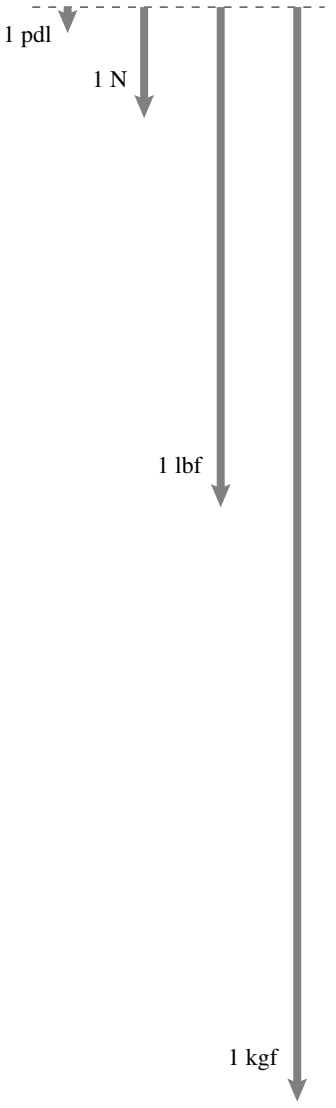



Figure A.2: Relative size of a poundal pdl, a Newton N, a pound force lbf, and a kilogram force kgf. 1 N = 7.24 pdl, 1 lbf = 4.45 N, and 1 kgf = 1 kilopond = 9.81 N = 2.2 lbf.

Filename:tfigure1-b

## An advertisement for careful use of units

Units and dimensions are part of scientific notation just as spelling, punctuation, and grammar are parts of English composition. If used properly, they

aid both thinking and the communication of these thoughts to others. If units and dimensions are used improperly they can impede communication, even with oneself, and convey the wrong meaning.

Example: **Breaking load**
A gadget that breaks with a 300 N (300 Newtons) load instead of a needed 300 lbf (300 pounds force) load is exactly as bad as one that breaks with a 67 lbf load instead of a needed 300 lbf load. An unsatisfied consumer will not be placated by learning that the engineer's calculation was 'numerically correct'.

If anybody is ever to use your calculation, giving them the wrong units is just as bad as giving them the wrong numerical value.

Although using units properly often seems annoyingly tedious, it also often pays. If units are carried through honestly, *not just tagged on to the end of an equation for appearance*, you can check your work for dimensional consistency. If you are trying to find a speed and your answer comes out 13 kg· m/ s, you know you have made a mistake — kg· m/ s just isn't a speed. Such dimensional errors in a calculation often reveal corresponding algebraic or conceptual mistakes. Also, if a problem is based on data with mixed units, such as cm and meters, or pound force and pound mass, you may often not know the units of your answer unless you properly 'carry' your units*.

## Three ways to be fussy about units.

People are most pleased if you speak their language, speak correctly, and make sense. Similarly, scientists and engineers with whom you communicate will be most comfortable if you use the units they use and use them with correct notation. But most importantly, you should use units in a way that makes physical sense. Just as the United Nations argues over which language to use for communication, educators, editors, and makers of standards have argued for decades over conventions for units: whether they should come in multiples of 10, whether they should use the standard international scientific conventions, and whether they will be clear to someone who has worked in the stock room of a supplier of $\frac{1}{2}$-inch bolts for 35 years and thinks SI might be a friend of his cousin Amil.

Even if you are not fluent in someone's favorite language, you can still say sensible things. Similarly, no matter what you or your work place's choice of units (SI, English, or hodge-podge), no matter whether you use upper case and lower case correctly, you should make sense. Physically sensible units — that is, balanced units — should be used to make your equations dimensionally correct. Then you should work on refining your notation so as to be more professional.

So, in order of importance,

1. use balanced units.
2. use units of the type that are liked by your colleagues.
3. spell and punctuate these units correctly.

If you are in a situation where your only problem is the third item on the list you are doing fine, unless you are *really* fussy, or work for someone who is

* You can easily generate errors of approximately a factor of 1000 with English units if you wishfully multiply or divide answers by 32.2 (the value of $g$ in ft/s$^2$) at the end of a sloppy calculation. If you do it wrong you get an error of a factor of $32.2^2$ which is 3% greater than 1000. Following sloppiness with unscrupulousness, some are tempted to then slide a decimal point three places to the right or left to 'fix' things. The decrepit insecurity that provokes such crimes is avoided by going to a church (temple or mosque) regularly, or just by carrying units.

really fussy. (*e.g.*, the authors of this book only hope to be good at the first two items on this list.)

Not everyone will take the care that we advise for you. You will find that, in both school and work, there are a variety of ways in which people use and abuse units, all within the context of productive engineering. So you will have to be aware and tolerant of the various conventions, even if they sometimes seem somewhat vague and imprecise.

## Units with calculators and computers

Calculators and computers generally do not keep track of units for you. In order for your numerical calculations to make sense you have the following choices.

**Use dimensionless variables.** Using dimensionless variables is the preferred method of scientists and theoretical engineers. The approach requires that you define a new set of dimensionless variables in terms of your original dimensional variables.

**Use a consistent unit system.** Express all quantities in terms of units that are consistent. *For example, all lengths should be in the same units and the unit of force should equal the unit of mass times the unit of distance divided by the unit of time squared. Each row of the table below defines a consistent set of units for mechanics.

| Name | length | mass | force | time | angle |
|---|---|---|---|---|---|
| mks | meter | kilogram | Newton | second | radian |
| cgs | centimeter | gram | dyne | second | radian |
| English | foot | lbm | poundal | second | radian |
| English2 | foot | slug | lbf | second | radian |

The radian is the unit of angle in all consistent unit systems. Whether or not a radian is a proper unit or not is an issue of some philosophical debate. Practically speaking, you can generally replace 1 radian with the number 1.

**Use numerical equations.** If you are using the computer to evaluate a formula that you trust, and you have balanced the units in a way that makes you secure, you can have the computer do the arithmetic part of the calculation. It is easy to make mistakes, however, unless the formula is expressed in consistent units.

Example: **Force units conversion**
What, in the SI system, is the net braking force when a 2000 lbm car skids to a stop on level ground? For this units problem we skip the careful mechanics and just work with the formula

$$F = \mu m g$$

where $m$ is the mass of the car, $g$ is the local gravitational constant and $\mu$ is the coefficient of friction for sliding between the tire and the road. We won't be off by more than a quarter of a percent using the standard rather than the local value of the gravitational constant, $g = 32.2\,\mathrm{ft/s^2}$. The coefficient of friction for rubber and dry road is about one, so we use $\mu = 1$. We proceed by plugging in values into the formula and then multiplying by 1 until things are

* **Caution:** Doing a computer calculation using quantities from an inconsistent unit system can easily lead to wrong results. To be safe make sure that all quantities are expressed in terms of only one row of the table shown.

in standard SI (Systémé Internationale) form. We use a table of units to make the various substitutions for 1. A few of the detailed steps could be contracted. The approach below is only one, albeit an awkward one, of many routes to the answer.

$$\begin{aligned}
F &= \mu m g \\
F &= 1 \cdot 2000\,\text{lbm} \cdot (32.2\,\text{ft/s}^2) \\
&= (2000{\cdot}32.2)\,\frac{\text{lbm}\cdot\text{ft}}{\text{s}^2} \\
&= (2000{\cdot}32.2)\,\frac{\text{lb}\!\!\not{\text{m}}\cdot\not{\text{ft}}}{\text{s}^2}\cdot\underbrace{\left(\frac{1\,\text{kg}}{2.2\,\text{lb}\!\!\not{\text{m}}}\right)}_{1} \\
&\qquad\cdot\underbrace{\left(\frac{30.48\,\not{\text{cm}}}{1\,\not{\text{ft}}}\right)}_{1}\cdot\underbrace{\left(\frac{1\,\text{m}}{100\,\not{\text{cm}}}\right)}_{1} \\
&= 8917\,\frac{\not{\text{kg}}\cdot\not{\text{m}}}{\not{\text{s}}^2}\cdot\underbrace{\left(\frac{1\,\not{\text{N}}}{1\,\not{\text{kg}}\cdot\not{\text{m}}/\not{\text{s}}^2}\right)}_{1}\cdot\underbrace{\left(\frac{1\,\text{kN}}{1000\,\not{\text{N}}}\right)}_{1} \\
&= 8.92\ \text{kN}
\end{aligned}$$

The net braking force is 8.92 kN.

## A.1 Advised and ill-advised use of units

**Good use of units** Say a car has a constant speed of $v = 50$ mi/ hr for half an hour. The following is true and expressed correctly.

The distance traveled in time $t$ is $x = vt$, so

$$\begin{aligned}
x &= vt \\
&= (50\,\text{mi/hr})(30\,\text{min}) = 50\cdot 30\,\text{mi}\cdot\,\text{min/hr} \\
&\qquad\text{(Awkward but true!)} \\
&= 50\cdot 30\,\text{mi}\cdot\not{\text{min}}/\not{\text{hr}}\underbrace{\left(\frac{1\,\not{\text{hr}}}{60\,\not{\text{min}}}\right)}_{1} \\
&= 25\,\text{mi}
\end{aligned}$$

That is, unsurprisingly, the distance covered in half an hour is 25 mi.

**Another good use of units.** If we start with the dimensionally correct formula $x = (50\,\text{mi/hr})t$ we can differentiate to get

$$v = \frac{dx}{dt} = 50\,\text{mi/hr}.$$

The answer is dimensionally correct without having to think about the units. $v$ is speed and contains its units, $x$ is distance and contains its units. In any formula that contains $t$, $x$ or $v$ we can substitute *any* time, distance or speed. How far does the car go in one minute? As in the previous example,

$$\begin{aligned}
x &= vt \\
&= (50\,\text{mi/hr})(1\,\text{min}) \\
&= (50\,\text{mi/}\not{\text{hr}})(1\,\not{\text{min}})\underbrace{\left(\frac{1\,\not{\text{hr}}}{60\,\not{\text{min}}}\right)}_{1} \\
&= \frac{5}{6}\,\text{mi}
\end{aligned}$$

**Not such good use of units** It is common practice to write sentences like 'the distance the car travels is

$$x = 50t,$$

where $x$ is the distance in miles and $t$ is the time of travel in hours', although we discourage it. Why? Because the variables $x$ and $t$ are ambiguously defined. We would like to use the fact that speed $v$ is the derivative of distance with respect to time:

$$v = \frac{dx}{dt} = \frac{d}{dt}(50t) = 50.$$

But now we have a speed equal to a pure number, 50, rather than a dimensional quantity. In this simple example, common sense tells us that the speed $v$ is measured in mi/hr. But if we want to think of $v$ as a speed, a variable with dimensions of length divided by time, the formula misleads us and requires us to add the units. For this simple example it is not much of a problem to determine what units to add.

But better is if units are included correctly in the equations; then they take care of themselves whenever they are needed. The 'not such good' use of units above is sometimes called using numerical equations, that is equations that have numbers in them only. The good use of units uses quantity equations, that is equations that use dimensional quantities.

In engineering we do math not just with numbers, but with dimensional quantities. The bad habits of many of us not withstanding, there are good and useful standards for how to deal with units in calculations*.Here we describe how some people use units and also present our biases.

## Use of units in old-style handbooks.

Many standard empirical formulas, formulas based on experience and not theory, are presented in an undimensional or numerical form. The units are not part of the equations. We present the approach here, not because we want to promote it, but because we don't want your more formal approach to units to stop you from reading and using empirical sources.

For example, Mark's *Handbook for Mechanical Engineers (8th edition, page 8-138)* presents the following useful formula to describe the working life of commercially manufactured ball bearings:

$$L_{10} = \frac{16,700}{N}\left(\frac{C}{P}\right)^K,$$

where

$L_{10}$ = the number of hours that pass before 10% of the bearings fail,
$N$ = the rotational speed in revolutions per minute
$C$ = the rated load capacity of the bearing in lbf,
$P$ = the actual load on the bearing in lbf, and
$K$ = 3 for ball bearings, 10/3 for roller bearings.

In this approach the idea of dimensional consistency has been disguised for the sake of brevity. $L_{10}$, $N$, $C$, and $P$ are just numbers. Such an equation is sometimes called a 'numerical equation'. It is a relation between numerical quantities. If you happen to know the rotation speed of the shaft in radians per second instead of revolutions per minute you will have to first convert before plugging in the formula. Unlike a dimensional formula, the formula does not help you to convert these units.

## Units with calculators and computers

Unfortunately, most calculators and computers are not equipped to carry units. They are only equipped to carry numbers. How do we handle this problem? The best and clearest option is only to do calculations with dimensionless variables.

The simplest way to use dimensionless variables, though not necessarily the best, is to do something that involves notational compromise. For example, let $x$ represent dimensionless distance rather than distance. That is, $x$ represents distance divided by 1 mi. Similarly, $t$ is time divided by 1 hr. And $dx/dt$ is dimensionless distance differentiated with respect to dimensionless time, which is, evidently, dimensionless speed. In this example, recovering the dimensional speed is common sense: speed is in mi/hr. The notational compromise is that $v$ is being used to represent both dimensional and dimensionless speed, with the precise meaning depending on context

* An excellent description of good practice is the "Guide for the Use of the International System of Units (SI)" by Barry Taylor, 1998. This is NIST (National Institute of Standards and Technology) publication # 811.

- **Example:** Using notational compromise we can use the formula $x = vt$ with $v = 50\,\text{mi/hr}$ to do a set of calculations. Say we want to know the distance $x$ every quarter of an hour for two hours. So we multiply 50 by .25, .5, .75, ... and thus make a table with two columns labeled $t$ (hr) and $x$ (mi).

| $t$ (hr) | $x$ (mi) |
|---|---|
| 0 | 0 |
| .25 | 12.5 |
| .5 | 25 |
| .75 | 37.5 |

- **Example:** The exact meaning of the columns in the above example are a little ambiguous. We can make it more precise by labeling the columns as follows

| $t/(\text{hr})$ | $x/(\text{mi})$ |
|---|---|
| 0 | 0 |
| .25 | 12.5 |
| .5 | 25 |
| .75 | 37.5 |

That is, the columns of numbers are dimensionless. The first column, is the time divided by one hour the second is distance divided by one mile.

- **Example:**If we take $x$ to be dimensional distance, $t$ to be dimensional time, and $v$ to be dimensional speed, we can define new dimensionless variables. $t^* = t/(1\,\text{hr})$, $x^* = x/(1\,\text{mi})$, and $v^* = v/(1\,\text{mi/hr})$. Now there is no ambiguity: $x$ is dimensional and $x^*$ is dimensionless. This approach is more precise, if cumbersome, than using $x$ to be both dimensional and dimensionless depending on context. Dividing the

## A.2 An improvement to the old-style handbook approach

An alternative to the standard approach to empirical formulas is to write a formula that makes sense with any dimensional variables. The bearing life formula would be replaced with the formula below:

$$l_{10} = \frac{16,700}{n}\left(\frac{c}{p}\right)^K \text{hr}\cdot\text{rev/min}$$

where

$l_{10}$ = the time that passes before 10% of the bearings fail,
$n$ = the rotational speed,
$c$ = the rated load capacity of the bearing,
$p$ = the actual load on the bearing, and
$K$ = 3 for ball bearings, 10/3 for roller bearings.

and the variables $l_{10}$, $n$, $c$, and $p$ are dimensional quantities. One can use any dimensions one wants for all of the variables. For example, using

$n$ = 50 rev/sec
$c$ = 1 kN
$p$ = 100 lbf, and
$K$ = 3 for the given ball bearing,

we can calculate the life of the bearing by plugging these values in to the formula directly.

$$\begin{aligned} l_{10} &= \frac{16,700}{50\,\frac{\text{rev}}{\text{sec}}}\left(\frac{1\,\text{kN}}{100\,\text{lbf}}\right)^3 \text{hr}\cdot\text{rev/min} \\ &= \frac{16,700}{50\,\text{rev/sec}}\underbrace{\left(\frac{60\text{sec}}{\text{min}}\right)}_{1}\cdot \\ &\qquad \left(\frac{1\,\text{kN}}{100\,\text{lbf}}\underbrace{\left(\frac{1\,\text{lbf}}{4.448\,\text{N}}\right)}_{1}\underbrace{\left(\frac{1\,\text{N}}{1000\,\text{kN}}\right)}_{1}\right)^3 \text{hr}\cdot\text{min/rev} \\ &\approx 45000\,\text{hr} \end{aligned}$$

This approach has the advantage of precision if mixed units are used. Any of the quantities can be measured with any units and the answer always comes out right.

equation $x = vt$ on both sides by one mile, and multiplying the right side by 1, in the form of $1 = (1\,\text{hr}/1\,\text{hr})$ we get:

$$\frac{x}{1\,\text{mi}} = \frac{v}{1\,\text{mi/hr}} \cdot \frac{t}{1\,\text{hr}}$$

which is, using the dimensionless variables,

$$x^* = v^* t^*.$$

Because $v$ is 50 mi/ hr, $v^* = 50$. We can show this reasoning somewhat formally as follows.

$$v^* = v/(1\,\text{mi/hr}) = (50\,\text{mi/hr})/(1\,\text{mi/hr}) = 50.$$

The dimensionless speed $v^*$ is just the dimensionless number 50. Now we can make a table by multiplying 50 by .25, .5, .75, .... The columns of the table can be labeled $t^*$ and $x^*$ and all variables are clearly defined.

| $t^*$ | $x^*$ |
|---|---|
| 0 | 0 |
| .25 | 12.5 |
| .5 | 25 |
| .75 | 37.5 |

Most often, most people will not go to such trouble unless they have confused themselves by not being careful. But it is easy to get in doubt if problems get complicated, if you loose track of what the difference is between a pound force and a pound mass, or if some variables are measured in meters and others in feet, etc.

## A.3 Force, Weight and English Units

The force of gravity on an object is its weight — well, almost. A given object has different weight on different parts of the earth, with up to 0.5% variation. That is, $g$, the earth's gravitational 'constant,' varies from about $9.78\,\mathrm{m/s^2}$ at the equator to about $9.83\,\mathrm{m/s^2}$ at the North Pole. The official value of the 'constant' $g$ is in between exactly $9.80665\,\mathrm{m/s^2}$ (this is about $32.1740486\,\mathrm{ft/s^2}$). Multiplying the official $g$ by the mass $m$ will give you almost exactly the force it takes to hold it up if you are in exactly the official place, somewhere in Potsdam. Outside of Potsdam you have to accept an error of up to 1/4% when calculating gravitational forces, unless you happen to know the value of $g$ in your neighborhood.

Historically, people understood weight before they understood mass: bigger things are harder to hold up so they have more weight. This relationship is easier to perceive than that bigger things are harder to accelerate, i.e., have more mass. So people defined the quantity of matter by weight. 'How much flour?' one would ask. 'A pound of flour,' meaning one pound weight, might be the answer. A one pound weight is pulled with a 1 lbf by gravity, or in the older notation where one did not worry about mass, by 1 lb. People didn't notice that it was a little harder, i.e.. would stretch a given spring more, to hold something up on the north pole than at the top of Mount Everest, so the earth's gravity force on an object was a fine measure of quantity.

When it became important to talk about mass, as opposed to weight, the pound mass was defined as the mass of something that weighed a pound. That is,

$$1\,\mathrm{lbm} \equiv 1\,\mathrm{lbf}/g.$$

Then people thought 'what is the mass that accelerates one foot per second squared if a one-pound force is applied?' They found

$$\begin{aligned} m &= F/a \\ &= (1\,\mathrm{lbf})/(1\,\mathrm{ft/s^2}) = 1\left(\mathrm{lbf}/\,\mathrm{ft/s^2}\right)\underbrace{\left(\frac{32.174\,\mathrm{lbm\,ft/s^2}}{\mathrm{lbf}}\right)}_{1} \\ &= 32.174\,\mathrm{lbm}. \end{aligned}$$

But this 32.174 was awkward. People felt that if a unit force causes something to accelerate at a unit rate that thing should have a unit mass. So they invented the slug. $1\,\mathrm{slug} \equiv 1\,\mathrm{lbf}/(1\,\mathrm{ft/s^2})$. So what do we get for the mass in the previous equation?

$$\begin{aligned} m &= F/a \\ &= (1\,\mathrm{lbf})/(1\,\mathrm{ft/s^2}) = 1\left(\mathrm{lbf/ft/s^2}\right)\underbrace{\left(\frac{32.174\,\mathrm{lbm\,ft/s^2}}{\mathrm{lbf}}\right)}_{1} \\ &\stackrel{def}{=} 1\,\mathrm{slug} \end{aligned}$$

That is, 1 slug accelerates $1\,\mathrm{ft/s^2}$ when 1 lbf is applied. How much does a slug weigh? The force of gravity on a slug, in Potsdam, is 32.174 lbf.

Now the invention of the slug did not make people happy enough. They thought, 'what is the force required to accelerate 1 lbm at an acceleration of $1\,\mathrm{ft/s^2}$?' It is

$$\begin{aligned} F &= ma \\ &= (1\,\mathrm{lbm})(1\,\mathrm{ft/s^2}) = 1\left(\mathrm{lbm\,ft/s^2}\right)\underbrace{\left(\frac{1\,\mathrm{lbf}}{32.174\,\mathrm{lbm\,ft/s^2}}\right)}_{1} \\ &= \left(\frac{1}{32.174}\right)\mathrm{lbf}. \end{aligned}$$

People found *this* 1/32.174 awkward also, so in order to simplify some arithmetic and confuse many generations of engineers, they invented the poundal. They defined the poundal to be the force it takes to accelerate one pound mass at one foot per second squared. So they got

$$\begin{aligned} F &= ma \\ &= (1\,\mathrm{lbm})(1\,\mathrm{ft/s^2}) \stackrel{def}{=} 1\,\mathrm{pdl}. \end{aligned}$$

So, because scientists and engineers of old liked the number 1 better than both the number 32.174 and the number 1/32.174 they left us two new units to worry about: the poundal $= 1\,\mathrm{lbm\,ft/s^2} = (1/32.174)\,\mathrm{lbf}$, and the slug $= 1\,\mathrm{lbf}/(\,\mathrm{ft/s^2}) = 32.174\,\mathrm{lbm}$. If you are used to the internationally acceptable units for force and mass $1\,\mathrm{pdl} = .138255\,\mathrm{N}$ and $1\,\mathrm{slug} = 14.5939\,\mathrm{kg}$. Fortunately, the slug and the poundal are used less and less as the decades roll by. Certainly there are more people who laugh at their confusion about slugs and poundals than there are people who use them seriously.

**Don't laugh if you are from Europe** Unfortunately for dimensional purists, engineers using the SI system have copied one of the confusing traditions that the SI system was designed to avoid. They invented the kilogram-force, kgf, also called a kilopond, which is 1 kg times the official value of $g$. That is $1\,\mathrm{kgf} = 1\,\mathrm{kilopond} = 9.80665\,\mathrm{N}$. A kilopond is the force of gravity on a kilogram, exactly so somewhere in Potsdam — well, almost.

**Well, almost** Why do we say 'well, almost' about '$g$' being the acceleration due to gravity? Because, unfortunately and confusingly, $mg$ is not the force due to gravity, it is the force of the spring which holds up the mass on a rotating earth! What is called $g$ is the 'effective' gravity which is the acceleration due to gravity minus a centripetal term due to the earth's rotation.

# Answers to *'d problems

**2.55)** $r_x = \vec{r} \cdot \hat{\imath} = (3\cos\theta + 1.5\sin\theta)$ ft, $r_y = \vec{r} \cdot \hat{\jmath} = (3\sin\theta - 1.5\cos\theta)$ ft.

**2.77)** No partial credit.

**2.78)** To get chicken road sin theta.

**2.83)** $\vec{N}\frac{1000\text{N}}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} + \hat{k})$.

**2.86)** $d = \sqrt{\frac{3}{2}}$.

**2.90a)** $\hat{\lambda}_{OB} = \frac{1}{\sqrt{50}}(4\hat{\imath} + 3\hat{\jmath} + 5\hat{k})$.

**b)** $\hat{\lambda}_{OA} = \frac{1}{\sqrt{34}}(3\hat{\jmath} + 5\hat{k})$.

**c)** $\vec{F}_1 = \frac{5\text{N}}{\sqrt{34}}(3\hat{\jmath} + 5\hat{k})$, $\vec{F}_2 = \frac{7\text{N}}{\sqrt{50}}(4\hat{\imath} + 3\hat{\jmath} + 5\hat{k})$.

**d)** $\angle AOB = 34.45\,\text{deg}$.

**e)** $F_{1_x} = 0$

**f)** $\vec{r}_{DO} \times \vec{F}_1 = \left(\frac{100}{\sqrt{34}}\hat{\jmath} - \frac{60}{\sqrt{34}}\hat{k}\right)$ N·m.

**g)** $M_\lambda = \frac{140}{\sqrt{50}}$ N·m.

**h)** $M_\lambda = \frac{140}{\sqrt{50}}$ N·m.(same as (7))

**2.92a)** $\hat{n} = \frac{1}{3}(2\hat{\imath} + 2\hat{\jmath} + \hat{k})$.

**b)** $d = 1$.

**c)** $\frac{1}{3}(-2, 19, 11)$.

**2.94)** $\ell/\sqrt{2}$

**2.110)** Yes.

**2.122a)** $\vec{r}_2 = \vec{r}_1 + \vec{F}_1 \times \hat{k}M_1/|\vec{F}_1|^2$, $\vec{F}_2 = \vec{F}_1$.

**b)** $\vec{r}_2 = \vec{r}_1 + \vec{F}_1 \times \hat{k}M_1/|\vec{F}_1|^2 + c\vec{F}_1$ where $c$ is any real number, $\vec{F}_2 = \vec{F}_1$.

**c)** $\vec{F}_2 = \vec{0}$ and $\vec{M}_2 = \vec{M}_1$ applied at any point in the plane.

**2.123a)** $\vec{r}_2 = \vec{r}_1 + \vec{F}_1 \times \vec{M}_1/|\vec{F}_1|^2$, $\vec{F}_2 = \vec{F}_1$, $\vec{M}_2 = \vec{M}_1 \cdot \vec{F}_1\vec{F}_1/|\vec{F}_1|^2$. If $\vec{F}_1 = \vec{0}$ then $\vec{F}_2 = \vec{0}$, $\vec{M}_2 = \vec{M}_1$, and $\vec{r}_2$ is any point at all in space.

**b)** $\vec{r}_2 = \vec{r}_1 + \vec{F}_1 \times \vec{M}_1/|\vec{F}_1|^2 + c\vec{F}_1$ where $c$ is any real number, $\vec{F}_2 = \vec{F}_1$, $\vec{M}_2 = \vec{M}_1 \cdot \vec{F}_1 \vec{F}_1/|\vec{F}_1|^2$. See above for the special case of $\vec{F}_1 = \vec{0}$.

**2.124)** $(0.5\,\text{m},\ -0.4\,\text{m})$

**3.1a)** The forces and moments that show on a free body diagram, the *external* forces and moments.

**b)** The forces and moments that show on a free body diagram, the *external* forces and moments. No "inertial" or "acceleration" forces show.

**3.2)** You don't.

**3.12)** Note, no couples show on any of the free body diagrams requested.

**4.5)** $T_1 = Nmg$, $T_2 = (N-1)mg$, $T_N = (1)mg$, and in general $T_n = (N+1-n)mg$

**4.23)** (a)$T_{AB} = 30\,\text{N}$, (b) $T_{AB} = \frac{300}{17}\,\text{N}$, (c)$T_{AB} = \frac{5\sqrt{26}}{2}\,\text{N}$

**4.59)** $\theta \geq \tan^{-1}\left((1-\mu^2)/2\mu\right)$

**4.62)** For this device to hold, $\mu \geq 1$. (Demanding $\mu \geq 1$ is large for a practical device because typical rock friction has $\mu \approx 0.5$. The too-large number follows from the simplified geometry and numbers chosen for a homework problem.)

**4.66)** $T_{AB} = \sqrt{10}\mu mg/(3+\mu)$

**4.66)** Minimum tension if rope slope is $\mu$ (instead of $1/3$)

**4.68a)** $\frac{m}{M} = \frac{R\sin\theta}{R\cos\theta + r} = \frac{2\sin\theta}{1+2\cos\theta}$.

**b)** $T = mg = 2Mg\frac{\sin\theta}{1+2\cos\theta}$.

**c)** $\vec{F}_C = Mg\left[-\frac{2\sin\theta}{2\cos\theta+1}\hat{\imath}' + \hat{\jmath}'\right]$ (where $\hat{\imath}'$ and $\hat{\jmath}'$ are aligned with the horizontal and vertical directions)

**c)** $\tan\phi = \frac{\sin\theta}{2+\cos\theta}$. Needs somewhat involved trigonometry, geometry, and algebra.

**d)** $\tan\psi = \frac{m}{M} = \frac{2\sin\theta}{1+2\cos\theta}$.

**4.69a)** $\frac{m}{M} = \frac{R\sin\theta}{R\cos\theta r} = \frac{2\sin\theta}{2\cos\theta - 1}$.

**b)** $T = mg = 2Mg\frac{\sin\theta}{2\cos\theta - 1}$.

**c)** $\vec{F}_C = \frac{Mg}{1-2\cos\theta}\left[\sin\theta\hat{\imath} + (\cos\theta - 2)\hat{\jmath}\right]$.

**4.70a)** $\frac{F_1}{F_2} = \frac{R_o + R_i\sin\phi}{R_o - R_i\sin\phi}$

**b)** For $R_o = 3R_i$ and $\mu = 0.2$, $\frac{F_1}{F_2} \approx 1.14$.

**4.75)** None are true. The tension is 100 N.

**4.90)** Maximum overhang when $n \to \infty is\ell$.

**4.93)** Assuming no side-loads from floor the support from leg AB is 250 N, $T_{AB} = -250\,\text{N}$.

**4.94)** $T_{IE} = mg/2$, $T_{CH} = \sqrt{2}mg/2$, $T_{BH} = -mg/2$, $A_x = mg/2$, $A_y = mg/2$, $A_z = mg$

**4.97g)** $T_{EH} = 0$ as you can find a number of ways.

**4.98a)** Use axis EC.

**b)** Use axis AH.

**c)** Use $\hat{\boldsymbol{j}}$ axis through B.

**d)** Use axis DE.

**e)** Use axis EH.

**f)** Can't do in one shot.

**4.99)** $T_{AC} = -\sqrt{2}mg = -1000\sqrt{2}\,\text{N} \approx -1410\,\text{N}$ (the bar is in compression)

**4.99)** $T_{IP} = 0$

**4.99)** $T_{KL} = \sqrt{2}mg/6 = \left(1000\sqrt{2}/6\right) N \approx 408\,\text{N}$ (the bar is in tension)

**4.101)** Hint: With reference to a free body diagram of the robot, use moment balance about axis BC.

**5.9)** $T_{AC} = -1000\,\text{N}$, (AC is in compression)

**5.10)** $T_{AB} = 173\,\text{N}$

**5.13)** 12 of the 15 bars are zero-force members; all but BD, DG, and GJ. The others carry no load but are needed for stability.

**5.36)** $T_{EB} = -11F/2$

**5.36)** $T_{HI} = -11bF/2a$

**5.36)** $T_{JK} = -35bF/2a$, (more than 3 times the compression of HI)

**6.1)** 1000 N

**6.2)** 0.08 cm

**6.3)** 1160 N

**6.4)** 5 cm

**6.5)** $k_e = 66.7\,\text{N/cm},\ \delta = 0.75\,\text{cm}$

**6.7)** $k = 20\,\text{N/cm}$

**6.8)** Middle spring: $\delta = 1\,\text{cm}$; side-springs $\delta = 0.5\,\text{cm}$

**6.12)** Surprise! This pendulum is in equilibrium for all values of $\theta$.

**6.37)** 200 N

**6.48)** $N = (h(w+d)/d\ell)\,F_h$

**6.55)** Either by looking at part KAP or at part BAQ, if we think of moment balance about A we see that the cutting force has to fight about twice the torque in the gear mechanism as in the ungeared mechanism. For example KAP is aided in its cutting by the torque from the force at G.

**6.56)** The mechanism multiplies the force at B and C by a factor of 2 compared to having the handle hinged at A. The force at G also gets (a shade less than) this force but with half the lever arm. Together they give a force multiplication of (a shade less than) 2+1=3.

**6.57)** $F_P = 125\,\text{N}$

**6.57)** $F_P = 125\,\text{N}$

**6.57)** For the load at I, $F_P = 75\,\text{N}$. For the load at J, $F_P = 250N$.

**6.57)** With the welded handle there is just a simple lever and the mechanical advantage comes from the horizontal distance between the load and hinge A. For the 4 bar mechanism the force at C is the applied vertical load, no matter where it is applied. So the lever arm is the horizontal distance from A to C.

**6.58)** $F_A = 500\,\text{lbf}$

**6.59d)** reduce the dimension marked "2 inches". The smaller the less the friction needed.

**e)** As the "2 inch" dimension is reduced to zero, the needed coefficient of friction goes to zero and the forces squeezing the pipe go to infinity. This is bad because it can damage the pipe. It is also bad because a small pipe deformation will cause the hinge on the wrench to snap through, like a so called "toggle mechanism" and thus not grab at all.

**6.60)** $\vec{\boldsymbol{R}}_A = \vec{\mathbf{0}}$

**6.60)** $T = 200\,\text{lbf}$

**6.62)** $F_D = \ell_{EC}(\ell_{EH} - d)F/d\ell_{CD}$

**6.62)** $T_{CC'} = (\ell_{EH}/d - 1)\,(\ell_{EC}/\ell_{CD} + 1)\,F$

**6.62)** As $d \to 0$, $F_D \to \infty$. Two problems: the amount of motion goes to zero and the assumption of rigidity becomes non-negligibly inaccurate.

**6.63)** $F_N\left(b(a^2 + b^2)/a^2)\right)F = 130F = 1300\,\text{lbf}$

**6.63)** The mechanism uses three tricks to multiply the force: a lever, a wedge, and a toggle. Each of these multiplies by about 5. Thus the nut-force $F_N$ is on the order of $5^3 = 125$ times as big as $F$.

**7.3)** $(117\gamma/2)\,\text{m}^3 = 5.85 * 10^5\,\text{N}$

**7.4)** Water starts to spill at $h = 3r_{AB} = 3\,\text{m}$.

**7.4)** Assuming no friction at B, $\vec{\boldsymbol{F}}_A = 2.25 * 10^5\hat{\boldsymbol{\imath}}\,\text{N}$

**7.9a)** $\rho g\pi r^2\ell$

**b)** $-\rho g\pi r^2(h - \ell)$, note the minus sign, it now takes force to lift the can.

**8.14)** $F_{Ay} = -500\,\text{N}$, $M_A = -500/3\,\text{N·m}$

**8.15)** $V(\ell/2) = -w\ell/8$, $M(\ell/2) = w\ell^2/16$, $M_{max} = M(3\ell/8) = 9wl^2/128$

**8.17b)** [Hint: at every height $y$ the cross sectional area must be big enough to hold the weight plus the wire below that point. From this you can set up and a differential equation for the cross sectional area $A$ as a function of $y$. Find appropriate initial conditions and solve the equation. Once solved, the volume of wire can be calculated as $V = \int_0^1 0\,\text{mi}A(y)dy$ and the mass as $\rho V$.]

**9.11)** $x(3\,\text{s}) = 20\,\text{m}$

**9.15)** (a) $v(3\,\text{s}) = 2\,\text{m/s}$ in each case. (b) $x(3\,\text{s}) = 3\,\text{m}$ for case (a), $x(3\,\text{s}) = 4\,\text{m}$ for case (b).

**9.16)** $F_s = \frac{\pi}{4} F_T$

**9.48)** Time span $= 3\pi\sqrt{m/k}/2$

**9.51)** (a) $m\ddot{x} + kx = F(t)$, (b) $m\ddot{x} + kx = F(t)$, and (c) $m\ddot{y} + 2ky - 2k\ell_0 \frac{y}{\sqrt{\ell_0^2+y^2}} = F(t)$

**9.53b)** $mg - k(x - \ell_0) = m\ddot{x}$

**c)** $\ddot{x} + \frac{k}{m}x = g + \frac{k\ell_0}{m}$

**e)** This solution is the static equilibrium position; i.e., when the mass is hanging at rest, its weight is exactly balanced by the upwards force of the spring at this constant position $x$.

**f)** $\ddot{\hat{x}} + \frac{k}{m}\hat{x} = 0$

**g)** $x(t) = \left[D - \left(\ell_0 + \frac{mg}{k}\right)\right]\cos\sqrt{\frac{k}{m}}\,t + (\ell_0 + \frac{mg}{k})$

**h)** period=$2\pi\sqrt{\frac{m}{k}}$.

**i)** If the initial position $D$ is more than $\ell_0 + 2mg/k$, then the spring is in compression for part of the motion. A floppy spring would buckle.

**9.55a)** period= $\frac{2\pi}{\sqrt{\frac{k}{m}}} = 0.96\,\text{s}$

**b)** maximum amplitude=0.75 ft

**c)** period= $2\sqrt{\frac{2h}{g}} + \sqrt{\frac{m}{k}}\left[\pi + 2\tan^{-1}\sqrt{\frac{mg}{2kh}}\right] \approx 1.64\,\text{s}$.

**9.56)** LHS of Linear Momentum Balance: $\sum \vec{F} = -(kx + b\dot{x})\hat{\imath} + (N - mg)\hat{\jmath}$.

**9.70a)** Two normal modes.

**b)** $x_2 = const * x_1 = const * (A\sin(ct) + B\cos(ct))$,where $const = \pm 1$.

**c)** $\omega_1 = \sqrt{\frac{3k}{m}}$, $\omega_2 = \sqrt{\frac{k}{m}}$.

**9.71b)** If we start off by assuming that each mass undergoes simple harmonic motion at the same frequency but different amplitudes, we will find that this two-degree-of-freedom system has two natural frequencies. Associated with each natural frequency is a fixed ratio between the amplitudes of each mass. Each mass will undergo simple harmonic motion at one of the two natural frequencies only if the initial displacements of the masses are in the fixed ratio associated with that frequency.

**9.73)** $\vec{a}_B = \ddot{x}_B\hat{\imath} = \frac{1}{m_B}[-k_4x_B - k_2(x_B - x_A) + c_1(\dot{x}_D - \dot{x}_B) + k_3(x_D - x_B)]\hat{\imath}$.

**9.74)** $\vec{a}_B = \ddot{x}_B\hat{\imath} = \frac{1}{m_B}[-k_4x_B - c_1(\dot{x}_B - \dot{x}_A) + (k_2 + k_3)(x_D - x_B)]$.

**9.77a)** $\omega = \sqrt{\frac{2k}{m}}$.

**9.81a)** One normal mode: [1, 0, 0].

**b)** The other two normal modes: [0, 1, $\frac{1\pm\sqrt{17}}{4}$].

TABLE I
**Momenta and energy**

| What system | Linear Momentum $\vec{L}$ (a) | Linear Momentum $\dot{\vec{L}} = \frac{d}{dt}\vec{L}$ (b) | Angular Momentum $\vec{H}_{\rm C}$ (c) | Angular Momentum $\dot{\vec{H}}_{\rm C} = \frac{d}{dt}\vec{H}_{\rm C}$ (d) | Kinetic Energy $E_{\rm K}$ (e) | |
|---|---|---|---|---|---|---|
| In General | $L_x\hat{\imath} + L_y\hat{\jmath} + L_z\hat{k}$ $= m_{\rm tot}\vec{v}_{\rm cm}$ $= \frac{d}{dt}.m_{\rm tot}\vec{r}_{\rm cm}/$ | $\dot{L}_x\hat{\imath} + \dot{L}_y\hat{\jmath} + \dot{L}_z\hat{k}$ $= m_{\rm tot}\vec{a}_{\rm cm}$ "F = ma" | $H_{Cx}\hat{\imath} + H_{Cy}\hat{\jmath} + H_{Cz}\hat{k}$ $= \vec{r}_{\rm cm/C} \times \vec{v}_{\rm cm} m_{\rm tot} + \vec{H}_{\rm cm}$ $= \frac{d}{dt}.$no such thing/ | $\dot{H}_{Cx}\hat{\imath} + \dot{H}_{Cy}\hat{\jmath} + \dot{H}_{Cz}\hat{k}$ $= \vec{r}_{\rm cm/C} \times \vec{a}_{\rm cm} m_{\rm tot} + \dot{\vec{H}}_{\rm cm}$ = (no simple general expression) | $\frac{1}{2} m_{\rm tot} v_{\rm cm}^2 + E_{{\rm K}/cm}$ | (1) |
| One Particle P | $m_{\rm P}\vec{v}_{\rm P}$ | $m_{\rm P}\vec{a}_{\rm P}$ | $\vec{r}_{\rm P/C} \times \vec{v}_P m_{\rm P}$ | $\vec{r}_{\rm P/C} \times \vec{a}_{\rm P} m_{\rm P}$ | $\frac{1}{2} m_{\rm P} v_{\rm P}^2$ | (2) |
| System of Particles | $\sum_{\text{all particles i}} m_i \vec{v}_i$ | $\sum_{\text{all particles i}} m_i \vec{a}_i$ | $\sum_{\text{all particles}} \vec{r}_{i/\rm C} \times \vec{v}_i m_i$ | $\sum_{\text{all particles}} \vec{r}_{i/\rm C} \times \vec{a}_i m_i$ | $\frac{1}{2} \sum_{\text{all particles}} v_i^2 m_i$ | (3) |
| Continuum | $\int_{\text{all mass}} \vec{v}\, dm$ | $\int_{\text{all mass}} \vec{a}\, dm$ | $\int_{\text{all mass}} \vec{r}_{/\rm C} \times \vec{v}\, dm$ | $\int_{\text{all mass}} \vec{r}_{/\rm C} \times \vec{a}\, dm$ | $\frac{1}{2}\int_{\text{all mass}} v^2\, dm$ | (4) |
| System of Systems (eg. rigid bodies) | $\sum_{\text{all sub-systems}} m_i \vec{v}_i$ | $\sum_{\text{all sub-systems}} m_i \vec{a}_i$ | $\sum_{\text{all sub-systems}} \vec{H}_{{\rm C}i}$ | $\sum_{\text{all sub-systems}} \dot{\vec{H}}_{{\rm C}i}$ | $\sum_{\text{all sub-systems}} E_{{\rm K}i}$ | (5) |
| **Rigid Bodies** | | | | | | |
| One rigid body (2D and 3D) | $m_{\rm tot}\vec{v}_{\rm cm}$ | $m_{\rm tot}\vec{a}_{\rm cm}$ | $\vec{r}_{\rm cm/C} \times \vec{v}_{\rm cm} m_{\rm tot} + \underbrace{[\mathbf{I}^{cm}] \cdot \vec{\omega}}_{\vec{H}_{\rm cm}}$ | $\vec{r}_{\rm cm/C} \times \vec{a}_{\rm cm} m_{\rm tot} + \underbrace{[\mathbf{I}^{cm}] \cdot \dot{\vec{\omega}} + \vec{\omega} \times \vec{H}_{\rm cm}}_{\dot{\vec{H}}_{\rm cm}}$ | $\frac{1}{2} m_{\rm tot} v_{\rm cm}^2 + \underbrace{\frac{1}{2}\vec{\omega} \cdot [\mathbf{I}^{cm}] \cdot \vec{\omega}}_{E_{{\rm K}/cm}}$ | (6) |
| 2D rigid body in $xy$ plane with $\vec{\omega} = \omega\hat{k}$ | $m_{\rm tot}\vec{v}_{\rm cm}$ | $m_{\rm tot}\vec{a}_{\rm cm}$ | $\vec{r}_{\rm cm/C} \times \vec{v}_{\rm cm} m_{\rm tot} + \underbrace{I_{zz}^{cm}\omega\hat{k}}_{\vec{H}_{\rm cm}}$ | $\vec{r}_{\rm cm/C} \times \vec{a}_{\rm cm} m_{\rm tot} + \underbrace{I_{zz}^{cm}\dot{\omega}\hat{k}}_{\dot{\vec{H}}_{\rm cm}}$ | $\frac{1}{2} m_{\rm tot} v_{\rm cm}^2 + \underbrace{\frac{1}{2} I_{zz}^{cm}\omega^2}_{E_{{\rm K}/cm}}$ | (7) |
| One rigid body *if* C is a fixed point (2D and 3D) | $m_{\rm tot}\vec{v}_{\rm cm}$ | $m_{\rm tot}\vec{a}_{\rm cm}$ | $[\mathbf{I}^C] \cdot \vec{\omega} = \vec{H}_C$ | $[\mathbf{I}^C] \cdot \dot{\vec{\omega}} + \vec{\omega} \times \vec{H}_C$ | $\frac{1}{2}\vec{\omega} \cdot [\mathbf{I}^C] \cdot \vec{\omega}$ | (8) |
| 2D rigid body *if* C is a fixed point with $\vec{\omega} = \omega\hat{k}$ | $m_{\rm tot}\vec{v}_{\rm cm}$ | $m_{\rm tot}\vec{a}_{\rm cm}$ | $I_{zz}^C\omega\hat{k}$ | $I_{zz}^C\dot{\omega}\hat{k}$ "$M = I\alpha$" | $\frac{1}{2} I_{zz}^C\omega^2$ | (9) |

The table has used the following terms:
$m_{\rm tot}$ =total mass of system,
$m_i$ = mass of body or subsystem i,
$\vec{r}_{\rm cm/C}$ = the position of the center of mass relative to point $C$,
$\vec{v}_i$ = velocity of the center of mass of sub-system or particle $i$,
$\vec{a}_i$ = acceleration of the center of mass of sub-system $i$,
$\vec{H}_{{\rm C}i}$ = angular momentum of subsystem $i$ relative to point C.
$\dot{\vec{H}}_{{\rm C}i}$ = rate of change of angular momentum of sub-system $i$ relative to point C.
$\vec{H}_{\rm cm} = \sum \vec{r}_{i/\rm cm} \times .m_i \vec{v}_i/$ angular momentum about the center of mass
$\dot{\vec{H}}_{\rm cm} = \sum \vec{r}_{i/\rm cm} \times .m_i \vec{a}_i/$ rate of change of angular momentum about the center of mass
$\vec{\omega}$ is the angular velocity of a rigid body,
$\dot{\vec{\omega}} = \vec{\alpha}$ is the angular acceleration of the rigid body,
$[\mathbf{I}^{cm}]$ is the moment of inertia matrix of the rigid body relative to the center of mass, and
$[\mathbf{I}^o]$ is the moment of inertia matrix of the rigid body relative to a fixed point (not moving point) on the body.

Table II
## Summary of methods of calculating velocity and acceleration

| Method | Position | Velocity | Acceleration |
|---|---|---|---|
| In general, as measured relative to the fixed frame $\mathcal{F}$. | $\vec{\boldsymbol{r}}$ or $\vec{\boldsymbol{r}}_P$ or $\vec{\boldsymbol{r}}_{P/O}$ | $\vec{\boldsymbol{v}}$ or $\vec{\boldsymbol{v}}_P$ or $\vec{\boldsymbol{v}}_{P/\mathcal{F}}$ | $\vec{\boldsymbol{a}}$ or $\vec{\boldsymbol{a}}_P$ or $\vec{\boldsymbol{a}}_{P/\mathcal{F}}$ |
| Cartesian Coordinates | $r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}} + r_z\hat{\boldsymbol{k}}$ | $v_x\hat{\boldsymbol{\imath}} + v_y\hat{\boldsymbol{\jmath}} + v_z\hat{\boldsymbol{k}}$ $= \dot{r}_x\hat{\boldsymbol{\imath}} + \dot{r}_y\hat{\boldsymbol{\jmath}} + \dot{r}_z\hat{\boldsymbol{k}}$ | $a_x\hat{\boldsymbol{\imath}} + a_y\hat{\boldsymbol{\jmath}} + a_z\hat{\boldsymbol{k}}$ $= \dot{v}_x\hat{\boldsymbol{\imath}} + \dot{v}_y\hat{\boldsymbol{\jmath}} + \dot{z}_z\hat{\boldsymbol{k}}$ $= \ddot{r}_x\hat{\boldsymbol{\imath}} + \ddot{r}_y\hat{\boldsymbol{\jmath}} + \ddot{r}_z\hat{\boldsymbol{k}}$ |
| Polar Coordinates/ Cylindrical Coordinates | $R\hat{\boldsymbol{e}}_R + z\hat{\boldsymbol{k}}$ | $v_R\hat{\boldsymbol{e}}_R + v_\theta\hat{\boldsymbol{e}}_\theta + v_z\hat{\boldsymbol{k}}$ $= \dot{R}\hat{\boldsymbol{e}}_R + R\dot{\theta}\hat{\boldsymbol{e}}_\theta + \dot{z}\hat{\boldsymbol{k}}$ | $a_R\hat{\boldsymbol{e}}_R + a_\theta\hat{\boldsymbol{e}}_\theta + a_z\hat{\boldsymbol{k}}$ $= .\ddot{R} \quad R\dot{\theta}^2/\hat{\boldsymbol{e}}_R + .R\ddot{\theta} + 2\dot{R}\dot{\theta}/\hat{\boldsymbol{e}}_\theta + \ddot{z}\hat{\boldsymbol{k}}$ |
| Path Coordinates | not used | $v\hat{\boldsymbol{e}}_t$ | $a_t\hat{\boldsymbol{e}}_t + a_n\hat{\boldsymbol{e}}_n$ $= \dot{v}\hat{\boldsymbol{e}}_t + .v^2/\rho/\hat{\boldsymbol{e}}_n$ |
| Using data from a moving frame $\mathcal{B}$ with origin at $O'$ and angular velocity relative to the fixed frame of $\vec{\boldsymbol{\omega}}_\mathcal{B}$. The point $P'$ is glued to $\mathcal{B}$ and instantaneously coincides with $P$. | $\vec{\boldsymbol{r}}_{O'/O} + \vec{\boldsymbol{r}}_{P/O'}$ | $\vec{\boldsymbol{v}}_{P'/\mathcal{F}} + \vec{\boldsymbol{v}}_{P/\mathcal{B}} =$ $\underbrace{\dot{\vec{\boldsymbol{r}}}_{O'/O} + \vec{\boldsymbol{\omega}}_\mathcal{B} \times \vec{\boldsymbol{r}}_{P'/O'}}_{\vec{\boldsymbol{v}}_{P'/\mathcal{F}}} + \underbrace{{}^{\mathcal{B}}\dot{\vec{\boldsymbol{r}}}_{P/O'}}_{\vec{\boldsymbol{v}}_{P/\mathcal{B}}}$ | $\vec{\boldsymbol{a}}_{P'/\mathcal{F}} + \vec{\boldsymbol{a}}_{P/\mathcal{B}} + 2\vec{\boldsymbol{\omega}}_\mathcal{B} \times \vec{\boldsymbol{v}}_{P/\mathcal{B}} =$ $\overbrace{\ddot{\vec{\boldsymbol{r}}}_{O'/O} + \dot{\vec{\boldsymbol{\omega}}}_\mathcal{B} \times \vec{\boldsymbol{r}}_{P'/O'} + \vec{\boldsymbol{\omega}}_\mathcal{B} \times \vec{\boldsymbol{\omega}}_\mathcal{B} \times \vec{\boldsymbol{r}}_{P'/O'}}^{\vec{\boldsymbol{a}}_{P'/\mathcal{F}}}$ $+ \underbrace{{}^{\mathcal{B}}\ddot{\vec{\boldsymbol{r}}}_{P/O'}}_{\vec{\boldsymbol{a}}_{P/\mathcal{B}}} + 2\vec{\boldsymbol{\omega}}_\mathcal{B} \times \vec{\boldsymbol{v}}_{P/\mathcal{B}}$ 'the 5-term acceleration formula' |

## Some facts about path coordinates

The path of a particle is $\vec{\boldsymbol{r}}.t/$.

$$\hat{\boldsymbol{e}}_t \equiv \frac{d\vec{\boldsymbol{r}}.s/}{ds}, \quad \hat{\boldsymbol{e}}_t = \frac{d\vec{\boldsymbol{r}}.t/}{dt}\frac{dt}{ds} = \frac{\vec{\boldsymbol{v}}}{v}, \quad \vec{\boldsymbol{\kappa}} \equiv \frac{d\hat{\boldsymbol{e}}_t}{ds} = \frac{d\hat{\boldsymbol{e}}_t}{dt}\frac{1}{v}, \quad \hat{\boldsymbol{e}}_n = \frac{\vec{\boldsymbol{\kappa}}}{|\vec{\boldsymbol{\kappa}}|}, \quad \boldsymbol{e}_b \equiv \hat{\boldsymbol{e}}_t \times \hat{\boldsymbol{e}}_n, \quad \rho = \frac{1}{|\vec{\boldsymbol{\kappa}}|}.$$

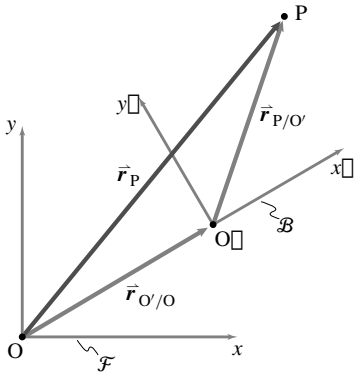
## Summary of the direct differentiation method

In the direct differentiation method, using moving frame $\mathcal{B}$, we calculate $\vec{\boldsymbol{v}}_P$ by using a combination of the product rule of differentiation and the facts that

$$\dot{\hat{\boldsymbol{\imath}}}' = \vec{\boldsymbol{\omega}}_\mathcal{B} \times \hat{\boldsymbol{\imath}}', \quad \dot{\hat{\boldsymbol{\jmath}}}' = \vec{\boldsymbol{\omega}}_\mathcal{B} \times \hat{\boldsymbol{\jmath}}', \text{ and } \dot{\hat{\boldsymbol{k}}}' = \vec{\boldsymbol{\omega}}_\mathcal{B} \times \hat{\boldsymbol{k}}',$$

as follows:

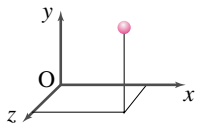
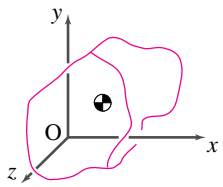
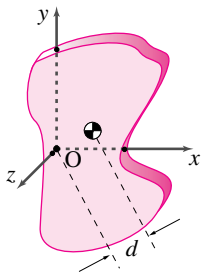
$$\begin{aligned}
\vec{\boldsymbol{v}}_P &= \frac{d}{dt}\vec{\boldsymbol{r}}_P \\
&= \frac{d}{dt}\left[\vec{\boldsymbol{r}}_{O'/O} + \vec{\boldsymbol{r}}_{P/O'}\right. \\
&= \frac{d}{dt}\left[.x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}/ + .x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}' + z'\hat{\boldsymbol{k}}'/\right] \\
&= .\dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}} + \dot{z}\hat{\boldsymbol{k}}/ + .\dot{x}'\hat{\boldsymbol{\imath}}' + \dot{y}'\hat{\boldsymbol{\jmath}}' + \dot{z}'\hat{\boldsymbol{k}}'/ + \\
&\quad \left[x'.\vec{\boldsymbol{\omega}}_\mathcal{B} \times \hat{\boldsymbol{\imath}}'/ + y'.\vec{\boldsymbol{\omega}}_\mathcal{B} \times \hat{\boldsymbol{\jmath}}'/ + z'.\vec{\boldsymbol{\omega}}_\mathcal{B} \times \hat{\boldsymbol{k}}'/\right]
\end{aligned}$$




but stop short of identifying these three groups of three terms as $\vec{\boldsymbol{v}}_P = \vec{\boldsymbol{v}}_{O'/O} + \dot{\vec{\boldsymbol{r}}}_{rel} + \vec{\boldsymbol{\omega}}_\mathcal{B} \times \vec{\boldsymbol{r}}_{P/O}$.

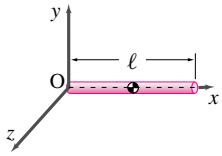
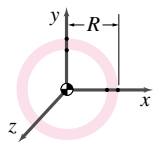
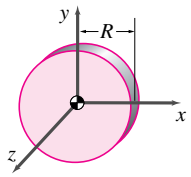
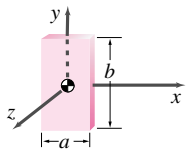
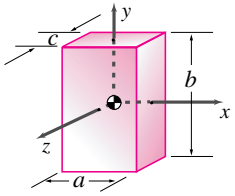
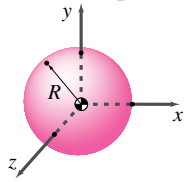
We could calculate $\vec{\boldsymbol{a}}_P$ similarly and would get a similar formula with 15 non-zero terms (3 for each term in the 'five-term' acceleration formula).

Table III

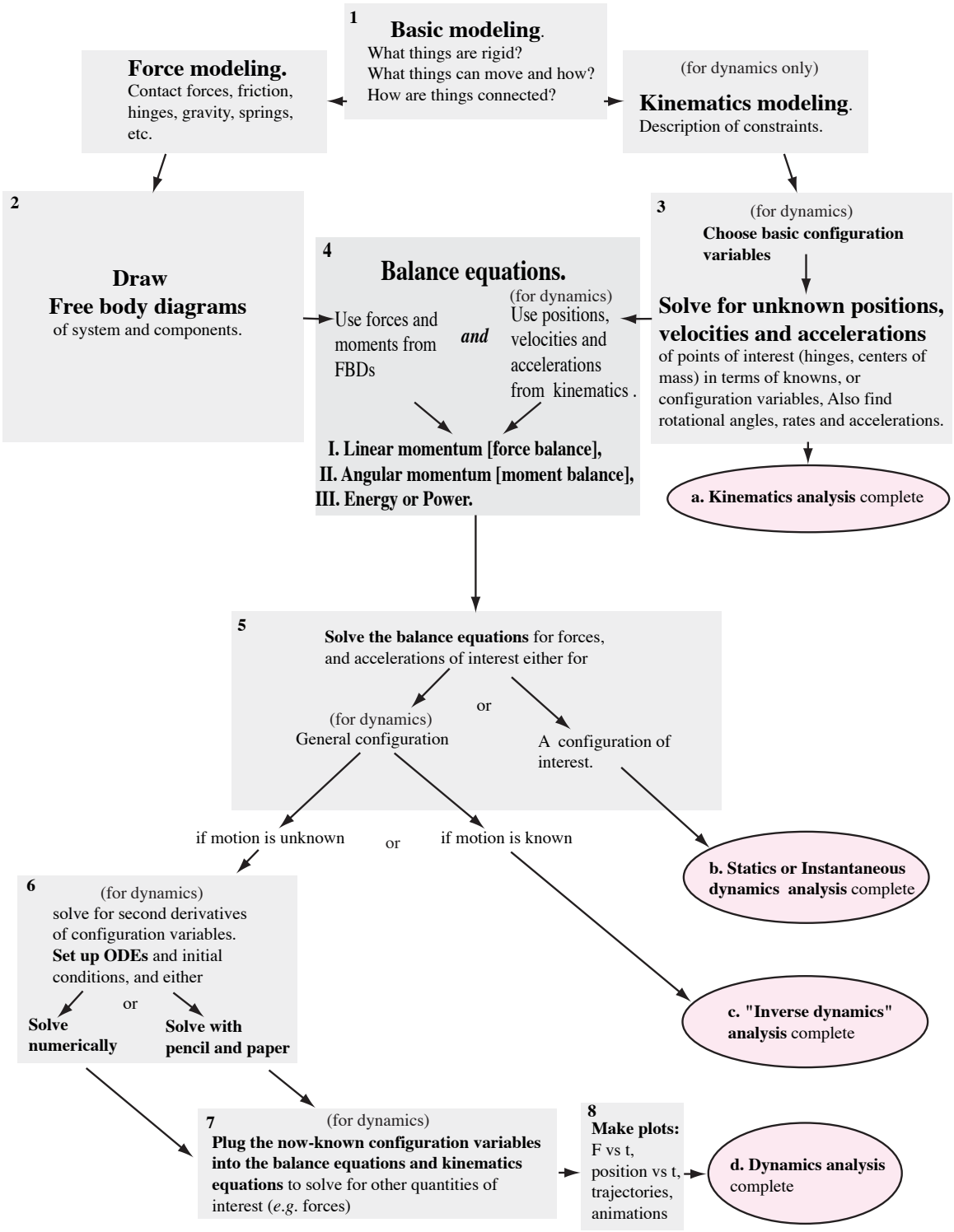
| Object | $[I]$ |
|---|---|
| Point mass   | $[I^{cm}] = m \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ <br> $[I^{O}] = m \begin{bmatrix} y^2+z^2 & xy & xz \\ xy & x^2+z^2 & yz \\ xz & yz & x^2+y^2 \end{bmatrix}$ |
| General 3D body   | $[I^{cm}] = \int \begin{bmatrix} y_{/cm}^2+z_{/cm}^2 & x_{/cm}y_{/cm} & x_{/cm}z_{/cm} \\ x_{/cm}y_{/cm} & x_{/cm}^2+z_{/cm}^2 & y_{/cm}z_{/cm} \\ x_{/cm}z_{/cm} & y_{/cm}z_{/cm} & x_{/cm}^2+y_{/cm}^2 \end{bmatrix} dm$ <br> If the axes are principal axes of the body. $[I^{cm}] = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}$ With $A, B, C \geq 0$ and $A+B \geq C$, $B+C \geq A$, and $A+C \geq B$. <br> $[I^{o}] = \int \begin{bmatrix} y_{/o}^2+z_{/o}^2 & x_{/o}y_{/o} & x_{/o}z_{/o} \\ x_{/o}y_{/o} & x_{/o}^2+z_{/o}^2 & y_{/o}z_{/o} \\ x_{/o}z_{/o} & y_{/o}z_{/o} & x_{/o}^2+y_{/o}^2 \end{bmatrix} dm$ <br> $[I^{o}] = [I^{cm}] + m \begin{bmatrix} y_{cm/o}^2+z_{cm/o}^2 & x_{cm/o}y_{cm/o} & x_{cm/o}z_{cm/o} \\ x_{cm/o}y_{cm/o} & x_{cm/o}^2+z_{cm/o}^2 & y_{cm/o}z_{cm/o} \\ x_{cm/o}z_{cm/o} & y_{cm/o}z_{cm/o} & x_{cm/o}^2+y_{cm/o}^2 \end{bmatrix}$ The 3D Parallel Axis Theorem |
| General 2D Body   | $[I^{cm}] = \int \begin{bmatrix} y_{/cm}^2 & x_{/cm}y_{/cm} & 0 \\ x_{/cm}y_{/cm} & x_{/cm}^2 & 0 \\ 0 & 0 & \underbrace{x_{/cm}^2+y_{/cm}^2}_{I_{zz}^{cm}} \end{bmatrix} dm$ <br> If the axes are principal axes of the body. $[I^{cm}] = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}$ With $A+B=C$ (The perpendicular axis theorem). Also, $A \geq 0$, $B \geq 0$. <br> $[I^{o}] = \int \begin{bmatrix} y_{/o}^2 & x_{/o}y_{/o} & 0 \\ x_{/o}y_{/o} & x_{/o}^2 & 0 \\ 0 & 0 & x_{/o}^2+y_{/o}^2 \end{bmatrix} dm$ <br> $[I^{o}] = [I^{cm}] + m \begin{bmatrix} y_{cm/o}^2 & x_{cm/o}y_{cm/o} & 0 \\ x_{cm/o}y_{cm/o} & x_{cm/o}^2 & 0 \\ 0 & 0 & \underbrace{x_{cm/o}^2+y_{cm/o}^2}_{d^2} \end{bmatrix}$ The 3D Parallel Axis Theorem. The 2D thm concerns the lower right terms of these 3 matrices. |

**General moments of inertia.** The tableshows a point mass, a general 3-D body, and a general 2-D body. The most general cases of the perpeendicual axis theorem and the parallel axis theorem are also shown..

Table IV
**Examples of Moment of Inertia**

| Object | $[\boldsymbol{I}]$ |
|---|---|
| Uniform rod   | $I_{zz}^{\text{cm}} = \frac{1}{12}m\ell^2, \quad [\boldsymbol{I}^{cm}] = \frac{1}{12}m\ell^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ <br> $I_{zz}^{\text{O}} = \frac{1}{3}m\ell^2, \quad [\boldsymbol{I}^{O}] = \frac{1}{3}m\ell^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| Uniform hoop   | $I_{zz}^{\text{cm}} = mR^2, \quad [\boldsymbol{I}^{cm}] = mR^2 \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| Uniform disk   | $I_{zz}^{\text{cm}} = \frac{1}{2}mR^2, \quad [\boldsymbol{I}^{cm}] = mR^2 \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$ |
| Rectangular plate   | $I_{zz}^{\text{cm}} = \frac{1}{12}m.a^2 + b^2/, \quad [\boldsymbol{I}^{cm}] = \frac{1}{12}m \begin{bmatrix} b^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{bmatrix}$ |
| Solid Box   | $[\boldsymbol{I}^{cm}] = \frac{1}{12}m \begin{bmatrix} b^2 + c^2 & 0 & 0 \\ 0 & a^2 + c^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{bmatrix}$ |
| Uniform sphere   | $[\boldsymbol{I}^{cm}] = \frac{2}{5}mR^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |

**Moments of inertia of some simple objects.** For the rod both the $[\boldsymbol{I}^{cm}]$ and $[\boldsymbol{I}^{O}]$ (for the end point at O) are shown. In the other cases only $[\boldsymbol{I}^{cm}]$ is shown. To calculate $[\boldsymbol{I}^{O}]$ relative to other points one has to use the parallel axis theorem. In all the cases shown the coordinate axes are principal axes of the objects.




Basic flow chart for solving the various types of dynamics problems.
Statics only uses the solution route 1 → 2 → 4 → 5 → b.
Dynamics uses other boxes as needed.
At first reading this chart shows you the logic of the subject.
Later it is self-evident and internalized as the approach to solve problems.