CHAPTER 11

Two or more particles in space (unconstrained)

This more advanced chapter concerns the motion of two or more particles in space. We will use $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ for each particle. We will use Cartesian coordinates only. The start is the set up of "two-body" type problems which are easily generalized to 3 or more particles. The first section concerns smooth motions due to forces from gravity, springs, smoothly applied forces and friction. The second section concerns the sudden change in velocities when impulsive forces are applied.

Contents

In the previous chapter you saw that once you know the forces on a particle, or how to find those forces given a particle's position, velocity and time, you can easily set up the equations of motion. That is, the linear momentum balance equation for a particle

$$\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}},$$

with initial conditions, gives a well defined mathematical problem. The solution of this math problem gives the position and velocity of the particle as a function of time. The solution may be hard or impossible to find with pencil and paper, but can usually be found quite directly using numerical integration.

Now we generalize this idea to two, three or more particles. In one model of the universe every one of its parts is made of particles, and each particle obeys Newton's laws. We could think of all materials as made of atoms, and of all the atoms moving in deterministic ways governed by Newton's laws and known force laws. If we knew the initial positions and velocities accurately enough, then we could accurately predict the motions of all things for all time. *

To put it in other words, given a simple atomic view of the world and a big computer, we could end a course on dynamics here. You know how to use $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ for each atom, so you could then simulate anything by simulating the motions of the atoms which make it up.

Of course there are some serious limitations to this point of view, so before proceeding, we list some serious caveats:

- there are no computers big enough to keep track of the 10^{23} or so atoms needed to describe macroscopic objects or the 10^{79} or so atoms in the universe;
- the laws of interaction between the most fundamental particles are not given by Newton's laws but by quantum chromodynamics, or whatever;
- one feature of the rules of the world, as physicists now understand them, is that they are not deterministic, quantum mechanics says that you *cannot* know the state of the world perfectly;
- the state of the world (the positions and velocities of all the bits is not that well known);
- the solutions of dynamics equations are often unstable in that the smallest of errors in the initial conditions propagates into a large error in the predicted motion (so called "chaos theory");

* The mathematician and mechanician Laplace (1749-1827) imagined a 'vast intellect' that could solve the differential equations that describe the universe. "Laplace's demon" was a hyper mega super computer with access to perfect data: *"We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at any given moment knew all of the forces that animate nature and the mutual positions of the beings that compose it, if this intellect were vast enough to submit the data to analysis, could condense into a single formula the movement of the greatest bodies of the universe and that of the lightest atom; for such an intellect nothing could be uncertain and the future just like the past would be present before its eyes."*

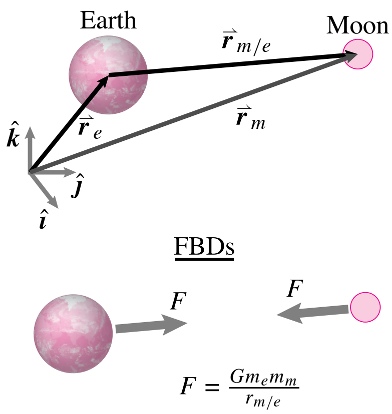

Figure 11.1: The earth and moon. Position is measured relative to some "fixed" point $\boldsymbol{C}$.

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- some common descriptions of mechanical interactions, particularly those for contact between nominally rigid objects, are genuinely non-deterministic in that the governing equations do not have unique solutions; and finally
- massive simulations, even if accurate, are not always the best way to understand how things work.

Despite these limitations, in this chapter we look at the nature of systems of interacting particles. Using this particle model we can, for example, derive some results about angular momentum that turn out to be reliable, despite the questionable microscopic physics. Also, the multi-particle model of the systems is good for intuition and is also useful for modeling machines with many parts as well as of galaxies.

11.1 Coupled motions of particles in space

Assume you know enough about a system so that you know the forces on each particle if someone tells you the time and the positions and velocities of all the particles. This means you can write the governing equations for the system of particles like this:

$$\begin{aligned} \vec{a}_1 &= \frac{1}{m_1}\vec{F}_1 \\ \vec{a}_2 &= \frac{1}{m_2}\vec{F}_2 \\ \vec{a}_3 &= \frac{1}{m_3}\vec{F}_3 \\ &\text{etc.} \end{aligned} \tag{11.1}$$

where $\vec{F}_1$, $\vec{F}_2$ *etc.* are the total of the forces on the corresponding particles. The force on each particle may come from air-friction, from springs or dashpots connected here and there, or from gravity interactions with other particles, from known applied loads, *etc.*. One way or another, all the forces on all the particles are known given the time, the positions and velocities of the particles. Thus *eqn.* (11.1) can be written as a system of first order differential equations in standard form, ready for computer simulation. Given accurate initial conditions and a good computer then the motions of all the particles can be found accurately.

Example: **Coupled motion of the earth and moon in three dimensions.** Let's neglect the sun and just look at the coupled motions of the earth and moon. They attract each other by the same law of gravity that we used for the sun and earth. The difference between this problem and a "central-force" problem is that we now need to look at the 'absolute' positions of the sun and the moon ($\vec{r}_e$ and $\vec{r}_m$), as well as the 'relative' position $\vec{r}_{m/e} \equiv \vec{r}_m - \vec{r}_e$ (*Fig.* 11.1).

The linear momentum balance equations are now

$$m_e \ddot{\vec{r}}_e = \frac{-G m_e m_m \vec{r}_{m/e}}{|\vec{r}_{m/e}|^3} \quad \text{and} \tag{11.2}$$

$$m_m \ddot{\vec{r}}_m = \frac{+G m_e m_m \vec{r}_{m/e}}{|\vec{r}_{m/e}|^3}, \tag{11.3}$$

which, when broken into x, y, and z components give 6 second order ordinary differential equations. These equations can be written as 12 first order equations by defining a list of 12 z variables: $z_1 = x_e$, $z_2 = \dot{x}_e$, $z_3 = \dot{y}_e$, $z_4 = y_e$, $etc.$

After you find solutions, using various initial conditions you can check if the computer finds such truths (that is, features of the exact solution of the differential equations) as:

1. that the line between the earth and moon always lies on one fixed plane,
2. the center-of-mass moves at constant speed on a straight line,
3. relative to the center-of-mass both the earth and moon travel on paths that are conic sections (circle, ellipse, parabola, hyperbola or a straight line).
4. the total energy ($E_\text{K} + E_\text{P}$) of the system is constant,
5. and that the angular momentum of the system about the center-of-mass is a constant.

These facts are discussed further below in the subsection on 'Two-particle central force motion'.

Momentum and energy of systems

There are a plethora of theorems about the momentum and energy of systems of particles. These are discussed in section **??**. The simplest of these are just the ones that you get from adding up the results for a single particle from section 10.2:

Linear momentum balance. $\sum_\text{all forces} \vec{F}_j = \sum m_i \vec{a}_i$. Either because the forces between particles in a system are usually assumed to come in equal and opposite pairs or because it is an independent postulate of mechanics for general systems, the force sum can be replaced with a sum over all the external forces.

Angular momentum balance.
$\sum_\text{all forces} \vec{r}_{j/C} \times \vec{F}_j = \sum m_i \vec{r}_{i/C} \vec{a}_i$. As for linear momentum, the force sum can be replaced with only the forces that act externally on the system.

Power balance. $\sum_\text{all forces} \vec{F}_j \cdot \vec{v}_j = \frac{d}{dt} \sum m_i \vec{v}_i^2/2$. In this case the sum is over *all* the forces, internal and external. The simplification to just external forces doesn't apply to system kinetic energy like it does for momentum and angular momentum.

The one-body problem

Lets review one special problem from the previous section. The 'one-body' problem should properly be about the mechanics of a single particle interacting with nothing else. Such a particle moves at constant velocity and is too boring to get a name. Instead, when people refer to the 'one-body' problem they are talking about a particle flying around a stationary point mass to which it is attracted. That stationary point mass is held in place by, well, who knows what. Its just an idealized thing anchored by a massless structure. The 'one-body' problem is to find the motion of the particle flying around.

As we discussed in the previous sections, if the gravitational attraction follows an inverse square law then the particle moves on a plane on a curve which is either an elipse, a circle, a parabola or a hyperbola. These are, quite accurately, the trajectories of the planets and comets around the sun.

The two-body problem: two mutually attracting particles

If two particles are attracted equally to each other by mutually central forces, and no other forces act, this is called 'the two-body problem'*. Assume the two particles are m_1 and m_2 with positions $\vec{r}_1$ and $\vec{r}_2$ (relative to the origin of a coordinate system fixed in a Newtonian frame). The force on particle 1 from particle 2 is

$$\vec{F}_{12} = F(r_{12})\frac{\vec{r}_{12}}{r_{12}}$$

where $\vec{r}_{12} = \vec{r}_{2/1}$ is the position of particle 2 relative to particle 1, r_{12} is the distance $|\vec{r}_{12}|$ between the particles and F is the magnitude of the attractive force. We assume the force on particle 2 is the opposite of this

$$\vec{F}_{21} = -\vec{F}_{12}.$$

The instantaneous velocities are $\vec{v}_1$ and $\vec{v}_2$. We can find the center of mass G of the pair of particles as

$$m_{tot}\vec{r}_G = m_1\vec{r}_1 + m_2\vec{r}_2$$

with $m_{tot} = m_1 + m_2$.

Either by system linear momentum balance or by adding up $\vec{F} = m\vec{a}$ for each of the particles it is easy to see that

$$\vec{a}_G = \vec{0} \qquad \text{and} \qquad \vec{v}_G = \text{constant}.$$

Thus we could put the origin of a good Newtonian reference frame at the center of mass $\vec{r}_G$. The positions, velocities and accelerations relative to G, indicated with a prime ('), are

$$\begin{aligned}
\vec{r}\,'_1 &= \vec{r}_1 - \vec{r}_G \\
\vec{r}\,'_2 &= \vec{r}_2 - \vec{r}_G \\
\vec{v}\,'_1 &= \vec{v}_1 - \vec{v}_G \\
\vec{v}\,'_2 &= \vec{v}_2 - \vec{v}_G \\
\vec{a}\,'_1 &= \vec{a}_1 \\
\vec{a}\,'_2 &= \vec{a}_2
\end{aligned}$$

*This is such a famous problem in the history of science that people use it for word play to describe certain social situations. For example if two people in a couple are having trouble finding jobs in the same city they are said to have a 'two-body problem'.

where we can skip use of the prime for the acceleration. Now some facts.

- $m_1\vec{r}'_1 + m_2\vec{r}'_2 = \vec{0}$ so $\vec{r}'_1 = -(m_2/m_1)\vec{r}'_2$. For all time the two positions (relative to the center of mass) are in the opposite direction and proportional. Similarly $\vec{v}'_1 = -(m_2/m_1)\vec{v}'_2$ and $\vec{a}'_1 = -(m_2/m_1)\vec{a}'_2$.
- At a given instant there is a single plane defined by $\vec{r}'_1$, $\vec{r}'_2$, $\vec{v}'_1$, $\vec{v}'_2$, $\vec{a}'_1$ and $\vec{a}'_2$ because the positions and accelerations are all parallel (or antiparallel) and the two velocities are (anti) parallel.
- The plane above is constant in time. This is because neither the velocity nor the acceleration has a component orthogonal to the plane, thus there is no tendency to leave the plane.
- Each particle moves as if it was a single particle attracted to a central force at G. Why? Lets look at the force on mass 1

$$\vec{F}_{12} = F(r_{12})\frac{\vec{r}_{12}}{r_{12}} = -F(r_{12})\frac{\vec{r}'_1}{r'_1}$$

  because the relative position of the masses passes through the origin G.
- In the special case of inverse-square gravitational attraction

$$F = \frac{Gm_1m_2}{r_{12}^2} = \frac{Gm_1m_2}{(r'_1 + r'_2)^2} = \frac{Gm_1M}{r_1'^2}$$

  where $M = m_2/(1 + 2(m_1/m_2) + (m_1/m_2)^2)$ is a fictitious mass at G we find using the substitution $r'_2 = (m_1/m_2)r_1$.

What we have found here is somewhat remarkable. Two particles are flying around in space attracted to each other by inverse-square gravitational attraction. Instead of doing something wild, they each move, relative to their joint center of mass, as if they were in central force motion with a fixed mass. That is, the 3D two-body problem reduces, exactly, to the 2D one body problem. You just have to use a coordinate system that is on the plane of motion and whose origin is at the center of mass.

Thus, the moon doesn't really go around the earth. Rather the moon and earth go around their common center of mass (a point about 3/4 of the way out towards the earth's surface from its center). And Jupiter doesn't go around the sun, the sun and Jupiter go around their combined center of mass just outside the sun. But both of these examples are, in detail, wrong. Because the earth-moon system is affected by the sun and jupiter. And the Jupiter-sun system is affected by the earth and moon.

The three-body problem

With inverse-square attraction, one body goes around a fixed point on one or another conic section. Two bodies go around each other in exactly the same way as one body about a fixed point. The two-body problem reduced to the one-body problem. What about lots of bodies? Let's start with three. How, in general, do three bodies move that are all mutually attracted with inverse-square gravitation? Great question. Lots of people have asked it. And no-one knows the answer. Given any three masses and their initial conditions we could use a computer program to find out their subsequent positions and velocities. But no-one knows how to categorize all the possible motions of such systems.

Some things are known about 'the three body problem.' One is that it is hard, the best minds haven't been able to solve it in general. Another is that the solutions can be pretty wild. For example, three particles might tumble around each other for a long time and, with no change in the equations, all of a sudden one of the particles will be ejected at high speed and never return (as if on a hyperbolic trajectory relative to the other two particles). A few special solutions of the three-body problem are known. For example, with the right initial conditions, three identical particles can move in either a circle or in Montgomery's figure 8.

Despite the difficulty of analytic description, there is no special impediment to finding solutions to any 3-body problem with computer simulation.

The n-body problem

With many particles all manner of complicated motion is possible. And there are few solutions which are known analytically. One solution has the n particles chasing each-other around in a circle, with the particles forming a regular polygon. Another amazing approximate solution, the Buck solution, is that a string of thousands of particles will all chase each-other around an arbitrary curve in 3-dimensional space. At least approximately, for a while.

By applying $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ to 3 or 1000 interacting particles you can see all manner of n-body solutions on your computer.

SAMPLE 11.1 **Location of the center-of-mass.** A structure is made up of three point masses, $m_1 = 1\,\mathrm{kg}$, $m_2 = 2\,\mathrm{kg}$ and $m_3 = 3\,\mathrm{kg}$. At the moment of interest, the coordinates of the three masses are $(1.25\,\mathrm{m}, 3\,\mathrm{m})$, $(2\,\mathrm{m}, 2\,\mathrm{m})$, and $(0.75\,\mathrm{m}, 0.5\,\mathrm{m})$, respectively. At the same instant, the velocities of the three masses are $2\,\mathrm{m/s}\hat{\imath}$, $2\,\mathrm{m/s}(\hat{\imath} - 1.5\hat{\jmath})$ and $1\,\mathrm{m/s}\hat{\jmath}$, respectively.

1. Find the coordinates of the center-of-mass of the structure.
2. Find the velocity of the center-of-mass.

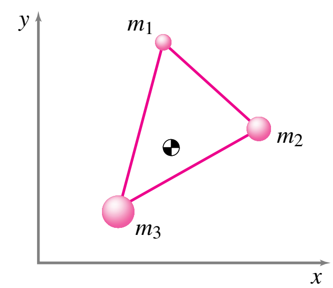

Figure 11.2:

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Solution

1. Let $(\bar{x}, \bar{y})$ be the coordinates of the mass-center. Then from the definition of mass-center

$$\begin{aligned}\bar{x} &= \frac{\sum m_i x_i}{\sum m_i} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} \\ &= \frac{1\,\mathrm{kg}\cdot 1.25\,\mathrm{m} + 2\,\mathrm{kg}\cdot 2\,\mathrm{m} + 3\,\mathrm{kg}\cdot 0.75\,\mathrm{m}}{1\,\mathrm{kg} + 2\,\mathrm{kg} + 3\,\mathrm{kg}} \\ &= \frac{7.25\,\not{\mathrm{kg}}\cdot\mathrm{m}}{6\,\not{\mathrm{kg}}} = 1.25\,\mathrm{m}.\end{aligned}$$

Similarly,

$$\begin{aligned}\bar{y} &= \frac{\sum m_i y_i}{\sum m_i} \\ &= \frac{1\,\mathrm{kg}\cdot 3\,\mathrm{m} + 2\,\mathrm{kg}\cdot 2\,\mathrm{m} + 3\,\mathrm{kg}\cdot 0.5\,\mathrm{m}}{1\,\mathrm{kg} + 2\,\mathrm{kg} + 3\,\mathrm{kg}} \\ &= \frac{8.55\,\not{\mathrm{kg}}\cdot\mathrm{m}}{6\,\not{\mathrm{kg}}} = 1.42\,\mathrm{m}.\end{aligned}$$

Thus the center-of-mass is located at the coordinates $(1.25\,\mathrm{m}, 1.42\,\mathrm{m})$.

$$\boxed{(1.25\,\mathrm{m},\ 1.42\,\mathrm{m})}$$

2. For a system of particles, the linear momentum

$$\begin{aligned}\vec{L} &= \sum m_i \vec{v}_i = m_{\mathrm{tot}}\vec{v}_{cm} \\ \Rightarrow \quad \vec{v}_{cm} &= \frac{\sum m_i \vec{v}_i}{m_{\mathrm{tot}}} \\ &= \frac{1\,\mathrm{kg}\cdot(2\,\mathrm{m/s}\hat{\imath}) + 2\,\mathrm{kg}\cdot(2\hat{\imath} - 3\hat{\jmath})\,\mathrm{m/s} + 3\,\mathrm{kg}\cdot(1\,\mathrm{m/s}\hat{\jmath})}{6\,\mathrm{kg}} \\ &= \frac{(6\hat{\imath} - 3\hat{\jmath})\,\not{\mathrm{kg}}\cdot\mathrm{m/s}}{6\,\not{\mathrm{kg}}} \\ &= 1\,\mathrm{m/s}\hat{\imath} + 0.5\,\mathrm{m/s}\hat{\jmath}.\end{aligned}$$

$$\boxed{\vec{v}_{cm} = 1\,\mathrm{m/s}\hat{\imath} + 0.5\,\mathrm{m/s}\hat{\jmath}}$$

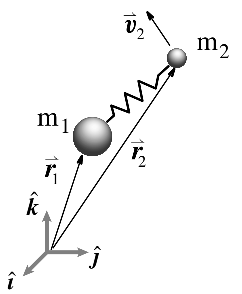

Figure 11.3:
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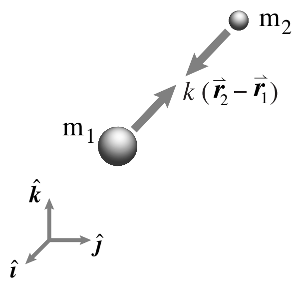

Figure 11.4: Free-body diagram of the two masses.
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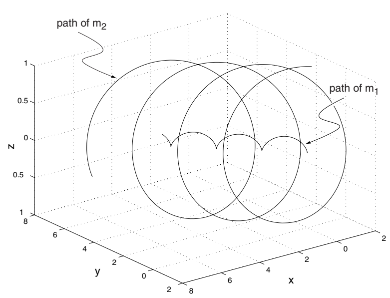

Figure 11.5: 3-D trajectory of m_1 and m_2 plotted from numerical solution of the equations of motion.
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SAMPLE 11.2 A spring-mass system in space. A spring-mass system consists of two masses, $m_1 = 10\,\text{kg}$ and $m_2 = 1\,\text{kg}$, and a weak spring with stiffness $k = 1\,\text{N/m}$. The spring has zero relaxed length. The system is in 3-D space where there is no gravity. At the moment f observaion, *i.e.*, at $t = 0$, $\vec{r}_1 = \vec{0}$, $\vec{r}_2 = 1\,\text{m}(\hat{\imath} + \hat{\jmath} + \hat{k})$, $\dot{\vec{r}}_1 = \vec{0}$, and $\dot{\vec{r}}_2 = \sqrt{6}\,\text{m/s}(-\hat{\imath} + \hat{\jmath})$. Track the motion of the system for the next 20 seconds. In particular,

1. Plot the trajectory of the two masses in space.
2. Plot the trajectory of the center-of-mass of the system.
3. Plot the trajectory of the two masses as seen by an observer sitting at the center-of-mass.
4. Compute and plot the total energy of the system and show that it remains constant during the entire motion.

Solution The free-body diagrams of the two masses are shown in *Fig*. 11.4. The only force acting on each mass is the force due to the spring which is directed along the line joining the two masses. Thus, the system represents a central force problem. From the linear momentum balance of the two masses, we can write the equations of motion as follows.

$$
\begin{aligned}
m_1 \ddot{\vec{r}}_1 &= k(\vec{r}_2 - \vec{r}_1) \\
m_2 \ddot{\vec{r}}_2 &= -k(\vec{r}_2 - \vec{r}_1)
\end{aligned}
$$

Let $\vec{r}_1 = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$ and $\vec{r}_2 = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$. Substituting above and dotting the two equations with $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, we get

$$
\begin{aligned}
\ddot{x}_1 &= \frac{k}{m_1}(x_2 - x_1); & \ddot{x}_2 &= -\frac{k}{m_2}(x_2 - x_1) \\
\ddot{y}_1 &= \frac{k}{m_1}(y_2 - y_1); & \ddot{y}_2 &= -\frac{k}{m_2}(y_2 - y_1) \\
\ddot{z}_1 &= \frac{k}{m_1}(z_2 - z_1); & \ddot{z}_2 &= -\frac{k}{m_2}(z_2 - z_1)
\end{aligned}
$$

Thus we get six second order coupled linear ODEs as equations of motion.

1. To plot the trajectory of the two masses, we need to solve for $\vec{r}_1(t)$ and $\vec{r}_2(t)$, *i.e.*, for $x_1(t), y_1(t), z_1(t)$, and $x_2(t), y_2(t), z_2(t)$. We can do this by first writing the six second order equations as a set of 12 first order equations and then solving them using a numerical ODE solver. Here is a pseudocode to accomplish this task.

```
ODEs = {x1dot = u1,
        u1dot = k/m1*(x2-x1),
        y1dot = v1,
        v1dot = k/m1*(y2-y1),
        z1dot = w1,
        w1dot = k/m1*(z2-z1),
        x2dot = u2,
        u2dot = -k/m2*(x2-x1),
        y2dot = v2,
        v2dot = -k/m2*(y2-y1),
        z2dot = w2,
        w2dot = -k/m2*(z2-z1) }
IC   = {x1(0)=0, y1(0)=0, z1(0)=0,
        u1(0)=0, v1(0)=0, w1(0)=0,
```

```
        x2(0)=1, y2(0)=1, z2(0)=1,
        u2(0)=-sqrt(6), v2(0)=sqrt(6), w2(0)=0}
Set  k=1, m1=10, m2=1

Solve ODEs with IC for t=0 to t=20
Plot {x1,y1,z1} and {x2,y2,z2}
```

The 3-D plot showing the trajectory of the two masses obtained from the numerical solution is shown in *Fig*. 11.5. From the plot, it seems like the smaller mass goes around the bigger mass as the bigger mass moves on its trajectory.

2. We can find the trajectory of the center-of-mass using the following relationships.

$$x_{\text{cm}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, \quad y_{\text{cm}} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}, \quad z_{\text{cm}} = \frac{m_1 z_1 + m_2 z_2}{m_1 + m_2}.$$

Since there is no external force on the system if we consider the two masses and the spring together, the center-of-mass of the system has zero acceleration. Therefore, we expect the center-of-mass to move on a straight path with constant velocity. The center-of-mass coordinates x_{cm}, y_{cm}, and z_{cm} are plotted against time in *Fig*. 11.6 which show that the center-of-mass moves on a straight line in a plane parallel to the xy-plane (z is constant). This is expected since the initial velocity of the center of has no z-component:

$$\begin{aligned} \vec{v}_{\text{cm}} &= \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2} \\ &= \frac{m_1 \cdot \vec{0} + 1\,\text{kg} \cdot \sqrt{(6)}\,\text{m/s}(-\hat{\imath} + \hat{\jmath})}{10\,\text{kg} + 1\,\text{kg}} \\ &= 0.22\,\text{m/s}(-\hat{\imath} + \hat{\jmath}). \end{aligned}$$

3. The trajectory of the two masses with respect to the center-of-mass can be easily obtained by the following relationships.

$$\begin{aligned} x_{1/cm} &= x_1 - x_{\text{cm}}, & y_{1/cm} &= y_1 - y_{\text{cm}}, & z_{1/cm} &= z_1 - z_{\text{cm}} \\ x_{2/cm} &= x_2 - x_{\text{cm}}, & y_{2/cm} &= y_2 - y_{\text{cm}}, & z_{2/cm} &= z_2 - z_{\text{cm}} \end{aligned}$$

The trajectories thus obtained are shown in *Fig*. 11.6. It is clear that the two masses have closed orbits with respect to the center-of-mass. These closed orbits are actually conic sections as we would expect in a central force problem.

4. We can calculate the kinetic energy of the two masses and the potential energy of the spring at each instant during the motion and add them up to find the total energy.

$$\begin{aligned} (E_k)_{m_1} &= \frac{1}{2} m_1 (u_1^2 + v_1^2 + w_1^2) \\ (E_k)_{m_2} &= \frac{1}{2} m_2 (u_2^2 + v_2^2 + w_2^2) \\ E_p &= \frac{1}{2} k[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2] \\ E_{\text{total}} &= (E_k)_{m_1} + (E_k)_{m_2} + E_p \end{aligned}$$

The energies so calculated are plotted in *Fig*. 11.7. It is clear from the plot that the total energy remains constant during the entire motion.

□

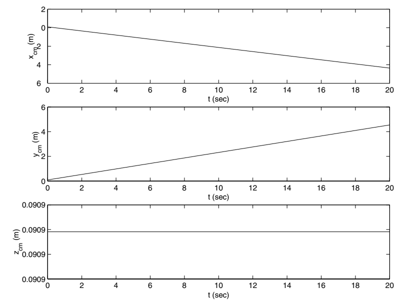

Figure 11.6: The center-of-mass coordinates $x_{\text{cm}}(t)$, $y_{\text{cm}}(t)$, and $z_{\text{cm}}(t)$. The center-of-mass moves on a straight line in a plane parallel to the xy-plane.

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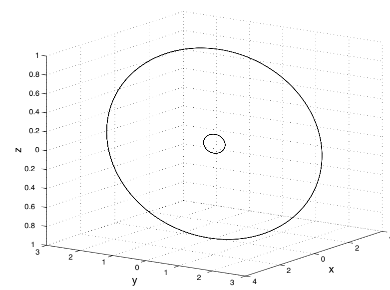


Figure 11.7: The paths of m_1 and m_2 as seen from the center-of-mass. The two masses are on closed orbits with respect to the center-of-mass.

Filename:sfig5-10-hopper-d

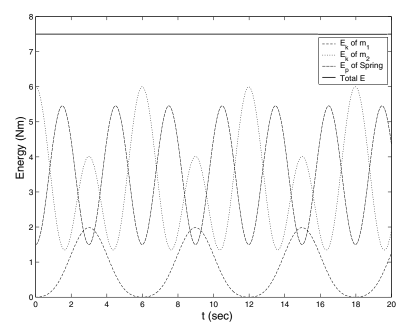


Figure 11.8: The kinetic energy of the two masses and the potential energy of the spring sum up to the constant total energy of the system.

Filename:sfig5-10-hopper-e

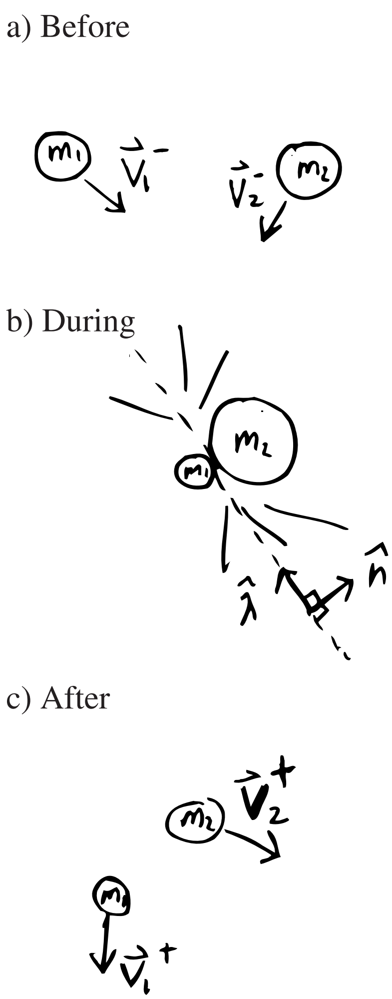


Figure 11.9: Two particles collide. Their velocities before the collision are $\vec{\boldsymbol{v}}_1^-$ and $\vec{\boldsymbol{v}}_2^-$. When they collide their common tangent plane has normal $\hat{\boldsymbol{n}}$. After the collision their velocities are $\vec{\boldsymbol{v}}_1^+$ and $\vec{\boldsymbol{v}}_2^+$.

Filename:tfig11-collision-general

11.2 Collisions and explosions of particles in 2D and 3D

When two things bump into each other there is often a big interaction force. Think about a ball bouncing off the ground, two pool balls collidng, a baseball hitting a bat, two cars crashing, or the big forces when a satellite gravitationally slingshots around a planet it passes close by. Similarly there are big short-lived forces when things explode into two or more pieces. A big and short-lived force is often described by

$$\text{its net impulse } \vec{\boldsymbol{P}} = \int \vec{\boldsymbol{F}} dt$$

rather than its detailed time-history $\vec{\boldsymbol{F}}(t)$. The collision modeling assumption is that these interaction forces are so big that all other forces on the particles can be ignored. For a two-particle collision the impulses are $\vec{\boldsymbol{P}}_1 = \vec{\boldsymbol{P}}$ and $\vec{\boldsymbol{P}}_2 = -\vec{\boldsymbol{P}}$ acting on m_1 and m_2. Rather than looking at the acceleration of mass during the collision one just calculates

$$\text{the net change in velocity} = \Delta \vec{\boldsymbol{v}}.$$

Before the collision two particles m_1 and m_2 have velocities $\vec{\boldsymbol{v}}_1^-$ and $\vec{\boldsymbol{v}}_2^-$ (see Fig. 11.9). The superscript "$^-$" means *just* before the collision. Then the particles collide. Even though we ignore the spatial extent of the particles for most of the mechanics analysis, we note that the two particles have a common tangent plane. The normal of that plane, pointing out of particle 1, say, is $\hat{\boldsymbol{n}}$. Just after the collision the particles have velocities $\vec{\boldsymbol{v}}_1^+$ and $\vec{\boldsymbol{v}}_2^+$ with the superscript "$^+$" indicating *just* after the collision.

The general collision problem is

> Given some information about the motion before the collision, the motion after the collision, and the collisional impulse, find other information about these same quantities.

We find the unknowns using

- Momentum balance for each particle: $\vec{\boldsymbol{P}} = \int \vec{\boldsymbol{F}}(t)\, dt = m\Delta \vec{\boldsymbol{v}}$; and
- Some information about the collisional impulse, usually a constitutive law for the collision.

For collisional modeling the constitutive law for interaction involves impulse and change of velocity. We only consider two such constitutive models:

- *plastic sticking* collisions where $\vec{\boldsymbol{v}}_1^+ = \vec{\boldsymbol{v}}_2^+$.

- frictionless restitution with $(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+) \cdot \hat{\boldsymbol{n}} = -e(\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-) \cdot \hat{\boldsymbol{n}}$ and $\vec{\boldsymbol{P}} \cdot \hat{\lambda} = 0$.

The constitutive models are discussed further below in the context of the the three idealized collisions we treat here:

- sticking collisions
- frictionless collisions with restitution
- explosions.

The only expansion in this section over the 1D collissions in section 9.5 is the need for 2D and 3D geometry.

Sticking collisions

The conceptually simplest collision is a *sticking* collision also called a *perfectly plastic no-slip* collision (see *Fig.* 11.10). Here the word 'plastic' is used in its old latin meaning malleable or 'clay like'. Imagine two lumps of wet clay colliding in space and just sticking together. This model might be used when a projectile gets imbedded in its target, when two cars crash and get entangled so move together after the collision, or when two machine parts engage at contact because of a mechanism like a door catch.

In short, the constitutive law for plastic collisions is

$$\vec{\boldsymbol{v}}_1^+ = \vec{\boldsymbol{v}}_2^+$$

And the impulse is what it is, as determined by momentum balance for the two particles. Here's the simplest collision problem.

Example: **A particle collides with an immovable object.**
The impulse on the particle is

$$\vec{\boldsymbol{P}} = m\Delta\vec{\boldsymbol{v}} = m(\vec{0} - \vec{\boldsymbol{v}}^-) = -m\vec{\boldsymbol{v}}^-.$$

And here is the general two-particle sticking collision problem.

Example: **Two particles collide and stick.**
There are three velocities to consider, the before-collision velocities $\vec{\boldsymbol{v}}_1^-$ and $\vec{\boldsymbol{v}}_2^-$ and the common after-collision velocity $\vec{\boldsymbol{v}}^+$. Also relevant is the interaction impulse $\vec{\boldsymbol{P}}$. That's 4 vector quantities (8 scalars in 2D, 12 in 3D). The governing equations are momentum balance for the two particles

$$\vec{\boldsymbol{P}} = m_1(\vec{\boldsymbol{v}}^+ - \vec{\boldsymbol{v}}_1^-) \qquad \text{and} \qquad \vec{\boldsymbol{P}} = m_2(\vec{\boldsymbol{v}}^+ - \vec{\boldsymbol{v}}_2^-)$$

making up 2 vector equations (4 scalar equations in 2D, 6 in 3D). Thus to solve a problem in 2D, 4 scalar quantities need to be given so that the other quantities can be found from the momentum balance equations. In 3D, 6 scalar quantities have to be given.

There are all different ways to involute such problems, say by taking one of the masses as unknown. Here is the most straightforward example.

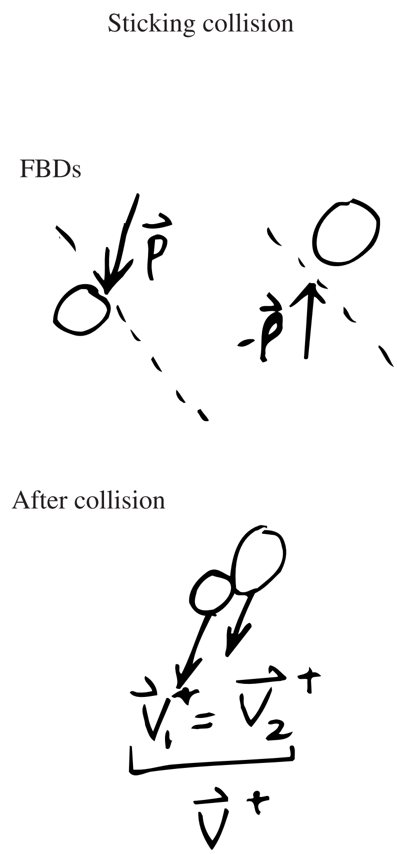


Figure 11.10: Two particles collide and stick together so in their subsequent motion $\vec{\boldsymbol{v}}_1^+ = \vec{\boldsymbol{v}}_1^- = \vec{\boldsymbol{v}}^+$. The action-reaction impulse pair is in whatever direction it needs to be to get this sticking; the impulses need not be in just the $\hat{\boldsymbol{n}}$ direction.

Filename:tfig11-collision-sticking

Frictionless collision with restitution

FBDs:

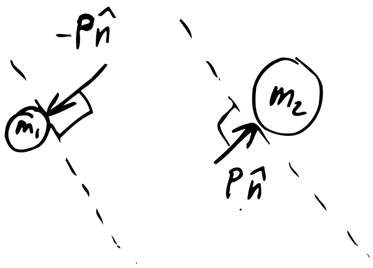


Figure 11.11: Two particles collide and bounce off each other frictionlessly. The assumption is that their relative separation speed is the coefficient of restitution e times their approach speed. Even though for momentum balance we treat the masses as particles, for considering the collision we look at the normal $\hat{\boldsymbol{n}}$ of the common contacting tangent plane. The separation speed is $(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+)\cdot\hat{\boldsymbol{n}}$ and the approach speed is $-(\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-)\cdot\hat{\boldsymbol{n}}$.
Filename:tfig11-collision-restitution

Example: **Find the post-collision velocities for a sticking collision.** Given $m_1, m_2, \vec{\boldsymbol{v}}_1^-$ and $\vec{\boldsymbol{v}}_2^-$ we find by solving the momentum balance equations that

$$\vec{\boldsymbol{v}}^+ = \frac{m_1\vec{\boldsymbol{v}}_1^- + m_2\vec{\boldsymbol{v}}_2^-}{m_1+m_2} \quad \text{and} \quad \vec{\boldsymbol{P}}_1 = -\vec{\boldsymbol{P}}_2 = \frac{m_1 m_2}{m_1+m_2}\left(\vec{\boldsymbol{v}}_2^- - \vec{\boldsymbol{v}}_1^-\right).$$

The answer can be interpreted like this. The final velocity is the same as the pre-collision average velocity. This is also the system's initial (and final) center of mass velocity. The impulsive interaction is associated with the change of velocity of $\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-$ of an effective 'reduced mass' with a value of $m_{red} = m_1 m_2/(m_1+m_2)$ (see box 11.1 on page 612).

Frictionless collisions with restitution

This is the most common model used in elementary mechanics courses. It is originally due to Newton, at least in the 1D case we discussed in section 9.5. Two particles collide and then separate. There is no interaction force in their common contact tangent plane (hence 'frictionless'). See *Fig.* 11.11. The impulse is such that the particles separate at a speed that is a fixed ratio e of the speed at which they approached. The speed of approach and separation are measured in the $\hat{\boldsymbol{n}}$ direction.

The speed of approach is the rate at which the distance between the particles decreases just before the collision. Really, this only makes precise sense if

- the masses are round, or
- the masses are not rotating.

The approach speed is the relative velocity dotted with the $\hat{\boldsymbol{n}}$ direction

$$v_{approach} = \left(\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-\right)\cdot\hat{\boldsymbol{n}}.$$

11.1 *THEORY*
Effective mass

In two-particle collisions the forces of interest are in the action-reaction pair between the particles: $\vec{\boldsymbol{F}}_1$ acts on m_1 and $\vec{\boldsymbol{F}}_2 = -\vec{\boldsymbol{F}}_1$ acts on particle 2. If we know one we know the other, so let's call $\vec{\boldsymbol{F}}$ the force $\vec{\boldsymbol{F}}_1$ on m_1. The two-particle system has no net acceleration, meaning the center of mass does not accelerate. All that the interaction force does is affect the relative motion of the particles.

So consider the relative acceleration of particle 1, say, relative to particle 2:

$$\vec{\boldsymbol{a}}_{\rm rel} = \vec{\boldsymbol{a}}_1 - \vec{\boldsymbol{a}}_2 = \frac{\vec{\boldsymbol{F}}_1}{m_1} - \frac{\vec{\boldsymbol{F}}_2}{m_2} = \frac{\vec{\boldsymbol{F}}_1}{m_1} - \frac{-\vec{\boldsymbol{F}}_1}{m_2} = \vec{\boldsymbol{F}}\left(\frac{1}{m_1}+\frac{1}{m_2}\right).$$

Thus we can write

$$\vec{\boldsymbol{F}} = m_{\rm eff}\vec{\boldsymbol{a}}_{\rm rel}$$

$$\text{with} \quad m_{\rm eff} = \left(\tfrac{1}{m_1}+\tfrac{1}{m_2}\right) = \tfrac{m_1 m_2}{m_1+m_2}$$

The 'reduced mass' or 'effective mass' $m_{\rm eff}$ is that which connects the interaction force with the relative acceleration $\vec{\boldsymbol{a}}_{\rm rel} = \vec{\boldsymbol{a}}_1 = \vec{\boldsymbol{a}}_2$ of the masses. To personalize it, imagine you have negligible mass and are floating in space between two big masses holding a handle on each. Then the relation between the tension in your arms and the relative acceleration of the masses is determined by the reduced effective mass $m_{\rm eff}$. The effective mass is less than either of the masses separately (because the relative acceleration comes from the addition of the two accelerations). For two equal masses $m = m_1 = m_2$ the effective mass is $m_{\rm eff} = m/2$.

Integrating in time the effective mass also relates the interaction impulse with the change in relative velocity.

$$\vec{\boldsymbol{P}} = m_{\rm eff}\Delta\vec{\boldsymbol{v}}_{\rm rel}.$$

where $\vec{\boldsymbol{P}} = \int \vec{\boldsymbol{F}}dt$ acts on m_1 and $\vec{\boldsymbol{v}}_{\rm rel} = \vec{\boldsymbol{v}}_1 - \vec{\boldsymbol{v}}_2$.

The separation speed, measured just after the collision, has the same definition but with a sign change

$$v_{\text{sep}} = -\left(\vec{\boldsymbol{v}}_1^{+} - \vec{\boldsymbol{v}}_2^{+}\right) \cdot \hat{\boldsymbol{n}}.$$

Newton's law of collisional restitution* is

$$v_{\text{sep}} = e v_{approach} \quad \text{or} \quad \left(\vec{\boldsymbol{v}}_1^{+} - \vec{\boldsymbol{v}}_2^{+}\right) \cdot \hat{\boldsymbol{n}} = -e\left(\vec{\boldsymbol{v}}_1^{-} - \vec{\boldsymbol{v}}_2^{-}\right) \cdot \hat{\boldsymbol{n}}. \quad (11.4)$$

We use the coefficient of restitution for approximate collisional modeling but,

> the 'coefficient of restitution' restitution equation is not an accurate law of nature.

Example: **Two-particle elastic collision.**
Two particles m_1 and m_2 have pre-collision velocities of $\vec{\boldsymbol{v}}_1^{-}$ and $\vec{\boldsymbol{v}}_2^{-}$ and collide frictionlessly with coefficient of restitution e on the tangent plane with normal $\hat{\boldsymbol{n}}$. The post collision velocities $\vec{\boldsymbol{v}}_1^{+}$ and $\vec{\boldsymbol{v}}_2^{+}$, as well as the impulse $\vec{\boldsymbol{P}}$ are found by simultaneously solving these equations.

$$\begin{aligned} \vec{\boldsymbol{P}} &= m_1\left(\vec{\boldsymbol{v}}_1^{+} - \vec{\boldsymbol{v}}_1^{-}\right) \\ -\vec{\boldsymbol{P}} &= m_2\left(\vec{\boldsymbol{v}}_2^{+} - \vec{\boldsymbol{v}}_2^{-}\right) \\ \left(\vec{\boldsymbol{v}}_1^{+} - \vec{\boldsymbol{v}}_2^{+}\right) \cdot \hat{\boldsymbol{n}} &= -e\left(\vec{\boldsymbol{v}}_1^{-} - \vec{\boldsymbol{v}}_2^{-}\right) \cdot \hat{\boldsymbol{n}} \end{aligned}$$

In 2D this makes up 5 scalar equations for 5 scalar unknowns. In 3D its 7 equations for 7 unknowns. The most direct solution is to set up and solve these equations using a computer.

Rather it is an approximate empirical observation. Or, to put it another way, the value of the coefficient of restitution e depends on the material, the shape, the orientation and the speed of the colliding particles. It is not a true constant. Nonetheless, $eqn.$ (11.4) is a reasonable approximation for some engineering purposes. Just don't assume that predictions it makes will generally be highly accurate.

The 'frictionless' part of this collision law is expressed by the assumption that the net impulse of interaction is in the $\hat{\boldsymbol{n}}$ direction. So $\vec{\boldsymbol{P}} = P\hat{\boldsymbol{n}}$ with no component in the $\hat{\lambda}$ direction.

Generally one assumes that the coefficient of restitution is between zero and one:

$$0 \leq e \leq 1.$$

For $e < 0$ the masses have to pass through each other. For $e > 1$ the collision would involve a gain in energy. This might happen if there was explosive gunpowder in the contact region of the collision.

In the case $e = 0$ the collision is perfectly plastic but still frictionless. This is generally not a sticking collision because the masses can

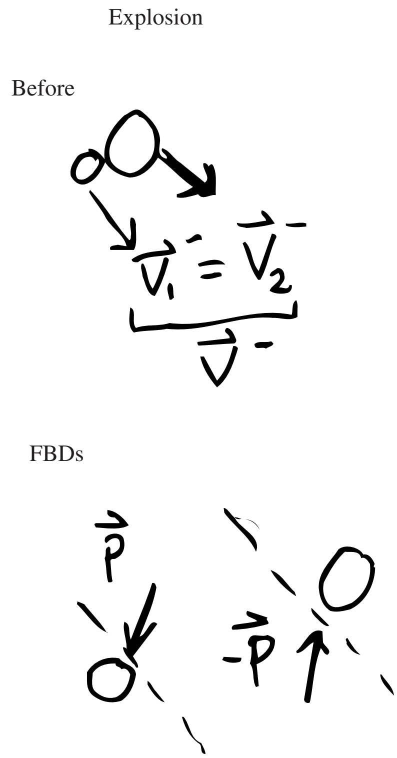


Figure 11.12: An explosion is like a sticking collision run backwards in time. The particles initially have the same velocity $\vec{\boldsymbol{v}}_1^{-} = \vec{\boldsymbol{v}}_2^{-} = \vec{\boldsymbol{v}}^{-}$ and then separate due to an action-reaction impulse pair in any direction.
Filename:tfig11-collision-explosion

*Why the word *restitution*? The particles approach each other with some momentum relative to their common center of mass. At some point during the collision they have none. Then some of the momentum is restituted, paid back, and they bounce. No restitution, $e = 0$, and there is no bounce. Full restitution, $e = 1$, and they separate with the as much relative momentum as they had when they approached.

enter and hence leave the collision with some relative velocity in the common contact plane (the $\hat{\lambda}$ direction).

Explosions

If one particle explodes into pieces it's as if the pieces had a collision. It's just that the initial velocities of the pieces were all the same and the total kinetic energy of the system increases during the 'collision'. See *Fig.* 11.12. The overall treatment is extremely similar to that for sticking collisions, but in some sense backwards. Instead of the particles entering the collision with different velocities and leaving with the same velocity, they enter with the same velocity and leave with different velocities. But the same momentum principles apply. There is no colllision law or coefficient of restitution to apply, all of the post-collision relative velocity is restituted from nothing. Rather one just has to know (or find) the action-reaction impulse between the masses.

Example: **An explosion.**
Two particles m_1 and m_2 are stuck together and moving at $\vec{\boldsymbol{v}}^-$ when they explode and an impulse $\vec{\boldsymbol{P}}$ separates them. After the collision

$$\vec{\boldsymbol{v}}_1^+ = \vec{\boldsymbol{v}}^- + \vec{\boldsymbol{P}}/m_1 \qquad \text{and} \qquad \vec{\boldsymbol{v}}_2^+ = \vec{\boldsymbol{v}}^- - \vec{\boldsymbol{P}}/m_2.$$

11.2 *THEORY*
Energetics of collisions

Often one thinks of collisions as passive and energetically dissipative. However, as noted in the text, an explosion is a collision of sorts in which the system kinetic energy increases. We'd like to treat these cases in a unified way. First let's calculate the total kinetic energy.

$$\begin{aligned} 2E_{\mathrm{K}} &= m_1 v_1^2 + m_2 v_2^2 \\ &= m_{\mathrm{tot}} v_{\mathrm{cm}}^2 + m_1|\vec{\boldsymbol{v}}_1 - \vec{\boldsymbol{v}}_{\mathrm{cm}}|^2 + m_2|\vec{\boldsymbol{v}}_2 - \vec{\boldsymbol{v}}_{\mathrm{cm}}|^2 \\ &= m_{\mathrm{tot}} v_{\mathrm{cm}}^2 + m_{\mathrm{eff}}|\vec{\boldsymbol{v}}_1 - \vec{\boldsymbol{v}}_2|^2 \\ &= m_{\mathrm{tot}} v_{\mathrm{cm}}^2 + m_{\mathrm{eff}} v_{\mathrm{rel}}^2 \end{aligned}$$

where $m_{\mathrm{tot}} = m_1 + m_2$, $m_{\mathrm{eff}} = m_1 m_2/(m_1 + m_2)$ and $v_{\mathrm{rel}} = |\vec{\boldsymbol{v}}_1 - \vec{\boldsymbol{v}}_2|$. There are a few algebra steps needed to go from line to line above (see section **??** for related calculations). The concept of effective mass m_{eff} is introduced in box 11.1. The key result is that the kinetic energy of a two-particle system can be written as the sum of two terms, one involving center of mass velocity and one involving the relative velocity of the two masses.

This is a special result for two-particle systems. For any system the kinetic energy is a center of mass term $(mv_{\mathrm{cm}}^2/2)$ plus a term for motion relative to the center of mass. But generally the relative motion term is written as a sum of terms, one for each particle, and the motion of each particle is measured relative to the center of mass $(mv_{i/\mathrm{cm}})$. What is special for two-particle systems is that the relative motion part can be written in terms of the motion of the two particles relative to each other. Because that is not the velocity of any real thing, it only gives the right kinetic energy when used with the corrected effective mass (m_{eff}).

What about energy and collisions? The center of mass velocity and energy do not change in the collision. So the only change in kinetic energy is that associated with changes in $m_{\mathrm{eff}} v_{\mathrm{rel}}^2$.

$$\begin{aligned} 2\Delta E_{\mathrm{K}} &= m_{\mathrm{eff}}\left((v_{\mathrm{rel}}^+)^2 - (v_{\mathrm{rel}}^-)^2\right) \\ &= 2\vec{\boldsymbol{v}}_{\mathrm{rel}}^- \cdot \vec{\boldsymbol{P}} + |\vec{\boldsymbol{P}}|^2/m_{\mathrm{eff}} \end{aligned}$$

where we used that $\vec{\boldsymbol{P}} = m_{\mathrm{eff}}(\vec{\boldsymbol{v}}_{\mathrm{rel}}^+ - \vec{\boldsymbol{v}}_{\mathrm{rel}}^-)$. This formula applies for both sticking collisions, in which case $\vec{\boldsymbol{P}} = -m_{\mathrm{eff}}\vec{\boldsymbol{v}}_{\mathrm{rel}}^-$ and $2\Delta E_{\mathrm{K}} = -m_{\mathrm{eff}}(v_{\mathrm{rel}}^-)^2$, and to explosions where $\vec{\boldsymbol{P}} = m_{eff}\vec{\boldsymbol{v}}_{\mathrm{rel}}^+$ and $2\Delta E_{\mathrm{K}} = m_{\mathrm{eff}}(v_{\mathrm{rel}}^+)^2$. It also applies to interactions in-between.

All that enters the change-of-energy equations above is the projection of the relative velocity in the $\vec{\boldsymbol{P}}$ direction. Thus the issue of energy loss or gain is determined by whether the projection of the relative velocity in the $\vec{\boldsymbol{P}}$ direction decreases or increases in magnitude. Thus a collision with $-1 < e < 1$ loses energy and a collision with $|e| > 1$ increases energy. We included $e < 0$ for completeness even though it is sometimes considered 'non-physical' in that it involves the particles passing by or passing through each other.

The full range of behavior for sticking collisions to explosions can be captured with a single restitution coefficient e_g (see box 11.3).

Frictional collisions

Our avoiding of frictional collisions is not because there generally is no friction during collisions. Friction is a fact of the mechanical world. We avoid friction here because a host of special assumptions are needed to make frictional problems deterministic. And no given set of assumptions is known to yield accurate predictions. Frictional collision models have too dis-satisfyingly low a ratio of accuracy to complexity for inclusion in a book at this level.

Simultaneous collisions

If one particle is involved in two collisions at one time then we have not explained how to calculate the resulting motion. In an attempt to make the situation clear one is tempted to say "Let's make it ideal and assume the collisions are exactly instantaneous and at *exactly* the same time." Then, unfortunately, one is making the situation *exactly* ambiguous.

Unfortunately for our hope of making reliable predictions, simultaneous collisions are *not* rare events. Why? Imagine B is touching C and both are stationary. Then A comes and bangs into B. Because B and C are already touching one must assume that there are impulsive forces not just between A and B, but also between B and C. And we have no reliable rules for sorting out the result. Nor will we find such rules if we make it a life's work.

Example: **A triangular array of identical spheres.**
Imagine 15 accurately-machined nominally-identical spheres laid out in a tight triangle (5 in one row, 4 in the next, then 3,2, and 1) on a very flat smooth surface. Then imagine a 16th ball rolls in and hits the apex of the triangle. How do the 15 balls move?

This experiment is performed in smoky rooms full of intoxicated people night after night. Its the 'break' in a pool game. And the game depends on the result being unpredictable. Each time, due to tiny differences, the results are different.

And, according to theory, the more rigid and perfect the balls are, the more sensitive are the results to the smallest of differences in the initial conditions.

What is the source of the problem?

Example: **Three balls in a line.**
Consider the one-dimensional collision of three identical particles. B and C are in line, stationary and touching and then A comes along with $v_{A0} = v^-$. Let's assume that the collision(s) whatever they are, are completely elastic and conserve energy ($e = 1, e_g = 0$). Here are two ways to predict the outcome:

- A hits B and C, being all the way at the other side of B, is oblivious to the interaction between A and B until it is complete. Thus A comes to rest and B is moving to the right with v^-. Then B colllides elastically with C and B comes to rest and C shoots off with the v^-.
- B and C are touching and act as a single rigid object throughout the collision with A. Thus the result is like that between a particle with mass m and another with mass $2m$. Such an elastic collision would leave B and C going forwards at $2v^-/3$ and A going the other way with $v_A^+ = -v^-/3$. A different result.

- actually there is a one-parameter family of results that are consistent with energy conservation and momentum balance. We have three outcomes (the velocities of the three particles) and only 2 scalar equations restricting them (momentum and energy balance).

But what would *really* happen? That would depend on details that are not stated. Of course if the *exact* shape and configuration of the balls was known, and the exact rules for elastic and inelastic deformation, then one could calculate the resulting motion solving partial differential equations or with atomic simulations. In principle. But we generally do not know such details nor have have such calculation abilities. And crowding that which we don't know into concepts like 'rigid-object' and 'exactly simultaneous' crowds the prediction of the outcome to dependence on infinitesimal things.

So, as an engineer, what are you supposed to do when calculating in situations involving simultaneous collisions?

- first relax and remember that no collision calculation is likely to be very accurate (unless the result only depends on balance of momentum). So simultaneous collisions, while philosophically worse in that even the equations are indeterminate, are not that much worse than the usual deterministic, but not accurate, collisional relations.
- do experiments, and
- take account the range of outcomes depending on assumptions about the collision details.

Samples 11.4 and 11.5 starting on page 619 illustrate the ambiguity of simultaneous collisions in 2D.

Final comments

This is the second of three sections about collisions. Section 9.5 was about collisions in 1D, then this section about particles in 2D and 3D, and finally the ideas in this section will be extended from particles to rigid objects in section **??**.

SAMPLE 11.3 **Projectile hits a slanted floor:** A ball of mass $m = 0.2\,\mathrm{kg}$ is thrown in the air at an angle $\theta = 60°$ with initial speed $v_0 = 10\,\mathrm{m/s}$. The ball lands on a hard, frictionless floor that is tilted at angle $\alpha = 20°$ with the horizontal. The coefficient of restitution between the floor and the ball is $e = 0.85$. Ignore air resistance. Find the height of the ball after the rebound from the floor.

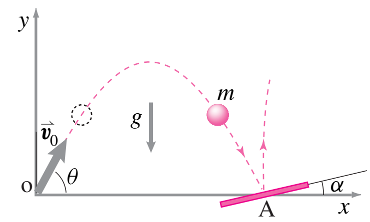


Figure 11.13:

Filename:sfig11-2-tiltedfloor

Solution This problem has two parts to it. In order to figure out the height after rebound, we need to find the rebound velocity. But to find the rebound velocity, we need to know the velocity of the ball before impact with the floor. Let the velocity just before the impact be $\vec{\boldsymbol{v}}^-$ and the velocity of rebound (just after impact be $\vec{\boldsymbol{v}}^+$. Let us first find $\vec{\boldsymbol{v}}^-$.

The ball undergoes projectile motion before it lands at A. Its initial (launch) velocity is $\vec{\boldsymbol{v}}_0 = v_0(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath})$. From energy conservation, we know that the kinetic energy just before impact at A, $m|\vec{\boldsymbol{v}}^-|^2/2$, must be the same as kinetic energy at launch, $mv_0^2/2$. Thus $|\vec{\boldsymbol{v}}^-| = v_0$. And, from the symmetry of the flight, we can conclude that $\vec{\boldsymbol{v}}^-$ must make the same angle θ with the horizontal that $\vec{\boldsymbol{v}}_0$ does. Thus, using $Fig.$ 11.14, we have

$$\vec{\boldsymbol{v}}^- = v_0(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}).$$

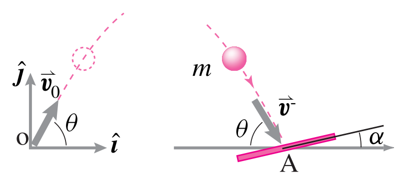


Figure 11.14: Trajectory of the ball through air: From the symmetry of the trajectory, we can conclude that the angle the velocity vector makes with the horizontal at point A is the same as the angle of launch, θ.

Filename:sfig11-2-tiltedfloor-a

Now we are ready to do collision mechanics at point A. We need to determine $\vec{\boldsymbol{v}}^+$ given $\vec{\boldsymbol{v}}^-$ and the coefficient of restitution for the collision at A. From collision law, we now that the velocity component normal to the floor changes because of the normal impulse during collision, while the tangential velocity remains the same because there is no force or impulse parallel to the floor. Thus,

$$\begin{aligned} \vec{\boldsymbol{v}}^+ \cdot \hat{\boldsymbol{n}} &= -e(\vec{\boldsymbol{v}}^- \cdot \hat{\boldsymbol{n}}) \\ \vec{\boldsymbol{v}}^+ \cdot \hat{\lambda} &= \vec{\boldsymbol{v}}^- \cdot \hat{\lambda}. \end{aligned}$$

Writing out $\vec{\boldsymbol{v}}^+ = v_x^+\hat{\imath} + v_y^+\hat{\jmath}$, and noting that $\hat{\lambda} = \cos\alpha\hat{\imath} + \sin\alpha\hat{\jmath}$ and $\hat{\boldsymbol{n}} = -\sin\alpha\hat{\imath} + \cos\alpha\hat{\jmath}$, we get, from the equations above,

$$\begin{aligned} \cos\alpha v_x^+ + \sin\alpha v_y^+ &= v_0\cos\theta\cos\alpha - v_0\sin\theta\sin\alpha = v_0\cos(\theta+\alpha) \\ -\sin\alpha v_x^+ + \cos\alpha v_y^+ &= -e\,v_0(-\cos\theta\sin\alpha - \sin\theta\cos\alpha) = e\,v_0\sin(\theta+\alpha). \end{aligned}$$

These are two equations in two unknowns, v_x^+ and v_y^+. Writing them in a matrix form and solving the matrix equation, we get

$$\begin{pmatrix} v_x^+ \\ v_y^+ \end{pmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{pmatrix} v_0\cos(\theta+\alpha) \\ e\,v_0\sin(\theta+\alpha) \end{pmatrix} = \begin{pmatrix} v_0\cos(\theta+2\alpha) \\ e\,v_0\sin(\theta+2\alpha) \end{pmatrix}.$$

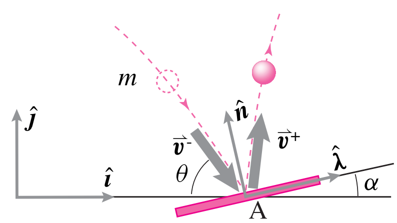


Figure 11.15: Collision of the ball with the tilted floor: The ball strikes the floor with velocity $\vec{\boldsymbol{v}}^-$. It rebounds with velocity $\vec{\boldsymbol{v}}^+$. The impulse during the collision acts in the normal direction to the floor at A. The unit normal and the unit tangent vectors at A are $\hat{\boldsymbol{n}} = -\sin\alpha\hat{\imath} + \cos\alpha\hat{\jmath}$, and $\hat{\lambda} = \cos\alpha\hat{\imath} + \sin\alpha\hat{\jmath}$ respectively.

Filename:sfig11-2-tiltedfloor-b

Thus, we know the rebound velocity $\vec{\boldsymbol{v}}+ = v_0[\cos(\theta+2\alpha)\hat{\imath} + \sin(\theta+2\alpha)\hat{\jmath}]$.

To find the maximum height reached by the ball on the rebound, we only need the vertical component of the rebound velocity. Since the ball has a constant deceleration g, we can use the formula $(v_y)^2 = (v_{y_0})^2 - 2gh$ with $v_y = 0$ at the maximum height h_{max} to get,

$$h_{\mathrm{max}} = \frac{(v_y^+)^2}{2g} = \frac{e^2\,v_0^2\,\sin^2(\theta+2\alpha)}{2g}$$

Substituting the given values of v_0, e, θ, and α, and using $g = 9.81\,\mathrm{m/s^2}$, we get,

$$h_{\mathrm{max}} = 3.57\,\mathrm{m}.$$

$\boxed{h_{\mathrm{max}} = 3.57\,\mathrm{m}}$

You can see that the inclined plane helps in getting the ball reach higher on the bounce. If the floor were flat ($\alpha = 0$), we will get $h_{\max} = 2.76\,\text{m}$. It should be obvious that for maximum height, we should have $\sin(\theta + 2\alpha) = 1$ which gives $\alpha = \frac{1}{2}(\frac{\pi}{2} - \theta)$.

11.3 *THEORY*
Coefficient of generation

Often one thinks of collisions as passive and energetically dissipative. However, as noted in the text, an explosion is a collision of sorts in which the system kinetic energy increases. For passive frictionless collisions one can characterize the collision by the coefficient of restitution

$$e \equiv \frac{\text{(separation speed)}}{\text{(approach speed)}} = \frac{-(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+)\cdot\hat{\boldsymbol{n}}}{(\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-)\cdot\hat{\boldsymbol{n}}} \tag{11.5}$$

However, for explosions the coefficient of restitution is $e = \infty$. If one is equally interested in energy absorbing or energy creating collisions one can use a more democratic coefficient of generation

$$e_g \equiv \frac{\text{(separation speed)-(approach speed)}}{\text{(separation speed)+(approach speed)}} = \frac{-(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+)\cdot\hat{\boldsymbol{n}} - (\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-)\cdot\hat{\boldsymbol{n}}}{-(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+)\cdot\hat{\boldsymbol{n}} + (\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-)\cdot\hat{\boldsymbol{n}}} \tag{11.6}$$

We can write the collisional coefficient of generation in terms of the restitution coefficient, and *vice versa*, as

$$e_g = \frac{e-1}{e+1} \quad \text{and} \quad e = \frac{1+e_g}{1-e_g}.$$

The generation coefficient is -1 for sticking collisions and 1 for explosions. This coefficient is zero for energetically neutral collisions (no gain, no loss, $e = 1$). And the coefficient of generation does not allow for passing-through or passing-by collisions ($e < 0$).

As a replacement for the conventional coefficient of restitution the coefficient of generation e_g is more complex to use in simple calculations in that ***eqn.* (11.6)** is more complex than ***eqn.* (11.5)**. On the other hand the coefficient of generation is convenient for describing situations which are a mix of passive ($e < 1$ and $e_g < 0$) and active ($e > 1$ and $e_g > 0$). Such is the case, for example, in simple models of legged locomotion (see box 11.4 on page 620).

Note that in all the collisional restitution formulas we could replace $\hat{\boldsymbol{n}}$ with $-\hat{\boldsymbol{n}}$ without affecting the validity of the equations. Similarly all the subscript 2's could be replaced with 1's and *vice versa* without affecting the validity of the equations. Knowing this relieves anxiety about the choice of normal $\hat{\boldsymbol{n}}$ (towards m_1 or towards m_2?) or which particle to call 1 and which to call 2.

SAMPLE 11.4 Simultaneous collisions: This problem involves two simultaneous collisions. In general, such problems are hard to solve. We are going to show one way of solving such problems by treating the collisions successively. However, this leads to nonuniqueness of solution. Here we solve the problem in one way and in the next sample, we solve the same problem in another way.

A 12 kg cart with an inclined face rests on a frictionless floor. A ball of mass 3 kg is shot horizontally with speed 30 m/s at the inclined face of the cart. The coefficient of restitution between the cart and the ball is 0.9. The cart subsequently moves horizontally on the floor. Find the velocity of the ball and that of the cart after the collision.

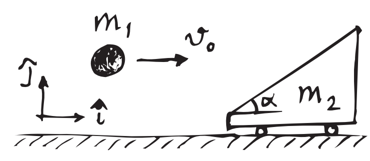


Figure 11.16:

Filename:sfig11-2-two-collisions

Solution There are two simultaneous collisions in this problem. One collision is between the ball and the cart and the other is between the cart and the ground. Here, we will treat the two collisions one after the other, the one between the ball and the cart preceding the one between the cart and the ground. In Sample 11.5, we treat the ground collision first.

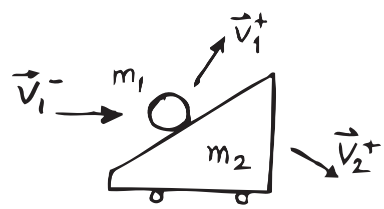


Figure 11.17: The ball and the cart as a system during the collision. There are no external forces or impulses acting on the system. Therefore, the linear momentum must be conserved during the collision; *i.e.*, $\vec{\boldsymbol{L}}^{-} = \vec{\boldsymbol{L}}^{+}$.

Filename:sfig11-2-two-collisions-a

Collision between the ball and the cart: Here we assume that the ball hits the cart and both are free to move in any direction immediately after the collision. Let the mass of the ball be m_1 and that of the cart be m_2. Let their after collision velocities be $\vec{\boldsymbol{v}}_1^{+}$ and $\vec{\boldsymbol{v}}_2^{+}$, respectively. Let the impulse during this collision be P_1.

Let us consider the cart and the ball as a single system during the collision. Then, the impulse becomes internal to this system and there is no net impulse on this system. Therefore, the linear momentum is conserved; that is, $\vec{\boldsymbol{L}}^{+} = \vec{\boldsymbol{L}}^{-}$. From this relationship, we have,

$$\mathrm{m}_1 \vec{\boldsymbol{v}}_1^{+} + \mathrm{m}_2 \vec{\boldsymbol{v}}_2^{+} = \mathrm{m}_1 \vec{\boldsymbol{v}}_1^{-} + \mathrm{m}_2 \vec{\boldsymbol{v}}_2^{-} = m_1 v_0 \hat{\imath}.$$

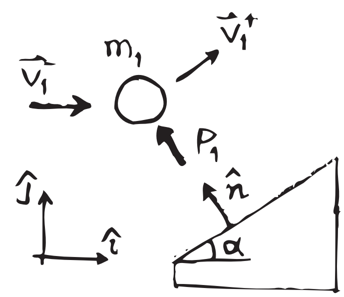


Figure 11.18: Free body diagram of the ball during the collision. The impulse $\boldsymbol{P}_1$ acts normal to the plane of collision which is the inclined surface of the cart.

Filename:sfig11-2-two-collisions-b

Writing out the unknown velocities in terms of their x and y components and dotting the resulting equation with $\hat{\imath}$ and $\hat{\jmath}$ separately, we get the following two scalar equations:

$$m_1 v_{1x}^{+} + m_2 v_{2x}^{+} = m_1 v_0 \qquad (11.7)$$

$$m_1 v_{1y}^{+} + m_2 v_{2y}^{+} = 0. \qquad (11.8)$$

We have four unknowns here, v_{1x}^{+}, v_{1y}^{+}, v_{2x}^{+}, and v_{2y}^{+}. So, far we have just two equations. We need more equations. We can write restitution equation relating the relative velocities of the ball and the cart in the normal direction before and after the collision:

$$(\vec{\boldsymbol{v}}_2^{+} - \vec{\boldsymbol{v}}_1^{+}) \cdot \hat{\boldsymbol{n}} = -e(\vec{\boldsymbol{v}}_2^{-} - \vec{\boldsymbol{v}}_1^{-}) \cdot \hat{\boldsymbol{n}} = e(v_0 \hat{\imath}) \cdot \hat{\boldsymbol{n}}.$$

Now, writing $\hat{\boldsymbol{n}} = n_x \hat{\imath} + n_y \hat{\jmath}$ and carrying out the dot products (after writing $\vec{\boldsymbol{v}}_1^{+}$ and $\vec{\boldsymbol{v}}_2^{+}$ in terms of their components), we get,

$$(v_{2x}^{+} - v_{1x}^{+}) n_x + (v_{2y}^{+} - v_{1y}^{+}) n_y = -e\, v_0\, n_x. \qquad (11.9)$$

We still need another equation. Let us now consider the impulse acting on the ball during the collision. From the free body diagram shown in $Fig.$ 11.2, we can write the change in momentum of the ball as,

$$P_1 \hat{\boldsymbol{n}} = m_1 \vec{\boldsymbol{v}}_1^{+} - m_1 \vec{\boldsymbol{v}}_1^{-}$$

$$\text{or} \qquad P_1 (n_x \hat{\imath} + n_y \hat{\jmath}) = m_1 (v_{1x}^{+} \hat{\imath} + v_{1y}^{+} \hat{\jmath} - v_0 \hat{\imath}).$$

11.4 *THEORY*
A particle collision model of running

At every step a running person flies through the air, hits the ground with a foot and pushes on the ground. By action and reaction, the ground pushes back on the foot which pushes on the leg which pushes on the body which causes the body to slow its descent and then go from moving forward and somewhat down to moving forward and somewhat up. Then the foot leaves the ground and the person flies through the air again readying for the next foot contact.

Human bodies are somewhat bigger than human legs so one approximation is that the legs have negligible mass. Human bodies don't tumble about much during a running step, so a next approximation is to neglect all distortion and rotation of the body and think of it as a particle. Finally, one might imagine that the ground contact time is short, and that the step on the ground is like a bounce. Thus running is like a sequence of collisions between a body and the ground. Obviously a running person is not a bouncing particle in all regards. Nonetheless, this model gives a means for making various estimates about running.

In the flight phase of running, neglecting air friction, the body moves in a parabolic arc according to:

$$\vec{F} = m\vec{a} \quad \Rightarrow \quad -mg\,\hat{\jmath} = m\vec{a}$$

This has solution that the time of flight is

$$t_f = 2v_{y0}/g$$

where v_{y0} is the vertical component of the velocity at the start of flight. The distance of flight is

$$d = v_x t_f = 2v_x v_y/g$$

where v_x is the constant horizontal component of velocity.

What happens in the 'collision' with the ground?

The horrible leap-frog model of running

We could think of each step as independent. Each running step would be a jump at the end of which the body would come to rest and then jump again. That is, each step would start with an explosion and, after a period of flight, end with a plastic no-slip collision. Then immediately after there would be another jump. How much energy would it take to run like that? Each jump would involve an impulse to get the body from zero velocity to $\vec{v} = v_x\hat{\imath} + v_{y0}\hat{\jmath}$. The work of the legs would be the increase in kinetic energy.

$$W = mv^2/2 = m(v_x^2 + v_{y0}^2)/2$$

Then the legs would absorb that much energy at landing. Muscles, unlike generators, are not regenerative. If muscles were regenerative you would feel especially peppy after you walked down a long stair case. On the other hand, walking down stairs is not that tiring. So let's approximate that there is no metabolic cost for absorbing work. So the energetic cost of locomotion per unit time would be

$$P = W/t_f = \frac{m\left(v_x^2 + v_{y0}^2\right)/2}{2v_{y0}/g} = mg\,v_x\frac{\tan\theta + \cot\theta}{4}$$

where $\tan\theta = v_{y0}/v_x$ is the angle of the trajectory at liftoff. The function $\tan\theta + \cot\theta$ has its minimum value of 2 at $\theta = 45°$ so the cost of such locomotion, in terms of average power, is $mg\,v_x/2$. Muscles use about 4 time as much chemical energy as they can produce work (ie, about 25% efficient at best) so the chemical energy to run by jumping and landing, over and over again, would be about

$$\text{Metabolic Cost per unit time} = P_{\text{met}} = 2mg\,v_x,$$

that twice the weight times the speed. The chemical energy needed per unit distance would be about $2\cdot$(weight).

Obviously this seems like a tiring way to run. You shouldn't stop and start your horizontal motion at every step. Real people don't do that. Furthermore, the energy cost we have just predicted is bigger than what people use by a factor of about 5; the rate at which people use chemical energy to run is more like $mg\,v_x/2$ or $mg\,v_x/3$. Notice that the energetic cost of this mode of 'running' does not depend on the step length or flight time but only on the initial angle of the trajectories. Smaller steps involve smaller collisions and hence smaller energy cost per collision. But with smaller jumps there are more collisions per unit distance. The two effects exactly cancel in this model. Only the angle of liftoff matters, not the length of the jumps.

Frictionless collision model of running

Although shoes generally have high friction, the legs pivot under the body during ground contact. The result is that the main force transmitted by the leg to the body is vertical. In effect the leg mediates an effectively frictionless collision. At least that's an extreme idealization of what a leg does. Perhaps a better model of running is then a sequence of vertical frictionless collisions.

At each step there is, in effect, a plastic frictionless collision which absorbs energy immediately followed by an energetically generative collision that sends the body back up again. Together they look like a single frictionless elastic collision, but in this model we want to take account of the work absorbed in landing and the work needed to take off again. To start we will neglect that humans do have springs in their legs (e.g., tendons).

Thus at each step the energy needed to take off is

$$W = mv_{y0}^2/2.$$

The time of flight is again $t_f = 2v_{y0}/g$ and so, for this model the average work per unit time is

$$P = W/t_f = \frac{mv_{y0}^2/2}{2v_{y0}/g} = mg\,v_{y0}/4$$

and the work per unit distance would be

$$\text{work per unit distance} = mg\frac{v_{y0}/v_x}{4} = mg\tan\theta/4$$

and the metabolic cost per unit distance, taking muscle efficiency as 25% again, would be the weight times $\tan\theta$. So, at a given horizontal speed, the energy cost per unit distance can be made arbitrarily small by having the flight angle small and there being, consequently, more and more small collisions. But for a person to try to save energy that way she would have to swing her legs in impossibly tiring small rapid steps. To complete this model so thqat it would not predict that people should choose infinite frequency and infinitesimal steps we would have to add in a formula for the cost of swinging the legs rapidly. If we evaluate this model with the step length of real human running, and the consequent launch angle θ we over-estimate the actual energetic cost of running by about a factor of 2. Why is that? Probably because people do use their springs to bounce. They don't just throw away all their energy at each ground landing and then jump vertically. Rather their tendons store energy and release it at each step, doing something like half of the work needed to get airborne again.

Again, separating out this equation in scalar equations (by dotting the equation with $\hat{\imath}$ and $\hat{\jmath}$ separately), we get,

$$P_1 n_x - m_1 v_{1x}^+ = -m_1 v_0 \qquad (11.10)$$

$$P_1 n_y - m_1 v_{1y}^+ = 0. \qquad (11.11)$$

Now, we have added another unknown P_1, but fortunately, we have got an extra equation too. We now have five unknowns and five independent equations. So we should be able to solve for all the unknowns.

For solving these equations, we first write them in matrix form and then use a computer to solve them. We write $eqn.$ (11.7)–$eqn.$ (11.11) as,

$$\begin{bmatrix} m_1 & 0 & m_2 & 0 & 0 \\ 0 & m_1 & 0 & m_2 & 0 \\ -n_x & -n_y & n_x & n_y & 0 \\ -m_1 & 0 & 0 & 0 & n_x \\ 0 & -m_1 & 0 & 0 & n_y \end{bmatrix} \begin{pmatrix} v_{1x}^+ \\ v_{1y}^+ \\ v_{2x}^+ \\ v_{2y}^+ \\ P_1 \end{pmatrix} = \begin{pmatrix} m_1 v_0 \\ 0 \\ -e v_0 n_x \\ -m_1 v_0 \\ 0 \end{pmatrix}.$$

Here is the pseudo computer code to solve this matrix equation:

```
m1 = 3, m2 = 12
theta = pi/6                   % angle in radians
nx = -sin(theta), ny = cos(theta) % components of the normal
v0 = 30
e =  0.9

A = [ m1   0   m2  0    0    % x comp of lin mom bal
      0    m1  0   m2   0    % y comp of lin mom bal
     -nx  -ny  nx  ny   0    % restitution equation
     -m1   0   0   0   -nx   % impulse-momentum for m1, x comp
      0   -m1  0   0   -ny]  % impulse-momentum for m1, y comp
b = [m1*v0  0  -e*v0*nx  -m1*v0  0]' % the known right hand side
solve A*x = b for x
```

The solution thus computed gives us

$$v_{1x}^+ = 18.60\,\mathrm{m/s}, \qquad v_{1y}^+ = 19.74\,\mathrm{m/s},$$
$$v_{2x}^+ = 2.85\,\mathrm{m/s}, \qquad v_{2y}^+ = -4.94\,\mathrm{m/s},$$
$$P_1 = -68.40\,\mathrm{N\cdot s}.$$

$$\boxed{\begin{aligned} \vec{v}_1^{\,+} &= (18.60\,\mathrm{m/s})\hat{\imath} + (19.74\,\mathrm{m/s})\hat{\jmath} \\ \vec{v}_2^{\,+} &= (2.85\,\mathrm{m/s})\hat{\imath} + (-4.94\,\mathrm{m/s})\hat{\jmath} \end{aligned}}$$

Collision between the cart and the ground: Now, we consider the collision between the cart and the ground, taking $\vec{v}_2^{\,+}$ as the velocity of the cart just before the collision. Figure 11.19 shows the impulse from the ground acting on the cart. We know the final velocity of the cart has to be in the $\hat{\imath}$ direction. Just to keep our notations straight, let us denote the velocity of the cart after collision as $\vec{v}_2^{\,++}$ (after the second collision) and keep the incoming velocity as $\vec{v}_2^{\,+}$. Then, from impulse momentum, we have,

$$P_2 \hat{\jmath} = m_2 \vec{v}_2^{\,++} - m_2 \vec{v}_2^{\,+}.$$

This is a vector equation which we can write as two scalar equations in the $\hat{\imath}$ and $\hat{\jmath}$ directions. Note that $\vec{v}_2^{\,++} = v_2^{++}\hat{\imath}$ and we already know $\vec{v}_2^{\,+} = (2.85\,\mathrm{m/s})\hat{\imath} + (-4.94\,\mathrm{m/s})\hat{\jmath}$ as found before. Thus,

$$v_2^{++} = \; = \vec{v}_2^{\,+} \cdot \hat{\imath} = 2.85\,\mathrm{m/s}$$

$$P_2 = -m_2 \vec{v}_2^{\,+} \cdot \hat{\jmath} = -(12\,\mathrm{kg})(-4.94\,\mathrm{m/s}) = 59.24\,\mathrm{kg\cdot m/s} = 59.24\,\mathrm{N\cdot s}.$$

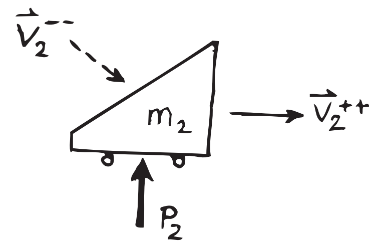


Figure 11.19:

Filename:sfig11-2-two-collisions-c

$$\boxed{\vec{\boldsymbol{v}}_2^{++} = 2.85\,\mathrm{m/s}\hat{\imath}}$$

SAMPLE 11.5 Simultaneous collisions again: Consider the ball and the cart collision problem of Sample 11.4 again. This time, consider the ball and the cart together to have a collision with the ground first. Then consider the collision between the cart and the ball. Once again, you are to find the final horizontal velocity of the cart. The problem parameters are the same — mass of the ball $m_1 = 3\,\text{kg}$, mass of the cart $m_2 = 12\,\text{kg}$, $e = 0.9$ between the ball and the cart, and the velocity of the ball before impact, $v_0 = 30\,\text{m/s}$.

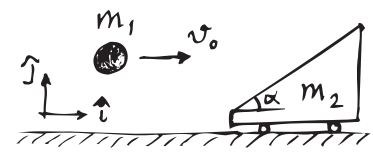


Figure 11.20:

Filename:sfig11-2-two-collisions2

Solution Let us consider the ball and the cart as a system colliding with the ground as shown in *Fig.* 11.21. There is an unknown external impulse P_2 from the ground acting on this system in the $\hat{\jmath}$ direction. Using this information, we now write impulse-momentum equation for this system:

$$P_2\hat{\jmath} = \vec{L}_2 - \vec{L}_1 = m_1\vec{v}_1^{\,+} + m_2\vec{v}_2^{\,+} - m_1\vec{v}_1^{\,-}$$

Assuming that $\vec{v}_1^{\,+} = v_{1x}^+\hat{\imath} + v_{1y}^+\hat{\jmath}$ and $\vec{v}_2^{\,+} = v_2^+\hat{\imath}$, and using the given information $\vec{v}_1^{\,-} = v_0\hat{\imath}$, we obtain the following two scalar equations from the vector impulse-momentum equation:

$$m_1 v_{1x}^+ + m_2 v_2^+ = m_1 v_0 \qquad (11.12)$$

$$m_1 v_{1y}^+ - P_2 = 0 \qquad (11.13)$$

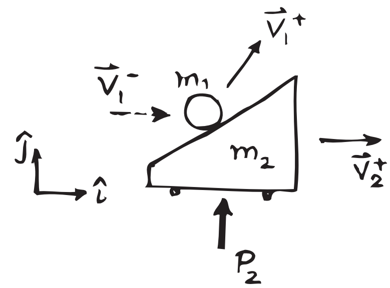


Figure 11.21: Free body diagram of the ball-cart system during the collision with the ground. The ground impulse is $\boldsymbol{P}$, the velocities of the ball and the cart before the collision are given — $\vec{v}_1^{\,-} = v_0\hat{\imath}$, $\vec{v}_2^{\,-} = \vec{0}$, and the velocities after the collision $\vec{v}_2^{\,+} = v_2\hat{\imath}$ and $\vec{v}_1^{\,+} = v_{1x}^+\hat{\imath} + v_{1y}^+\hat{\jmath}$ are unknown.

Filename:sfig11-2-two-collisions2-a

So far, we have two equations and four unknowns — v_{1x}^+, v_{1y}^+, v_2^+ and P_2. Obviously, we need more equations. Now, let us consider the collision between the cart and the ball. Let the impulse of this collision be P_1. Then the impulse-momentum equation for the ball gives us,

$$P_1\hat{\boldsymbol{n}} = m_1(v_{1x}^+\hat{\imath} + v_{1y}^+\hat{\jmath}) - m_1 v_0\hat{\imath}.$$

Once again, we separate out the scalar equations from this vector equation, using the information $\hat{\boldsymbol{n}} = n_x\hat{\imath} + n_y\hat{\jmath}$:

$$m_1 v_{1x}^+ - Pn_x = m_1 v_0 \qquad (11.14)$$

$$m_1 v_{1y}^+ - Pn_y = 0 \qquad (11.15)$$

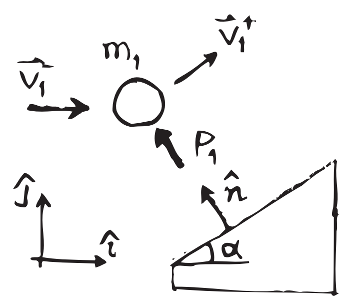


Figure 11.22: Free body diagram of the ball during the collision with the cart. The cart impulse $\boldsymbol{P}_1$ acts along the normal $\hat{\boldsymbol{n}}$. From geometry, we find that $\hat{\boldsymbol{n}} \equiv n_x\hat{\imath} + n_y\hat{\jmath} = -\sin\alpha\hat{\imath} + \cos\alpha\hat{\jmath}$.

Filename:sfig11-2-two-collisions2-b

Thus, we have now four equations; we still need one more. We now use the restitution equation to relate the normal components of the relative velocities of approach and departure of the ball and the cart:

$$(\vec{v}_1^{\,+} - \vec{v}_2^{\,+})\cdot\hat{\boldsymbol{n}} = -e(v_1^- - v_2^-)\cdot\hat{\boldsymbol{n}}$$

$$\Rightarrow \quad v_{1x}^+ n_x + v_{1y}^+ n_y - v_2 n_x = -e\,v_0 n_x. \qquad (11.16)$$

Now we have five equations in five unknowns. All we need to do now is to solve these linear equations for all the unknowns. We do so by first writing the five equations (*eqn*. (11.12) to *eqn*. (11.16) in matrix form and then solving the matrix equation on a computer. The matrix equation is:

$$\begin{bmatrix} m_1 & 0 & m_2 & 0 & 0 \\ 0 & m_1 & 0 & 0 & -1 \\ m_1 & 0 & 0 & -n_x & 0 \\ 0 & m_1 & 0 & -n_y & 0 \\ n_x & n_y & -n_x & 0 & 0 \end{bmatrix} \begin{Bmatrix} v_{1x}^+ \\ v_{1y}^+ \\ v_2^+ \\ P_1 \\ P_2 \end{Bmatrix} = \begin{Bmatrix} m_1 v_0 \\ 0 \\ m_1 v_0 \\ 0 \\ -e\,v_0 n_x. \end{Bmatrix}.$$

Solving this equation as in the previous sample, we get,

$$v_{1x}^+ = 16.59\,\text{m/s},\ v_{1y}^+ = 23.23\,\text{m/s},\ v_2^+ = 3.35\,\text{m/s},\ P_1 = 80.47\,\text{N}\cdot\text{s},\ P_2 = 69.69\,\text{N}\cdot\text{s}.$$

$$\boxed{\vec{v}_2^{\,+} = (3.35\,\text{m/s})\hat{\imath}}$$

Note that the answer obtained here is not the same as that found in Sample 11.4; the cart moves a bit faster to the right in this answer. Depending on the mass ratios and the angle of impact, the two methods can give very different answers or very close answers. Welcome to the world of modeling!

Problems for Chapter 11

Coupled motions for particles in space

11.1 Coupled motions of particles in space

Preparatory Problems

11.1 Linear momentum balance applied to the whole of a system consisting of multiple interacting particles reduces to $\vec{F} = m\vec{a}$ *if you interpret the terms correctly.* What are the correct interpretations of $\vec{F}$, m and $\vec{a}$?

11.2 A particle of mass $m_1 = 6\,\text{kg}$ and a particle of mass $m_2 = 10\,\text{kg}$ are moving in the xy-plane. At a particular instant of interest, particle 1 has position $\vec{r}_1 = 3\,\text{m}\hat{\imath} + 2\,\text{m}\hat{\jmath}$, velocity $\vec{v}_1 = -16\,\text{m/s}\hat{\imath} + 6\,\text{m/s}\hat{\jmath}$, and acceleration $\vec{a}_1 = 10\,\text{m/s}^2\hat{\imath} - 24\,\text{m/s}^2\hat{\jmath}$; and particle 2 has position $\vec{r}_2 = -6\,\text{m}\hat{\imath} - 4\,\text{m}\hat{\jmath}$, velocity $\vec{v}_2 = 8\,\text{m/s}\hat{\imath} + 4\,\text{m/s}\hat{\jmath}$, and acceleration $\vec{a}_2 = 5\,\text{m/s}^2\hat{\imath} - 16\,\text{m/s}^2\hat{\jmath}$.

a) Find the linear momentum $\vec{L}$ and its rate of change $\dot{\vec{L}}$ of each particle at the instant of interest.

b) Find the linear momentum $\vec{L}$ and its rate of change $\dot{\vec{L}}$ of the system of the two particles at the instant of interest.

c) Find the center of mass of the system at the instant of interest.

d) Find the velocity and acceleration of the center of mass.

11.3 A particle of mass $m_1 = 5\,\text{kg}$ and a particle of mass $m_2 = 10\,\text{kg}$ are moving in space. At a particular instant of interest, particle 1 has position, velocity, and acceleration

$$\begin{aligned}\vec{r}_1 &= 1\,\text{m}\hat{\imath} + 1\,\text{m}\hat{\jmath}\\ \vec{v}_1 &= 2\,\text{m/s}\hat{\jmath}\\ \vec{a}_1 &= 3\,\text{m/s}^2\hat{k}\end{aligned}$$

respectively, and particle 2 has position, velocity, and acceleration

$$\begin{aligned}\vec{r}_2 &= 2\,\text{m}\hat{\imath}\\ \vec{v}_2 &= 1\,\text{m/s}\hat{k}\\ \vec{a}_2 &= 1\,\text{m/s}^2\hat{\jmath}\end{aligned}$$

respectively. For the system of particles at the instant of interest, find its

a) linear momentum $\vec{L}$,

b) rate of change of linear momentum $\dot{\vec{L}}$,

c) angular momentum about the origin $\vec{H}_{/O}$,

d) rate of change of angular momentum about the origin $\dot{\vec{H}}_{/O}$,

e) kinetic energy E_K, and

f) rate of change of kinetic energy.

11.4 If you are given the total mass, the position, the velocity, and the acceleration of the center of mass of a system of particles can you find the angular momentum $\vec{H}_{/O}$ of the system, where O is not at the center of mass? If so, how and why? If not, then give a reason and/or a counter example.

11.5 Seventeen particles are interaction with the force on particle i from particle j being $\vec{F}_{ij}$ with all $\vec{F}_{ij}$ known.

a) What is the commonly assumed assumption about the relation between, say, $\vec{F}_{36}$ and $\vec{F}_{63}$?

b) What is the total force on particle 5?

More-Involved Problems

11.6 Two particles each of mass m are connected by a massless elastic spring of spring constant k and unextended length $2R$. The system slides without friction on a horizontal table, so that no net external forces act.

a) Is the total linear momentum conserved? Justify your answer.

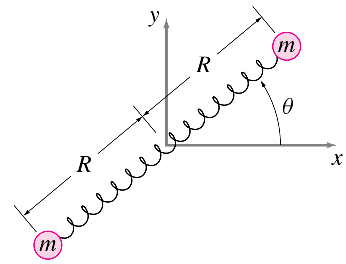
b) Can the center of mass accelerate? Justify your answer.

c) Draw free body diagrams for each mass.

d) Derive the equations of motion for each mass in terms of cartesian coordinates.

e) What are the total kinetic and potential energies of the system?

f) For constant values and initial conditions of your choosing, plot the trajectories of the two particles and of the center of mass (on the same plot).



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11.7 Two ice skaters whirl around one another. They are connected by a linear elastic cord whose center is stationary in space. We wish to consider the motion of one of the skaters by modeling her as a mass m held by a cord that exerts k Newtons for each meter it is extended from the central position.

a) Draw a free body diagram showing the forces that act on the mass is at an arbitrary position.

b) Write the differential equations that describe the motion.

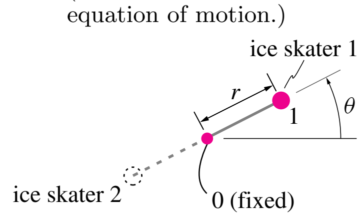
c) Describe in physical and mathematical terms the nature of the motion for the three cases

a) $\omega < \sqrt{k/m}$;

b) $\omega = \sqrt{k/m}$;-

c) $\omega > \sqrt{k/m}$.

(You are not asked to solve the equation of motion.)



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11.8 n identical particles with mass m are on the vertices of an n sided regular polygon. Equivalently, n particles are equally spaced on a circle with radius R. At $t = 0$ they all have velocities tangent to the circle and equal in magnitude v_0. All the particles are attracted to each other with an inverse square gravitational attraction. For the numerical simulations below pick values of n, m, G and R any way that pleases you.

a) Find an initial value for v_0 so that all the masses spiral in and then bounce out again. Plot the trajectories of all the masses on one plot for a long-enough time so the plot is pleasing to the eye.

b) Find a value for v_0 so all the particles travel on circular trajectories.

c) Can you find a formula for v_0 above in terms of the other parameters in the problem?

11.9 Two masses, both with $M = 1000m$ travel in circles on the xy plane according two

$$\vec{r}_1 = -\vec{r}_2 = R(\cos\omega t\hat{\imath} + \sin\omega t\hat{\imath})$$

a) Assume inverse square attraction and find a set of values for m, G, R and ω so the assumed circular path is a solution of the equations of motion.

b) A third mass m is introduced which is gravitationally attracted to the other two. Pick initial conditions for the two big masses that are consistent with their circular motion solution. For the third mass use initial conditions $[\vec{r}_0]_{xyz} = [0 0 z_0]$. Run a simulation for some time.

- Does the third mass stay *exactly* on the z axis for all time in the simulation? Would it if the simulation was exact? Is so, why? If not, why not?
- Is the motion of the third mass exactly periodic in the computer simulation? Would it be if the solution was exact? If so, why? If not, why not?

11.10 Three equal masses, say $m = 1$, are attracted by an inverse-square gravity law with $G = 1$. That is, each mass is attracted to the other by $F = Gm_1m_2/r^2$ where r is the distance between them. Use these unusual and special initial positions:

$$\begin{aligned}(x1, y1) &= (-0.97000436, 0.24308753)\\ (x2, y2) &= (-x1, -y1)\\ (x3, y3) &= (0, 0)\end{aligned}$$

and initial velocities

$$\begin{aligned}(vx3, vy3) &= (0.93240737, 0.86473146)\\ (vx1, vy1) &= -(vx3, vy3)/2\\ (vx2, vy2) &= -(vx3, vy3)/2.\end{aligned}$$

a) Use computer integration to find and plot the motions of the particles. Plot each with a different color. Run the program for 2.1 time units.

b) Same as above, but run for 10 time units.

c) Same as above, but change the initial conditions slightly.

d) Same as above, but change the initial conditions more and run for a much longer time.

11.2 Collisions and explosions

Preparatory Problems

11.11 Assuming θ, v_0, and e to be known quantities, write the following equations in matrix form set up to solve for v'_{Ax} and v'_{Ay}:

$$\begin{aligned}\sin\theta v'_{Ax} + \cos\theta v'_{Ay} &= ev_0\cos\theta\\ \cos\theta v'_{Ax} - \sin\theta v'_{Ay} &= v_0\sin\theta.\end{aligned}$$

11.12 The equation $(\vec{v}'_1 - \vec{v}'_2)\cdot\hat{\boldsymbol{n}} = e(\vec{v}_2 - \vec{v}_1)\cdot\hat{\boldsymbol{n}}$ relates relative velocities of two point masses before and after frictionless impact in the normal direction $\hat{\boldsymbol{n}}$ of the impact. If $\vec{v}'_1 = v_{1_x}\hat{\imath} + v_{1_y}\hat{\jmath}$, $\vec{v}'_2 = -v_0\hat{\imath}$, $e = 0.5$, $\vec{v}_2 = \vec{0}$, $\vec{v}_1 = 2\,\text{ft/s}\hat{\imath} - 5\,\text{ft/s}\hat{\jmath}$, and $\hat{\boldsymbol{n}} = \frac{1}{\sqrt{2}}(\hat{\imath} + \hat{\jmath})$, find the scalar equation relating the velocities in the normal direction.

11.13 The following three equations are obtained by applying the principle of conservation of linear momentum on some system.

$$\begin{aligned}m_0v_0 &= 24.0\,\text{m/s}\; m_a - 0.67m_Bv_B - 0.58m_Cv_C\\ 0 &= 36.0\,\text{m/s}\; m_A + 0.33m_Bv_B + 0.3m_Cv_C\\ 0 &= 23.3\,\text{m/s}\; m_A - 0.67m_Bv_B - 0.58m_Cv_C.\end{aligned}$$

Assume v_0, v_B, and v_C are the only unknowns. Write the equations in matrix form set up to solve for the unknowns.

11.14 The following three equations are obtained to solve for v'_{Ax}, v'_{Ay}, and v'_{Bx}:

$$\begin{aligned}(v'_{Bx} - v'_{Ax})\cos\theta &= v'_{Ay}\sin\theta - 10\,\text{m/s}\\ v'_{Ax}\sin\theta &= v'_{Ay}\cos\theta - 36\,\text{m/s}\\ m_Bv'_{Bx} + m_Av'_{Ax} &= (-60\,\text{m/s})\,m_A.\end{aligned}$$

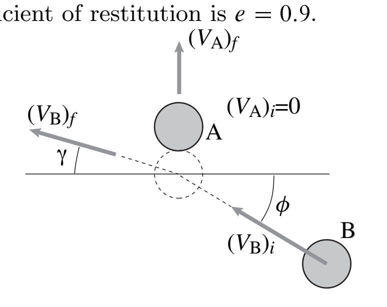
Set up these equations in matrix form.

11.15 Solve for the unknowns v'_{A_x}, v'_{A_y}, and v'_{B_x} in problem 11.14 taking $\theta = 50^\circ$, $m_A = 1.5\,\text{m}_B$ and $m_B = 0.8\,\text{kg}$. Use any computer program.

11.16 Using the matrix form of equations in Problem 11.11, solve for v'_{Ax} and v'_{Ay} if $\theta = 20^\circ$ and $v_0 = 5\,\text{ft/s}$.

More-Involved Problems

11.17 Two frictionless pucks sliding on a plane collide as shown in the figure. Puck A is initially at rest. Given that $(V_B)_i = 1.0\,\text{m/s}$, $(V_A)_i = 0$, and $(V_A)_f = 0.5\,\text{m/s}$, find the approach angle ϕ and rebound angle γ. The coefficient of restitution is $e = 0.9$.



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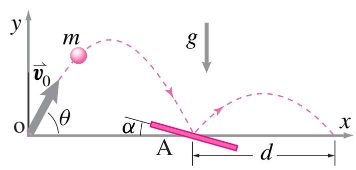
11.18 Reconsider problem 11.17. Given instead that $\gamma = 30°$, $(V_A)_i = 0$, and $(V_A)_f = 0.5\,\text{m/s}$, find the initial velocity of puck B.

11.19 A ball of mass $m = 0.5\,\text{kg}$ is thrown up in the air with initial speed $v_0 = 50\,\text{m/s}$ at an angle $\theta = 60°$. The ball lands on and bounces off a slanted floor that makes an angle $\alpha = 15°$ with the horizontal. Assume the collision with the floor to be elastic and ignore air drag on the ball.

a) Find the impulse of the collision of the ball,

b) After bouncing off the slanted floor, how much horizontal distance does the ball travel before landing on the ground again? Is this distance more, less, or the same as it would have travelled had the floor been not slanted?



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11.20 Solve the general two-particle frictionless collision problem. For example, write computer code that has lines like this near the start :

`m1=3; m2=19`	Set values of masses
`v1zero=[10 20]`	Initial velocity of mass 1
`v2zero=[-5 3]`	Initial velocity of mass 2
`e=.5`	Set coefficient of restitution
`theta=pi/4`	Angle that the normal to contact plane makes, measured CCW from +x axis, in radians

Your program (function, code, script) should calculate the impulse of mass 1 on mass 2, and the velocities of the two masses after the collision. Your program should assume consistent units for all quantities.

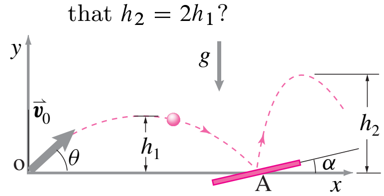
a) You should demonstrate that your program works by solving at least 4 different problems for which you can check your answer by simple pencil-and-paper calculations. These problems should have as much variety as possible. Sketch these problems clearly, show their analytic solution, and show that the computer agrees.

b) Solve the problem given above.

11.21 A projectile is launched at $\theta = 40°$ with speed $v_0 = 25\,\text{m/s}$. The projectile lands on a steel plate that can be adjusted to make any angle α with the horizontal. The projectile bounces off the steel plate without loosing any energy. The projectile is required to reach a height after rebound twice as much it did during its flight before hitting the plate. Ignore air resistance.

a) Find the required angle α of the plate.

b) Can you always find some α for any launch angle $\theta < \pi/2$ such that $h_2 = 2h_1$?



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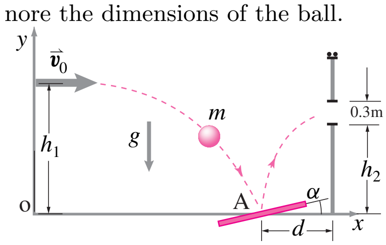
11.22 Two equal mass cars approach an intersection at right angles. They crash and stick together. One of the cars was going at 30 mph before the crash. The other car's path gets deflected by 15°. How fast was it going?

A ball m is thrown horizontally at height h and speed v_0. It then has a sequence of bounces on the horizontal ground. Treating each collision as frictionless with restitution coefficient e how far has the ball travelled horizontally when it just finishes bouncing? Answer in terms of some or all of m, g, h, v_0 and e. A ball m is thrown horizontally at height h and speed v_0. It then has a sequence of bounces on the horizontal ground. Treating each collision as frictionless with restitution coefficient e how far has the ball travelled horizontally when it just finishes bouncing? Answer in terms of some or all of m, g, h, v_0 and e. **11.23** A game involves using a pedal to direct a falling ball into a fixed vertical slot by simply rotating the pedal when the ball hits the pedal. A model of this game is shown in the figure. The ball is thrown horizontally with an initial speed $v = 10\,\text{m/s}$ from a height $h_{\text{ball}} = 3\,\text{m}$. The pedal is located at $d = 2\,\text{m}$ from the wall that houses the slot at height $h = 2\,\text{m}$. The slot itself is 0.3 m in extent. The coefficient of restitution between the pedal and the ball is $e = 0.9$. The air resistance is negligible. Find the angle α or the range of this angle, so that the ball makes it through the slot. You can ignore the dimensions of the ball.



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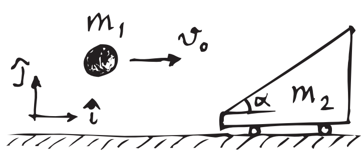
11.24 An airplane is flying steadily at an altitutude of 30,000 ft at a speed of 500 mph. It explodes into two equal pieces. One piece is found to the right of the airplane's initial trajectory and 8 miles forward of the explosion point. Where should you look for the other piece? Assume the interaction impulse is in the horizontal plane and make the approximation that the two pieces fly in frictionless parabolic trajectories.

11.25 Consider the simultaneous collisions problem discussed in Sample **??**. Consider the ball to be much more massive than the cart; $m_1 = 20\,\text{kg}$ and $m_2 = 1\,\text{kg}$. The angle of the inclined face is very shallow, $\alpha = 2°$. The ball hits the cart with the velocity $\vec{v}_1 = 50\,\text{m/s}\hat{\imath}$. The impact is elastic and frictionless. Find the subsequent velocities of the ball and the cart using the two methods discussed in Sample **??** and Sample **??**. Comment on the answers you get. How will your answers change if you reversed the mass ratio?



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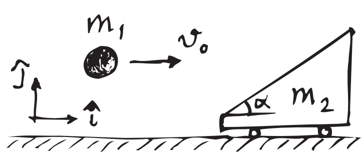
11.26 Consider the simultaneous collisions problem discussed in Sample **??** again. Assume that $m_1 = 5\,\mathrm{kg}$, $m_2 = 10\,\mathrm{kg}$, $\vec{\boldsymbol{v}}_1 = 50\,\mathrm{m/s}$, $e = 0.75$, and the angle $\alpha = 88°$. Find the final velocities of the cart and the ball assuming that the cart must move in the $\hat{\imath}$ direction only. What is the the net loss of energy in the impacts?



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CHAPTER 12

Constrained straight-line motion

Here is an introduction to kinematic constraint in its simplest context, systems that are constrained to move without rotation in a straight line. In one dimension pulley problems provide the main example. Two and three dimensional problems are covered, such as finding structural support forces in accelerating vehicles and the slowing or incipient capsize of a braking car or bicycle. Angular momentum balance is introduced as a needed tool but without the complexities of rotatioinal kinematics.

Contents

In the previous chapters you learned to write the equations of motion for a particle, or for a collection of a few particles, if you have a model for the forces on the particles in terms of their positions, velocities, and time. Some caveats to using that approach for engineering systems that don't seem to behave like isolated particles, but rather are composed of many particles were listed at the start of section 11 One way to finesse these problems is to make kinematic assumptions about how particles and collections of particles move. Why? Sometimes, often actually, the simplest model of mechanical interaction is not a law for force as a function of position, velocity and time, but just a geometric description of the relative positions or velocities of points. The reasons for this geometric, instead of force-based, approach are two-fold:

- **The minute details of the motion are often not of interest and therefore not worth tracking.** For example, the vibrations of a solid, or relative motions of atoms in a solid might be of a smaller scale than the overall motion of interest, and
- **Often one does not know an accurate force law.** For example, at the microscopic level one does not know the details of atomic interactions; or, at the machine level, one may not know exactly the relations between the small motions of one part relative to another with which it makes contact. For example, even though one knows that the axle being in a hole restricts the relative motion of the axle with the train one may not know in detail how the contact forces depend on the exact position of the axle in its hole.).

Much mechanical modeling involves the replacement of force-interaction rules with assumptions about the geometry of the motions. Idealizing an interaction force as causing a definite geometric restriction on motion is called imposing a *kinematic constraint.*

A kinematic constraint is an equation that describes a restriction on allowed positions, velocities or accelerations of parts in a system. Kinematic constraints are always accompanied by one or more *a priori* unknown 'constraint' forces that maintain the geometric constraint relations.

Figure 12.1: A truck or car running on straight level road is in straight-line motion, neglecting, of course, the wheel rotation, the bouncing, the moving engine parts, and the wandering eyes of the passengers.

Filename:TruckStraightLine5030

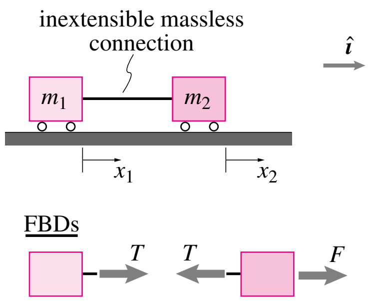

Figure 12.2: A schematic of one car pulling another, or of a boat pulling a barge. Also shown are FBDs of the bodies separately. Because our analysis is only in one spatial dimension, forces with no component in $\hat{\imath}$ direction are not shown.

Filename:tfigure-boatpullsbarge

The basic laws of forces and mechanics apply to all systems, no matter how they are or are not constrained. But, if objects are treated as kinematically constrained the methods in mechanics have a slightly different flavor. To get the idea we start with simple systems that have simple constraints and that move in simple ways. In this short chapter, we will discuss the mechanics of things where every point in each object has the same velocity and acceleration as every other point (so called *parallel motion*) and with the further restriction that every point moves in a straight line.

Example: **Train on Straight Level Tracks**
Consider a train on straight level tracks. If we focus on the body of the train, we can approximate the motion as parallel straight-line motion. All parts move the same amount, with the same velocities and accelerations in the same fixed direction.

We start with 1-D mechanics and constraint with string and pulleys, and then move on to 2-D and 3-D mechanics (of systems in 1D motion).

12.1 1-D constrained motion and pulleys

This section concerns things connected together with bars or ropes which are idealized as being inextensible. Consider a car towing another with a strong light chain. We may not want to consider the elasticity of the chain but instead idealize the chain as having a fixed length. This idealization of zero deformation is a simplification. But it is a simplification that requires special treatment. It is the simplest example of a kinematic constraint.

Figure 12.2 shows a schematic of one car pulling another. One-dimensional free body diagrams are also shown. The force F is the force transmitted from the road to the front car through the tires. The tension T is the tension in the connecting chain. From linear momentum balance for each of the objects (modeled as particles):

$$T = m_1\ddot{x}_1 \quad \text{and} \quad F - T = m_2\ddot{x}_2. \tag{12.1}$$

These equations are exactly the same as for cars connected by a spring, a dashpot, or any idealized-as-massless connector. And all these systems have the same free body diagrams but different motions. If the connection were with a spring or dashpot the equations above would be supplemented with

$$T = k(x_2 - x_1 - \ell_0) \quad \text{or} \quad T = c(\dot{x}_2 - \dot{x}_1)$$

In this case we need our equations to somehow indicate that the two particles are not allowed to move independently. We need a constraint equation to replace these constitutive laws.

Kinematic constraint: two approaches

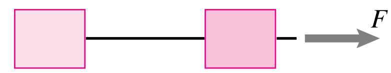

Figure 12.3: A free body diagram of the whole system. Note that the unknown tension (constraint) force does not show. As usual for 1D mechanics, vertical forces are left off for simplicity (although it would be more correct to include them).

Filename:tfigure-twocarstogether

There are two basic ways of dealing with kinematic constraints:

1. Use separate free body diagrams and equations of motion for each particle and then add extra kinematic constraint equations, or
2. do something clever to avoid having to find the constraint forces.

Methad 1: Finding the constraint force with the accelerations

The geometric (or kinematic) restriction that two masses must move in lock-step is

$$x_1 = x_2 + \text{Constant}.$$

We can differentiate the kinematic constraint twice to get

$$\ddot{x}_1 = \ddot{x}_2. \tag{12.2}$$

If we take F and the two masses as given, equations 12.1 and 12.2 are three equations for the unknowns $\ddot{x}_1, \ddot{x}_2$, and T. In matrix form,we have:

$$\begin{bmatrix} m_1 & 0 & -1 \\ 0 & m_2 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ T \end{bmatrix} = \begin{bmatrix} 0 \\ F \\ 0 \end{bmatrix}.$$

We can solve these equations to find $\ddot{x}_1, \ddot{x}_2$, and T in terms of F.

Method 2: Do something clever and finess the finding of the constraint force

On the other hand, if all we are interested in are the accelerations of the cars it would be nice to avoid even having to think about the constraint force. One way to avoid dealing with the constraint force is to draw a free body diagram of the entire system as in figure 12.3. If we just call the acceleration of the system $\ddot{x}$ we have, from linear momentum balance, that

$$F = (m_1 + m_2)\ddot{x},$$

which is one equation in one unknown.

Kinematic constraints

A generalization of the 1D inextensible-cable constraint example above is the rigid-object constraint where not just two, but many particles are assumed to keep constant distance from one another, and in one, two or three dimensions. Another important constraint is an ideal hinge connection between two objects. Much of the theory of mechanics after Newton has been motivated by a desire to deal easily with these and other kinematic constraints. In fact, one way of characterizing the primary difficulty of dynamics as a subject is the difficulty of dealing with kinematic constraints.

* See figure 4.24 on page 211 and the related text which shows why $T_1 = T_2$ for one round pulley idealized as frictionless and massless.

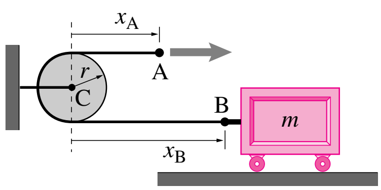

Figure 12.4: One mass, one pulley, and one string

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Pulleys

Pulleys are used to redirect force to amplify or attenuate force and to amplify or attenuate motion. Like a lever, a pulley system is an example of a mechanical transmission. Objects connected by inextensible ropes around ideal pulleys are also examples of kinematic constraint.

Constant length and constant tension

Problems with pulleys are solved by using two facts about idealized strings. First, an ideal string is inextensible so the sum of the string lengths, over the different inter-pulley sections, adds to a constant (not varying in time).

$$\ell_1 + \ell_2 + \ell_3 + \ell_4 + ... = \text{constant} \tag{12.3}$$

Second, for round pulleys of negligible mass and no bearing friction, tension is constant along the length of the string*.The tension on one side of a pulley is the same as the tension on the other side. And this can carry on if a rope is wrapped around several pulleys.

$$T_1 = T_2 = T_3 \dots \tag{12.4}$$

We use the trivial pulley example in figure 12.4 to show how to analyze the relative motion of various points in a pulley system.

Example: **Length of string calculation**

Starting from point A, we add up the lengths of string

$$\ell_{tot} = x_A + \pi r + x_B \equiv constant. \tag{12.5}$$

The portion of string wrapped around the pulley contacts half of the pulley so that it's length is half the pulley circumference, πr. Even if x_A and x_B change in time and different portions of string wrap around the pulley, the length of string touching the pulley is always πr.

We can now formally deduce the intuitively obvious relations between the velocities and accelerations of points A and B. Differentiating equation 12.5 with respect to time once and then again, we get

$$\begin{aligned} \dot{\ell}_{tot} = 0 &= \dot{x}_A + 0 + \dot{x}_B \\ \Rightarrow \dot{x}_A &= -\dot{x}_B \\ \Rightarrow \ddot{x}_A &= -\ddot{x}_B \end{aligned} \tag{12.6}$$

When point A is displaced to the right by an amount Δx_A, point B is displaced exactly the same amount but to the left; that is, $\Delta x_A = -\Delta x_B$. Note that in order to derive the kinematic relations 12.6 for the pulley system, we never need to know the total length of the string, only that it is constant in time. The constant-in-time quantities (the pulley half-circumference and the string length) get 'killed' in the process of differentiation.

Commonly we think of pulleys as small and thus never account for the pulley-contacting string length. Luckily this approximation generally leads to no error because we most often are interested in displacements, velocities, and accelerations in which cases the pulley contact length drops out of the equations anyway.

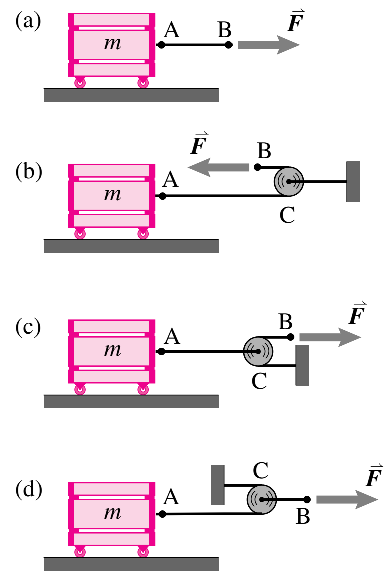


Figure 12.5: The four classic cases: (a) no pulley, (b) a pulley system with no mechanical advantage, (c) a pulley system that multiplies force and attenuates motion, and (d) a pulley system that attenuates force and amplifies motion.

Filename:tfigure-pulley1

The classic simple uses of pulleys

First imagine trying to move a load with no pulley as in *Fig*. 12.5a. The force you apply goes right to the mass. This is like direct drive with no transmission.

Now you would like to use pulleys to help you move the mass. In the cases we consider here the mass is on a frictionless support and we are trying to accelerate it. But the concepts are the same if there are also resisting forces on the mass. What can we do with one pulley? Three possibilities are shown in *Fig*. 12.5b-d which might, at a blinking glance, look roughly the same. But they are quite different. Here we discuss each design qualitatively. The details of the calculations are a homework problem.

In *Fig*. 12.5b we pull one direction and the mass accelerates the other way. This illustrates one use of a pulley, to redirect an applied force. The force on the mass has magnitude $|\vec{F}|$ and there is no mechanical advantage.

Fig. 12.5

c shows the most classic use of a pulley. A free body diagram of the pulley at C will show you that the tension in rope AC is $2|\vec{F}|$ and we have thus doubled the force acting on the mass. However, counting string length and displacement you will see that point A moves only half the distance that point B moves. Thus the force at B is multiplied by two to give the force at A and the displacement at B is divided by two to give the displacement at A.

Power balance

This result for *Fig*. 12.5c is most solidly understood using energy balance. The power of the force at B goes eventually entirely into the mass; the string and pulley do not absorb any energy. On the other hand if we cut the string AC, the same amount of power must be applied to the mass (it gains the same energy). Thus the product of the tension and velocity at A must equal the product of the tension and velocity at B,

$$T_A v_A = T_B v_B.$$

This is a general property of ideal transmissions, from levers to pulleys to gear boxes:

If force is amplified then motion is equally attenuated.

Fig. 12.5d shows a use of a pulley opposite to the use in *Fig*. 12.5d.

A free body diagram of the pulley shows that the tension in AC is $\frac{1}{2}|\vec{F}|$. Thus the force is attenuated by a factor of 2. A kinematic analysis reveals that the motion of A is twice that of B. Thus, as expected from energy considerations, the motion is amplified when the force is attenuated.

Summary of simple pulley uses

Summarizing,

> Relative to Fig. 12.5a the design Fig. 12.5b does nothing and the designs Fig. 12.5c and Fig. 12.5d are opposite in their effects. Fig. 12.5c amplifies motion and attenuates force, and Fig. 12.5d attenuates motion and amplifies force.

12.1 *THEORY*

The 'effective mass' of a point of force application

The feel of the machine is of concern for machines that people handle. One aspect of feel is the effective mass. The *effective mass* is defined by the response of a point when a force is applied.

$$m_{\text{eff}} = \frac{|\vec{F}_{\text{B}}|}{|\vec{a}_{\text{B}}|}.$$

For the case of Fig. 12.5a and Fig. 12.5b the effective mass of point B is the mass of the block, m. For the case of Fig. 12.5c the block at A has $2|\vec{F}|$ acting on it and point B has twice the acceleration of point A. So the acceeration of poing B is $4F/m = F/(m/4)$ and the effective mass of point B is $m/4$. For the case of Fig. 12.5d, the mass only has $|\vec{F}|/2$ acting on it and point B only has half the acceleration of point A, so the effective mass is $4m$.

These special cases exemplify the general rule:

> The effective mass of one end of a transmission is the mass of the other end multiplied by the square of the motion amplification ratio.

In terms of the effective mass, the systems shown inFig. 12.5c and Fig. 12.5d which look so similar to a novice, actually differ by a factor of $2^2 \cdot 2^2 = 16$. With a given F and m point B in Fig. 12.5c has 16 times the acceleration of point B in Fig. 12.5d.

SAMPLE 12.1 Find the motion of two cars. One car is towing another of equal mass on level ground. The thrust of the wheels of the first car is F. The second car rolls frictionlessly. Find the acceleration of the system two ways:

1. using separate free body diagrams,
2. using a system free body diagram.

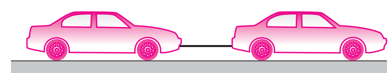


Figure 12.6:

Filename:sfig4-1-twocars

Solution

1. The free body diagram of each car is shown below, in *Fig*. 12.7.

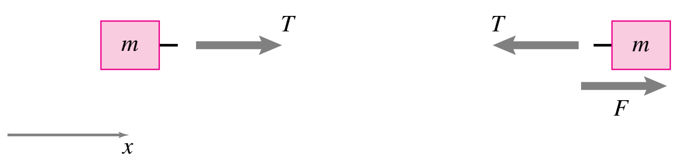


Figure 12.7: Partial free body diagrams of the two cars (the vertical ground reactions are not shown as they are of no interest to us for the horizontal motion.

Filename:sfig4-1-twocars-fbda

From the linear momentum balance of each car, we get

$$m\ddot{x}_1 = T \quad (12.7)$$
$$F - T = m\ddot{x}_2 \quad (12.8)$$

The kinematic constraint of towing (the cars move together, *i.e*., no relative displacement between the cars) gives

$$\ddot{x}_1 - \ddot{x}_2 = 0 \quad (12.9)$$

Solving eqns. (12.7), (12.8), and (12.9) simultaneously, we get

$$\ddot{x}_1 = \ddot{x}_2 = \frac{F}{2m} \quad (T = \frac{F}{2})$$

2. The free body diagram of the two cars together is shown below, in *Fig*. 12.8.

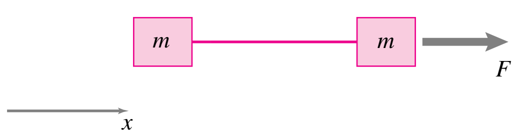


Figure 12.8:

Filename:sfig4-1-twocars-fbdb

From the linear momentum balance of the two cars as one system, we get

$$m\ddot{x} + m\ddot{x} = F$$
$$\ddot{x} = F/2m$$

$$\boxed{\ddot{x} = \ddot{x}_1 = \ddot{x}_2 = F/2m}$$

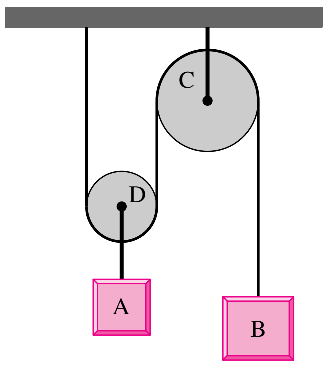


Figure 12.9:

Filename:sfig3-3-DH1

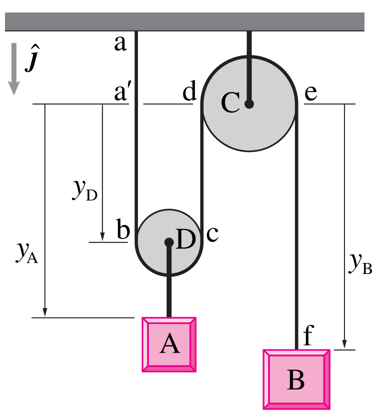


Figure 12.10:

Filename:sfig3-3-DH2

SAMPLE 12.2 Pulley kinematics. For the masses and ideal-massless pulleys shown in figure 12.9, find the acceleration of mass A in terms of the acceleration of mass B. Pulley C is fixed to the ceiling and pulley D is free to move vertically. All strings are inextensible.

Solution Let us measure the position of the two masses from a fixed point, say the center of pulley C. (Since C is fixed, its center is fixed too.) Let y_A and y_B be the vertical distances of masses A and B, respectively, from the chosen reference (C). Then the position vectors of A and B are:

$$\vec{r}_A = y_A \hat{\jmath} \quad \text{and} \quad \vec{r}_B = y_B \hat{\jmath}.$$

Therefore, the velocities and accelerations of the two masses are

$$\begin{aligned} \vec{v}_A &= \dot{y}_A \hat{\jmath}, & \vec{v}_B = \dot{y}_B \hat{\jmath}, \\ \vec{a}_A &= \ddot{y}_A \hat{\jmath}, & \vec{a}_B = \ddot{y}_B \hat{\jmath}. \end{aligned}$$

Since all quantities are in the same direction ($\hat{\jmath}$), we can drop $\hat{\jmath}$ from our calculations and just do scalar calculations. We are asked to relate $\ddot{y}_A$ to $\ddot{y}_B$.

In all pulley problems, the trick in doing kinematic calculations is to relate the variable positions to the fixed length of the string. Here, the length of the string ℓ_{tot} is:*

$$\begin{aligned} \ell_{tot} &= \text{ab} + \text{bc} + \text{cd} + \text{de} + \text{ef} = \text{constant} \\ \text{where} \quad \text{ab} &= \underbrace{\text{aa}'}_{\text{constant}} + \underbrace{\text{a}'\text{b}}_{(=\text{cd}=y_D)} \\ \text{bc} &= \text{string over the pulley D} = \text{constant} \\ \text{de} &= \text{string over the pulley C} = \text{constant} \\ \text{ef} &= y_B \\ \text{thus} \quad \ell_{tot} &= 2y_D + y_B + \overbrace{\text{constant}}^{(\text{aa}'+\text{bc}+\text{de})}. \end{aligned}$$

Taking the time derivative on both sides, we get

$$\overbrace{\frac{d}{dt}(\ell_{tot})}^{=0,\ \text{because } \ell_{tot} \text{ does not change with time}} = 2\dot{y}_D + \dot{y}_B \quad \Rightarrow \quad \dot{y}_D = -\frac{1}{2}\dot{y}_B \tag{12.10}$$

$$\Rightarrow \quad \ddot{y}_D = -\frac{1}{2}\ddot{y}_B. \tag{12.11}$$

$$\begin{aligned} \text{But} \quad & y_D = y_A - \text{AD and AD} = \text{constant} \\ \Rightarrow \quad & \dot{y}_D = \dot{y}_A \quad \text{and} \quad \ddot{y}_D = \ddot{y}_A. \end{aligned}$$

Thus, substituting $\dot{y}_A$ and $\ddot{y}_A$ for $\dot{y}_D$ and $\ddot{y}_D$ in (12.10) and (12.11) we get

$$\dot{y}_A = -\frac{1}{2}\dot{y}_B \quad \text{and} \quad \ddot{y}_A = -\frac{1}{2}\ddot{y}_B$$

$$\boxed{\ddot{y}_A = -\tfrac{1}{2}\ddot{y}_B}$$

* We have done an elaborate calculation of ℓ_{tot} here. Usually, the constant lengths over the pulleys and some constant segments such as aa$'$ are ignored in calculating ℓ_{tot}. These constant length segments can be ignored because they drop out of the equation when we take time derivatives to relate velocities and accelerations of different points, such as B and D here.

SAMPLE 12.3 **A two-mass pulley system.** The two masses shown in Fig. 12.11 have frictionless bases and round frictionless pulleys. The inextensible cord connecting them is always taut. Given that $F = 130\,\text{N}$, $m_A = m_B = m = 40\,\text{kg}$, find the acceleration of the two blocks using:

1. linear momentum balance and
2. energy balance.

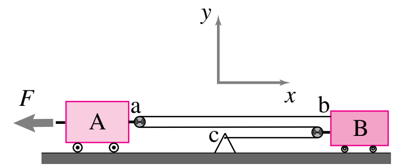


Figure 12.11: A two-mass pulley system.

Filename:sfig3-3-1

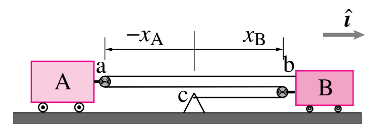


Figure 12.12: Pulley kinematics. Note that the variable distance from c to a is minus the x coordinate of a.

Filename:sfig3-3-1b

Solution

1. **Using Linear Momentum Balance:**

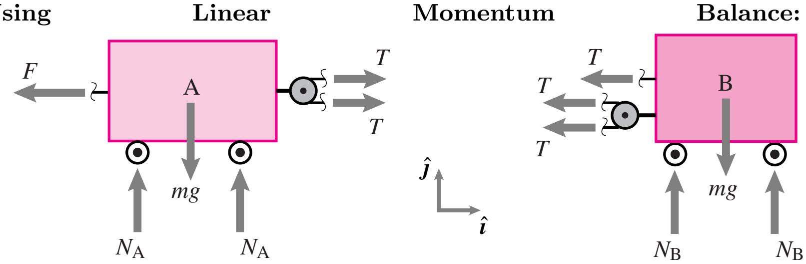


Figure 12.13:

Filename:sfig3-3-1a

The free-body diagrams of the two masses A and B are shown in Fig. 12.13 above. Linear momentum balance for mass A gives (assuming $\vec{a}_A = a_A\hat{\imath}$ and $\vec{a}_B = a_B\hat{\imath}$):

$$
\begin{aligned}
(2T - F)\hat{\imath} + (2N_A - mg)\hat{\jmath} &= m\vec{a}_A = -ma_A\hat{\imath} \\
\text{(dotting with } \hat{\jmath}) \quad \Rightarrow \quad 2N_A &= mg \\
\text{(dotting with } \hat{\imath}) \quad \Rightarrow \quad 2T - F &= ma_A
\end{aligned} \tag{12.12}
$$

Similarly, linear momentum balance for mass B gives:

$$
\begin{aligned}
-3T\hat{\imath} + (2N_B - mg)\hat{\jmath} &= m\vec{a}_B = ma_B\hat{\imath} \\
\Rightarrow \quad 2N_B &= mg \\
\text{and} \quad -3T &= ma_B.
\end{aligned} \tag{12.13}
$$

From (12.12) and (12.13) we have three unknowns: T, a_A, a_B, but only 2 equations!. We need an extra equation to solve for the three unknowns.*

We can get the extra equation from kinematics. Since A and B are connected by a string of fixed length, their accelerations must be related. For simplicity, and since these terms drop out anyway, we neglect the radius of the pulleys and the lengths of the little connecting cords. Using the fixed point C as the origin of our xy coordinate system we can write

$$
\begin{aligned}
\ell_{tot} &\equiv \text{length of the string connecting A and B} \\
&= 3x_B + 2(-x_A) \\
\Rightarrow \quad \overbrace{\dot{\ell}_{tot}}^{0} &= 3\dot{x}_B + 2(-\dot{x}_A) \\
\Rightarrow \quad \dot{x}_B &= -\frac{2}{3}(-\dot{x}_A) \quad \Rightarrow \quad \ddot{x}_B = \frac{2}{3}\ddot{x}_A
\end{aligned} \tag{12.14}
$$

* You may be tempted to use angular momentum balance (AMB) to get an extra equation. In this case AMB could help determine the vertical reactions, but offers no help in finding the rope tension or the accelerations.

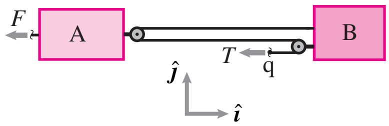


Figure 12.14: 1D free body diagram of the whole system. Note that except $\boldsymbol{F}$, no other forces do any work.

Filename:sfig3-3-1c

* It is probably not obvious why T shown in the FBD in *Fig.* 12.14 does no work. Since 'q' is a material point somewhere on the string from the fixed point 'c' to the pulley on mass B, point 'q' has no displacement (since point 'c' is stationary). So, although mass B moves, the selected cut point 'q' remains stationary (length of 'cq' cannot change) and therefore, the work done by T is zero.

Since

$$\begin{aligned}\vec{v}_A &= v_A\hat{\imath} = -\dot{x}_A\hat{\imath},\\ \vec{a}_A &= a_A\hat{\imath} = \ddot{x}_A\hat{\imath},\\ \vec{v}_B &= v_B\hat{\imath} = \dot{x}_B\hat{\imath}, \text{ and}\\ \vec{a}_B &= a_B\hat{\imath} = \ddot{x}_B\hat{\imath},\end{aligned}$$

we get

$$a_B = \frac{2}{3}a_A. \tag{12.15}$$

Substituting (12.15) into (12.13), we get

$$9T = -2m_B a_A. \tag{12.16}$$

Now solving (12.12) and (12.16) for T, we get

$$T = \frac{2F}{13} = \frac{2\cdot 130\,\mathrm{N}}{13} = 20\,\mathrm{N}.$$

Therefore,

$$\begin{aligned}a_A &= -\frac{9T}{2m} = -\frac{9\cdot 20\,\mathrm{N}}{2\cdot 40\,\mathrm{kg}} = -2.25\,\mathrm{m/s^2}\\ a_B &= \frac{2}{3}a_A = -1.5\,\mathrm{m/s^2}\end{aligned}$$

$$\boxed{\vec{a}_A = -2.25\,\mathrm{m/s^2}\hat{\imath},\ \vec{a}_B = -1.5\,\mathrm{m/s^2}\hat{\imath}.}$$

2. **Using Power Balance (III):** We have,

$$P = \dot{E}_K.$$

The power balance equation becomes

$$\sum \vec{F}\cdot\vec{v} = m\,a_A\,v_A + m_B\,a_B\,v_B.$$

Because the force at A is the only force that does work on the system, * when we apply power balance to the whole system (see the FBD in *Fig.* 12.14), we get,

$$\begin{aligned}-Fv_A - T\overbrace{v_q}^{=v_c=0} &= m_A v_A a_A + m_B v_B a_B\\ \text{or}\qquad F &= -ma_A - m\frac{v_B}{v_A}a_B\\ &= -a_A\Big(m + m\frac{v_B}{v_A}\frac{a_B}{a_A}\Big).\end{aligned}$$

Substituting $a_B = 2/3a_A$ and $v_B = 2/3v_A$ from eqn. (12.15),

$$a_A = \frac{-F}{m+\frac{4}{9}m} = \frac{-130\,\mathrm{N}}{40\,\mathrm{kg}(1+\frac{4}{9})} = -2.25\,\mathrm{m/s^2},$$

and since $a_B = 2/3a_A$,

$$a_B = -1.5\,\mathrm{m/s^2},$$

which are the same accelerations as found before.

$$\boxed{a_A = -2.25\,\mathrm{m/s^2}\hat{\imath},\ a_B = -1.5\,\mathrm{m/s^2}\hat{\imath}}$$

SAMPLE 12.4 In static equilibrium the spring in *Fig*. 12.15 is compressed by y_s from its unstretched length ℓ_0. Now, the spring is compressed by an additional amount y_0 and released with no initial velocity.

1. Find the force on the top mass m exerted by the lower mass M.
2. When does this force become minimum? Can this force become zero?
3. Can the force on m due to M ever be negative?

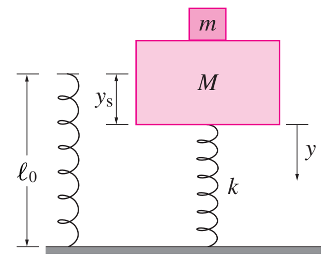


Figure 12.15:

Filename:sfig10-1-5

Solution

1. The free body diagram of the two masses is shown in Figure 12.16 when the system is in static equilibrium. From linear momentum balance we have

$$\sum \vec{F} = \vec{0} \qquad \Rightarrow \qquad k y_s = (m+M)g. \tag{12.17}$$

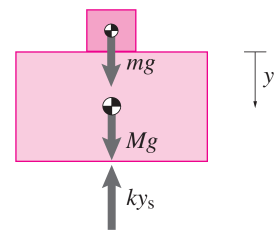


Figure 12.16: Free body diagram of the two masses as one system in static equilibrium (this special case could be skipped as it follows from the free body diagram below).

Filename:sfig10-1-5a

The free body diagrams of the two masses at an arbitrary position y during motion are shown in Figure 12.17. Since the two masses oscillate together, they have the same acceleration. From linear momentum balance for mass m we get (note that we have chosen y to be positive downwards),

$$mg - N = m\ddot{y}. \tag{12.18}$$

We are interested in finding the normal force N. Clearly, we need to find $\ddot{y}$ to calculate N. Now, from linear momentum balance for mass M we get

$$Mg + N - k(y + y_s) = M\ddot{y}. \tag{12.19}$$

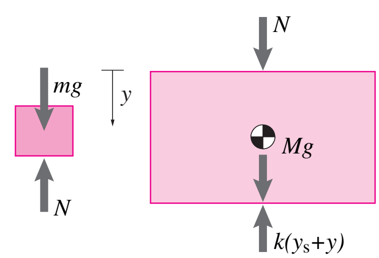


Figure 12.17: Free body diagrams of the individual masses.

Filename:sfig10-1-5b

Adding eqn. (12.18) with eqn. (12.19) we get

$$(m+M)g - ky - ky_s = (m+M)\ddot{y}.$$

But $ky_s = (m+M)g$ from eqn. (12.17). Therefore, the equation of motion of the system is

$$-ky = (m+M)\ddot{y}$$

$$\text{or} \quad \ddot{y} + \frac{k}{(m+M)}y = 0. \tag{12.20}$$

As you recall from your study of the harmonic oscillator, the general solution of this differential equation is

$$y(t) = A\sin\lambda t + B\cos\lambda t \tag{12.21}$$

$$\text{where} \quad \lambda = \sqrt{\frac{k}{m+M}}. \tag{12.22}$$

The constants A and B are to be determined from the initial conditions. From eqn. (12.21) we obtain

$$\dot{y}(t) = A\lambda\cos\lambda t - B\lambda\sin\lambda t. \tag{12.23}$$

Substituting the given initial conditions $y(0) = y_0$ and $\dot{y}(0) = 0$ in eqns. (12.21) and (12.23), respectively, we get

$$\overbrace{y(0)}^{y_0} = A\sin(\lambda\cdot 0) + B\cos(\lambda\cdot 0) \quad \Rightarrow \quad B = y_0$$

$$\underbrace{\dot{y}(0)}_{0} = A\lambda\cos(\lambda\cdot 0) - B\lambda\sin(\lambda\cdot 0) \quad \Rightarrow \quad A = 0.$$

Thus,

$$y(t) = y_0 \cos \lambda t. \tag{12.24}$$

Now we can find the acceleration by differentiating eqn. (12.24) twice :

$$\ddot{y} = -y_0 \lambda^2 \cos \lambda t.$$

Substituting this expression in eqn. (12.18) we get the force applied by mass M on the smaller mass m:

$$\begin{aligned} mg - N &= m \overbrace{(-y_0 \lambda^2 \cos \lambda t)}^{\ddot{y}} \\ \Rightarrow \quad N &= mg + m y_0 \lambda^2 \cos \lambda t \\ &= mg \left(1 + \frac{y_0 \lambda^2}{g} \cos \lambda t\right) \end{aligned} \tag{12.25}$$

$$\boxed{N = mg \left(1 + \frac{y_0 \lambda^2}{g} \cos \lambda t\right)}$$

2. Since $\cos \lambda t$ varies between ± 1, the value of the force N varies between $mg \pm y_0 \lambda^2$. Clearly, N attains its minimum value when $\cos \lambda t = -1$, *i.e.*, when $\lambda t = \pi$. This condition is met when the spring is fully stretched and the mass is at its highest vertical position. At this point,

$$N \equiv N_{\text{min}} = mg \left(1 - \frac{y_0 \lambda^2}{g}\right).$$

If y_0, the initial displacement from the static equilibrium position, is chosen such that $y_0 \lambda^2 = g$ (that is, the amplitude of the harmonically varying acceleration equals g), then $N = 0$ when $\cos \lambda t = -1$, *i.e.*, at the topmost point in the vertical motion. This condition, $N = 0$, means that the two masses momentarily lose contact with each other; and it happens precisely when they are about to begin their downward motion.

◁

3. From eqn. (12.25) we can get a negative value of N when $\cos \lambda t = -1$ and $y_0 \lambda^2 > g$. However, a negative value for N is nonsense unless the blocks are glued. Without glue the bigger mass M cannot apply a negative force (or a compression) on m, *i.e.*, it cannot "suck" m. When $y_0 \lambda^2 > g$ then N becomes zero before $\cos \lambda t$ decreases to -1. That is, assuming no bonding, the two masses lose contact on their way to the highest vertical position but before reaching the highest point. Beyond that point, the equations of motion derived above are no longer valid for unglued blocks because the equations assume contact between m and M. Equation (12.25) is inapplicable when $N \leq 0$. ◁

SAMPLE 12.5 **Driving a pile into the ground.** A cylindrical wooden pile of mass 10 kg and cross-sectional diameter 20 cm is driven into the ground with the blows of a hammer. The hammer is a block of steel with mass 50 kg which is dropped from a height of 2 m to deliver the blow. At the nth blow the pile is driven into the ground by an additional 5 cm. Assuming the impact between the hammer and the pile to be totally inelastic (*i.e.*, the two stick together), find the average resistance of the soil to penetration of the pile.

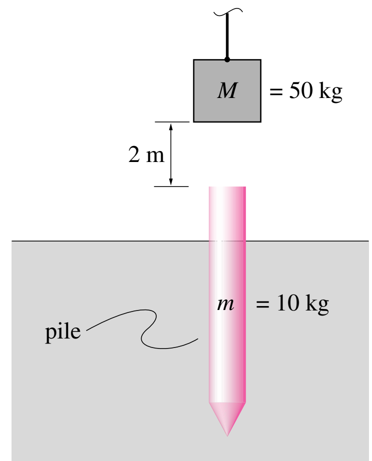


Figure 12.18:

Filename:sfig3-5-DH1

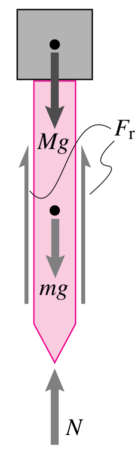


Figure 12.19: Free body diagram of the hammer and pile system. F_r is the total resistance of the ground.

Filename:sfig3-5-DH2

Solution Let F_r be the average (constant over the period of driving the pile by 5 cm) resistance of the soil. From the free body diagram of the pile and hammer system, we have

$$\sum \vec{F} = -mg\hat{\jmath} - Mg\hat{\jmath} + N\hat{\jmath} + F_r\hat{\jmath}.$$

But N is the normal reaction of the ground, which from static equilibrium, must be equal to $mg + Mg$. Thus,

$$\sum \vec{F} = Fr\hat{\jmath}.$$

Therefore, from linear momentum balance ($\sum \vec{F} = m\vec{a}$),

$$\vec{a} = \frac{F_r}{M+m}\hat{\jmath}.$$

Now we need to find the acceleration from given conditions. Let v be the speed of the hammer just before impact and V be the combined speed of the hammer and the pile immediately after impact. Then, treating the hammer and the pile as one system, we can ignore all other forces *during* the impact (none of the external forces: gravity, soil resistance, ground reaction, is comparable to the impulsive impact force, see page **??**). The impact force is internal to the system. Therefore, during impact, $\sum \vec{F} = \vec{0}$ which implies that linear momentum is conserved. Thus

$$\begin{aligned} -Mv\hat{\jmath} &= -(m+M)V\hat{\jmath} \\ \Rightarrow \quad V &= \left(\frac{M}{m+M}\right)v = \frac{50\,\text{kg}}{60\,\text{kg}}v = \frac{5}{6}v. \end{aligned}$$

The hammer speed v can be easily calculated, since it is the free fall speed from a height of 2 m:

$$v = \sqrt{2gh} = \sqrt{2\cdot(9.81\,\text{m/s}^2)\cdot(2\,\text{m})} = 6.26\,\text{m/s} \quad \Rightarrow \quad V = \frac{5}{6}v = 5.22\,\text{m/s}.$$

The pile and the hammer travel a distance of $s = 5\,\text{cm}$ under the deceleration a. The initial speed $V = 5.22\,\text{m/s}$ and the final speed $= 0$. Plugging these quantities into the one-dimensional kinematic formula

$$v^2 = v_0^2 + 2as,$$

we get,

$$\begin{aligned} 0 &= V^2 - 2as \quad \text{(Note that } a \text{ is negative)} \\ \Rightarrow \quad a &= \frac{V^2}{2s} = \frac{(5.22\,\text{m/s})^2}{2\times 0.05\,\text{m}} = 272.48\,\text{m/s}^2. \end{aligned}$$

Thus $\vec{a} = 272.48\,\text{m/s}^2\hat{\jmath}$. Therefore,

$$F_r = (m+M)a = (60\,\text{kg})\cdot(272.48\,\text{m/s}^2) = 1.635\times 10^4\,\text{N}$$

$$\boxed{F_r \approx 16.35\,\text{kN}}$$

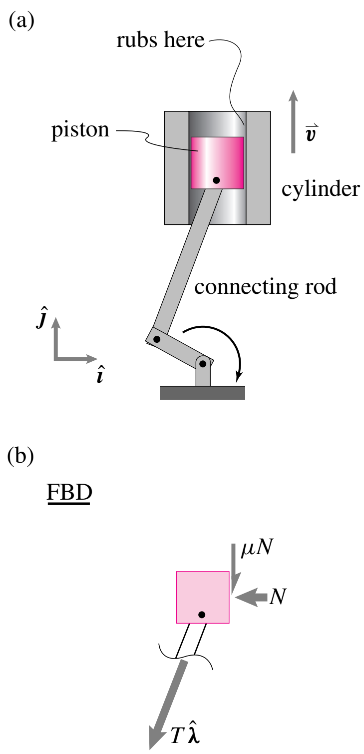


Figure 12.20: (a) shows a piston in a cylinder. (b) shows a free body diagram of the piston. To draw this FBD, we have assumed: (1) a coefficient of friction μ between the piston and cylinder wall, and (2) negligible mass for the connecting rod, and (3) ignored the spatial extent of the cylinder.

Filename:tfigure3-1

12.2 1D motion with 2D and 3D forces

Even if all the motion is in a single direction, an engineer may still have to consider two- or three-dimensional forces.

Example: **Piston in a cylinder.**
Consider a piston sliding vertically in a cylinder. For now neglect the spatial extent of the cylinder. Let's assume a coefficient of friction μ between the piston and the cylinder wall and that the connecting rod has negligible mass so it can be treated as a two-force member as discussed in section 4.2b.The free body diagram of the piston (with a bit of the connecting rod) is shown in figure 12.20. We have assumed that the piston is moving up so the friction force is directed down, resisting the motion. Linear momentum balance for this system is:

$$\begin{aligned} \sum \vec{F}_i &= \dot{\vec{L}} \\ -N\hat{\imath} - \mu N\hat{\jmath} + T\hat{\lambda}_{rod} &= m_{piston}\, a\hat{\jmath}. \end{aligned}$$

If we assume that the acceleration $a\hat{\jmath}$ of the piston is known, as is its mass m_{piston}, the coefficient of friction μ, and the orientation of the connecting rod $\hat{\lambda}_{rod}$, then we can solve for the rod tension T and the normal reaction N.
Even though the piston moves in one direction, the momentum balance equation is a two-dimensional vector equation.

The kinematically simple 1-D motions we assume in this chapter simplify the evaluation of the right hand sides of the momentum balance equations. But, unlike the 1D mechanics of the previous chapter, in this section the momentum balance equations are 2D and 3D vector equations.

Highly constrained bodies

This chapter is about rigid objects that move in straight lines. Most objects will not agree to be the topic of such discussion without being forced into doing so. In general, one expects bodies to rotate or move along a curved path. To keep an object that is subject to various forces from rotating or curving takes some constraint. The object needs to be rigid and held by wires, rods, rails, hinges, welds, etc. that keep it from spinning, keeping it in parallel motion. Of course the presence of constraint is not always associated with the disallowance of rotation — constraints could even cause rotation. But to keep a rigid object in the straight-line motion of interest here requires some kind of constraint.

Constraint forces are of interest

Of common interest for constrained structures is making sure that static and dynamic loads do not cause failure of the parts that enforce the constraints. For example, suppose a truck hauls a very heavy load that is held down by chains or straps. When the truck accelerates, what is the tension in the chains, and will it exceed the strength limit of the chains so that they might break? Thus the constraint forces needed to impose the assumed motion are of interest.

1D mechanics

This is in contrasts with the situation in 1D "unconstrained" dynamics of the previous chapter. For one-dimensional mechanics, we assume that, in addition to the restricted kinematics, everything of interest mechanically happens in, say, the $\hat{\imath}$ (x) direction. That is, we ignored *all* torques and angular momenta, and only consider the $\hat{\imath}$ components of the forces (*i.e.*, $\vec{F} \cdot \hat{\imath}$) and linear momentum ($\vec{L} \cdot \hat{\imath}$), namely F_x and L_x. Here we want to go beyond that 1D mechanics.

Kinematics of straight line motion

Let's consider a set of points in the system of interest. Let's call them A to G, or generically, P. For convenience we distinguish a reference point O'. O' may be the center-of-mass, the origin of a local coordinate system, or a fleck of dirt that serves as a marker. By *parallel motion*, we mean that the system happens to move in such a way that $\vec{a}_P = \vec{a}_{O'}$, and $\vec{v}_P = \vec{v}_{O'}$ ($Fig.$ 12.21). That is,

$$\vec{a}_A = \vec{a}_B = \vec{a}_C = \vec{a}_D = \vec{a}_E = \vec{a}_F = \vec{a}_G = \vec{a}_P = \vec{a}_{O'}$$

at every instant in time. We also assume that $\vec{v}_A = \ldots = \vec{v}_P = \vec{v}_{O'}$.

A special case of parallel motion is straight-line motion.

a system moves with straight-line motion if it moves like a non-rotating rigid body, in a straight line.

For straight-line motion, the velocity of the body is in a fixed unchanging direction. If we call a unit vector in that direction $\hat{\lambda}$, then we have

$$\vec{v}(t) = v(t)\hat{\lambda}, \qquad \vec{a}(t) = a(t)\hat{\lambda} \qquad \text{and} \qquad \vec{r}(t) = \vec{r}_0 + s(t)\hat{\lambda}$$

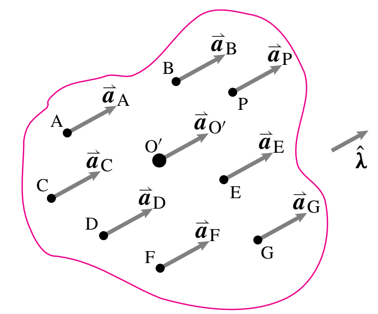
for every point in the system. $\vec{r}_0$ is the position of a point at time 0 and s is the distance the point moves in the $\hat{\lambda}$ direction. Every point in the system has the same s, v, a, and $\hat{\lambda}$ as the other points. There are a variety of problems of practical interest that can be idealized as fitting into this class, notably, the motions of things constrained to move on belts, roads, and rails, like the train in figure **??**.

Example: **Parallel swing is not straight-line motion**
The swing shown does not rotate — all points on the swing have the same velocity. The velocity of all particles are parallel but, since paths are curved, this motion is not straight-line motion. Such curvilinear parallel motion will be discussed later in the book.

Velocity of a point

The velocity of any point P on a non-rotating rigid body (such as for straight-line motion) is the same as that of any reference point on the body (see Fig. 12.23).

$$\vec{v}_{\mathrm{P}} = \vec{v}_{\mathrm{O'}}$$



$\vec{a}_A = \vec{a}_B = \vec{a}_C = \vec{a}_D = \vec{a}_E = \vec{a}_F = \vec{a}_G = \vec{a}_{O'}$

Figure 12.21: Parallel motion: all points on the body have the same acceleration $\vec{a} = a\hat{\lambda}$. For straight-line motion: $\hat{\lambda}(t)$=constant in time and $\vec{v} = v\hat{\lambda}$.
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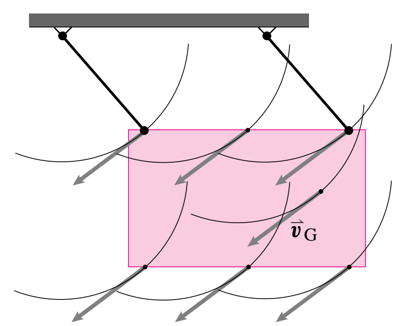

Figure 12.22: A swing showing instantaneous parallel motion which is *curvilinear*. At every instant, each point has the same velocity as the others, but the motion is not in a straight line.
Filename:tfigure3-swing

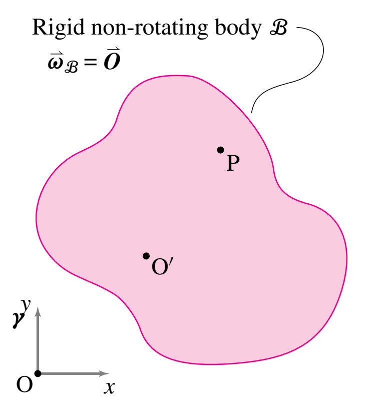

Figure 12.23: A non-rotating body $\mathcal{B}$ with points O' and P.
Filename:tfigure3-2-1

A more general case, which you will learn in later chapters, is shown as 5b in Table II at the back of the book. This formula concerns rotational rate which we will measure with the vector $\vec{\omega}$. For now all you need to know is that $\vec{\omega} = \vec{0}$ when something is not rotating. In 5b in Table II, if you set $\vec{\omega}_{\mathcal{B}} = 0$ and $\vec{\boldsymbol{v}}_{P/\mathcal{B}} = \vec{0}$ it says that $\vec{\boldsymbol{v}}_P = \dot{\vec{\boldsymbol{r}}}_{O'/O}$ or in shorthand, $\vec{\boldsymbol{v}}_P = \vec{\boldsymbol{v}}_{O'}$, as we have written above.

Acceleration of a point

Similarly, the acceleration of every point on a non-rotating rigid body is the same as every other point. The more general case, not needed in this chapter, is shown as entry 5c in Table II at the back of the book.

General results

Before we proceed with discussion of the details of the mechanics of straight-line motion we present some ideas that are also more generally applicable. That is, the concept of the center-of-mass allows some useful simplifications of the general expressions for $\vec{\boldsymbol{L}}$, $\dot{\vec{\boldsymbol{L}}}$, $\vec{\boldsymbol{H}}_{/C}$, $\dot{\vec{\boldsymbol{H}}}_{/C}$ and E_K.

Linear momentum $\vec{\boldsymbol{L}}$ and its rate of change $\dot{\vec{\boldsymbol{L}}}$ for straight-line motion

Although we are dealing with zillions of atoms in a given object, the linear momentum and angular momentum are simple to evaluate:

$$\vec{\boldsymbol{L}} = m_{tot} v_{cm} \qquad \text{and} \qquad \dot{\vec{\boldsymbol{L}}} = m_{tot} a_{cm}.$$

Actually, as the front inside cover states, these formulas are good for any motion of any system. The nice simplification for the straight-line

12.2 *THEORY*

Calculation of $\vec{\boldsymbol{H}}_{/C}$ and $\dot{\vec{\boldsymbol{H}}}_{/C}$ for straight-line motion

For straight-line motion, and parallel motion in general, we can derive the simplification in the calculation of $\vec{\boldsymbol{H}}_{/C}$ as follows:

$$\begin{aligned}
\vec{\boldsymbol{H}}_{/C} &\equiv \sum \vec{\boldsymbol{r}}_{i/C} \times m_i \vec{\boldsymbol{v}}_i \text{ (definition)} \\
&= \sum \vec{\boldsymbol{r}}_{i/C} \times m_i \vec{\boldsymbol{v}}_{cm} \quad (\text{since, } \vec{\boldsymbol{v}}_i = \vec{\boldsymbol{v}}_{cm}) \\
&= \left(\sum \vec{\boldsymbol{r}}_{i/C}\, m_i\right) \times \vec{\boldsymbol{v}}_{cm}, \\
&= \vec{\boldsymbol{r}}_{cm/C} \times (m_{tot} \vec{\boldsymbol{v}}_{cm}), \\
&\quad (\text{ since, } \textstyle\sum \vec{\boldsymbol{r}}_{i/C}\, m_i \equiv m_{tot} \vec{\boldsymbol{r}}_{cm/C}).
\end{aligned}$$

The derivation that $\dot{\vec{\boldsymbol{H}}}_{/C} = \vec{\boldsymbol{r}}_{cm/C} \times (m \vec{\boldsymbol{a}}_{cm})$ follows from $\dot{\vec{\boldsymbol{H}}}_{/C} \equiv \sum \vec{\boldsymbol{r}}_{i/C} \times m_i \vec{\boldsymbol{a}}_i$ by the same reasoning.

motion of this chapter is that all points on a given object have the same velocity and acceleration. So we don't need to find or track the center of mass, but can track the motion of any point on the object.

Angular momentum $\vec{\boldsymbol{H}}_{/\mathrm{C}}$ and its rate of change, $\dot{\vec{\boldsymbol{H}}}_{C}$ for straight-line motion

For the motions in this chapter, where $\vec{\boldsymbol{a}}_i = \vec{\boldsymbol{a}}_{cm}$ and thus $\vec{\boldsymbol{a}}_{i/cm} = \vec{0}$, angular momentum considerations are simplified, as explained in Box 12.2 on page 646*.

$$\vec{\boldsymbol{H}}_{/\mathrm{C}} = \vec{\boldsymbol{r}}_{\mathrm{cm/C}} \times v_{\mathrm{cm}} m_{\mathrm{tot}} \qquad \text{and} \qquad \dot{\vec{\boldsymbol{H}}}_{\mathrm{C}} = \vec{\boldsymbol{r}}_{\mathrm{cm/C}} \times a_{\mathrm{cm}} m_{\mathrm{tot}}$$

But for straight-line motion (and, slightly more generally, for any parallel motion), the calculations turn out to be the same as we would get if we put a single point mass at the center-of-mass*:

$$\begin{aligned} \vec{\boldsymbol{H}}_{/\mathrm{C}} &\equiv \sum(\vec{\boldsymbol{r}}_{i/\mathrm{C}} \times m_i \vec{\boldsymbol{v}}_i) &= \vec{\boldsymbol{r}}_{cm/C} \times (m_{\mathrm{total}} \vec{\boldsymbol{v}}_{\mathrm{cm}}), \\ \dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}} &\equiv \sum(\vec{\boldsymbol{r}}_{i/\mathrm{C}} \times m_i \vec{\boldsymbol{a}}_i) &= \vec{\boldsymbol{r}}_{cm/C} \times (m_{\mathrm{total}} \vec{\boldsymbol{a}}_{cm}). \end{aligned}$$

Note, there is some subtlety in the definition of $\vec{\boldsymbol{H}}_{/\mathrm{C}}$, as explained in section ??.

Kinetic energy

Generally things will not be so simple, but for straight-line motion, or any parallel motion where all points on an object have the same velocity and acceleration, kinetic energy and its rate of change are also easy to calculate:

$$E_{\mathrm{K}} = m_{\mathrm{tot}} v_{\mathrm{cm}}^2/2 \qquad \text{and} \qquad \dot{E}_{\mathrm{K}} = m_{\mathrm{tot}} v_{\mathrm{cm}} a_{\mathrm{cm}}.$$

The kinetic energy works the same as if all the mass was concentrated at the center of mass. This result does not generalize to more complex motions.

Approach

To study systems in straight-line motion (as always) we:

- draw a free body diagram, showing the appropriate forces and couples at places where connections are 'cut',
- state reasonable kinematic assumptions based on the motions that the constraints allow,
- write linear and/or angular momentum balance equations and/or energy balance, and

* Calculating rate of change of angular momentum will get more difficult as the book progresses. For a rigid body $\mathcal{B}$ in more general motion, the calculation of rate of change of angular momentum involves the angular velocity $\vec{\omega}_{\mathcal{B}}$, its rate of change $\dot{\vec{\omega}}_{\mathcal{B}}$, and the moment of inertia matrix $[\boldsymbol{I}^{\mathrm{cm}}]$. If you look in the back of the book at Table I, entries 6c and 6d, you will see formulas that reduce to the formulas below if you assume no rotation and thus use $\vec{\omega} = \vec{0}$ and $\dot{\vec{\omega}} = \vec{0}$.
But rate of change of linear momentum is simple, at least in concept, in this chapter, as well as in the rest of this book, where

$$\dot{\vec{\boldsymbol{L}}} = m_{tot} \vec{\boldsymbol{a}}_{cm}$$

always applies.

* **Caution:** The special motions in this chapter are almost the only cases where the angular momentum and its rate of change are so easy to calculate.

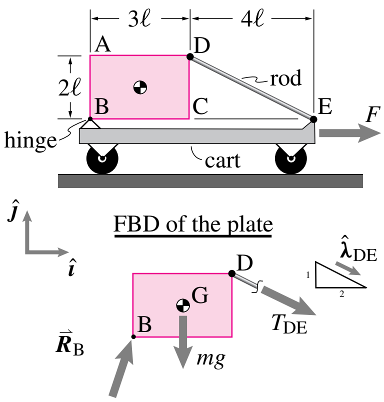

Figure 12.24: Uniform plate supported by a hinge and a rod on an accelerating cart.

Filename:tfigure3-2D-guyed

- solve for quantities of interest.

Angular momentum balance about a judiciously chosen axis is a particularly useful tool for reducing the number of equations that need to be solved.

Example: **Plate on a cart**

A uniform rectangular plate $ABCD$ of mass m is supported by a light rigid rod DE and a hinge joint at point B. The dimensions are as shown. The cart has acceleration $a_x\hat{\imath}$ due to a force $F\hat{\imath}$ and the constraints of the wheels. Referring to the free body diagram in figure 12.24 and writing angular momentum balance for the plate about point B, we can get an equation for the tension in the rod T_{DE} in terms of m and a_x:

$$\sum \vec{M}_{/B} = \dot{\vec{H}}_{/B}$$

$$\left\{\vec{r}_{D/B}\times(T_{DE}\hat{\lambda}_{DE}) + \vec{r}_{G/B}\times(-mg\,\hat{\jmath}) = \vec{r}_{G/B}\times(ma_x\hat{\imath})\right\}$$

$$\{\}\cdot\hat{k} \Rightarrow \quad T_{DE} = \frac{\sqrt{5}}{7}m\left(a_x-\frac{3}{2}g\right).$$

Summarizing note:

angular momentum balance is important even when there is no rotation.

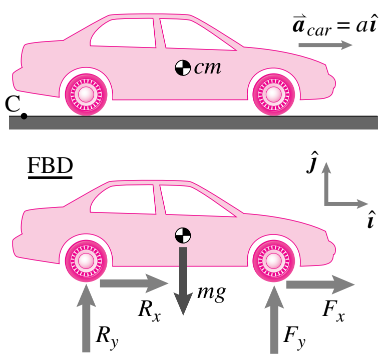

Figure 12.25: A four-wheel drive car accelerating but not tipping. See fig. 3.32 on page 179 for more about FBDs involving wheel contact.

Filename:tfigure3-4wd-car

Sliding and pseudo-sliding objects

A car coming to a stop can be roughly modeled as a rigid body that translates and does not rotate. That is, at least for a first approximation, the rotation of the car due to the suspension and tire deformation, can be neglected. The free body diagram will show various forces with lines of action that do not all act through a single point so that angular momentum balance must be used to analyze the system. Similarly, a bicycle which is braking or a box that is skidding (if not tipping) may be analyzed by assuming straight-line motion.

Example: **Car skidding**

Consider the accelerating four-wheel drive car in figure 12.25. The motion quantities for the car are $\dot{\vec{L}} = m_{car}\vec{a}_{car}$ and $\dot{\vec{H}}_{/C} = \vec{r}_{cm/C}\times\vec{a}_{car}m_{car}$. We could calculate angular momentum balance relative to the car's center of mass in which case $\sum\vec{M}_{cm} = \dot{\vec{H}}_{cm} = \vec{0}$ (because the position of the center-of-mass relative to the center-of-mass is $\vec{0}$).

As mentioned, it is often useful to calculate angular momentum balance of sliding objects about points of contact (such as where tires contact the road) or about points that lie on lines of action of applied forces when writing angular momentum balance to solve for forces or accelerations. To do so usually eliminates some unknown reactions from the equations to be solved. For example, the angular momentum balance equation about the rear-wheel contact of a car does not contain the rear-wheel contact forces.

Wheels

The function of wheels is to allow easy sliding-like (pseudo-sliding) motion between objects, at least in the direction they are pointed. On the other hand, wheels do sometimes slip due to:

- being overpowered (as in a screeching accelerating car),
- being braked hard, or
- having very bad bearings (like a rusty toy car).

How wheels are treated when analyzing cars, bikes, and the like depends on both the application and on the level of detail one requires. In *this chapter*, we will always assume that wheels have negligible mass. Thus, when we treat the special case of un-driven and un-braked wheels our free body diagrams will be as in figure 3.33 on page 180 and *not* like the one in figure **??** on page **??**. With the ideal wheel approximation, all of the various cases for a car traveling to the right are shown with partial free body diagrams of a wheel in figure 3.32. For the purposes of actually solving problems, we have accepted Coulomb's law of friction as a model for contacting interaction (see pages **??**-177).

3-D forces in straight-line motion

The ideas we have discussed apply as well in three dimensions as in two. As you learned from doing statics problems, working out the details in 3D, where vector methods must be used carefully, is more involved than in 2D. As for statics, three dimensional problems often yield simple results and simple intuitions by considering angular momentum balance about an axis.

Angular momentum balance about an axis

The simplest way to think of angular momentum balance about an axis is to look at angular momentum balance about a point and then take a dot product with a unit vector along an axis:

$$\hat{\lambda} \cdot \left\{ \sum \vec{\boldsymbol{M}}_{/\mathrm{C}} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}} \right\}.$$

Note that the axis need not correspond to any mechanical device in any way resembling an axle. The equation above applies for any point C and any vector $\hat{\lambda}$. If you choose C and $\hat{\lambda}$ judiciously many terms in your equations may drop out.

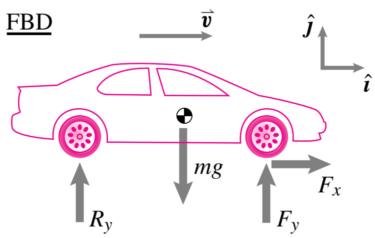

Figure 12.26: Free body diagram of a front-wheel-drive car during braking. Note that we have (arbitrarily) pointed F_x to the right. The algebra in this problem will tell us that $F_x < 0$.

Filename:sfig2-6-3a

SAMPLE 12.6 Force in braking. A front-wheel-drive car of mass $m = 1200\,\text{kg}$ is cruising at $v = 60\,\text{mph}$ on a straight road when the driver slams on the brake. The car slows down to 20 mph in 4 s while maintaining its straight path.

1. What is the average force (average in time) applied on the car during braking?
2. What is the average power of breaking?

Solution

1. Let us assume that we have an xy coordinate system in which the car is traveling along the x-axis during the entire time under consideration. Then, the velocity of the car before braking, $\vec{v}_1$, and after braking, $\vec{v}_2$, are

$$\vec{v}_1 = v_1\hat{\imath} = 60\,\text{mph}\;\hat{\imath} \quad \text{and} \quad \vec{v}_2 = v_2\hat{\imath} = 20\,\text{mph}\;\hat{\imath}.$$

The linear impulse during braking is $\vec{F}_{\text{ave}}\Delta t$ where $\vec{F} \equiv F_x\hat{\imath}$ (see free body diagram of the car). Now, from the impulse-momentum relationship,

$$\vec{F}\Delta t = \vec{L}_2 - \vec{L}_1,$$

where $\vec{L}_1$ and $\vec{L}_2$ are linear momenta of the car before and after braking, respectively, and $\vec{F}$ is the average applied force. Therefore,

$$\begin{aligned}
\vec{F} &= \frac{1}{\Delta t}(\vec{L}_2 - \vec{L}_1) = \frac{m}{\Delta t}(\vec{v}_2 - \vec{v}_1) \\
&= \frac{1200\,\text{kg}}{4\,\text{s}}(20 - 60)\,\text{mph}\hat{\imath} \\
&= -12000\frac{\text{kg}}{\text{s}}\cdot\frac{\not{\text{m}}\text{i}}{\not{\text{h}}\text{r}}\cdot\frac{1600\text{ m}}{1\,\not{\text{m}}\text{i}}\cdot\frac{1\,\not{\text{h}}\text{r}}{3600\,\text{s}}\hat{\imath} \\
&= -\frac{16,000}{3}\,\text{kg}\cdot\text{m/s}^2\hat{\imath} = -5.33\,\text{kN}\hat{\imath}.
\end{aligned}$$

Thus

$$F_x\hat{\imath} = -5.33\,\text{kN}\hat{\imath} \quad \Rightarrow \quad F_x = -5.33\,\text{kN}.$$

$$\boxed{F_x = -5.33\,\text{kN}}$$

2. Let the average power during breaking be P_{ave}. Then the work done during breaking is $W = \int P_{\text{ave}}dt$. From work-energy principle, we have

$$\begin{aligned}
W &= \Delta E_{\text{K}} \\
\int_{t_1}^{t_2} P_{\text{ave}}dt &= \frac{1}{2}m(v_2^2 - v_1^2) \\
P_{\text{ave}}(t_2 - t_1) &= \frac{1}{2}m(v_2^2 - v_1^2) \\
P_{\text{ave}} &= \frac{m}{2\Delta t}(v_2^2 - v_1^2)
\end{aligned}$$

Substituting $m = 1200\,\text{kg}$, $\Delta t = 4\,\text{s}$, $v_1 = 60\,\text{mph} = 26.67\,\text{m/s}$ and $v_2 = 20\,\text{mph} = 8.89\,\text{m/s}$, we get

$$P_{\text{ave}} = -94815\,\text{N}\cdot\text{m/s} = -94.815\,\text{kW}.$$

It is easy to check that if we take the average force F_{ave} calculated above and the average speed $v_{\text{ave}} = (v_1 + v_2)/2 = 40\,\text{mph} = 17.77\,\text{m/s}$, then

$$P_{\text{ave}} = F_{\text{ave}}v_{\text{ave}} = -5.33\,\text{kN}\cdot 17.77\,\text{m/s} = -94.815\,\text{kW},$$

as obtained above.

$$\boxed{P_{\text{ave}} = -94.815\,\text{kW}}$$

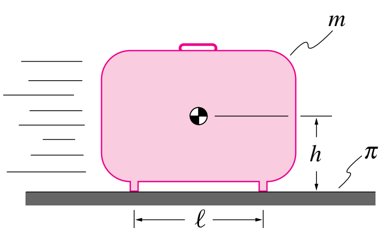

Figure 12.27: A suitcase in motion.
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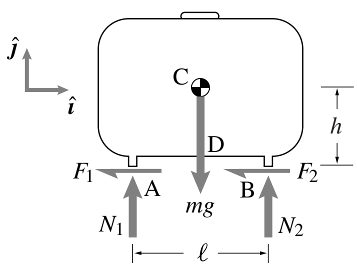


Figure 12.28: FBD of the suitcase.
Filename:sfig3-5-1a

SAMPLE 12.7 **A suitcase skidding on frictional ground.** A suitcase of mass m is pushed and sent sliding on a horizontal surface. The suitcase slides without any rotation. A and B are the only contact points of the suitcase with the ground. If the coefficient of friction between the suitcase and the ground is μ, find all the forces applied by the ground on the suitcase. Discuss the results obtained for normal forces.

Solution As usual, we first draw a free body diagram of the suitcase. The FBD is shown in Fig. 12.28. Assuming Coulomb's law of friction holds, we can write

$$\vec{F}_1 = -\mu N_1 \hat{\imath} \quad \text{and} \quad \vec{F}_2 = -\mu N_2 \hat{\imath}. \tag{12.26}$$

Now we write the balance of linear momentum for the suitcase:

$$\sum \vec{F} = m\vec{a}_{\text{cm}}$$

$$\Rightarrow \quad -(F_1 + F_2)\hat{\imath} + (N_1 + N_2 - mg)\hat{\jmath} = ma_C\hat{\imath} \tag{12.27}$$

where $\vec{a}_C = a_C\hat{\imath}$ is the unknown acceleration. Dotting eqn. (12.27) with $\hat{\imath}$ and $\hat{\jmath}$ and substituting for F_1 and F_2 from eqn. (12.26) we get

$$-\mu(N_1 + N_2) = ma_C \tag{12.28}$$

$$N_1 + N_2 = mg. \tag{12.29}$$

Equations (12.28) and (12.29) represent 2 scalar equations in three unknowns N_1, N_2 and a. Obviously, we need another equation to solve for these unknowns.

We can write the balance of angular momentum about any point. Points A or B are good choices because they each eliminate some reaction components. Let us write the balance of angular momentum about point A:

$$\sum \vec{M}_{\text{A}} = \dot{\vec{H}}_{\text{A}}$$

$$\begin{aligned}\sum \vec{M}_{\text{A}} &= \vec{r}_{B/A} \times N_2\hat{\jmath} + \vec{r}_{D/A} \times (-mg)\hat{\jmath} \\ &= \ell\hat{\imath} \times N_2\hat{\jmath} + \frac{\ell}{2}\hat{\imath} \times (-mg)\hat{\jmath} \\ &= (\ell N_2 - mg\frac{\ell}{2})\hat{k}\end{aligned} \tag{12.30}$$

and

$$\begin{aligned}\dot{\vec{H}}_{\text{A}} &= \vec{r}_{C/A} \times m\vec{a}_C \\ &= (\frac{\ell}{2}\hat{\imath} + h\hat{\jmath}) \times ma_C\hat{\imath} \\ &= -ma_C h\hat{k}.\end{aligned} \tag{12.31}$$

Equating (12.30) and (12.31) and dotting both sides with $\hat{k}$ we get the following third scalar equation:

$$\ell N_2 - mg\frac{\ell}{2} = -ma_C h. \tag{12.32}$$

Solving eqns. (12.28) and (12.29) for a we get

$$a_C = -\mu g$$

and substituting this value of a_C in eqn. (12.32) we get

$$\begin{aligned}N_2 &= \frac{m\mu gh + mg\ell/2}{\ell} \\ &= mg\left(\frac{1}{2} + \frac{h}{\ell}\mu\right).\end{aligned}$$

Substituting the value of N_2 in either of the equations (12.28) or (12.29) we get

$$N_1 = mg\left(\frac{1}{2} - \frac{h}{\ell}\mu\right).$$

$$\boxed{N_1 = mg(\tfrac{1}{2} - \tfrac{h}{\ell}\mu),\ N_2 = mg(\tfrac{1}{2} + \tfrac{h}{\ell}\mu),\ f_1 = \mu N_1,\ f_2 = \mu N_2.}$$

Discussion: From the expressions for N_1 and N_2 we see that

1. $N_1 = N_2 = \frac{1}{2}mg$ if $\mu = 0$ because without friction there is no deceleration. The problem becomes equivalent to a statics problem.
2. $N_1 = N_2 \approx \frac{1}{2}mg$ if $\ell >> h$. In this case, the moment produced by the friction forces is too small to cause a significant difference in the magnitudes of the normal forces. For example, take $\ell = 20h$ and calculate moment about the center-of-mass to convince yourself.

Graphically, N_1, N_2 and their difference $N_1 - N_2$ are shown in the plot below as a function of h/ℓ for a particular value of μ and mg. As the equations indicate, $N_1 - N_2$ increases steadily as h/ℓ increases, showing how the moment produced by the friction forces makes a bigger and bigger difference between N_1 and N_2 as this moment gets bigger.

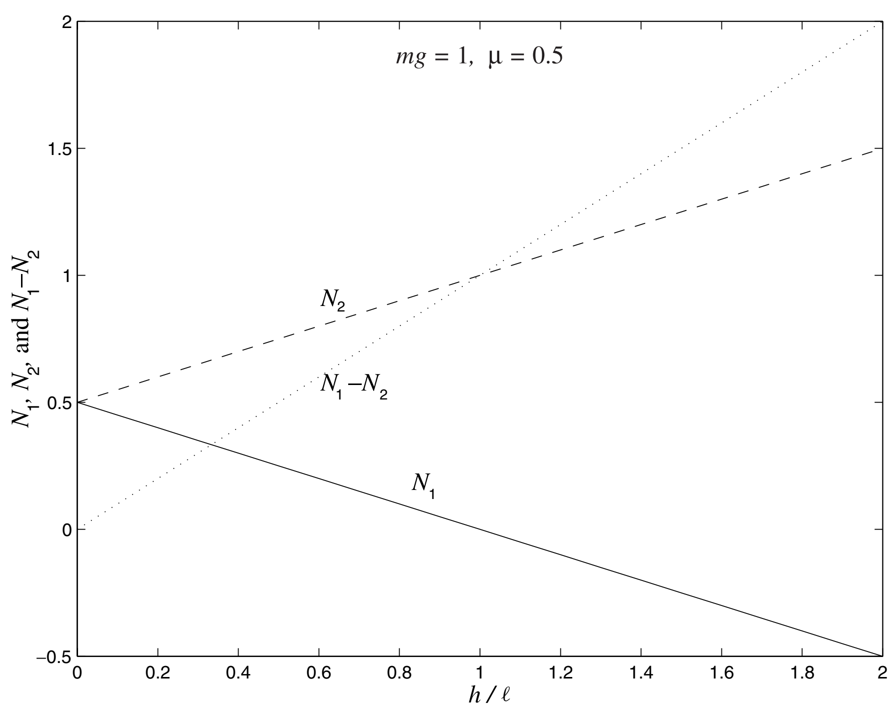


Figure 12.29: The normal forces N_1 and N_2 differ from each other more and more as h/ℓ increases.

Filename:sample6p8graph

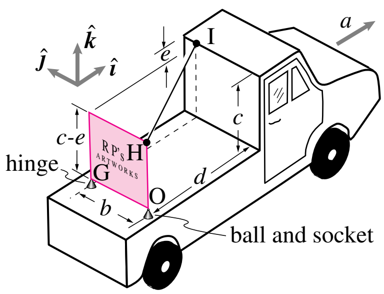


Figure 12.30: An accelerating board in 3-D

Filename:sfig3-5-2

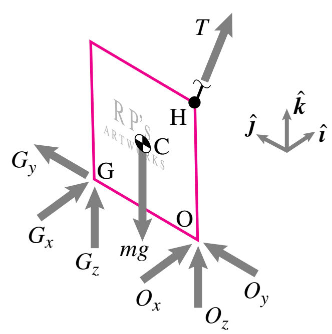


Figure 12.31: FBD of the board

Filename:sfig3-5-2a

SAMPLE 12.8 Uniform acceleration of a board in 3-D. A uniform sign-board of mass $m = 20\,\mathrm{kg}$ sits in the back of an accelerating flatbed truck. The board is supported with a ball-and-socket joint at O and a hinge at G. A light rod from H to I keeps the board from falling over. The truck is on level ground and has forward acceleration $\vec{a} = 0.6\,\mathrm{m/s^2}\hat{\imath}$. The relevant dimensions are $b = 1.5\,\mathrm{m}$, $c = 1.5\,\mathrm{m}$, $d = 3\,\mathrm{m}$, $e = 0.5\,\mathrm{m}$. There is gravity ($g = 10\,\mathrm{m/s^2}$).

1. Draw a free body diagram of the board.
2. Set up equations to solve for all the unknown forces shown on the FBD.
3. Use the balance of angular momentum about an axis to find the tension in the rod.

Solution

1. The free body diagram of the board is shown in Fig. 12.31.
2. Linear momentum balance for the board:

$$\sum \vec{F} = m\vec{a}, \quad \text{or}$$

$$(G_x + O_x)\hat{\imath} + (G_y + O_y)\hat{\jmath} + (G_z + O_z - mg)\hat{k} + T\hat{\lambda}_{HI} = ma\hat{\imath} \tag{12.33}$$

where

$$\hat{\lambda}_{HI} = \frac{d\hat{\imath} + b\hat{\jmath} + e\hat{k}}{\sqrt{d^2 + b^2 + e^2}} = \frac{d\hat{\imath} + b\hat{\jmath} + e\hat{k}}{\ell},$$

and ℓ is the length of the rod HI.

Dotting eqn. (12.33) with $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ we get the following three scalar equations:

$$G_x + O_x + T\frac{d}{\ell} = ma \tag{12.34}$$

$$G_y + O_y + T\frac{b}{\ell} = 0 \tag{12.35}$$

$$G_z + O_z + T\frac{e}{\ell} = mg \tag{12.36}$$

Angular momentum balance about point G:

$$\sum \vec{M}_G = \dot{\vec{H}}_G$$

$$\begin{aligned}
\sum \vec{M}_G &= \vec{r}_{C/G} \times (-mg\hat{k}) + \vec{r}_{O/G} \times (O_x\hat{\imath} + O_z\hat{k}) + \vec{r}_{H/G} \times T\hat{\lambda}_{HI} \\
&= (-\frac{b}{2}\hat{\jmath} + \frac{c-e}{2}\hat{k}) \times (-mg\hat{k}) - b\hat{\jmath} \times (O_x\hat{\imath} + O_z\hat{k}) \\
&\quad + [-b\hat{\jmath} + (c-e)\hat{k}] \times \frac{T}{\ell}(d\hat{\imath} + b\hat{\jmath} + e\hat{k}) \\
&= \left(\frac{b}{2}mg - bO_z - be\frac{T}{\ell} - (c-e)b\frac{T}{\ell}\right)\hat{\imath} \\
&\quad + (c-e)d\frac{T}{\ell}\hat{\jmath} + \left(bO_x + bd\frac{T}{\ell}\right)\hat{k}
\end{aligned} \tag{12.37}$$

and

$$\begin{aligned}\dot{\vec{H}}_G &= \vec{r}_{C/G} \times ma\hat{\imath} \\ &= (-\frac{b}{2}\hat{\jmath} + \frac{c-e}{2}\hat{k}) \times ma\hat{\imath} \\ &= \frac{b}{2}ma\hat{k} + \frac{c-e}{2}ma\,\hat{\jmath}. \end{aligned} \qquad (12.38)$$

Equating (12.37) and (12.38) and dotting both sides with $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ we get the following three additional scalar equations:

$$O_z + \frac{c}{\ell}T = \frac{1}{2}mg \qquad (12.39)$$

$$\frac{d}{\ell}T = \frac{1}{2}ma \qquad (12.40)$$

$$O_x + \frac{d}{\ell}T = \frac{1}{2}ma \qquad (12.41)$$

Now we have six scalar equations in seven unknowns — O_x, O_y, O_z, G_x, G_y, G_z, and T. From basic linear algebra, we know that we cannot find unique solutions for all these unknowns from the given equations. A closer inspection of eqns. (12.34–12.36) and (12.39–12.41) shows that we can easily solve for O_x, O_z, G_x, G_z, and T, but O_y and G_y cannot be determined uniquely because they appear together as the sum $G_y + O_y$. * Fortunately, we can find the tension in the wire HI without worrying about the values of O_y and G_y as we show below.

* Note that $\boldsymbol{G_y}$ and $\boldsymbol{O_y}$ will always appear together as the sum $\boldsymbol{G_y} + \boldsymbol{O_y}$ even if you took the angular momentum balance about some other point. This is because they have the same line of action. Thus, they cannot be found independently. This mathematical problem corresponds to the physical reality that the supports at points $\boldsymbol{O}$ and $\boldsymbol{G}$ could be squeezing the plate along the line $\boldsymbol{OG}$ with, say, $\boldsymbol{O_y} = \mathbf{1000}$ N and $\boldsymbol{G_y} = -\mathbf{1000}$ N even if there were no gravity, and the truck was not accelerating. To make prestress problems like this tractable, people often make assumptions like, 'Assume $\boldsymbol{G_y} = \mathbf{0}$', that is, they try to get rid of the redundancy in supports to make the problem statically determinate.

3. Balance of angular momentum about axis OG gives:

$$\begin{aligned}\hat{\lambda}_{OG} \cdot \sum \vec{M}_G &= \hat{\lambda}_{OG} \cdot \dot{\vec{H}}_G \\ &= \hat{\lambda}_{OG} \cdot (\vec{r}_{C/G} \times ma\hat{\imath}). \end{aligned} \qquad (12.42)$$

Since all reaction forces and the weight go through axis OG, they do not produce any moment about this axis (convince yourself that the forces from the reactions have no torque about the axis by calculation or geometry). Therefore,

$$\begin{aligned}\hat{\lambda}_{OG} \cdot \sum \vec{M}_G &= \hat{\jmath} \cdot (\vec{r}_{H/G} \times T\hat{\lambda}_{HI}) \\ &= T\frac{d(c-e)}{\ell}. \end{aligned} \qquad (12.43)$$

$$\begin{aligned}\hat{\lambda}_{OG} \cdot (\vec{r}_{C/G} \times ma\hat{\imath}) &= \hat{\jmath} \cdot \left[\left(\frac{b}{2}\hat{\jmath} + \frac{c-e}{2}\hat{k}\right) \times ma\hat{\imath}\right] \\ &= ma\frac{(c-e)}{2}. \end{aligned} \qquad (12.44)$$

Equating (12.43) and (12.44), as required by *eqn*. (12.42), we get

$$\begin{aligned} T &= \frac{ma\ell}{2d} \\ &= \frac{20\,\text{kg} \cdot 0.6\,\text{m/s}^2 \cdot 3.39\,\text{m}}{2 \cdot 3\,\text{m}} \\ &= 6.78\,\text{N}. \end{aligned}$$

$$\boxed{T_{HI} = 6.78\,\text{N}}$$

* Be careful with units. Most computer programs will not take care of your units. They only deal with numerical input and output. You should, therefore, make sure that your variables have proper units for the required calculations. Either do dimensionless calculations or use consistent units for all quantities.

SAMPLE 12.9 Computer solution of algebraic equations. In the previous sample problem (Sample 12.8), six equations were obtained to solve for the six unknown forces (assuming $G_y = 0$). (i) Set up the six equations in matrix form and (ii) solve the matrix equation on a computer. Check the solution by substituting the values obtained in one or two equations.

Solution

1. The six scalar equations — (12.34), (12.35), (12.36), (12.39), (12.40), and (12.41) are amenable to hand calculations. We, however, set up these equations in matrix form and solve the matrix equation on the computer. The matrix form of the equations is:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & \frac{d}{\ell} \\ 0 & 1 & 0 & 0 & 0 & \frac{b}{\ell} \\ 0 & 0 & 1 & 0 & 1 & \frac{e}{\ell} \\ 0 & 0 & 1 & 0 & 0 & \frac{c}{\ell} \\ 0 & 0 & 0 & 0 & 0 & \frac{d}{\ell} \\ 1 & 0 & 0 & 0 & 0 & \frac{d}{\ell} \end{bmatrix} \begin{Bmatrix} O_x \\ O_y \\ O_z \\ G_x \\ G_z \\ T \end{Bmatrix} = \begin{Bmatrix} ma \\ 0 \\ mg \\ mg/2 \\ ma/2 \\ ma/2 \end{Bmatrix}. \qquad (12.45)$$

The above equation can be written, in matrix notation, as

$$\mathbf{A}\,\mathbf{x} = \mathbf{b}$$

where $\mathbf{A}$ is the coefficient matrix, $\mathbf{x}$ is the vector of the unknown forces, and $\mathbf{b}$ is the vector on the right hand side of the equation. Now we are ready to solve the system of equations on the computer.

2. We use the following pseudo-code to solve the above matrix equation. *

```
m = 20, a = 0.6,
b = 1.5, c = 1.5, d = 3, e = 0.5, g = 10,
l = sqrt(b^2 + d^2 + e^2),

A = [1 0 0 1 0 d/l
     0 1 0 0 0 b/l
     0 0 1 0 1 e/l
     0 0 1 0 0 c/l
     0 0 0 0 0 d/l
     1 0 0 0 0 d/l]
b = [m*a, 0, m*g, m*g/2, m*a/2, m*a/2]'

{Solve A x = b for x}
x =                  % this is the computer output
        0
  -3.0000
  97.0000
   6.0000
 102.0000
   6.7823
```

The solution obtained from the computer means:

$$\boxed{O_x = 0,\; O_y = -3\,\text{N},\; O_z = 97\,\text{N},\; G_x = 6\,\text{N},\; G_z = 102\,\text{N},\; T = 6.78\,\text{N}.}$$

We now hand-check the solution by substituting the values obtained in, say, Eqns. (12.35) and (12.40). Before we substitute the values of forces, we need to calculate the length ℓ.

$$\begin{aligned} \ell &= \sqrt{d^2+b^2+e^2} \\ &= 3.3912\,\mathrm{m}. \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Eqn. (12.35):} \qquad O_y + T\frac{b}{\ell} &= -3\,\mathrm{N} + 6.78\,\mathrm{N}\cdot\frac{1.5\,\mathrm{m}}{3.3912\,\mathrm{m}} \\ &\overset{\checkmark}{\equiv} 0, \\ \text{Eqn. (12.40):} \qquad \frac{d}{\ell}T - \frac{1}{2}ma &= \frac{3\,\mathrm{m}}{3.3912\,\mathrm{m}}6.78\,\mathrm{N} - \frac{1}{2}20\,\mathrm{kg}\,0.6\,\mathrm{m/s^2} \\ &\overset{\checkmark}{\equiv} 0. \end{aligned}$$

Thus, the computer solution agrees with our equations.

Comments: We could have solved the six equations for seven unknowns without assuming $G_y = 0$ if our computer program or package allows us to do so. We will, of course, not get a unique solution. For example, by taking the following $\mathbf{A}$, a 6×7 matrix, and solving $\mathbf{A}\ \mathbf{x} = \mathbf{b}$ for $\mathbf{x} = [O_x\ O_y\ O_z; G_x\ G_y\ G_z\ T]^T$ with the same $\mathbf{b}$ as input above, we get the solution as shown below.

```
A = [1 0 0 1 0 0 d/l
     0 1 0 0 1 0 b/l
     0 0 1 0 0 1 e/l
     0 0 1 0 0 0 c/l
     0 0 0 0 0 0 d/l
     1 0 0 0 0 0 d/l]
b = [m*a, 0, m*g, m*g/2, m*a/2, m*a/2]'

{Solve A x = b for x}
x =                  % this is the computer output
         0
   -3.0000
   97.0000
    6.0000
         0
  102.0000
    6.7823
```

This is the same solution as we got before except that it includes $G_y = 0$ in the solution. Now, if we add a vector $\Delta\ \mathbf{x} = [0\ \alpha\ 0\ 0\ -\alpha\ 0\ 0]^T$ to $\mathbf{x}$ where α is any number, and compute $\mathbf{A}\ (\mathbf{x}+\Delta\mathbf{x})$, we get back $\mathbf{b}$. That is, the six equilibrium conditions are satisfied irrespective of the actual values of O_y and G_y as long as the value of $O_y + G_y$ remains the same.

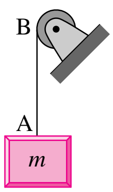
Problems for Chapter 12

One dimensional constrained dynamics

12.1 1D constrained motion and pulleys

Preparatory Problems

12.1 A motor at B allows the block of mass $m = 3\,\mathrm{kg}$ shown in the figure to accelerate downwards at $2\,\mathrm{m/s^2}$. There is gravity. What is the tension in the string AB?

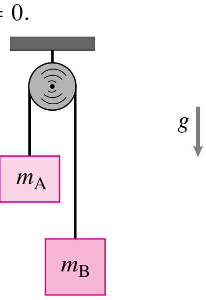


Filename:pfigure-blue-12-2

12.2 Two masses connected by an inextensible string hang from an ideal pulley.

a) Find the downward acceleration of mass B. Answer in terms of any or all of m_A, m_B, g, and the present velocities of the blocks. As a check, your answer should give $a_B = g$ when $m_A = 0$ and $a_B = 0$ when $m_A = m_B$.   .

b) Find the tension in the string. As a check, your answer should give $T = m_B g = m_A g$ when $m_A = m_B$ and $T = 0$ when $m_A = 0$.

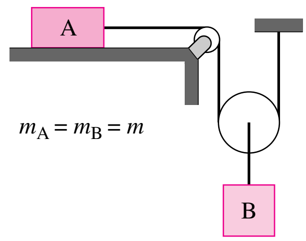


Filename:pfigure3-f95p1p2

12.3 The blocks shown are released from rest. Make reasonable assumptions about strings, pulleys, string lengths, and gravity.

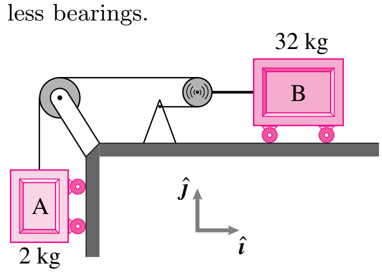
a) What is the acceleration of block A at $t = 0^+$ (just after release)?

b) What is the speed of block B after it has fallen 2 meters?



Filename:pfigure-blue-29-2

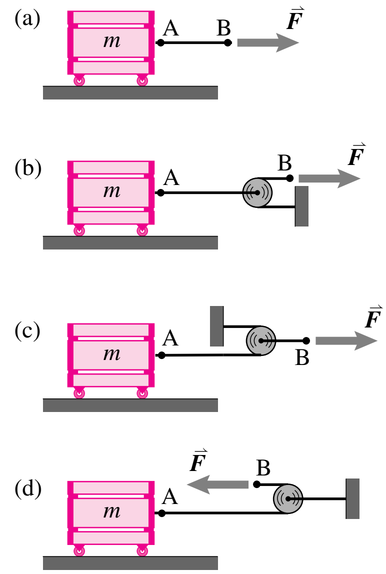
12.4 What is the acceleration of block A? Use $g = 10\,\mathrm{m/s^2}$. Assume the string is massless and that the pulleys are massless, round, and have frictionless bearings.



Filename:pfigure-f93q4

12.5 For the system shown in problem 12.2, find the acceleration of mass B using energy balance ($P = \dot{E}_K$).

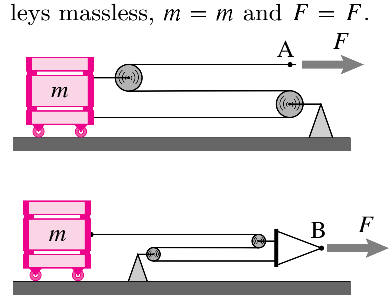
12.6 For the various situations pictured, find the acceleration of mass A and point B shown using balance of linear momentum. Define any variables, coordinates or sign conventions that you need to do your calculations and to define your solution.



Filename:pulley1

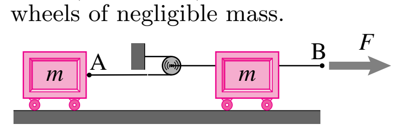
12.7 For each of the various situations pictured in problem 12.6 find the acceleration of the mass using energy balance ($P = \dot{E}_K$). Define any variables, coordinates, or sign conventions that you need to do your calculations and to define your solution.

12.8 What is the ratio of the acceleration of point A to that of point B in each configuration? In both cases, the strings are inextensible, the pulleys massless, $m = m$ and $F = F$.



Filename:sum95-p1-p3

12.9 Find the acceleration of points A and B in terms of F and m. Assume that the carts stay on the ground, have good (frictionless) bearings, and have wheels of negligible mass.

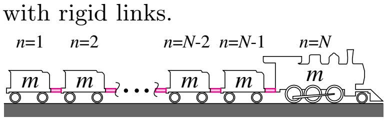


Filename:pfigure-s94q5p1

12.10 For the situation pictured in problem 12.9 find the accelerations of the two masses using energy balance ($P = \dot{E}_K$). Define any variables, coordinates, or sign conventions that you need to do your calculations and to define your solution.

More-Involved Problems

12.11 A train engine of mass m pulls and accelerates N cars, each of mass m. The power of the engine is P_t and its speed is v_t. Find the tension T_n between car n and car $n+1$. Assume there is no resistance and the ground is level. Assume the cars are connected with rigid links.



Filename:pfigure-newtrain

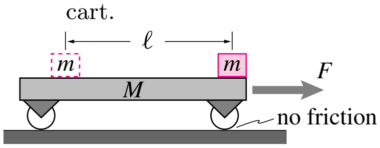
12.12 A cart of mass M, initially at rest, can move horizontally along a frictionless track. When $t = 0$, a force F is applied as shown to the cart. During the acceleration of M by the force F, a small box of mass m slides along the cart from the front to the rear. The coefficient of friction between the cart and the box is μ, and it is assumed that the acceleration of the cart is sufficient to cause sliding.

a) Draw free body diagrams of the cart, the box, and the cart and box together.

b) Write the equation of linear momentum balance for the cart, the box, and the system of cart and box.

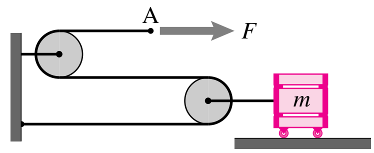
c) Show that the equations of motion for the cart and box can be combined to give the equation of motion of the mass center of the system of two bodies.

d) Find the displacement of the cart at the time when the box has moved a distance ℓ along the cart.



Filename:pfigure-blue-28-1

12.13 For the mass and pulley system shown in the figure, the point of application A of the force moves twice as fast as the mass. At some instant in time t, the speed of the mass is $\dot{x}$ to the left. Find the input power to the system at time t.



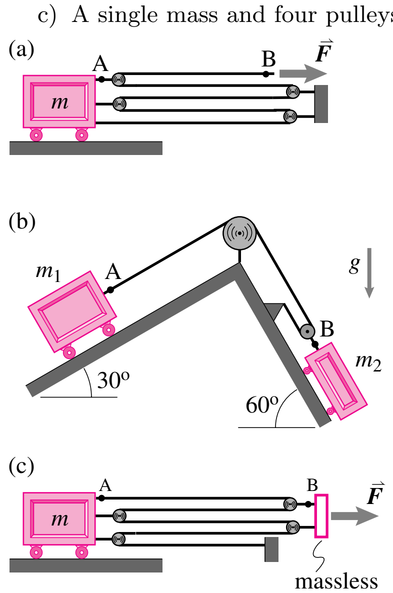
Filename:pfig2-3-rp8

12.14 For the various situations pictured, find the acceleration of mass A and point B shown using balance of linear momentum. Define any variables, coordinates or sign conventions that you need to do your calculations and to define your solution.

a) A single mass and four pulleys.

b) Two masses and two pulleys.

c) A single mass and four pulleys.



Filename:pulley4

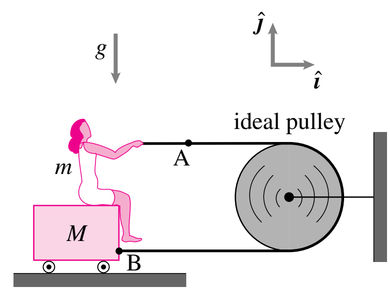
12.15 For the various situations pictured in problem 12.14, find the acceleration of the mass using energy balance ($P = \dot{E}_K$). Define any variables, coordinates, or sign conventions that you need to do your calculations and to define your solution.

12.16 A person of mass m, modeled as a rigid body is sitting on a cart of mass $M > m$ and pulling the massless inextensible string towards herself. The coefficient of friction between her seat and the cart is μ. All wheels and pulleys are massless and frictionless. Point B is attached to the cart and point A is attached to the rope.

a) If you are given that she is pulling rope in with acceleration a_0 relative to herself (that is, $\vec{a}_{A/B} \equiv \vec{a}_A - \vec{a}_B = -a_0\hat{\imath}$) and that she is not slipping relative to the cart, find $\vec{a}_A$. (Answer in terms of some or all of $m, M, g, \mu, \hat{\imath}$ and a_0.)

b) Find the largest possible value of a_0 without the person slipping off the cart? (Answer in terms of some or all of m, M, g and μ. You may assume her legs get out of the way if she slips backwards.)

c) If instead, $m < M$, what is the largest possible value of a_0 without the person slipping off the cart? (Answer in terms of some or all of m, M, g and μ. You may assume her legs get out of the way if she slips backwards.)



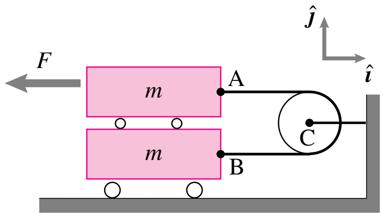
Pulley.

Filename:s97p2-2

12.17 Two blocks and a pulley. Two identical blocks are stacked and tied together by the pulley as shown. All bearings are frictionless. All rotating parts have negligible mass. Find

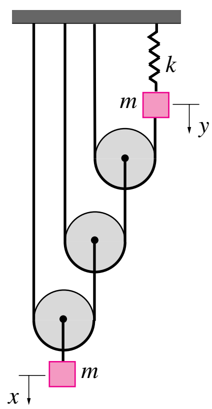
a) the acceleration of point A, and

b) the tension in the line.



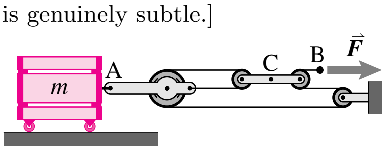
Filename:p-s96-p1-1

12.18 The pulleys are massless and frictionless. Include gravity. x measures the vertical position of the lower mass from equilibrium. y measures the vertical position of the upper mass from equilibrium. What is the natural frequency of vibration of this system?



Filename:pfigure-s95q4

12.19 For the situation pictured, find the acceleration of mass A and points B and C shown. [Hint: the situation with point C is tricky and the answer is genuinely subtle.]



Filename:pulley2

12.20 For the situation pictured in problem 12.19, find the acceleration of point A using energy balance ($P = \dot{E}_K$). Define any variables, coordinates, or sign conventions that you need to do your calculations and to define your solution.

12.21 Design a pulley system. You are to design a pulley system to move a mass. There is no gravity. Point A has a force $\vec{F} = F\hat{\imath}$ pulling it to the right. Mass B has mass m_B. You can connect point A to the mass with any number of ideal strings and ideal pulleys. You can make use of rigid walls or supports anywhere you like (say, to the right or left of the mass). You must design the system so that mass B accelerates to the left with $\frac{F}{2m_B}$ (i.e., $\vec{a}_B = -\frac{F}{2m_B}\hat{\imath}$).

a) Draw the system clearly. Justify your answer with enough words or equations so that a reasonable person, say a grader, can tell that you understand your solution.

b) Find the acceleration of point A.

12.22 Design a pulley system. You are to design a pulley system to move a mass. There is no gravity. Point A has a force $\vec{F} = F\hat{\imath}$ pulling it to the right. Mass B has mass m_B. You can connect the point A to the mass with any number of ideal strings and ideal pulleys. You can make use of rigid walls or supports anywhere you like (say, to the right or left of the mass). Draw the system clearly. Justify your answer with enough words or equations to convince a skeptical person that your solution is correct. You must design the system so that the mass B accelerates .

a) to the left with $\frac{F}{m_B}$ (i.e., $\vec{a}_B = -\frac{F}{m_B}\hat{\imath}$)

b) to the left with $\frac{2F}{m_B}$

c) to the left with $\frac{F}{2m_B}$

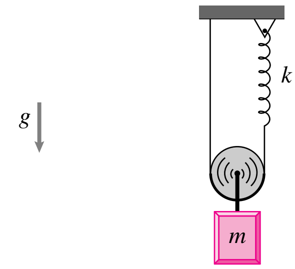
d) to the right with $\frac{2F}{m_B}$

e) to the right with $\frac{F}{2m_B}$

f) to the left with $\frac{8F}{m_B}$

g) to the right with $\frac{F}{5m_B}$

12.23 Pulley and spring. For the hanging mass find the period of oscillation. Assume a massless pulley with good bearings. The massless string is inextensible. Only vertical motion is of interest. There is gravity.

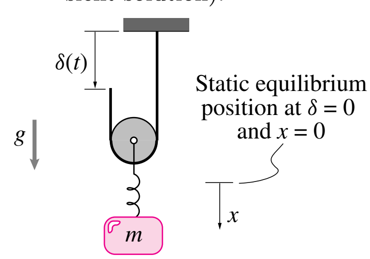


Filename:pfigure-s94h4p4

12.24 The spring-mass system shown (m = 10 slugs ($\equiv$ lb $\cdot\, sec^2/ft$), $k = 10\,\text{lb}/ft$) is excited by moving the free end of the cable vertically according to $\delta(t) = 4\sin(\omega t)$ in, as shown in the figure. Assuming that the cable is inextensible and massless and that the pulley is massless, do the following.

a) Derive the equation of motion for the block in terms of the displacement x from the static equilibrium position, as shown in the figure.

b) If $\omega = 0.9$ rad/s, check to see if the pulley is always in contact with the cable (ignore the transient solution).



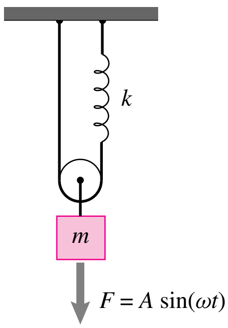
Filename:pfigure-blue-151-1

12.25 The block of mass m hanging on the spring with constant k and a string shown in the figure is forced by $F = A\sin(\omega t)$. Do not neglect gravity. The pulley is massless.

a) What is the differential equation governing the motion of the block? You may assume that the only motion is vertical motion.

b) Given A, m and k, for what values of ω would the string go slack at some point in the cyclical motion?
(The common assumption in such problems, which you can

use, is to neglect the homogeneous solution to the differential equation. It is assumed that the damping, small enough to be neglected in the governing equations is large enough so that the particular solution will have damped out at the time of observation.)



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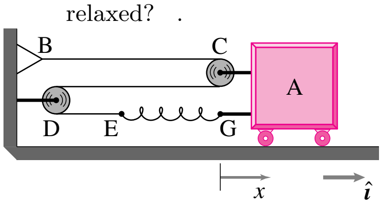
12.26 Block A, with mass m_A, is pulled to the right a distance d from the position it would have if the spring were relaxed. It is then released from rest. Assume ideal string, pulleys and wheels. The spring has constant k.

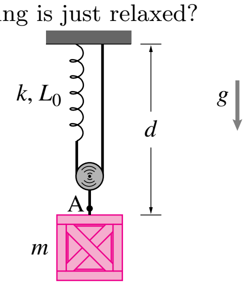
a) What is the acceleration of block A just after it is released (in terms of k, m_A, and d)?

b) What is the speed of the mass when the mass passes through the position where the spring is relaxed? .



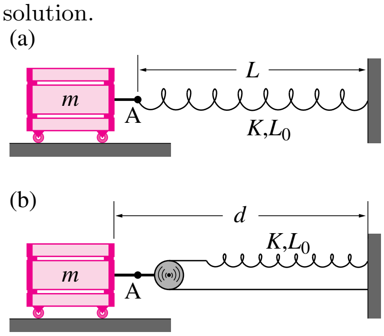
Filename:pfigure-f93q5

12.27 What is the static displacement of the mass from the position where the spring is just relaxed?



Filename:pulley3-a

12.28 For the two situations pictured, find the acceleration of point A shown using balance of linear momentum ($\sum \vec{F} = m\vec{a}$). Assuming both masses are deflected an equal distance from the position where the spring is just relaxed, how much smaller or bigger is the acceleration of block (b) than that of block (a). Define any variables, coordinate system origins, coordinates or sign conventions that you need to do your calculations and to define your solution.



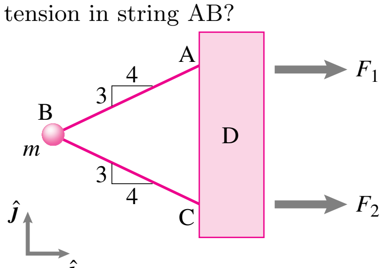
Filename:pulley3

12.29 For each of the situations pictured in problem 12.28, find the acceleration of the mass using energy balance ($P = \dot{E}_K$). Define any variables, coordinates, or sign conventions that you need to do your calculations and to define your solution.

12.2 1D motion with 2D and 3D forces

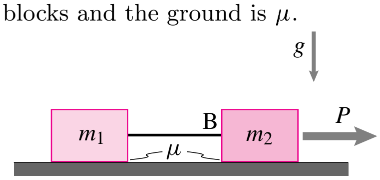
Preparatory Problems

12.30 Mass pulled by two strings. F_1 and F_2 are applied so that the system shown accelerates to the right at $5\,\text{m/s}^2$ (*i.e.*, $\boldsymbol{a} = 5\,\text{m/s}^2\hat{\imath} + 0\hat{\jmath}$) and has no rotation. The mass of D and forces F_1 and F_2 are unknown. What is the tension in string AB?



Filename:pg9-1

12.31 The two blocks, $m_1 = m_2 = m$, are connected by an inextensible string AB. The string can only withstand a tension T_{cr}. Find the maximum value of the applied force P so that the string does not break. The sliding coefficient of friction between the blocks and the ground is μ.



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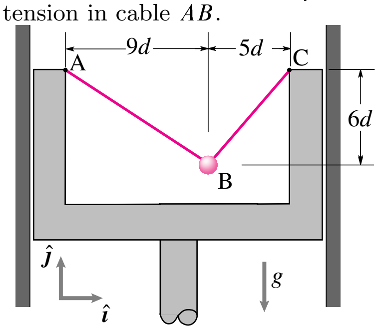
12.32 A point mass m is attached to a piston by two inextensible cables. The piston has upwards acceleration of $a_y\hat{\jmath}$. There is gravity. In terms of some or all of m, g, d, and a_y find the tension in cable AB.

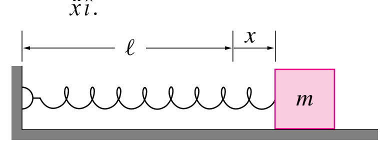


Filename:ch3-13

12.33 A point mass of mass m moves on a frictional surface with coefficient of friction μ and is connected to a spring with constant k and unstretched length ℓ. There is gravity. At the instant of interest, the mass is at a distance x to the right from its position where the spring is unstretched and is moving with $\dot{x} > 0$ to the right.

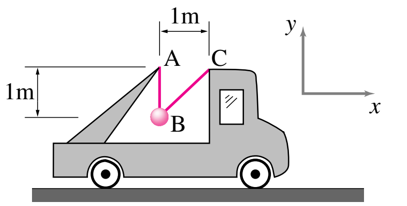
a) Draw a free body diagram of the mass at the instant of interest.

b) At the instant of interest, write the equation of linear momentum balance for the block evaluating the left hand side as explicitly as possible. Let the acceleration of the block be $\vec{a} = \ddot{x}\hat{\imath}$.



Filename:ch2-10a

12.34 Consider the mass at B (2 kg) supported by two strings in the back of a truck which has acceleration of $3\,\mathrm{m/s^2}$. Use $g = 10\,\mathrm{m/s^2}$. What is the tension T_{AB} in the string AB in Newtons?



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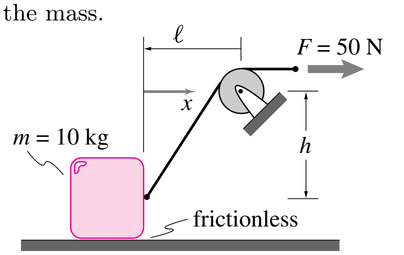
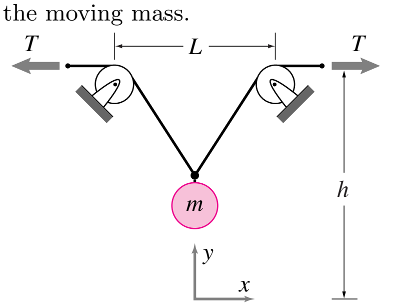
12.35 At the instant shown, the mass is moving to the right at speed $v = 3\,\mathrm{m/s}$. Find the rate of work done on the mass.



Filename:pfig2-3-rp9

12.36 A point mass 'm' is pulled straight up by two strings. The two strings pull the mass symmetrically about the vertical axis with constant and equal force T. At an instant in time t, the position and the velocity of the mass are $y(t)\hat{\jmath}$ and $\dot{y}(t)\hat{\jmath}$, respectively. Find the power input to the moving mass.



Filename:pfig2-3-rp2

More-Involved Problems

12.37 Two blocks, each of mass m, are connected together across their tops by a massless string of length S; the blocks' dimensions are small compared to S. They slide down a slope of angle θ. Do not neglect gravity but do neglect friction.

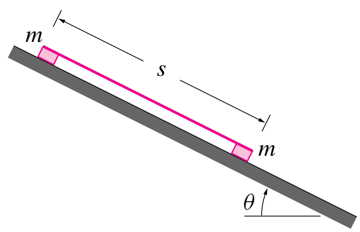
a) Draw separate free body diagrams of each block, the string, and the system of the two blocks and string.

b) Write separate equations for linear momentum balance for each block, the string, and the system of blocks and string.

c) What is the acceleration of the center of mass of the two blocks?

d) What is the force in the string?

e) What is the speed of the center of mass for the two blocks after they have traveled a distance d down the slope, having started from rest. [Hint: You need to dot your momentum balance equations with a unit vector along the ramp in order to reduce this problem to a problem in one dimensional mechanics.]



Filename:pfigure-blue-27-1

12.38 Two blocks, each of mass m, are connected together across their tops by a massless string of length S; the blocks' dimensions are small compared to S. They slide down a slope of angle θ. The materials are such that the coefficient of dynamic friction on the top block is μ and on the bottom block is $\mu/2$.

a) Draw separate free body diagrams of each block, the string, and the system of the two blocks and string.

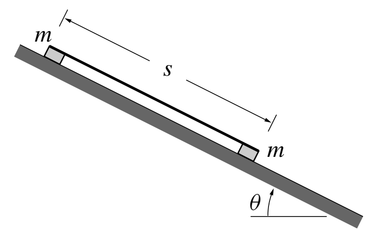
b) Write separate equations for linear momentum balance for each block, the string, and the system of blocks and string.

c) What is the acceleration of the center of mass of the two blocks?

d) What is the force in the string?

e) What is the speed of the center of mass for the two blocks after they have traveled a distance d down the slope, having started from rest.

f) How would your solutions to parts (a) and (c) differ in the following two variations: i.) If the two blocks were interchanged with the slippery one on top or ii.) if the string were replaced by a massless rod? Qualitative responses to this part are sufficient.



Filename:pfigure-blue-27-1a

12.39 Coin on a car on a ramp. A student engineering design course asked students to build a cart (mass = m_c) that rolls down a ramp with angle θ. A small weight (mass $m_w \ll m_c$) is placed on top of the cart on a surface tipped with respect to the cart (angle ϕ). Assume the small mass does not slide. Assume massless wheels with frictionless bearings. $\hat{\imath}$ is horizontal and $\hat{\jmath}$ is vertical up.

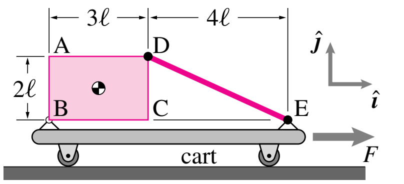
a) Find the acceleration of the cart. Answer in terms of some or all of $m_c, g, \hat{\imath}, \theta$ and $\hat{\jmath}$.

b) What coefficient of friction μ is required (the smallest that will work) to keep the small mass from sliding as the cart rolls down the slope? Answer in terms of some or all of m_c, m_w, g, θ, and ϕ.

c) What angle ϕ will allow a small mass to ride on the cart with the smallest coefficient of friction? Answer in terms of some or all of m_c, m_w, g, and θ.

12.40 Guyed plate on a cart A uniform rectangular plate $ABCD$ of mass m is supported by a rod DE and a hinge joint at point B. The dimensions are as shown. There is gravity.

What must the acceleration of the cart be in order for massless rod DE to be in tension?



Uniform plate supported by a hinge and a cable on an accelerating cart.

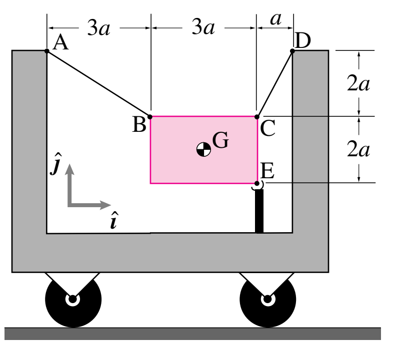
Filename:tfigure3-2D-a-guyed

12.41 A uniform rectangular plate of mass m is supported by two inextensible cables AB and CD and by a hinge at point E on the cart as shown. The cart has acceleration $a_x\hat{\imath}$ due to a force not shown. There is gravity.

a) Draw a free body diagram of the plate.

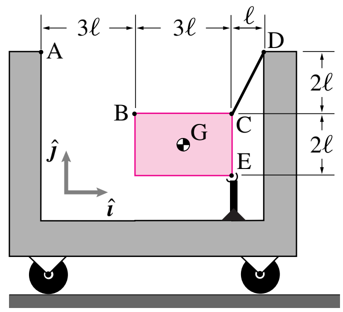
b) Write the equation of linear momentum balance for the plate and evaluate the left hand side as explicitly as possible.

c) Write the equation for angular momentum balance about point E and evaluate the left hand side as explicitly as possible.



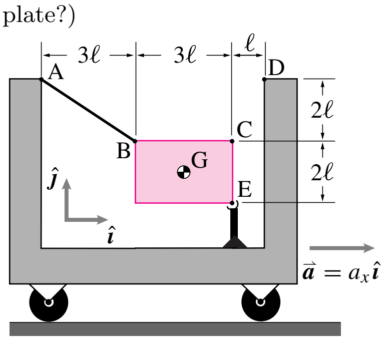
Filename:ch2-12

12.42 A uniform rectangular plate of mass m is supported by an inextensible cable CD and a hinge joint at point E on the cart as shown. The hinge joint is attached to a rigid column welded to the floor of the cart. The cart is at rest. There is gravity. Find the tension in cable CD.



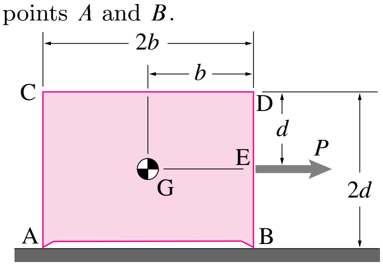
Filename:ch3-11-b

12.43 A uniform rectangular plate of mass m is supported by an inextensible cable AB and a hinge joint at point E on the cart as shown. The hinge joint is attached to a rigid column welded to the floor of the cart. The cart has acceleration $a_x\hat{\imath}$. There is gravity. Find the tension in cable AB. (What's 'wrong' with this problem? What if instead point B were at the bottom left hand corner of the plate?)



Filename:ch3-11a

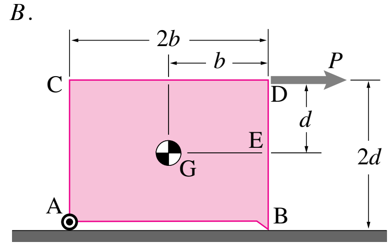
12.44 A block of mass m is sitting on a frictionless surface and acted upon at point E by the horizontal force P through the center of mass. Draw a free body diagram of the block. There is gravity. Find the acceleration of the block and reactions on the block at points A and B.



Filename:ch3-9

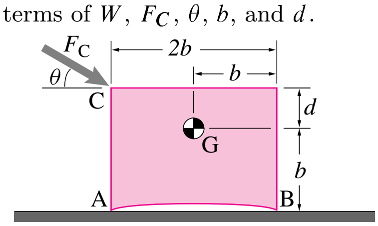
12.45 Reconsider the block in problem 12.44. This time, find the acceleration of the block and the reactions at A and B if the force P is applied instead at point D. Are the acceleration and the reactions on the block different from those found when P is applied at point E?

12.46 A block of mass m is sitting on a frictional surface and acted upon at point D by the horizontal force P. The block is resting on a sharp edge at point B and is supported by an ideal wheel at point A. There is gravity. Assuming the block is sliding with coefficient of friction μ at point B, find the acceleration of the block and the reactions on the block at points A and B.



Filename:ch3-12

12.47 A force F_C is applied to the corner C of a box of weight W with dimensions and center of gravity at G as shown in the figure. The coefficient of sliding friction between the floor and the points of contact A and B is μ. Assuming that the box slides when F_C is applied, find the acceleration of the box and the reactions at A and B in terms of W, F_C, θ, b, and d.

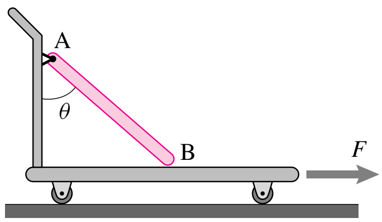


Filename:Mikes92p3

12.48 A uniform rod with mass m_r rests on a cart (mass m_c) which is being pulled to the right. The rod is hinged at one end (with a frictionless hinge) and has no friction at the contact with the cart. The cart is rolling on wheels that are modeled as having no mass and no bearing friction (ideal massless wheels). Answer in terms of g, m_r, m_c, θ and F. Find:

a) The force on the rod from the cart at point B.

b) The force on the rod from the cart at point A.



Filename:pfigure-s94h3p3

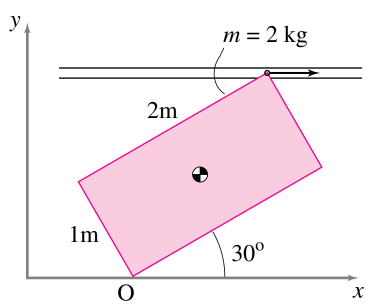
12.49 The box shown in the figure is dragged in the x-direction with a constant acceleration $\vec{a} = 0.5\,\mathrm{m/s^2}\,\hat{\imath}$. At the instant shown, the velocity of (every point on) the box is $\vec{v} = 0.8\,\mathrm{m/s}\hat{\imath}$.

a) Find the linear momentum of the box.

b) Find the rate of change of linear momentum of the box.

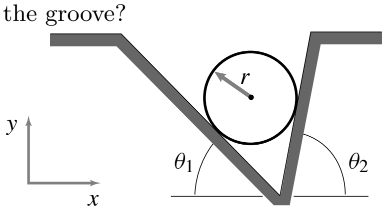
c) Find the angular momentum of the box about the contact point O.

d) Find the rate of change of angular momentum of the box about the contact point O.



Filename:pfigure3-mom-rp1

12.50 The groove and disk accelerate upwards, $\vec{a} = a\,\hat{\jmath}$. Neglecting gravity, what are the forces on the disk due to the groove?



Filename:ch2-5-ba

12.51 The following problems concern a box that is in the back of a pickup truck. The pickup truck is moving forward with acceleration of a_t. The truck's speed is v_t. The box has sharp feet at the front and back ends so the only place it contacts the truck is at the feet. The center of mass of the box is at the geometric center of the box. The box has height h, length ℓ and depth w (into the paper.) Its mass is m. There is gravity. The friction coefficient between the truck and the box edges is μ.

In the problems below you should express your solutions in terms of the variables given in the figure, ℓ, h, μ, m, g, a_t, and v_t. If any variables do not enter the expressions comment on why they do not. In all cases you may assume that the box does not rotate (though it might be on the verge of doing so).

a) Assuming the box does not slide, what is the total force that the truck exerts on the box (i.e. the sum of the reactions at A and B)?

b) Assuming the box does not slide what are the reactions at A and B? [Note: You cannot find both of them without additional assumptions.]

c) Assuming the box does slide, what is the total force that the truck exerts on the box?

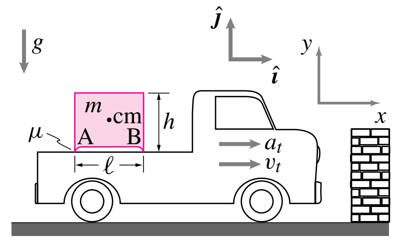
d) Assuming the box does slide, what are the reactions at A and B?

e) Assuming the box does not slide, what is the maximum acceleration of the truck for which the box will not tip over (hint: just at that critical acceleration what is the vertical reaction at B?)?

f) What is the maximum acceleration of the truck for which the block will not slide?

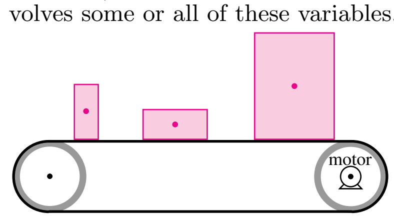
g) The truck hits a brick wall and stops instantly. Does the block tip over?

Assuming the block does not tip over, how far does it slide on the truck before stopping (assume the bed of the truck is sufficiently long)?



Filename:pfigure-blue-22-1

12.52 A collection of uniform boxes with various heights h and widths w and masses m sit on a horizontal conveyer belt. The acceleration $a(t)$ of the conveyer belt gets extremely large sometimes due to an erratic overpowered motor. Assume the boxes touch the belt at their left and right edges only and that the coefficient of friction there is μ. It is observed that some boxes never tip over. What is true about μ, g, w, h, and m for the boxes that always maintain contact at both the right and left bottom edges? (Write an inequality that involves some or all of these variables.)

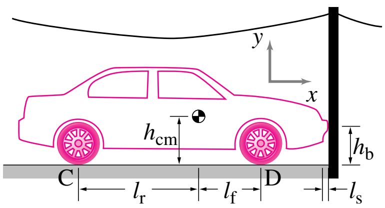


Filename:pfigure-f93q3

12.53 After failure of her normal brakes, a driver pulls the emergency brake of her old car. This action locks the rear wheels (friction coefficient = μ) but leaves the well lubricated and light front wheels spinning freely. The car, braking inadequately as is the case for rear wheel braking, hits a stiff and slippery phone pole which compresses the car bumper. The car bumper is modeled here as a linear spring (constant = k, rest length = l_0, present length = l_s). The car is still traveling forward at the moment of interest. The bumper is at a height h_b above the ground. Assume that the car, excepting the bumper, is a non-rotating rigid body and that the wheels remain on the ground (that is, the bumper is compliant but the suspension is stiff).

- What is the acceleration of the car in terms of g, m, μ, l_f, l_r,

k, h_b, h_{cm}, l_0, and l_s (and any other parameters if needed)?



Filename:pfigure-s94q4p1

12.54 Car braking: front brakes versus rear brakes versus all four brakes. There are a few puzzles in dynamics concerning the differences between front and rear braking of a car. Here is one you can deal with now. What is the peak deceleration of a car when you apply: the front brakes till they skid, the rear brakes till they skid, and all four brakes till they skid? Assume that the coefficient of friction between rubber and road is $\mu = 1$ (about right, the coefficient of friction between rubber and road varies between about .7 and 1.3) and that $g = 10\,\text{m/s}^2$ (2% error). Pick the dimensions and mass of the car, but assume the center of mass height h is above the ground. The height h, should be less than half the wheel base w, the distance between the front and rear wheel. Further assume that the *CM* is halfway between the front and back wheels (*i.e.*, $l_f = l_r = w/2$). Assume also that the car has a stiff suspension so the car does not move up or down or tip during braking; i. e., the car does not rotate in the xy-plane. Neglect the mass of the rotating wheels in the linear and angular momentum balance equations. Treat this problem as two-dimensional problem; i. e., the car is symmetric left to right, does not turn left or right, and that the left and right wheels carry the same loads. To organize your work, here are some steps to follow.

a) Draw a FBD of the car assuming rear wheel is skidding. The FBD should show the dimensions, the gravity force, what you know *a priori* about the forces on the wheels from the ground (i.e., that the friction force $F_r = \mu N_r$, and that there is no friction at the front wheels), and the coordinate directions. Label points of interest that you will use in your momentum balance equations. (Hint: also draw a free body diagram of the rear wheel.)

b) Write the equation of linear momentum balance.

c) Write the equation of angular momentum balance relative to a point of your choosing. Some particularly useful points to use are:
   - the point above the front wheel and at the height of the center of mass;
   - the point at the height of the center of mass, behind the rear wheel that makes a 45 degree angle line down to the rear wheel ground contact point; and
   - the point on the ground straight under the front wheel that is as far below ground as the wheel base is long.

d) Solve the momentum balance equations for the wheel contact forces and the deceleration of the car. If you have used any or all of the recommendations from part (c) you will have the pleasure of only solving one equation in one unknown at a time.

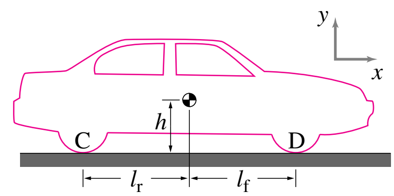
e) Repeat steps (a) to (d) for front-wheel skidding. Note that the advantageous points to use for angular momentum balance are now different. Does a car stop faster or slower or the same by skidding the front instead of the rear wheels? Would your solution to (e) be different if the center of mass of the car were at ground level(h=0)?

f) Repeat steps (a) to (d) for all-wheel skidding. There are some shortcuts here. You determine the car deceleration without ever knowing the wheel reactions (or using angular momentum balance) if you look at the linear momentum balance equations carefully.

g) Does the deceleration in (f) equal the sum of the decelerations in (d) and (e)? Why or why not?

h) What peculiarity occurs in the solution for front-wheel skidding if the wheel base is twice the height of the CM above ground and $\mu = 1$?

i) What impossibility does the solution predict if the wheel base is shorter than twice the CM height? What wrong assumption gives rise to this impossibility? What would really happen if one tried to skid a car this way?



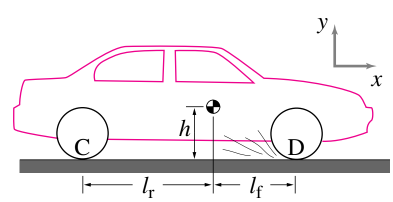
Filename:pfigure-s94h3p6

12.55 Assuming massless wheels, an infinitely powerful engine, a stiff suspension (*i.e.*, no rotation of the car) and a coefficient of friction μ between tires and road,

a) what is the maximum forward acceleration of this front wheel drive car?

b) what is the force of the ground on the rear wheels during this acceleration?

c) what is the force of the ground on the front wheels?



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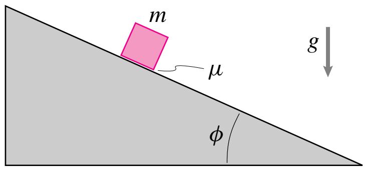
12.56 At time $t = 0$, the block of mass m is released from rest on the slope of angle ϕ. The coefficient of friction between the block and slope is μ.

a) What is the acceleration of the block for $\mu > 0$?

b) What is the acceleration of the block for $\mu = 0$?

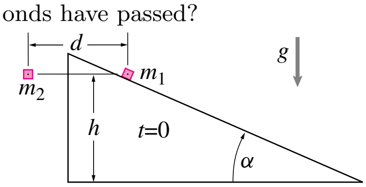
c) Find the position and velocity of the block as a function of time for $\mu > 0$.

d) Find the position and velocity of the block as a function of time for $\mu = 0$.



Filename:Danef94s1q5

12.57 A small block of mass m_1 is released from rest at altitude h on a frictionless slope of angle α. At the instant of release, another small block of mass m_2 is dropped vertically from rest at the same altitude. The second block does not interact with the ramp. What is the velocity of the first block relative to the second block after t seconds have passed?



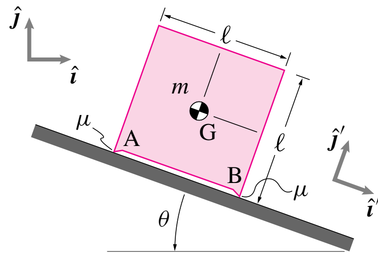
Filename:ch8-7

12.58 Block sliding on a ramp with friction. A square box is sliding down a ramp of angle θ with instantaneous velocity $v\hat{\imath}'$. Assume it does not tip over.

a) What is the force on the block from the ramp at point A? Answer in terms of any or all of θ, ℓ, m, g, μ, v, $\hat{\imath}'$, and $\hat{\jmath}'$. As a check, your answer should reduce to $\frac{mg}{2}\hat{\jmath}'$ when $\theta = \mu = 0$.

b) In addition to solving the problem by hand, see if you can write a set of computer commands that, if θ, μ, ℓ, m, v and g were specified, would give the correct answer.

c) Assuming $\theta = 80°$ and $\mu = 0.9$, can the box slide this way or would it tip over? Why?



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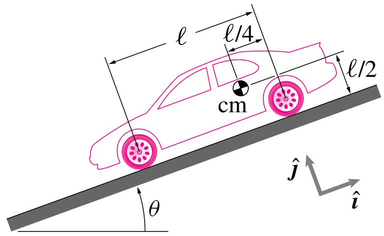
12.59 A coin is given a sliding shove up a ramp with angle ϕ with the horizontal. It takes twice as long to slide down as it does to slide up. What is the coefficient of friction μ between the coin and the ramp. Answer in terms of some or all of m, g, ϕ and the initial sliding velocity v.

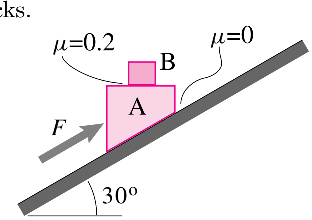
12.60 A skidding car. What is the braking acceleration of the front-wheel braked car as it slides down hill. Express your answer as a function of any or all of the following variables: the slope θ of the hill, the mass of the car m, the wheel base ℓ, and the gravitational constant g. Use $\mu = 1$.



A car skidding downhill on a slope of angle θ

Filename:pfigure3-car

12.61 Two blocks A and B are pushed up a frictionless inclined plane by an external force F as shown in the figure. The coefficient of friction between the two blocks is $\mu = 0.2$. The masses of the two blocks are $m_A = 5\,\text{kg}$ and $m_B = 2\,\text{kg}$. Find the magnitude of the maximum allowable force such that no relative slip occurs between the two blocks.



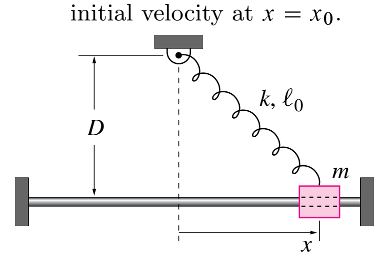
Filename:summer95f-1

12.62 A bead slides on a frictionless rod. The spring has constant k and rest length ℓ_0. The bead has mass m.

a) Given x and $\dot{x}$ find the acceleration of the bead (in terms of some or all of $D, \ell_0, x, \dot{x}, m, k$ and any base vectors that you define).

b) If the bead is allowed to move, as constrained by the slippery rod and the spring, find a differential equation that must be satisfied by the variable x. (Do not try to solve this somewhat ugly non-linear equation.)

c) In the special case that $\ell_0 = 0$* find how long it takes for the block to return to its starting position after release with no initial velocity at $x = x_0$.



Filename:p-s96-p1-2

12.63 A bead oscillates on a straight frictionless wire. The spring obeys the equation $F = k\,(\ell - \ell_0)$, where ℓ = length of the spring and ℓ_0 is the 'rest' length. Assume

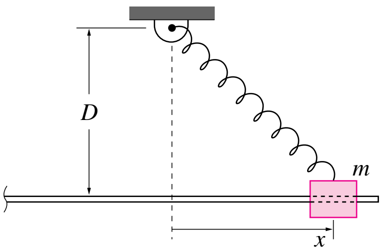
$$x(t = 0) = x_0, \;\; \dot{x}(t = 0) = 0.$$

a) Write a differential equation satisfied by $x(t)$.

b) What is $\dot{x}$ when $x = 0$? [hint: Don't try to solve the equation in (a)!]

c) What is the simplification in (a) if $\ell_o = 0$ (spring is then a so-called "zero–length" spring).

d) For this special case ($\ell_o = 0$) solve the equation in (a) and show the result agrees with (b) in this special case.



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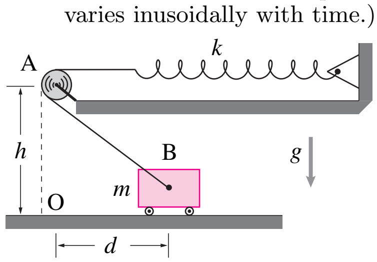
12.64 A cart on an elastic leash. A cart B (mass m) rolls on a frictionless level floor. One end of an inextensible string is attached to the the cart. The string wraps around a pulley at point A and the other end is attached to a spring with constant k. When the cart is at point O, it is in static equilibrium. The spring relaxed length, rope length, and room height h are such that the spring would be relaxed if the end of rope at B were diconnected from the cart and brought up to point A. The gravitational constant is g. The cart is pulled a horizontal distance d from the center of the room (at O) and released.

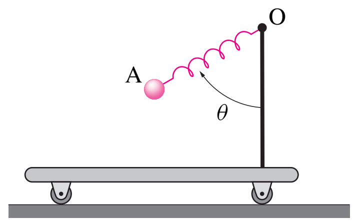
a) Assuming that the cart never leaves the floor, what is the speed of the cart when it passes through the center of the room, in terms of m, h, g and d.

b) Does the cart undergo simple harmonic motion for small or large oscillations (specify which if either)? (Simple harmonic motion occurs when position varies inusoidally with time.)



Filename:cartosc

12.65 The cart moves to the right with constant acceleration a. The ball has mass m. The spring has unstretched length ℓ_0 and spring constant k. Assuming the ball is stationary with respect to the cart find the distance from O to A in terms of k, ℓ_0, and a . [Hint: find θ first.]



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12.66 Consider a person, modeled as a rigid body, riding an accelerating motorcycle (2-D). The person is sitting on the seat and cannot slide fore or aft, but is free to rock in the plane of the motorcycle (as if there were a hinge connecting the motorcycle to the rider at the seat). The person's feet are off the pegs and the legs are sticking down and not touching anything. The person's arms are like cables (they are massless and only carry tension). Assume all dimensions and masses are known (you have to define them carefully with a sketch and words). Assume the forward acceleration of the motorcycle is known. You may use numbers and/or variables to describe the quantities of interest.

a) Draw a clear sketch of the problem showing needed dimensional information and the coordinate system you will use.

b) Draw a Free Body Diagram of the rider.

c) Write the equations of linear and angular momentum balance for the rider.

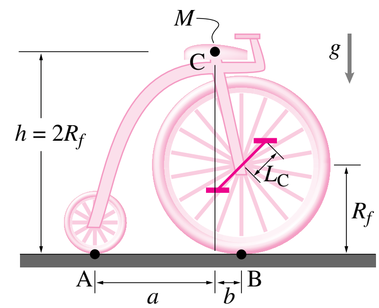
d) Find all forces on the rider from the motorcycle (i.e., at the hands and the seat).

e) What are the forces on the motorcycle from the rider?

12.67 Acceleration of a bicycle on level ground. 2-D . A very compact bicycler (modeled as a point mass M at the bicycle seat C with height h, and distance b behind the front wheel contact), rides a very light old-fashioned bicycle (all components have negligible mass) that is well maintained (all bearings have no frictional torque) and streamlined (neglect air resistance). The rider applies a force F_p to the pedal perpendicular to the pedal crank (with length L_c). No force is applied to the other pedal. The radius of the front wheel is R_f.

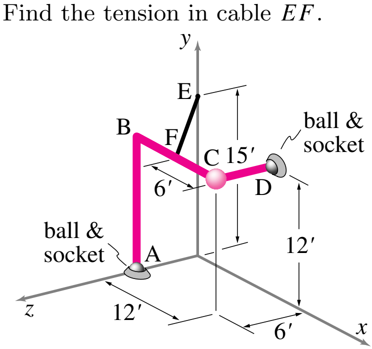
a) Assuming no slip, what is the forward acceleration of the bicycle? [Hint: draw a FBD of the front wheel and crank, and another FBD of the whole bicycle-rider system.]

b) (Harder) Assuming the rider can push arbitrarily hard but that $\mu = 1$, what is the maximum possible forward acceleration of the bicycle.



Filename:f-3

12.68 A 320 lbm mass is attached at the corner C of a light rigid piece of pipe bent as shown. The pipe is supported by ball-and-socket joints at A and D and by cable EF. The points A, D, and E are fastened to the floor and vertical sidewall of a pick-up truck which is accelerating in the z-direction. The acceleration of the truck is $\vec{a} = 5\,\text{ft/s}^2\hat{k}$. There is gravity. Find the tension in cable EF.

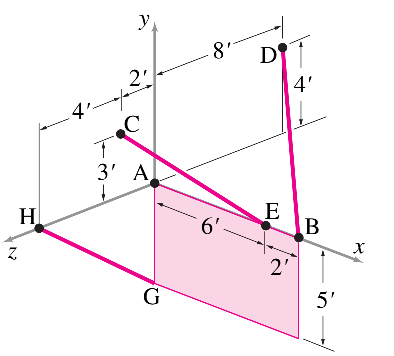


Filename:mikef91p1

12.69 A 5 ft by 8 ft rectangular plate of uniform density has mass $m = 10\,\text{lbm}$ and is supported by a ball-and-socket joint at point A and the light rods CE, BD, and GH. The entire system is attached to a truck which is moving with acceleration $\vec{a}_T$. The plate is moving without rotation or angular acceleration relative to the truck. Thus, the center of mass acceleration of the plate is the same as the truck's. Dimensions are as shown. Points A, C, and D are fixed to the truck but the truck is not touching the plate at any other points. Find the tension in rod BD.

a) If the truck's acceleration is $\boldsymbol{a}_{cm} = (5\,\mathrm{ft/s^2})\boldsymbol{k}$, what is the tension or compression in rod BD?

b) If the truck's acceleration is $\vec{\boldsymbol{a}}_{\mathrm{cm}} = (5\,\mathrm{ft/s^2})\hat{\jmath} + (6\,\mathrm{ft/s^2})\hat{\boldsymbol{k}}$, what is the tension or compression in rod GH?



Filename:Mikesp92p1

12.70 Hanging a shelf. A shelf with negligible mass supports a 0.5 kg mass at its center. The shelf is supported at one corner with a ball and socket joint and at the other three corners with strings. At the moment of interest the shelf is in a rocket in outer space and accelerating at $10\,\mathrm{m/s^2}$ in the $\boldsymbol{k}$ direction. The shelf is in the xy plane.

a) Draw a FBD of the shelf.

b) Challenge: without doing any calculations on paper can you find one of the reaction force components or the tension in any of the cables? Give yourself a few minutes of staring to try this approach. If you can't, then come back to this question after you have done all the calculations.

c) Write down the linear momentum balance equation (a vector equation).

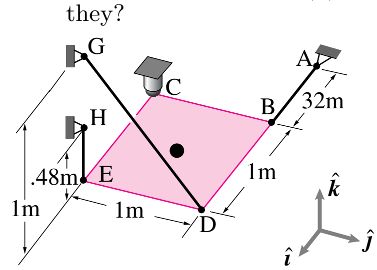
d) Write down the angular momentum balance equation using the center of mass as a reference point.

e) By taking components, turn (b) and (c) into six scalar equations in six unknowns.

f) Solve these equations by hand or on the computer.

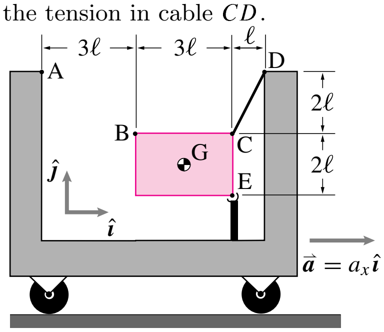
g) Instead of using a system of equations try to find a single equation which can be solved for T_{EH}. Solve it and compare to your result from before.

h) Challenge: For how many of the reactions can you find one equation which will tell you that particular reaction without knowing any of the other reactions? [Hint, try angular momentum balance about various axes as well as linear momentum balance in an appropriate direction. It is possible to find five of the six unknown reaction components this way.] Must these solutions agree with (d)? Do they?



Filename:pfigure-s94h2p10

12.71 A uniform rectangular plate of mass m is supported by an inextensible cable CD and a hinge joint at point E on the cart as shown. The hinge joint is attached to a rigid column welded to the floor of the cart. The cart has acceleration $a_x\hat{\imath}$. There is gravity. Find the tension in cable CD.

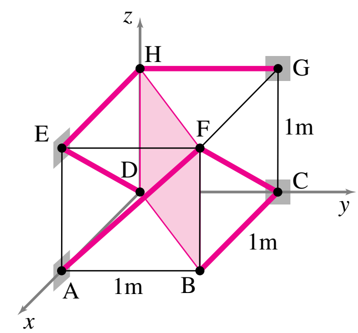


Filename:ch3-11

12.72 The uniform 2 kg plate DBFH is held by six massless rods (AF, CB, CF, GH, ED, and EH) which are hinged at their ends. The support points A, C, G, and E are all accelerating in the x-direction with acceleration $\boldsymbol{a} = 3\,\mathrm{m/s^2}\imath$. There is no gravity.

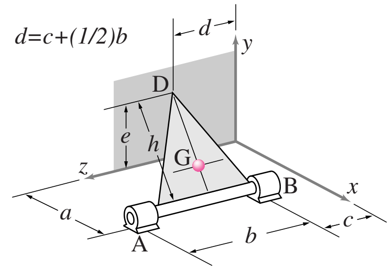
a) What is $\{\sum \vec{F}\}\cdot\hat{\imath}$ for the forces acting on the plate?

b) What is the tension in bar CB?



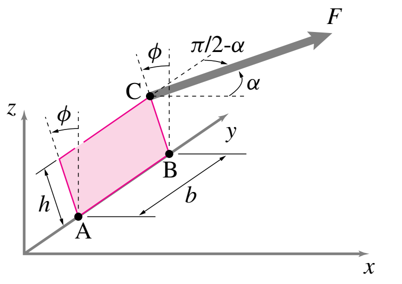
Filename:pfigure-s94q3p1

12.73 A massless triangular plate rests against a frictionless wall of a pick-up truck at point D and is rigidly attached to a massless rod supported by two ideal bearings fixed to the floor of the pick-up truck. A ball of mass m is fixed to the centroid of the plate. There is gravity. The pick-up truck skids across a road with acceleration $\vec{\boldsymbol{a}} = a_x\hat{\imath} + a_z\hat{\boldsymbol{k}}$. What is the reaction at point D on the plate?



Filename:ch3-1a

12.74 Towing a bicycle. A bicycle on the level xy plane is steered straight ahead and is being towed by a rope. The bicycle and rider are modeled as a uniform plate with mass m (for the convenience of the artist). The tow force F applied at C has no z component and makes an angle α with the x axis. The rolling wheel contacts are at A and B. The bike is tipped an angle ϕ from the vertical. The towing force F is the magnitude needed to keep the bike accelerating in a straight line (along the y axis) without tipping any more or less than the angle ϕ. What is the acceleration of the bicycle? Answer in terms of some or all of b, h, α, ϕ, m, g and $\hat{\jmath}$ (Note: F should *not* appear in your final answer.)



Filename:s97f3

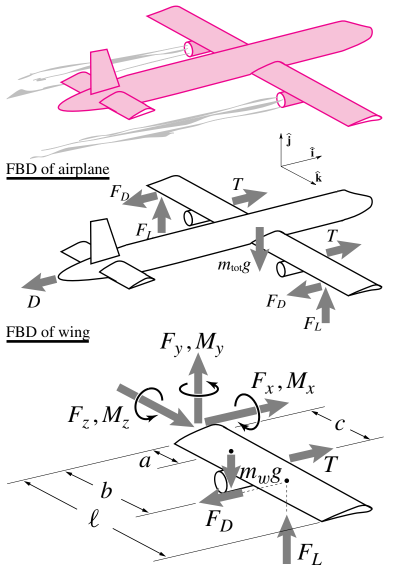
12.75 An airplane is in straight level flight but is accelerating in the forward direction. In terms of some or all of the following parameters,

- m_{tot} $\equiv$ the total mass of the plane (including the wings),
- D= the drag force on the fuselage,
- F_D= the drag force on each wing,
- g= gravitational constant, and,
- T = the thrust of one engine.

a) What is the lift on each wing F_L?

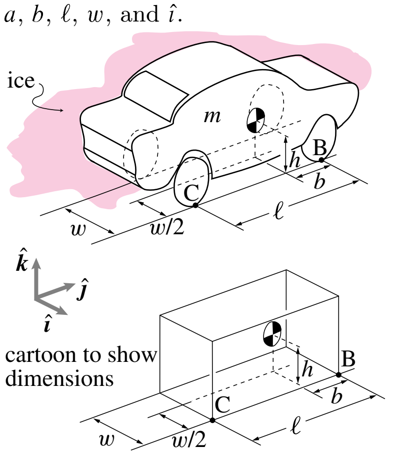
b) What is the acceleration of the plane $\vec{a}_P$?

c) A free body diagram of one wing is shown. The mass of one wing is m_w. What, in terms of m_{tot}, m_w, F_L, F_D, g, a,b, c, and ℓ are the reactions at the base of the wing (where it is attached to the plane), $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$ and $\vec{M} = M_x\hat{\imath} + M_y\hat{\jmath} + M_z\hat{k}$?



Filename:pfigure3-airplane

12.76 A rear-wheel drive car on level ground. The two left wheels are on perfectly slippery ice. The right wheels are on dry pavement. The negligible-mass front right wheel at B is steered straight ahead and rolls without slip. The right rear wheel at C also rolls without slip and drives the car forward with velocity $\vec{v} = v\hat{\jmath}$ and acceleration $\vec{a} = a\hat{\jmath}$. Dimensions are as shown and the car has mass m . What is the sideways force from the ground on the right front wheel at B? Answer in terms of any or all of m, g, a, b, ℓ, w, and $\hat{\imath}$.

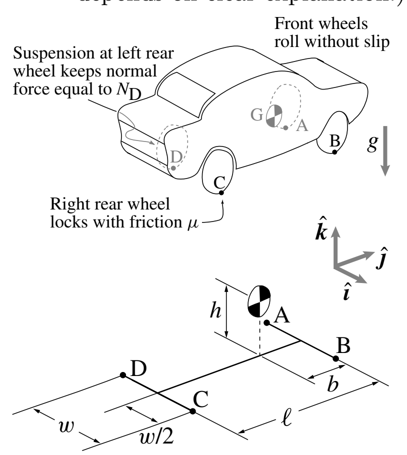


Filename:pfigure3-f95p1p3

12.77 A somewhat crippled car slams on the brakes. The suspension springs at A, B, and C are frozen and keep the car level and at constant height. The normal force at D is kept equal to N_D by the only working suspension spring which is on the left rear wheel at D. The only brake which is working is that of the right rear wheel at C which slides on the ground with friction coefficient μ. Wheels A, B, and D roll freely without slip. Dimensions are as shown.

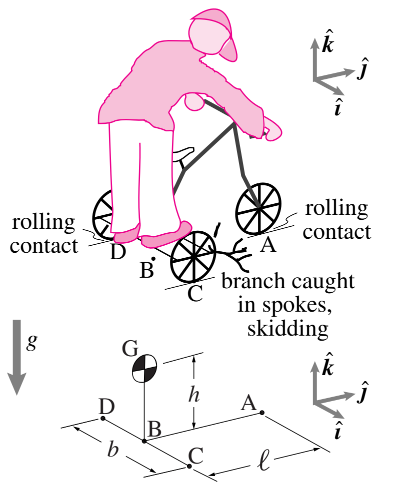
a) Find the acceleration of the car in terms of some or all of $m, w, \ell, b, h, g, \mu, \hat{\jmath}$, and N_D.

b) From the information given could you also find all of the reaction forces at all of the wheels? If so, why? If not, what can't you find and why? (No credit for correct answer. Credit depends on clear explanation.)



Filename:s97p2-3

12.78 Speeding tricycle gets a branch caught in the right rear wheel. A scared-stiff tricyclist riding on level ground gets a branch stuck in the right rear wheel so the wheel skids with friction coefficient μ. Assume that the center of mass of the tricycle-person system is directly above the rear axle. Assume that the left rear wheel and the front wheel have negligible mass, good bearings, and have sufficient friction that they roll in the $\hat{\jmath}$ direction without slip, thus constraining the overall motion of the tricycle. Dimensions are shown in the lower sketch. *Find the acceleration of the tricycle* (in terms of some or all of ℓ, h, b, m, $[I^{cm}]$, μ, g, $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$). [Hint: check your answer against special cases for which you might guess the answer, such as when $\mu = 0$ or when $h = 0$.]



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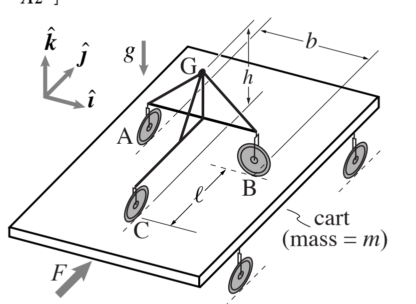
12.79 A 3-wheeled robot. A 3-wheeled robot with mass m is being transported on a level flatbed trailer also with mass m. The trailer is being pushed with a force $F\hat{\jmath}$. The ideal massless trailer wheels roll without slip. The ideal massless robot wheels also roll without slip. The robot steering mechanism has turned the wheels so that wheels at A and C are free to roll in the $\hat{\jmath}$ direction and the wheel at B is free to roll in the $\hat{\imath}$ direction. The center of mass of the robot at G is h above the trailer bed and symmetrically above the axle connecting wheels A and B. The wheels A and B are a distance b apart. The length of the robot is ℓ.

Find the force vector $\vec{F}_A$ of the trailer on the robot at A in terms of some or all of $m, g, \ell, F, b, h, \hat{\imath}, \hat{\jmath}$, and $\hat{k}$. [Hints: Use a free body diagram of the cart with robot to find their acceleration. With reference to a free body diagram of the robot, use angular momentum balance about axis BC to find F_{Az}.]



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