CHAPTER 14

General planar motion of a single rigid object

The goal here is to generate equations of motion for general planar motion of a (planar) rigid object that may roll, slide or be in free flight. Multi-object systems are also considered so long as they do not involve kinematic constraints between the bodies. Features of the solution that can be obtained from analysis are discussed, as are numerical solutions.

Contents

Many machine and structural parts move in straight-lines (Chapter 12) or circles (Chapters 13). But other things have with more general motions, like a plane in unsteady flight or a connecting rod in a car engine. Keeping track of sudch motion is a bit more difficult. To keep things simpler we only treat these more general motions in 2-D this chapter.

Mostly we will use these two modeling approximations:

- The objects are planar, or symmetric with respect to a plane; and
- They have planar motions in that plane.

A *planar object* is one where the whole object is flat and all its matter is confined to one plane, say the xy plane. This is a palatable approximation for a f piece cut out of flat sheet metal. For more substantial real objects, like a full car, the approximation seems at a glance to be terrible. But it turns out that so long as the *motion* is planar and the car is reasonably idealized as symmetrical (left to right) that treating the car as equivalent to its being squished into a plane does not introduce any more approximation. Thus, even in this 3-D world we live in with 3-D objects, it is fruitful to do 2-D analysis of the type you will learn in this chapter.

A *planar motion* is one where the velocities of all points are in the same constant plane, say a fixed xy plane, at all times and where points with, say, the same z coordinate have the same velocity. The positions of the points do not have to be in same plane for a planar motion. Each point stays in a plane, but different points can be in different planes, with each plane parallel to the others.

Figure 14.1: Planar motion of a 3D car. If the car is symmetrical it can be studied by the means of this chapter.

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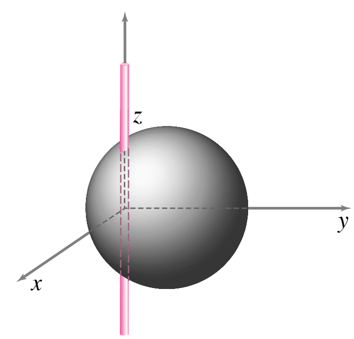

Figure 14.2: Planar motion of a skewered sphere. This can be studied by the means in this chapter.

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Example: **A car going over a hill**
Assume the road is straight in map view, say in the x direction. Assume the whole width of the road has the same hump. Although the car is clearly not planar, the car motion is probably close to planar, with the velocities of all points in the car in the xy plane (see Fig. 14.1)

Example: **Skewered sphere**
A sphere skewered and rotating about a fixed axes in the $\hat{\boldsymbol{k}}$ direction has a planar motion (see Fig. 14.2). The points on the object do not all lie in a common plane. But all of the velocities are orthogonal to $\hat{\boldsymbol{k}}$ and thus in the xy plane. This problem does fit in with the methods of this chapter. The symmetry of the sphere with respect to the xy plane makes it so that the three-dimensional mass distribution does not invalidate the two-dimensional analysis.

Example: **Skewered plate**
A flat rectangular plate with normal $\hat{\boldsymbol{n}}$ has a fixed axis of rotation in the direction $\hat{\boldsymbol{\lambda}}$ that makes a 45° to $\hat{\boldsymbol{n}}$ (see Fig. 14.3). This is a planar object (a plane normal to $\hat{\boldsymbol{n}}$) in planar motion (all velocities are in the plane normal to $\hat{\boldsymbol{\lambda}}$). But the plane of motion is not the plane of the mass distribution, the object is not symmetric with respect to a motion plane, o this example does not fit into the discussion of this chapter *.

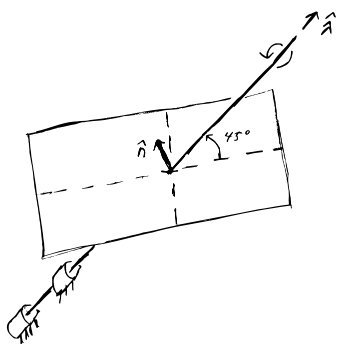

Figure 14.3: Planar motion of a planar object. But the plane of the motion is *not* the plane of the object. This is not a natural topic for this chapter.
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* Actually, a two-dimensional analysis of the plate in this example we would be legitimate in this sense. Project all the plates mass into the plane normal to the $\hat{\boldsymbol{\lambda}}$ direction. The projections of the forces on this plane would be correctly predicted, but three dimensional effects, like those associated with dynamic imbalance, would be lost in this projection.

No real object is exactly planar and no real motion is exactly a planar motion. But many objects are relatively flat and thin or symmetrical and many motions are approximately planar motions. Thus many, if not most, simple engineering analysis assume planar motion. For bodies that are approximately symmetric about the xy plane of motion (such as a car, if the asymmetrically placed driver's mass etc. is neglected), there is no loss in doing a two-dimensional planar rather than full three dimensional analysis.

The plan of this chapter. We start with planar kinematics. Then we evaluate and use expressions for the rates of change of linear and angular momentum for planar bodies. Finally we discuss rolling, sliding and collisions.

14.1 Description of motion: planar rigid-object kinematics

We start our study of planar motion with the kinematic question: How do points on a rigid object (or 'body') move? There are two reasons to ask this question. First, velocities and accelerations of mass points are needed to apply the momentum-balance equations. Second, formulas for positions, velocities and accelerations of points are useful to understand *mechanisms*, machines where various parts (each one usually idealized as a rigid object) are connected to each other with hinges and bearings of one type of another.

The central observation in all rigid-object kinematics is that

> all pairs of points on a single moving rigid object keep constant distance from each other.

This is the definition of a rigid object. In this section you will learn how to use rigidity to calculate positions, velocities and accelerations of all points (millions and billions of them) on a rigid body given only a few numbers (about 8 of them). This goal is achieved by putting together the ideas from Chapter 11 (arbitrary motion of one particle), Chapter 12 (straight-line motion), and chapter 13 (circular motion of a rigid body in a plane).

Displacement and rotation

When a planar object (read, say, body or machine part) $\mathcal{B}$ moves from one configuration in the plane it has a displacement and a rotation. For definiteness, we start in some reference position *. We mark a reference point on the body that, in the reference configuration, coincides with a fixed reference point, say 0. We also mark a (directed) line on the body that, in the reference configuration, coincides with a fixed reference line, say the positive x axis. The body never has to pass through this reference position, however. For example, the position of an airplane flying from New York to Mumbai is measured relative to a point in the Gulf of Guinea 1000 miles west of Gabon,* even though the airplane never goes there (nor does anyone want it to).

*That's the location of O° longitude and O° latitude.

We could measure rotation by measuring the rotations of any lines that connected any pair of points fixed to the object. For each line we keep track of the angle that line makes with a line fixed in space, say the positive x or y axis. Its simplest to stick to the convention that counter-clockwise rotations are positive (Fig. 14.4). The angles θ_1, $\theta_2 \ldots$, all change with time and are all different from each other. But all the angles change the same amount, just like in section 13.3. We can pick any one line we like for definiteness and measure the object rotation by the rotation of that line. So

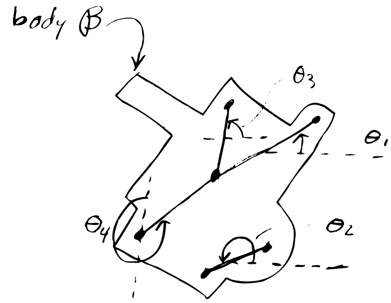

Figure 14.4: Rotation of object $\mathcal{B}$ is measured by the rotation of real or imagined lines marked on the object. The lines make different angles: $\theta_1 \neq \theta_2$, $\theta_2 \neq \theta_3$ etc, but $\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \ldots$. Angular velocity is defined as $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ with $\omega \equiv \dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \ldots$.

Filename:tfigure-2Drotation

> The net motion of a rigid planar object is described by *translation*, the vector displacement of a reference point from a reference position $\vec{\boldsymbol{r}}_{o'/o} = \vec{\boldsymbol{r}}_{oo'}$, and a *rotation* θ of the object from the reference orientation.

That is, the general planar motion of a rigid object is the general motion of a point plus circular motion about that point.

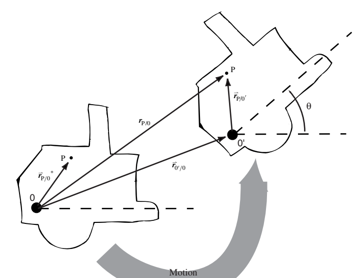

Figure 14.5: The displacement and rotation of a planar object relative to a reference configuration.

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The position of a point on a moving rigid object.

Let's denote the reference configuration with a star ($*$). Given that P on the object is at $\vec{\boldsymbol{r}}_{\text{P/0}}{}^*$ in the reference configuration, where is it (What is $\vec{\boldsymbol{r}}_{\text{P/0}}$?) after the object has been displaced by $\vec{\boldsymbol{r}}_{0'/0}$ and rotated an angle θ? An easy way to treat this is to write the new position of P as (see Fig. 14.5)

$$\vec{\boldsymbol{r}}_{\text{P/0}} = \vec{\boldsymbol{r}}_{0'/0} + \vec{\boldsymbol{r}}_{\text{P/0}'}.$$

This is the *base-independent* or *direct vector* representation of the position of P. The formula is correct no matter what base vectors are used to represent the vectors in the formula. The vector $\vec{\boldsymbol{r}}_{0'/0}$ describes *translation*, that's half the story. The other term $\vec{\boldsymbol{r}}_{\text{P/0}'}$ we find by rotating $\vec{\boldsymbol{r}}_{\text{P/0}}{}^*$ as we did in Section 13.3. Thus, we can describe the coordinates of a point as,

$$\left[\vec{\boldsymbol{r}}_{\text{P/0}}\right]_{xy} = \underbrace{\left[\vec{\boldsymbol{r}}_{0'/0}\right]_{xy}}_{\text{displacement}} + \underbrace{\left[R(\theta)\right]\left[\vec{\boldsymbol{r}}_{\text{P/0}'}\right]_{x'y'}}_{\text{rotation}} \tag{14.1}$$

or, writing out all the components of the vectors and matrices,

$$\begin{bmatrix} x_P \\ y_P \end{bmatrix} = \begin{bmatrix} x_{0'/0} \\ y_{0'/0} \end{bmatrix} + \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x^*_{P/0} \\ y^*_{P/0} \end{bmatrix}. \tag{14.2}$$

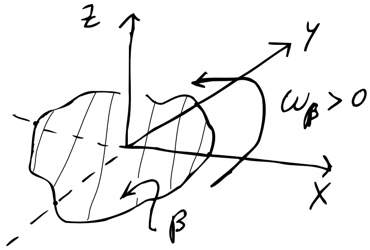

Figure 14.6: It is generally best to take positive ω to be counterclockwise when viewed from the positive z axis.
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As the motion progresses the displacement $\begin{bmatrix} x_{0'/0} \\ y_{0'/0} \end{bmatrix}$ changes with time as does the rotation angle θ. We call eqn. (14.2) the *fixed basis* or *component* representation of the motion. It gives the components of the position in terms of base vectors that are fixed in space.

Example:
If in the reference position a particle on a rigid object is at $\vec{r}_{P/0} = (1\hat{\imath} + 2\hat{\jmath})$ m and the object displaces by $\vec{r}_{0'/0} = (3\hat{\imath} + 4\hat{\jmath})$ m and rotates by $\theta\pi/3$ rad $= 60$ deg relative to that configuration, then its new position is:

$$\begin{aligned}
\left[\vec{r}_{P/0}\right]_{xy} &= \left[\vec{r}_{0'/0}\right]_{xy} + \left[R(\theta)\right]\left[\vec{r}_{P/0'}\right]_{x'y'} \\
&= \begin{bmatrix} x_{0'/0} \\ y_{0'/0} \end{bmatrix} + \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x^*_{P/0} \\ y^*_{P/0} \end{bmatrix} \\
&= \left\{\begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} \cos\pi/3 & \sin\pi/3 \\ -\sin\pi/3 & \cos\pi/3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} \text{m} \\
&= \begin{bmatrix} 3.5 + \sqrt{3} \\ 5 - \sqrt{3}/2 \end{bmatrix} \text{m} \\
\Rightarrow \vec{r}_{P/0} &= \left((3.5 + \sqrt{3})\hat{\imath} + (5 - \sqrt{3}/2)\hat{\jmath}\right) \text{m}
\end{aligned}$$

Finally, the *changing base* representation uses base vectors $\hat{\imath}'$, $\hat{\jmath}'$ that are aligned with $\hat{\imath}$, $\hat{\jmath}$ in the reference configuration but which are glued to the rotating object. If we define x' and y' as the x and y components of P in the reference (*) configuration we have that

$$\left[\vec{r}_{P/0}{}^*\right]_{xy} = \left[\vec{r}_{P/0'}{}^*\right]_{x'y'} = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad \text{so} \quad \vec{r}_{P/0} = \left(x_{0'/0}\hat{\imath} + y_{0'/0}\hat{\jmath}\right) + \left(x'\hat{\imath}' + y'\hat{\jmath}'\right).$$

Often the changing-base notation the clearest, the component or fixed base representation the best for computer calculations, and the base-independent or direct-vector notation the quickest and easiest.

Angular velocity

Because all lines object $\mathcal{B}$ rotate at the same rate (at a given instant) $\mathcal{B}$'s rotation rate is the single number we call $\omega_\mathcal{B}$ ('omega b'). In order to make various formulas work out we define a vector angular velocity with magnitude $\omega_\mathcal{B}$ which is perpendicular to the xy plane:

$$\vec{\omega}_\mathcal{B} = \underbrace{\omega_\mathcal{B}}_{\dot{\theta}} \hat{k}$$

where $\dot{\theta}$ is the rate of change of the angle of *any* line marked on object $\mathcal{B}$.

So long as you are careful to define angular velocity by the rotation of line segments and not by the motion of individual particles, the concept of angular velocity in general motion is defined exactly as for a object rotating about a fixed axis. A legitimate way to think about planar motion of a rigid object is that any given point is moving in circles about any other given point (relative to that point). When a rigid object moves it always has an angular velocity (possibly zero). If we call the object $\mathcal{B}$ (script B), we then call the object's angular velocity $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$. In general it is best to use the sign convention that when $\omega_{\mathcal{B}} > 0$ the object is rotating counterclockwise when viewed looking in from the positive z axis (see Fig. 14.6).

The angular velocity vector $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$ of a object $\mathcal{B}$ describes it's rate and direction of rotation. For planar motions $\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \omega_{\mathcal{B}}\hat{\boldsymbol{k}}$.

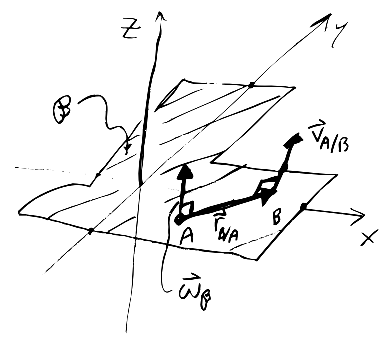

Figure 14.7: The relative velocity of points A and B is in the xy plane and perpendicular to the line segment AB.

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Relative velocity of two points on a rigid object

For any two points A and B glued to a rigid object $\mathcal{B}$ the relative velocity of the points ('the velocity of B relative to A') is given by the cross product of the angular velocity of the object with the relative position of the two points:

$$\vec{\boldsymbol{v}}_{B/A} \equiv \vec{\boldsymbol{v}}_B - \vec{\boldsymbol{v}}_A = \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A}. \tag{14.3}$$

This formula says that the relative velocity of two points on a rigid object is the same as would be predicted for one of the points if the other were stationary. The derivation of this formula is the same as for planar circular motion.

Note that even though we are doing planar kinematics, it is convenient to use three dimensional cross products. Generally we will call the plane of motion the xy plane and $\vec{\boldsymbol{\omega}}$ will be in the z direction. Because $\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}$ must be perpendicular to $\vec{\boldsymbol{\omega}}$ it is perpendicular to the z axis. So this three dimensional cross product always gives a vector in the xy plane that is perpendicular to $\vec{\boldsymbol{r}}$.

We can also represent the relative velocity in the changing base notation as

$$\begin{aligned}
\vec{\boldsymbol{v}}_{B/A} &= \frac{d}{dt}\left(x'_{B/A}\hat{\boldsymbol{\imath}}' + x'_{B/A}\hat{\boldsymbol{\jmath}}'\right)\\
&= x'_{B/A}\frac{d}{dt}\hat{\boldsymbol{\imath}}' + x'_{B/A}\frac{d}{dt}\hat{\boldsymbol{\jmath}}'\\
&= x'_{B/A}\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{\imath}}' + x'_{B/A}\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{\jmath}}'.
\end{aligned}$$

Finally, we can use the fixed-base or component notation:

$$\begin{aligned}
\left[\vec{v}_{\mathrm{B/A}}\right]_{xy} &= \frac{d}{dt}\begin{bmatrix} x_{B/A} \\ y_{B/A} \end{bmatrix} \\
&= \frac{d}{dt}\left\{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x^*_{\mathrm{B/A}} \\ y^*_{\mathrm{B/A}} \end{bmatrix}\right\} \\
&= \begin{bmatrix} -\dot{\theta}\sin\theta & \dot{\theta}\cos\theta \\ -\dot{\theta}\cos\theta & -\dot{\theta}\sin\theta \end{bmatrix}\begin{bmatrix} x^*_{\mathrm{B/A}} \\ y^*_{\mathrm{B/A}} \end{bmatrix} \\
&= \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x^*_{\mathrm{B/A}} \\ y^*_{\mathrm{B/A}} \end{bmatrix}
\end{aligned}$$

where $x^*_{B/A}$ and $y^*_{B/A}$ are the components of the position of B with respect to A in the reference configuration and hence do not change with time.

Absolute velocity of a point on a rigid object

If one knows the velocity of one point on a rigid object and one also knows the angular velocity of the object, then one can find the velocity of any other point. How? By addition. Say we know the velocity of point A, the angular velocity of the object, and the relative position of A and B, then

$$\begin{aligned}
\vec{v}_B &= \vec{v}_A + (\vec{v}_B - \vec{v}_A) \\
&= \vec{v}_A + \vec{v}_{B/A} \\
&= \vec{v}_A + \vec{\omega}_{\mathcal{B}} \times \underbrace{\vec{r}_{B/A}}_{\vec{r}_B - \vec{r}_A} .
\end{aligned} \tag{14.4}$$

That is, the absolute velocity of the point B is the absolute velocity of the point A plus the velocity of the point B relative to the point A. Because B and A are on the same rigid object, their relative velocity is given by formula 14.4 above. For ease of understanding one pretends one knows the quantities on the right and are trying to find the quantity on the left. But the equation is valid and useful no matter which quantities are known and which are not.

An alternative approach is to differentiate the coordinate expression eqn. (14.3) (see Box 14.1 on 699).

Angular acceleration

We define the angular acceleration $\vec{\alpha}$ ('alpha') of a rigid object as the rate of change of angular velocity, $\vec{\alpha} = \dot{\vec{\omega}}$. The angular acceleration of a object $\mathcal{B}$ is $\vec{\alpha}_{\mathcal{B}}$. For two-dimensional bodies moving in the plane both the angular velocity and the angular acceleration are always perpendicular to the plane. That is $\vec{\omega} = \omega\hat{k}$ and $\vec{\alpha} = \alpha\hat{k} = \dot{\omega}\hat{k}$. In 2-D the angular acceleration is only due to the speeding up or slowing down of the rotation rate; i.e., $\alpha = \dot{\omega} = \ddot{\theta}$.

Relative acceleration of two points on a rigid object

For any two points A and B glued to a rigid object $\mathcal{B}$, the acceleration of B relative to A is

$$\begin{aligned}
\vec{a}_{B/A} &= \frac{d}{dt}\vec{v}_{\mathrm{B/A}} \\
&= \frac{d}{dt}\left\{\vec{\omega}_{\mathcal{B}} \times \vec{r}_{\mathrm{B/A}}\right\} \\
&= \dot{\vec{\omega}}_{\mathcal{B}} \times \vec{r}_{B/A} + \vec{\omega}_{\mathcal{B}} \times (\vec{v}_{\mathrm{B/A}}), \\
&= \dot{\vec{\omega}}_{\mathcal{B}} \times \vec{r}_{B/A} + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{\mathrm{B/A}}), \\
&= \alpha_{\mathcal{B}}\hat{\boldsymbol{k}} \times \vec{r}_{B/A} + (-\omega_{\mathcal{B}}^2\vec{r}_{\mathrm{B/A}}), \qquad (14.5)
\end{aligned}$$

This is the base-independent or direct-vector expression for relative acceleration. If point A has no acceleration, this formula is the same as that for the acceleration of a point going in circles from chapter 7. On a rigid object in 2D all two points on rigid object can do relative to each other is to go in circles.

Equation (14.5) could also be derived, with some algebra, by taking two time derivatives of the relative position coordinate expression

$$\left[\vec{r}_{\mathrm{B/A}}\right]_{xy} = \left[R(\theta)\right]\left[\vec{r}_{\mathrm{B/A}}{}^{*}\right]_{x'y'}$$

or by taking two time derivatives of the changing base vector expression

$$\vec{r}_{\mathrm{B/A}} = x'_{B/A}\hat{\boldsymbol{\imath}}' + y'_{B/A}\hat{\boldsymbol{\jmath}}'.$$

14.1 *THEORY*

Using matrices to find velocity from position

An alternative derivation for the velocity eqn. (14.3) of a point on a rigid object comes from differentiating the matrix formula for the position (eqn. (14.3)). Denoting $\vec{r}_{\mathrm{P/O}}$ as the reference position of the particle and $\vec{r}_{\mathrm{P'/O'}}$ as the position relative to the reference point on the moved object at the time of interest, we have:

$$\begin{aligned}
\left[\vec{v}_{\mathrm{P/0}}\right]_{xy} &= \frac{d}{dt}\left[\vec{r}_{\mathrm{P/0}}\right]_{xy} \\
&= \frac{d}{dt}\begin{bmatrix} x_{0'/0} \\ y_{0'/0} \end{bmatrix} \\
&\quad + \frac{d}{dt}\left\{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x^*_{\mathrm{P/0}} \\ y^*_{\mathrm{P/0}} \end{bmatrix}\right\} \\
&= \begin{bmatrix} \dot{x}^*_{0'/0} \\ \dot{y}^*_{0'/0} \end{bmatrix} + \begin{bmatrix} -\dot{\theta}\sin\theta & \dot{\theta}\cos\theta \\ -\dot{\theta}\cos\theta & -\dot{\theta}\sin\theta \end{bmatrix}\begin{bmatrix} x^*_{\mathrm{P/0}} \\ y^*_{\mathrm{P/0}} \end{bmatrix} \\
&= \begin{bmatrix} \dot{x}_{0'/0} \\ \dot{y}_{0'/0} \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x^*_{\mathrm{P/0}} \\ y^*_{\mathrm{P/0}} \end{bmatrix} \\
&= \begin{bmatrix} \dot{x}_{0'/0} \\ \dot{y}_{0'/0} \end{bmatrix} + \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}\left[\vec{r}_{\mathrm{P'/0'}}\right]_{xy}.
\end{aligned}$$

Thus, matrix product $\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}\left[\vec{r}_{\mathrm{P'/0'}}\right]_{xy}$ is equivalent to the vector product $\vec{\omega} \times \vec{r}_{\mathrm{P'/0'}}$ and the matrix $\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$ is sometimes called the angular velocity matrix. It is an example of a so-called *skew symmetric* matrix because it is the negative of its own transpose.

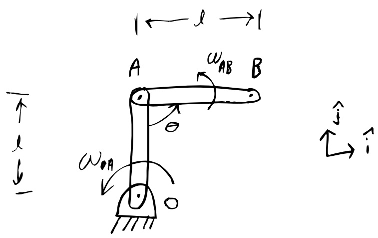

Figure 14.8: A two link robot arm.

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Absolute acceleration of a point on a rigid object

If one knows the acceleration of one point on a rigid body *and* the angular velocity and acceleration of the body, then one can find the acceleration of any other point. How?

$$\begin{aligned}\vec{a}_B &= \vec{a}_A + (\vec{a}_B - \vec{a}_A) = \vec{a}_A + \vec{a}_{B/A}\\ &= \vec{a}_A + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{B/A}) + \dot{\vec{\omega}}_{\mathcal{B}} \times \vec{r}_{B/A}\\ &= \vec{a}_A - \omega_{\mathcal{B}}^2 \vec{r}_{B/A} + \alpha_{\mathcal{B}} \hat{k} \times \vec{r}_{B/A} \qquad (14.6)\end{aligned}$$

This is the base-independent or direct-vector expression for acceleration. The fixed-base (component) and changing-base notations are somewhat more complex.

Equation 14.7 is often called the three term acceleration formula. The acceleration of a point B on a rigid object is the sum of three terms. The first, $\vec{a}_A$, is the acceleration of some point A on the object. The second term, $\vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{B/A})$, is the centripetal acceleration of B going in circles relative to A. It is directed from B towards A. The third term, $\dot{\vec{\omega}}_{\mathcal{B}} \times \vec{r}_{B/A}$, is due to the change in the magnitude of the angular velocity and is in the direction normal to the line from A to B.

Example: **Robot arm**

Given the configuration shown in Fig. 14.8 the acceleration of point B can be found by thinking of link AB as the object $\mathcal{B}$ in eqn. (14.7) and using what you know about circular motion to find the acceleration of A:

$$\begin{aligned}\vec{a}_B &= \vec{a}_A - \omega_{\mathcal{B}}^2 \vec{r}_{B/A} + \alpha_{\mathcal{B}} \hat{k} \times \vec{r}_{B/A}\\ &= \left(-\omega_{\mathrm{OA}}^2 \ell \hat{j} - \dot{\omega}_{\mathrm{OA}} \ell \hat{\imath}\right) - \left(\omega_{\mathrm{AB}}^2 \ell \hat{\imath}\right) + \left(\dot{\omega}_{\mathrm{AB}} \hat{k} \times (\ell \hat{\imath})\right)\\ &= -\left(\dot{\omega}_{\mathrm{OA}} \ell + \omega_{\mathrm{AB}}^2 \ell\right) \hat{\imath} + \left(-\omega_{\mathrm{OA}}^2 \ell + \dot{\omega}_{\mathrm{AB}} \ell\right) \hat{j}\end{aligned}$$

[Note that $\omega_{\mathrm{AB}} \neq \dot{\theta}$ where θ is the angle between the links. Rather $\omega_{\mathrm{AB}} = \omega_{0\mathrm{A}} + \dot{\theta}$.]

Computer graphics

Given one point given by the xy pair $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ we can find out what happens to it by rotation $[R]$ as

$$\begin{bmatrix} x \\ y \end{bmatrix} = [R] \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

For example the point $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ gets changed by a 45 deg rotation to

$$\begin{aligned}\begin{bmatrix} x \\ y \end{bmatrix} = [R] \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} &= \begin{bmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}\\ &\approx \begin{bmatrix} .7 & .7 \\ -.7 & .7 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \approx \begin{bmatrix} 1.4 \\ 1.4 \end{bmatrix}.\end{aligned}$$

A translation is just a vector addition. For example the point $\begin{bmatrix} 1.4 \\ 1.4 \end{bmatrix}$ gets translated a distance 2 in the y direction by the addition of $\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ like this

$$\begin{bmatrix} x \\ y \end{bmatrix}_{\text{translated}} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 1.4 \\ 1.4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.4 \\ 2.4 \end{bmatrix}.$$

Putting these together the point $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ gets rotated and translated by first multiplying by the rotation matrix and then adding the translation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = [R]\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} x_t \\ y_t \end{bmatrix} \approx \begin{bmatrix} .7 & .7 \\ -.7 & .7 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 1.4 \\ 2.4 \end{bmatrix}.$$

A collection of points all rotated the same amount and then all translated the same amount keep their relative distances.

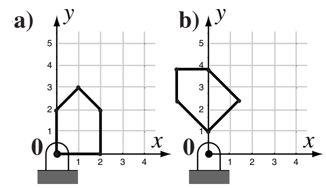


Figure 14.9: (a) A house drawn as 6 dots connected by line segments. The first and last point are the same. (b) The same house but rigidly rotated and translated.

Filename:tfigure-rotatedhouse2

A picture is a set of points on a plane. If all the points are rotated and translated the same amount the picture is rotated and translated. Thus a picture of a rigid object described by points is rigidly rotated and translated. On a computer line drawings are often represented as a connect-the-dots picture. The picture is represented by the x and y coordinates of the reference dots at the corners. These can be stored in an array with the first row being the x coordinates and the second row the y coordinates as explained on page 607. Each column of this matrix represents one point of the connect-the-dots picture. Thus a primitive picture of a house at the origin is given by the array

$$[P_0] \equiv \left[xy \text{ points originally}\right] = \begin{bmatrix} 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 2 & 1 \end{bmatrix}$$

with the lower left corner of the house at the origin.

To rotate this picture we rotate each of the columns of the matrix P_0]. But this is exactly what is accomplished by the matrix multiplication $[R][P_0]$. To translate the points you add the translation vector to each of the columns of the resulting matrix. Thus the whole picture rotated by 45° and translated up by 1 is given by

$$[P_{\text{new}}] = [R][P_0] + \begin{bmatrix} x_t \\ y_t \end{bmatrix} \approx \begin{bmatrix} .7 & .7 \\ -.7 & .7 \end{bmatrix}[P_0] + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

which gives a new array of points that, when connected give the picture shown. We have allowed the informal notation of adding a column matrix to a rectangular matrix, by which we mean adding to each column of the rectangular matrix.

To animate the motion of, say, a house flying in the Wizard of Oz you would first define the house as the set of points $[P_0]$. Then define, maybe by means of numerical solution of differential equations, a set of rotations and translations. Then for each rotation and translation the picture of the house should be drawn, one after the other. The sequence of such pictures is an animation of a flying and spinning house.

Summary of the kinematics of one rigid object in general 2D motion

You can use the position of one reference point and the rotation of the object as simple kinematic measures of the entire motion of the object. If you know the position, velocity, and acceleration of one point on a rigid object (represented by 6 scalars, say) , and you know the rotation angle, angular rate and angular acceleration (3 scalars) then you can find the position, velocity and acceleration of any point on the object. In 2D, just 9 numbers tell you the position, velocity, and acceleration of any of the billions of points whose initial positions you know*.

*In 1D it takes just 3 numbers and in 3D just 18. The unusual patter (3,9,18) comes from rotation being characterized by 0, 1, and 3 numbers in 1, 2, and 3 dimensions, respectively.

SAMPLE 14.1 Velocity of a point on a rigid body in planar motion. An equilateral triangular plate ABC is in motion in the x-y plane. At the instant shown in the figure, point B has velocity $\vec{v}_B = 0.3\,\text{m/s}\hat{\imath} + 0.6\,\text{m/s}\hat{\jmath}$ and the plate has angular velocity $\vec{\omega} = 2\,\text{rad/s}\hat{k}$. Find the velocity of point A.

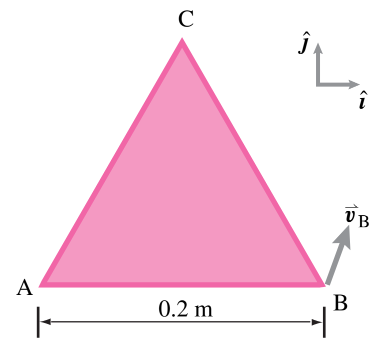


Figure 14.10:

Filename:sfig9-1-triang-1

Solution We are given $\vec{v}_B$ and ω, and we need to find $\vec{v}_A$, the velocity of point A on the same rigid body. We know that,

$$\vec{v}_A = \vec{v}_B + \vec{\omega} \times \vec{r}_{A/B}$$

Thus, to find $\vec{v}_A$, we need to find $\vec{r}_{A/B}$. Let us take an x-y coordinate system whose origin coincides with point A of the plate at the instant of interest and the x-axis is along AB. Then,

$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B = \vec{0} - (0.2\,\text{m}\hat{\imath}) = -0.2\,\text{m}\hat{\imath}$$

Thus,

$$\begin{aligned} \vec{v}_A &= \vec{v}_B + \vec{\omega} \times \vec{r}_{A/B} \\ &= (0.3\hat{\imath} + 0.6\hat{\jmath})\,\text{m/s} + 2\,\text{rad/s}\hat{k} \times (-0.2\hat{\imath})\,\text{m} \\ &= (0.3\hat{\imath} + 0.6\hat{\jmath})\,\text{m/s} - 0.4\hat{\jmath}\,\text{m/s} \\ &= (0.3\hat{\imath} + 0.2\hat{\jmath})\,\text{m/s}. \end{aligned}$$

$$\boxed{\vec{v}_A = (0.3\hat{\imath} + 0.2\hat{\jmath})\,\text{m/s}}$$

SAMPLE 14.2 The instantaneous center of rotation. A rigid body is in planar motion. At some instant t, the angular velocity of the body is $\vec{\omega} = 5\,\text{rad/s}\hat{k}$ and the linear velocity of a point C on the body is $\vec{v}_C = 2\,\text{m/s}\hat{\imath} - 5\,\text{m/s}\hat{\jmath}$. Find a point on the body, assuming it exists, that has zero velocity. *

* The point with zero velocity is called the instantaneous center of rotation. Sometimes this point may lie outside the body.

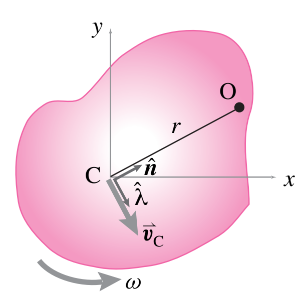


Figure 14.11:

Filename:sfig9-1-2-body

Solution Let the point of zero velocity be O, with position vector $\vec{r}_{O/C}$ with respect to point C. Since $\vec{v}_O = \vec{v}_C + \vec{\omega} \times \vec{r}_{O/C}$, for $\vec{v}_O$ to be zero, $\vec{\omega} \times \vec{r}_{O/C}$ must be parallel to and in the opposite direction of $\vec{v}_C$. Since $\vec{\omega}$ is out of plane, $\vec{r}_{O/C}$ must be normal to $\vec{v}_C$ for the cross product to be parallel to $\vec{v}_C$. Now, let $\vec{v}_C = v_C\hat{\lambda}$. Then, $\vec{r}_{O/C} = r\hat{n}$ where $\hat{n} \perp \hat{\lambda}$ and $r = |\vec{r}_{O/C}|$. Thus,

$$v_C\hat{\lambda} + \omega\hat{k} \times r\hat{n} = \vec{v}_O = \vec{0} \tag{14.7}$$

Dotting eqn. (14.7) with $\hat{\lambda}$, we get

$$v_C = \omega r \quad \Rightarrow \quad r = \frac{v_C}{\omega} = \frac{\sqrt{29}\,\text{m/s}}{5\,\text{rad/s}} = 1.08\,\text{m}.$$

Since $\hat{\lambda} = \vec{v}_C/|\vec{v}_C| = 0.37\hat{\imath} - 0.93\hat{\jmath}$, $\hat{n} = 0.93\hat{\imath} + 0.37\hat{\jmath}$. Thus

$$\vec{r}_{O/C} = r\hat{n} = 1.08\,\text{m}(0.93\hat{\imath} + 0.37\hat{\jmath}) = 1\,\text{m}\hat{\imath} + 0.4\,\text{m}\hat{\jmath}.$$

$$\boxed{\vec{r}_{O/C} = 1\,\text{m}\hat{\imath} + 0.4\,\text{m}\hat{\jmath}}$$

Note: It is also possible to find $\vec{r}_{O/C}$ purely by vector algebra. Assuming $\vec{r}_{O/C} = (x\hat{\imath} + y\hat{\jmath})\,\text{m}$ and plugging into $\vec{v}_O = \vec{v}_C + \vec{\omega} \times \vec{r}_{O/C}$ along with the given values, we get $\vec{0} = (2-5y)\,\text{m/s}\hat{\imath} + (-5+5x)\,\text{m/s}\hat{\jmath}$. Dotting this equation with $\hat{\imath}$ and $\hat{\jmath}$, we get $2 - 5y = 0$ and $-5 + 5x = 0$, which give $x = 1$ and $y = 0.4$. Thus, $\vec{r}_{O/C} = 1\,\text{m}\hat{\imath} + 0.4\,\text{m}\hat{\jmath}$ as obtained above.

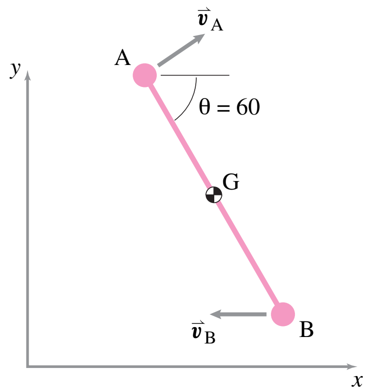


Figure 14.12:

Filename:sfig7-1-1

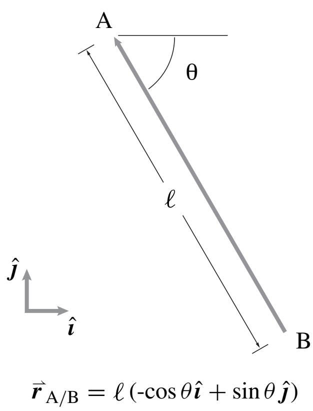


Figure 14.13:

Filename:sfig7-1-1a

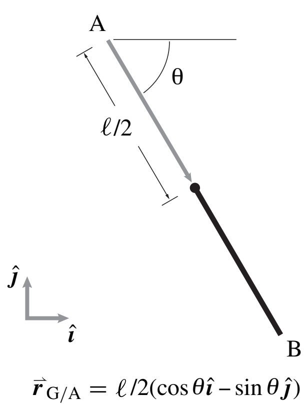


Figure 14.14:

Filename:sfig7-1-1b

SAMPLE 14.3 A cheerleader throws her baton up in the air in the vertical xy-plane. At an instant when the baton axis is at $\theta = 60^\circ$ from the horizontal, the velocity of end A of the baton is $\vec{v}_A = 2\,\text{m/s}\hat{\imath} + \sqrt{3}\,\text{m/s}\hat{\jmath}$. At the same instant, end B of the baton has velocity in the negative x-direction (but $|\vec{v}_B|$ is not known). If the length of the baton is $\ell = \frac{1}{2}$ m and the center-of-mass is in the middle of the baton, find the velocity of the center-of-mass.

Solution

$$\begin{aligned} \text{We are given:}\quad \vec{v}_A &= (2\hat{\imath} + \sqrt{3}\hat{\jmath})\,\text{m/s} \\ \text{and}\quad \vec{v}_B &= -v_B\hat{\imath} \end{aligned}$$

where $v_B = |\vec{v}_B|$ is unknown. We need to find $\vec{v}_G$. Using the relative velocity formula for two points on a rigid body, we can write:

$$\vec{v}_G = \vec{v}_A + \vec{\omega} \times \vec{r}_{G/A} \tag{14.8}$$

Here, $\vec{v}_A$ and $\vec{r}_{G/A}$ are known. Thus, to find $\vec{v}_G$, we need to find $\vec{\omega}$, the angular velocity of the baton. Since the motion is in the vertical xy-plane, let $\vec{\omega} = \omega\hat{k}$. Then,

$$\begin{aligned} \vec{v}_B &= \vec{v}_A + \vec{\omega} \times \vec{r}_{A/B} = \vec{v}_A + \omega\hat{k} \times \underbrace{\ell(-\cos\theta\hat{\imath} + \sin\theta\hat{\jmath})}_{\vec{r}_{A/B}} \\ \text{or}\quad -v_B\hat{\imath} &= (2\hat{\imath} + \sqrt{3}\hat{\jmath})\,\text{m/s} - \omega\ell(\cos\theta\hat{\jmath} + \sin\theta\hat{\imath}) \\ &= (2\hat{\imath} + \sqrt{3}\hat{\jmath})\,\text{m/s} - \omega\cdot\frac{1}{2}\,\text{m}\cdot(\frac{1}{2}\hat{\jmath} + \frac{\sqrt{3}}{2}\hat{\imath}) \end{aligned}$$

Dotting both sides of this equation with $\hat{\jmath}$ we get:

$$\begin{aligned} 0 &= \sqrt{3}\,\text{m/s} - \frac{\omega}{2}\,\text{m}\cdot\frac{1}{2} \\ \Rightarrow\quad \omega &= \sqrt{3}\frac{\cancel{\text{m}}}{\text{s}}\cdot\frac{4}{1\,\cancel{\text{m}}} = 4\sqrt{3}\,\text{rad/s}. \end{aligned}$$

Now substituting the appropriate values in Eqn 14.8 we get:

$$\begin{aligned} \vec{v}_G &= \vec{v}_A + \omega\hat{k} \times \underbrace{\frac{\ell}{2}(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath})}_{\vec{r}_{G/A}} \\ &= \vec{v}_A + \frac{\omega\ell}{2}(\cos\theta\hat{\jmath} + \sin\theta\hat{\imath}) \\ &= (2\hat{\imath} + \sqrt{3}\hat{\jmath})\,\text{m/s} + \sqrt{3}\,\text{m/s}\cdot(\frac{1}{2}\hat{\jmath} + \frac{\sqrt{3}}{2}\hat{\imath}) \\ &= (2 + \frac{3}{2})\,\text{m/s}\hat{\imath} + (\sqrt{3} + \frac{\sqrt{3}}{2})\,\text{m/s}\hat{\jmath} \\ &= 3.5\,\text{m/s}\hat{\imath} + 2.6\,\text{m/s}\hat{\jmath} \end{aligned}$$

$$\boxed{\vec{v}_G = (3.5\hat{\imath} + 2.6\hat{\jmath})\,\text{m/s}}$$

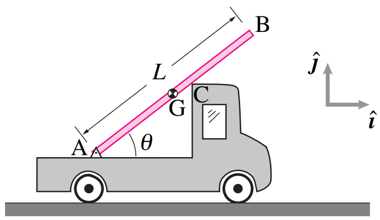


Figure 14.15:

Filename:sfig7-2-2a

SAMPLE 14.4 A board in the back of an accelerating truck. A 10 ft long board AB rests in the back of a flat-bed truck as shown in Fig. 14.15. End A of the board is hinged to the bed of the truck. The truck is going on a level road at 55 mph. In preparation for overtaking a vehicle in the front the trucker accelerates at a constant rate 3 mph/s. At the instant when the speed of the truck is 60 mph, the magnitude of the relative velocity and relative acceleration of end B with respect to the bed of the truck are 10 ft/s and 12 ft/s^2, respectively. There is wind and at this instant, the board has lost contact with point C. If the angle θ between the board and the bed is 45° at the instant mentioned, find

1. the angular velocity and angular acceleration of the board,
2. the absolute velocity and absolute acceleration of point B, and
3. the acceleration of the center-of-mass of the board.

Solution At the instant of interest

$$\begin{aligned}
\vec{\boldsymbol{v}}_A &= \text{velocity of the truck } = 60\,\text{mph}\,\hat{\boldsymbol{\imath}} = 88\,\text{ft/s}\,\hat{\boldsymbol{\imath}} \\
\vec{\boldsymbol{a}}_A &= \text{acceleration of the truck } = 3\,\text{mph}/s = 4.4\,\text{ft/s}^2\,\hat{\boldsymbol{\imath}} \\
|\vec{\boldsymbol{v}}_{B/A}| &= v_{B/A} = \text{magnitude of relative velocity of B } = 10\,\text{ft/s} \\
|\vec{\boldsymbol{a}}_{B/A}| &= a_{B/A} = \text{magnitude of relative acceleration of B } = 12\,\text{ft/s}^2.
\end{aligned}$$

Let $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ be the angular velocity and $\dot{\vec{\boldsymbol{\omega}}} = \dot{\omega}\hat{\boldsymbol{k}}$ be the angular acceleration of the board.

1. The relative velocity of end B of the board with respect to end A is

$$\begin{aligned}
\vec{\boldsymbol{v}}_{B/A} &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/A} = \omega\hat{\boldsymbol{k}} \times L(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) \\
&= \omega L(\cos\theta\,\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\boldsymbol{\imath}}) \\
\Rightarrow \quad \omega &= \frac{|\vec{\boldsymbol{v}}_{B/A}|}{L} = \frac{v_{B/A}}{L} = \frac{10\,\text{ft/s}}{10\,\text{ft}} = 1\,\text{rad/s}.
\end{aligned}$$

Note that we have taken the positive value for ω because the board is rotating counterclockwise at the instant of interest (it is given that the board has lost contact with point C).

Similarly, we can compute the angular acceleration:

$$\begin{aligned}
\vec{\boldsymbol{a}}_{B/A} &= \dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}}_{B/A} - \omega^2\vec{\boldsymbol{r}}_{B/A} \\
&= \dot{\omega}\hat{\boldsymbol{k}} \times L(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}) - \omega^2 L(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}) \\
&= \dot{\omega}L(\cos\theta\,\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\boldsymbol{\imath}}) - \omega^2 L(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}) \\
\Rightarrow \quad |\vec{\boldsymbol{a}}_{B/A}| &= \sqrt{(\dot{\omega}L)^2 + (\omega^2 L)^2} = a_{B/A} \text{ (given)} \\
\Rightarrow \quad a^2_{B/A} &= (\dot{\omega}L)^2 + (\omega^2 L)^2 \\
\Rightarrow \quad \dot{\omega} &= \sqrt{\frac{a^2_{B/A}}{L^2} - \omega^4} = \sqrt{\left(\frac{12\,\text{ft/s}^2}{10\,\text{ft}}\right)^2 - (1\,\text{rad/s})^4} \\
&= \pm 0.663\,\text{rad/s}^2.
\end{aligned}$$

Once again, we select the positive value for $\dot{\omega}$ since we assume that the board accelerates counterclockwise.

$$\boxed{\vec{\boldsymbol{\omega}} = 1\,\text{rad/s}\hat{\boldsymbol{k}}, \;\; \dot{\vec{\boldsymbol{\omega}}} = 0.663\,\text{rad/s}^2\hat{\boldsymbol{k}}}$$

2. The absolute velocity and the absolute acceleration of the end point B can be found as follows.

$$\begin{aligned}
\vec{\boldsymbol{v}}_B &= \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{v}}_{B/A} \\
&= v_A\hat{\boldsymbol{\imath}} + v_{B/A}(\cos\theta\,\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\boldsymbol{\imath}}) \\
&= 88\,\text{ft/s}\hat{\boldsymbol{\imath}} + 10\,\text{ft/s}(\frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}} - \frac{1}{\sqrt{2}}\hat{\boldsymbol{\imath}}) \\
&= 80.93\,\text{ft/s}\hat{\boldsymbol{\imath}} + 7.07\,\text{ft/s}\hat{\boldsymbol{\jmath}}.
\end{aligned}$$

$$\begin{aligned}
\vec{\boldsymbol{a}}_B &= \vec{\boldsymbol{a}}_A + \vec{\boldsymbol{a}}_{B/A} \\
&= \vec{\boldsymbol{a}}_A + \dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}}_{B/A} - \omega^2\vec{\boldsymbol{r}}_{B/A} \\
&= a_A\hat{\boldsymbol{\imath}} + \dot{\omega}\hat{\boldsymbol{k}} \times L(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}) - \omega^2 L(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}) \\
&= (a_A - \dot{\omega}L\sin\theta - \omega^2 L\cos\theta)\hat{\boldsymbol{\imath}} + (\dot{\omega}L\cos\theta - \omega^2 L\sin\theta)\,\hat{\boldsymbol{\jmath}} \\
&= \left(4.4\,\text{ft/s}^2 - \frac{0.66}{\text{s}^2}\cdot 10\,\text{ft}\cdot\frac{1}{\sqrt{2}} - \frac{1}{\text{s}^2}\cdot 10\,\text{ft}\cdot\frac{1}{\sqrt{2}}\right)\hat{\boldsymbol{\imath}} \\
&\quad + \left(\frac{0.66}{\text{s}^2}\cdot 10\,\text{ft}\cdot\frac{1}{\sqrt{2}} - \frac{1}{\text{s}^2}\cdot 10\,\text{ft}\cdot\frac{1}{\sqrt{2}}\right)\hat{\boldsymbol{\jmath}} \\
&= -7.34\,\text{ft/s}^2\hat{\boldsymbol{\imath}} - 2.40\,\text{ft/s}^2\,\hat{\boldsymbol{\jmath}}.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{v}}_B = (80.93\hat{\boldsymbol{\imath}} + 7.07\hat{\boldsymbol{\jmath}})\,\text{ft/s}, \quad \vec{\boldsymbol{a}}_B = -(7.34\hat{\boldsymbol{\imath}} + 2.40\hat{\boldsymbol{\jmath}})\,\text{ft/s}^2.}$$

3. Now, we can easily calculate the acceleration of the center-of-mass as follows.

$$\begin{aligned}
\vec{\boldsymbol{a}}_G &= \vec{\boldsymbol{a}}_A + \vec{\boldsymbol{a}}_{G/A} \\
&= a_A\hat{\boldsymbol{\imath}} + \dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}}_{G/A} - \omega^2\vec{\boldsymbol{r}}_{G/A} \\
&= a_A\hat{\boldsymbol{\imath}} + \dot{\omega}\hat{\boldsymbol{k}} \times \frac{L}{2}(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}) - \omega^2\frac{L}{2}(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}) \\
&= a_A\hat{\boldsymbol{\imath}} + \dot{\omega}\frac{L}{2}(\cos\theta\,\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\boldsymbol{\imath}}) - \omega^2\frac{L}{2}(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}) \\
&= 4.4\,\text{ft/s}^2\hat{\boldsymbol{\imath}} + 0.663\,\text{rad/s}^2\cdot\frac{10\,\text{ft}}{2}\cdot(\frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}} - \frac{1}{\sqrt{2}}\hat{\boldsymbol{\imath}}) \\
&\quad -(1\,\text{rad/s})^2\cdot\frac{10\,\text{ft}}{2}\cdot(\frac{1}{\sqrt{2}}\hat{\boldsymbol{\imath}} + \frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}}) \\
&= -1.48\,\text{ft/s}^2\hat{\boldsymbol{\imath}} - 1.19\,\text{ft/s}^2\,\hat{\boldsymbol{\jmath}}.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}}_G = -(1.48\hat{\boldsymbol{\imath}} + 1.19\hat{\boldsymbol{\jmath}})\,\text{ft/s}^2}$$

Comments: This problem is admittedly artificial. We, however, solve this problem to show kinematic calculations.

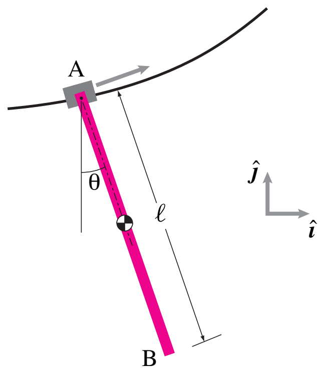


Figure 14.16:

Filename:sfig9-2-rodontrack

SAMPLE 14.5 Tracking motion. A cart moves along a suspended curved path. A rod AB of length $\ell = 1\,\text{m}$ hangs from the cart. End A of the rod is attached to a motor on the cart. The other end B hangs freely. The motor rotates the rod such that $\theta(t) = \theta_0 \sin(\lambda t)$ while the cart moves along the path such that $\vec{r}_A = t\hat{\imath} + \frac{t^3}{18}\hat{\jmath}$, where all variables ($r$, t, etc.) are nondimensional.

1. Find the velocity and acceleration of point B as a function of nondimensional time t.
2. Take $\theta_0 = \pi/3$ and $\lambda = 6$. Find and plot the position of the bar at $t = 0, 0.1, 0.3, 0.9, 1, 1.1, 1.2$, and 1.5. Find and draw $\vec{v}_B$ and $\vec{a}_B$ at the specified t.

Solution

1. The velocity and acceleration of point B are given by

$$\begin{aligned}\vec{v}_B &= \vec{v}_A + \vec{v}_{B/A} = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A} \\ \vec{a}_B &= \vec{a}_A + \dot{\vec{\omega}} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A}.\end{aligned}$$

Thus, in order to find the velocity and acceleration of point B, we need to find the velocity and acceleration of point A and the angular velocity and angular acceleration of the bar. We are given the position of point A and the angular position of the bar as functions of t. We can, therefore, find $\vec{v}_A$, $\vec{a}_A$, $\vec{\omega}$, and $\dot{\vec{\omega}}$ by differentiating the given functions with respect to t.

$$\vec{r}_A = t\hat{\imath} + \frac{t^3}{18}\hat{\jmath}$$

$$\Rightarrow \quad \vec{v}_A \equiv \dot{\vec{r}}_A = \hat{\imath} + (t^2/6)\hat{\jmath} \tag{14.9}$$

$$\Rightarrow \quad \vec{a}_A \equiv \dot{\vec{v}}_A = (t/3)\hat{\jmath} \tag{14.10}$$

and

$$\theta\hat{k} = \theta_0 \sin(\lambda t)\hat{k}$$

$$\Rightarrow \quad \vec{\omega} \equiv \dot{\theta}\hat{k} = \theta_0 \lambda \cos(\lambda t)\hat{k} \tag{14.11}$$

$$\Rightarrow \quad \dot{\vec{\omega}} \equiv \ddot{\theta}\hat{k} = -\theta_0 \lambda^2 \sin(\lambda t)\hat{k}. \tag{14.12}$$

So,

$$\begin{aligned}\vec{v}_B &= \vec{v}_A + \vec{\omega} \times \ell(\sin\theta\hat{\imath} - \cos\theta\hat{\jmath}) \\ &= \hat{\imath} + (t^2/6)\hat{\jmath} + \ell\dot{\theta}(\sin\theta\hat{\jmath} + \cos\theta\hat{\imath}) \\ &= (1 + \ell\dot{\theta}\cos\theta)\hat{\imath} + (t^2/6 + \ell\dot{\theta}\sin\theta)\hat{\jmath} \qquad (14.13) \\ \vec{a}_B &= \vec{a}_A + \ddot{\theta}\hat{k} \times \ell(\sin\theta\hat{\imath} - \cos\theta\hat{\jmath}) - \dot{\theta}^2\ell(\sin\theta\hat{\imath} - \cos\theta\hat{\jmath}) \\ &= (t/3)\hat{\jmath} + \ell\ddot{\theta}\sin\theta\hat{\jmath} + \ell\ddot{\theta}\cos\theta\hat{\imath} + \ell\dot{\theta}^2\sin\theta\hat{\imath} + \ell\dot{\theta}^2\cos\theta\hat{\jmath} \\ &= \ell(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\hat{\imath} + [t/3 + \ell(\ddot{\theta}\sin\theta + \ddot{\theta}^2\cos\theta)]\hat{\jmath}\end{aligned}$$

(14.14)

where $\theta = \theta_0 \sin(\lambda t)$, $\dot{\theta} = \theta_0\lambda\cos(\lambda t)$, and $\ddot{\theta} = -\theta_0\lambda^2\sin(\lambda t) = -\lambda^2\theta$. Thus $\vec{v}_B$ and $\vec{a}_B$ are functions of t.

2. The position of the rod at any time t is specified by the position of the two end points A and B (or alternatively, the position of A and the angle of the rod θ). The position of point A is easily determined by substituting the value of t in the given expression for $\vec{r}_A$. The position of end B is given by

$$\begin{aligned}\vec{r}_B &= \vec{r}_A + \vec{r}_{B/A} = t\hat{\imath} + (t^3/18)\hat{\jmath} + \ell(\sin\theta\hat{\imath} - \cos\theta\hat{\jmath}) \\ &= (t + \ell\sin\theta)\hat{\imath} + (t^3/18 - \ell\cos\theta)\hat{\jmath}.\end{aligned}$$

To compute the positions, velocities, and accelerations of end points A and B at the given instants, we first compute θ, $\dot{\theta}$, and $\ddot{\theta}$, and then substitute them in the expressions for $\vec{r}_A$, $\vec{r}_B$, $\vec{v}_A$, $\vec{v}_B$, $\vec{a}_A$, and $\vec{a}_B$. A pseudocode for computer calculation is given below.

```
t = [0 0.1 0.3 0.9 1.0 1.1 1.2 1.5]
theta0=pi/3, L=.5, lam=6
for each t, compute
  theta = theta0*sin(lam*t)
  w = lam*theta0*cos(lam*t)
  wdot = -lam^2*theta
       % Position of A and B
  xA=t, yA=t^3/18
  xB = xA + L*sin(theta)
  yB = yA - L*cos(theta)
       % Velocity of A and B
  uA = 1,   vA = t^2/6
  uB = uA + L*w.*cos(theta)
  vB = vA + L*w.*sin(theta)
       % Acceleration of A and B
  axA = 0,   ayA = t/3
  axB = L*wdot*cos(theta) - L*w^2*sin(theta)
  ayB = ayA + L*wdot*sin(theta) + L*w^2*cos(theta)
```

From the above calculation, we get the desired quantities at each t. For example, at $t = 0$ we get,

```
xA = 0, yA = 0, xB = 0, yB = -0.5
uA = 1, vA = 0, uB = 4.14, vB = 0, axB = 0, ayB = 19.74
```

which means,

$$\vec{r}_A = \vec{0}, \quad \vec{r}_B = -0.5\hat{\jmath}, \quad \vec{v}_A = \hat{\imath}, \quad \vec{v}_B = 4.14\hat{\imath}, \quad \vec{a}_B = 19.74\hat{\jmath}.$$

The position of the bar, the velocity vectors at points A and B, and the acceleration vector at B, thus obtained, are shown in Fig. 14.17 graphically. *

* We can take several values of t, say 400 equally spaced values between $t = 0$ and $t = 4$, and draw the bar at each t to see its motion and the trajectory of its end points. Fig. 14.18 shows such a graph.

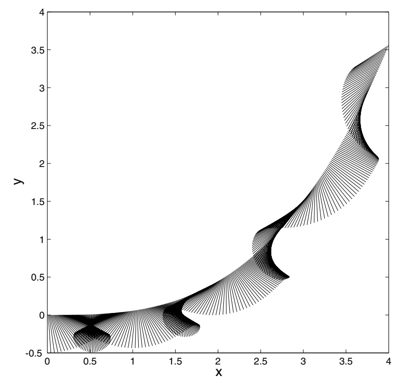


Figure 14.18: Graph of closely spaced configuration of the bar between $t = 0$ to $t = 4$.

Filename:sfig9-1-rodconfig

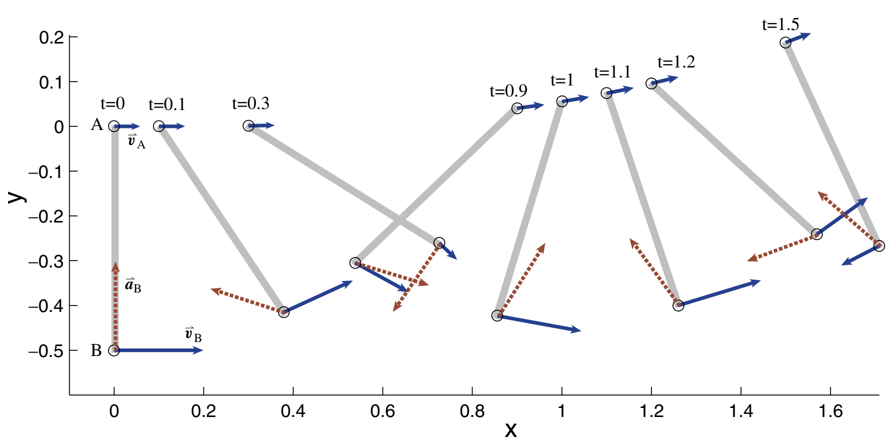


Figure 14.17: Position, velocity of the end points A and B, and acceleration of point B at various time instants.

Filename:rodvelacc

□

14.2 General planar mechanics of a rigid-object

We now apply the kinematics ideas of the last section to the general mechanics principles in Table I in the inside cover. The goal is to understand the relation between forces and motion for a planar object in general 2-D motion. The simple measures* of motion will be the displacement, velocity and acceleration of one reference point $0'$ on the object ($\vec{r}_{0'}$, $\vec{v}_{0'}$ and $\vec{a}_{0'}$) and the rotation, angular velocity, and angular acceleration of the object (θ, $\vec{\omega}$ and $\vec{\alpha}$).

We will treat all bodies as if they are squished into the plane and moving in the plane. But the analysis is sensible for a object that is symmetric with respect to the plane containing the velocities (see Box 14.2 on page 716).

*Advanced aside: What we call "simple measures" are examples of "generalized coordinates" in more advanced books. The idea sounds intimidating, but is simply this: If something can only move in a few ways, you should only keep track of the motion with that many variables. The kinematics of a rigid object (Sect. 14.1) allow us to "evaluate" the motion quantities, namely linear momentum, angular momentum, kinetic energy, and their rates of change in terms of these "simple measures". By "evaluate" we mean express the motion quantities in terms of these measures. The alternative is as sums over Avogadro's number of particles (There are on the order of 10^{23} atoms in a typical engineering part.). Even neglecting atoms and viewing matter as continuous we would still be stuck with integrals over complicated regions if we did not describe the motion with as few variables as possible. In the case of 2-D rigid object motion, the position of a reference point (x and y) with the rotation θ is called a set of *minimal coordinates*. These, and their time derivatives are the minimal information needed to describe all important mechanics motion quantities.

The balance laws for a rigid object

As always, once you have defined the system and the forces acting on it by drawing a free object diagram, the basic momentum balance equations are applicable (and exact for engineering purposes). Namely,

$$\text{Linear momentum balance:} \qquad \sum \vec{F}_i = \dot{\vec{L}} \qquad \text{and}$$

$$\text{Angular momentum balance:} \qquad \sum \vec{M}_{i/0} = \dot{\vec{H}}_{\text{O}}.$$

The same point 0, any point, is used on both sides of the angular momentum balance equation.

We also have power balance which, for a system with no internal energy or dissipation, is

$$\text{Power balance:} \qquad P = \dot{E}_{\text{K}}.$$

The left hand sides of the momentum balance equations are evaluated the same way, whether the system is composed of one object or many, whether the bodies are deformable or not, and whether the points move in straight lines, circles, hither and thither, or not at all. It is the right hand sides of the momentum equations that involve the motion. Similarly, in the energy balance equations the applied power P only depends on the position of the forces and the motions of the material points at those positions. But the kinetic energy E_{K} and its rate of change depend on the motion of the whole system. You already know how to evaluate the momenta and energy, and their rates of change, for a variety of special cases, namely

- Systems composed of particles where all the positions and accelerations are known (Chapter 5);
- Systems where all points have the same acceleration. That is, the system moves like a rigid object that does not rotate (Chapter 6); and
- Systems where all points move in circles about the same fixed axis. That is, the system moves like a rigid object that is rotating about a fixed skewer (Chapters 7 and 8).

Now we go on to consider the general 2-D motions of a planar rigid object. Its now a little harder to evaluate $\vec{\boldsymbol{L}}$, $\dot{\vec{\boldsymbol{L}}}$, $\vec{\boldsymbol{H}}_{\mathrm{O}}$, $\dot{\vec{\boldsymbol{H}}}_{\mathrm{O}}$, E_{K} and $\dot{E}_{\mathrm{K}}$. But not much.

Summary of the motion quantities

Table I in the back of the book describes the motion quantities for various special cases, including the planar motions we consider in this chapter. Most relevant is row (7).

The basic idea is that the momenta for general motion, which involves translation and rotation, is the sum of the momenta (both linear and angular, and their rates of change too) from those two effects. Namely, the linear momentum is described, as for any system with any motion, by the motion of the center-of-mass

$$\vec{\boldsymbol{L}} = m_{\mathrm{tot}}\vec{\boldsymbol{v}}_{\mathrm{cm}} \quad \text{and} \quad \dot{\vec{\boldsymbol{L}}} = m_{\mathrm{tot}}\vec{\boldsymbol{a}}_{\mathrm{cm}}, \tag{14.15}$$

and the angular momentum has two contributions, one from the motion of the center-of-mass and one from rotation of the object about the center of mass,

$$\vec{\boldsymbol{H}}_{\mathrm{O}} = \overbrace{\vec{\boldsymbol{r}}_{\mathrm{cm/O}} \times (m_{\mathrm{tot}}\vec{\boldsymbol{v}}_{\mathrm{cm}})}^{\text{Angular momentum due to motion of the center-of-mass}} + \overbrace{I_{zz}^{\mathrm{cm}}\vec{\boldsymbol{\omega}}}^{\text{Angular momentum due to motion relative to the center-of-mass}} \tag{14.16}$$

$$\text{and} \quad \dot{\vec{\boldsymbol{H}}}_{\mathrm{O}} = \vec{\boldsymbol{r}}_{\mathrm{cm/O}} \times (m_{\mathrm{tot}}\vec{\boldsymbol{a}}_{\mathrm{cm}}) + I_{zz}^{\mathrm{cm}}\dot{\vec{\boldsymbol{\omega}}}. \tag{14.17}$$

An important special case for the angular momentum evaluation is when the reference point is coincident with the center-of-mass. Then the angular momentum and its rate of change simplify to

$$\vec{\boldsymbol{H}}_{\mathrm{cm}} = I_{zz}^{\mathrm{cm}}\vec{\boldsymbol{\omega}} \qquad \text{and} \qquad \dot{\vec{\boldsymbol{H}}}_{\mathrm{cm}} = I_{zz}^{\mathrm{cm}}\dot{\vec{\boldsymbol{\omega}}}. \tag{14.18}$$

The kinetic energy and its rate of change are given by

$$E_{\mathrm{K}} = \overbrace{\frac{1}{2}m_{\mathrm{tot}}\underbrace{v_{cm}^2}_{\vec{\boldsymbol{v}}_{\mathrm{cm}}\cdot\vec{\boldsymbol{v}}_{\mathrm{cm}}}}^{\text{kinetic energy from center-of-mass motion}} + \overbrace{\frac{1}{2}I_{zz}^{\mathrm{cm}}\omega^2}^{\text{kinetic energy relative to the center-of-mass}} \tag{14.19}$$

$$\text{and} \quad \dot{E}_{\mathrm{K}} = m_{\mathrm{tot}}\underbrace{v_{cm}\dot{v}_{cm}}_{\vec{\boldsymbol{v}}_{\mathrm{cm}}\cdot\vec{\boldsymbol{a}}_{\mathrm{cm}}} + I_{zz}^{\mathrm{cm}}\omega\dot{\omega} \tag{14.20}$$

The relations above are easily derived from the general center of mass theorems at the end of chapter 5 (see box 14.2 on page 718 for some of these derivations).

Equations of motion

Putting together the general balance equations and the expressions for the motion quantities we can now write linear momentum balance, angular momentum balance and power balance as:

$$
\begin{aligned}
\text{LMB :} \quad & \sum \vec{F}_i &=& \; m_{\text{tot}} \vec{a}_{\text{cm}}, & \text{(a)} \\
\text{AMB :} \quad & \sum \vec{M}_{\text{O}} &=& \; \vec{r}_{\text{cm/O}} \times (m_{\text{tot}} \vec{a}_{\text{cm}}) + I_{zz}^{\text{cm}} \dot{\vec{\omega}} & \text{(b)} \\
\text{or} \quad & \sum \vec{M}_{\text{cm}} &=& \; I_{zz}^{\text{cm}} \dot{\vec{\omega}}, & \\
\text{and Power :} \quad & \vec{F}_{\text{tot}} \cdot \vec{v}_{\text{cm}} + \vec{\omega} \cdot \vec{M}_{\text{cm}} &=& \; m_{\text{tot}} v \dot{v} + I_{zz}^{\text{cm}} \omega \dot{\omega}. & \text{(c)}
\end{aligned}
\tag{14.21}
$$

Independent equations?

Equations are only independent if no one of them can be derived from the others. When counting equations and unknowns one needs to make sure one is writing independent equations. How many independent equations are in the set eqns. (14.21)abc applied to one free object diagram? The short answer is 3.

The linear momentum balance equation eqn. (14.21)a yields two independent equations by dotting with any two non-parallel vectors (say, $\hat{\imath}$ and $\hat{\jmath}$). Dotting with a third vector yields a dependent equation.

For any one reference point the angular momentum equation eqn. (14.21)a yields one scalar equation. It is a vector equation but always has zero components in the $\hat{\imath}$ and $\hat{\jmath}$ directions. But angular momentum equation can yield up to three independent equations by being applied to any set of three non-colinear points.

The power balance equation is one scalar equation.

In total, however, the full set of equations above only makes up a set of three independent equations.

To avoid thinking about what is or is not an independent set of equations some people prefer to stick with one of the canonical sets of independent equations:

- The coordinate based set ("old standard")
  - {LMB}$\cdot\hat{\boldsymbol{\imath}}$ or, equivalently, $\sum F_x = m_{\text{tot}} a_{\text{cm}_x}$,
  - {LMB}$\cdot\hat{\boldsymbol{\jmath}}$ or, equivalently, $\sum F_y = m_{\text{tot}} a_{\text{cm}_y}$, and
  - {AMB}$\cdot\hat{\boldsymbol{k}}$ or, equivalently, $\sum M_{\text{cm}} = I_{zz}^{\text{cm}}\dot{\omega}$.
- Moment only (good for eliminating unknown reaction forces)
  - {AMB about pt A}$\cdot\hat{\boldsymbol{k}}$ (A is any point, on or off the object)
  - {AMB about pt B}$\cdot\hat{\boldsymbol{k}}$ (B is any other point)
  - {AMB about pt C}$\cdot\hat{\boldsymbol{k}}$ (C is a third point not on the line AB)
- Two moments and a force component
  - {AMB about pt A}$\cdot\hat{\boldsymbol{k}}$ (A is any point, on or off the object)
  - {AMB about pt B}$\cdot\hat{\boldsymbol{k}}$ (B is any other point)
  - {LMB}$\cdot\hat{\boldsymbol{\lambda}}$ (where $\hat{\boldsymbol{\lambda}}$ is not perpendicular to the line AB)
- Two force components and a moment (also good for eliminating unknown forces)
  - {LMB}$\cdot\hat{\boldsymbol{\lambda}}_1$ (where $\hat{\boldsymbol{\lambda}}_1$ is any unit vector)
  - {LMB}$\cdot\hat{\boldsymbol{\lambda}}_2$ (where $\hat{\boldsymbol{\lambda}}_2$ is any other unit vector)
  - {AMB about pt A}$\cdot\hat{\boldsymbol{k}}$ qquad (A is any point, on or off the object)

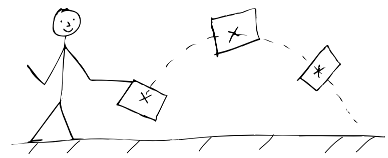


Figure 14.19: The X marked at the center-of-mass of a thrown spinning clipboard follows a parabolic trajectory.

Filename:tfigure-clipboard

Any of these will always do the job. The power balance equation is often used as a consistency check rather than an independent equation.

From a theoretical point of view one might ask the related question of which of the equations of motion can be derived from the others. There are many answers. Here are some of them:

- Power balance follows from LMB and AMB,
- AMB about three non-colinear points implies LMB, and
- LMB and power balance yield AMB

Interestingly, there is no way to derive angular momentum balance from linear momentum balance without the questionable microscopic assumptions discussed in box 11.5 on page 540.

Some simple examples

Here we consider some simple examples of unconstrained motion of a rigid object.

Example: **The simplest case: no force and no moment.**
If the net force and moment applied to a object are zero we have:

$$\begin{aligned} \text{LMB} \quad \Rightarrow \vec{\mathbf{0}} &= m_{\text{tot}}\vec{\boldsymbol{a}}_{\text{cm}} \quad \text{and} \\ \text{AMB} \quad \Rightarrow \vec{\mathbf{0}} &= I_{zz}^{\text{cm}}\dot{\omega}\hat{\boldsymbol{k}} \end{aligned}$$

so $\vec{\boldsymbol{a}}_{\text{cm}} = \vec{\mathbf{0}}$ and $\dot{\omega} = 0$ and the object moves at constant speed in a constant direction with a constant rate of rotation, all determined by the initial conditions. Throw an object in space and its center-of-mass goes in a straight line and it spins at constant rate (subject to the 2-D restrictions of this chapter).

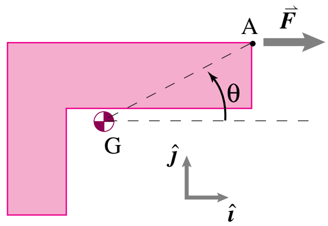


Figure 14.20: The only force applied to the object is the constant force $\vec{F} = F\hat{\imath}$ applied at point A. The resulting motion is a constant acceleration of the center-of-mass $\vec{a}_G = (F/m)\hat{\imath}$ and an oscillatory motion of θ identical to that of a pendulum hinged at G.

Filename:tfigure-constforce

Example: **Constant force applied to the center-of-mass.**
In this case angular momentum balance about the center-of-mass again yields that the rotation rate is constant. Linear momentum balance is now the same as for a particle at the center-of-mass, *i.e.*, the center-of-mass has a parabolic trajectory.

Near-earth (constant) gravity provides a simple example. An 'X' marked at the center-of-mass of a clipboard tossed across a room follows a parabolic trajectory (see Fig. 14.19).

Example: **Constant force not at the center-of-mass.**
Assume the only force applied to an object is a constant force $\vec{F} = F\hat{\imath}$ at A (see Fig. 14.20). Then linear momentum balance gives us that

$$\sum \vec{F}_i = \dot{\vec{L}} \quad \Rightarrow \quad F\hat{\imath} = m\vec{a}_G \quad \Rightarrow \quad \vec{a}_G = F/m\hat{\imath} = \text{constant}.$$

So if the object starts at rest, the point G will move in a straight line in the $\hat{\imath}$ direction (The common intuition that point G will be pulled up is incorrect). Angular momentum about the center-of-mass gives

$$\begin{aligned} \sum \vec{M}_{\mathrm{cm\,i}} = \dot{\vec{H}}_{\mathrm{cm}} \quad &\Rightarrow \quad \left\{\vec{r}_{A/G} \times F\hat{\imath} = I_{zz}^{\mathrm{cm}}\ddot{\theta}\hat{k}\right\} \\ \{\}\cdot\hat{k} \quad &\Rightarrow \quad \ddot{\theta} + \frac{F\ell}{I_{zz}^{\mathrm{cm}}}\sin\theta = 0, \end{aligned}$$

with $\ell = |\vec{r}_{A/G}|$, which is the pendulum equation. That is, the object can swing back and forth about $\theta = 0$ just like a pendulum, approximately sinusoidally if the angle θ starts small and with $\dot{\theta}$ initially also small. [One might wonder how to do this experiment. One way would be with a jet on a space craft that keeps re-orienting itself to keep in a constant spatial direction as the object changes orientation. Another would be with a string tied to A and pulled from a great distance.]

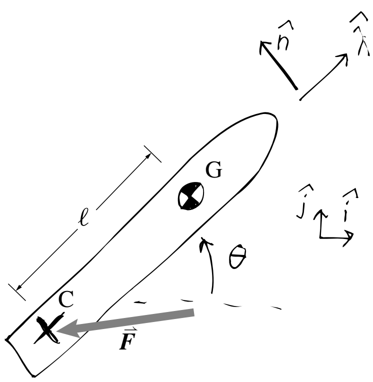


Figure 14.21: A rocket is pointed in the direction λ which makes and angle θ with the positive x axis. The position and velocity of the center-of-mass at G are called $\vec{r}$ and $\vec{v}$. The velocity of the tail is $\vec{v}_C$

Filename:tfigure-rocket

Example: **The flight of an arrow or rocket.**
As a primitive model of an arrow or rocket assume that the only force is from drag on the fins at C and that this force opposes motion according to

$$\vec{F} = -c\vec{v}_C$$

where c is a drag coefficient (see Fig. 14.21). From linear momentum balance we have:

$$\begin{aligned} \sum \vec{F}_i = \dot{\vec{L}} \quad \Rightarrow \quad \vec{F} &= m\vec{a} \\ -c\vec{v}_C &= m\dot{\vec{v}} \\ m\dot{\vec{v}} &= -c\left(\vec{v} + \vec{\omega} \times \vec{r}_{C/G}\right) \\ &= -c\left(\vec{v} + \dot{\theta}\hat{k} \times (-\ell\hat{\lambda})\right) \\ (\hat{k} \times \hat{\lambda} = \hat{n}) \quad \Rightarrow \quad \dot{\vec{v}} &= \frac{c}{m}\left(\dot{\theta}\ell\hat{n} - \vec{v}\right). \end{aligned}$$

So if $\vec{v}$, θ and $\dot{\theta}$ are known the acceleration $\dot{\vec{v}}$ is calculated by the formula above.

Similarly angular momentum balance about G gives

$$\begin{aligned} \sum \vec{M}_G = \dot{\vec{H}}_G \quad \Rightarrow \quad \{\vec{r}_{C/G} \times \vec{F} &= I_{zz}^{\mathrm{cm}}\dot{\omega}\hat{k}\} \\ \{\}\cdot\hat{k} \quad \Rightarrow \quad I_{zz}^{\mathrm{cm}}\dot{\omega} &= \vec{r}_{C/G} \times \vec{F}\cdot\hat{k}. \end{aligned}$$

Then, making the same substitutions as before for $\vec{r}_{C/G}$ and $\vec{F}$ we get

$$\dot{\omega} = \frac{c\ell}{I_{zz}^{\mathrm{cm}}}\left(\hat{\lambda} \times \vec{v}\cdot\hat{k} - \dot{\theta}\ell\right)$$

which determines the rate of change of ω if the present values of $\vec{v}$, θ and $\dot{\theta}$ are known.

Setting up differential equations for solution

If one knows the forces and torques on a object in terms of the bodies position, velocity, orientation and angular velocity one then has a ‘closed’ set of differential equations. That is, one has enough information to define the equations for a mathematician or a computer to solve them.

The full set of differential equations is gathered from linear and angular momentum balance and also from simple kinematics. Namely, one has the following set of 6 first order differential equations:

$$\begin{aligned}
\dot{x} &= v_x, \\
\dot{v}_x &= F_x/m, \\
\dot{y} &= v_y, \\
\dot{v}_y &= F_y/m, \\
\dot{\theta} &= \omega, \text{ and} \\
\dot{\omega} &= M_{\text{cm}}/I_{zz}^{\text{cm}},
\end{aligned}$$

where the positions and velocities are the positions and velocities of the center-of-mass. The expressions for F_x, F_y, and M_{cm} may well be complicated, as in the rocket example above.

14.2 *THEORY*
2-D mechanics makes sense in a 3-D world

The math for two-dimensional mechanics analysis is simpler than the math for three-dimensional analysis. And thus easier to learn first. But we do actually live in a three-dimensional world you might wonder at the utility of learning something that is not right. There are three answers.

1. Two dimensional analysis can give partial information about the three-dimensional system that is exactly the same as the three-dimensional analysis would give by *projection*, no matter what the motion;
2. if the motion is planar the 2-D kinematics can be used; and
3. if the object is planar or symmetric about the motion plane, and any constraints that hold the object are also symmetric about the motion plane, the 2-D analysis is not only correct, but complete.

Of course no machine is exactly planar or exactly symmetric, but if the approximation seems reasonable most engineers will accept a small loss in accuracy for great gain in simplicity.

a) Projection

First lets relax our assumption of 2-D motion. Consider arbitrary 3-D motions of an arbitrarily complex system. If we take the dot product of the linear momentum equations with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ and the angular momentum balance equation with $\hat{\boldsymbol{k}}$ we get

$$\begin{aligned}
&\left\{\sum \vec{F}_i = \sum m_i \vec{a}_i\right\} \cdot \hat{\imath} \;\Rightarrow \sum F_{ix} = \sum m_i a_{ix}, && \text{(a)}\\
&\left\{\sum \vec{F}_i = \sum m_i \vec{a}_i\right\} \cdot \hat{\jmath} \Rightarrow \sum F_{iy} = \sum m_i a_{iy}, \quad \text{and} && \text{(b)}\\
&\left\{\sum \vec{r}_i \times \vec{F}_i = \sum \vec{r}_i \times m_i \vec{a}_i\right\} \cdot \hat{k}\\
&\quad \Rightarrow \sum r_{ix}F_{iy} - r_{iy}F_{ix} = \sum m_i (r_{ix}a_{iy} - r_{iy}a_{ix}). && \text{(c)}
\end{aligned} \tag{14.22}$$

These are exactly the equations of 2-D mechanics. That is, if we only consider the planar components of the forces, the planar components of the positions, and the planar components of the motions, we get a correct but partial set of the 3-D equations. In this sense 2-D analysis is correct but incomplete.

b) Planar motion

If all the velocities of the parts of a 3-D system have no z component the motion is planar (in the xy plane). Thus the right-hand sides of eqns. (14.22) are not just projections, but the whole story. Further, in the case of rigid-object motion, the 2-D kinematics equation

$$\vec{v}_P = \vec{v}_G + \omega\hat{k} \times \vec{r}_{P/G} = \vec{v}_G + \omega\hat{k} \times \left(r_{P/G_x}\hat{\imath} + r_{P/G_y}\hat{\jmath}\right) \tag{14.23}$$

also applies (the z component of the position drops out of the cross product) and the expression for, say, the z component of the angular momentum of a object about its center-of-mass is

$$H_{\text{cm}z} = I_{zz}^{\text{cm}}\omega.$$

Differentiating, or adding up the $m_i\vec{a}_i$ terms we get,

$$H_{\text{cm}z} = I_{zz}^{\text{cm}}\omega.$$

Similarly, the z component of the full angular momentum balance equation for a 3-D rigid object in planar motion is the same as the z component of eqn. (14.21)b.

$$\sum \vec{M}_O \cdot \hat{k} = \left(\vec{r}_{\text{cm}/O} \times (m_{\text{tot}}\vec{a}_{\text{cm}})\right) \cdot \hat{k} + I_{zz}^{\text{cm}}\dot{\omega}$$

So for planar motion of 3-D rigid bodies one can do a correct 2-D analysis with the full ease of analyzing a planar object.

But this result is deceptively simple. The free object diagram in 3-D most likely shows forces in the z direction, pairs of forces in the x or y directions that are applied at points with the same x and y coordinates but different z values, or moments with components in the x or y directions. Full information about these force and moment components can't be found from 2-D analysis. That is,

> the nature of the forces that it takes to *keep* a system in planar motion can't be found from a planar analysis.

For example, a system rotating about the z axis which is statically balanced but is dynamically imbalanced (see section **??**) has no *net* x or y reaction force, as a planar analysis would reveal, yet the bearing reaction forces are not zero.

Another example would be a plan view of a car in a turn (assuming a stiff suspension). A 2-D analysis could be accurate, but would no be complete enough to describe the tire reaction forces needed to keep the car flat.

c) Symmetric bodies and planar bodies

If the rigid object has all its mass in the xy plane, or its mass is symmetrically distributed about the xy plane, and it is in planar motion in the xy plane then

$$\sum m_i a_{iz} = 0 \quad \text{and}$$
$$\left\{\sum \vec{r}_i \times m_i\vec{a}_i\right\} \cdot \hat{\imath} = 0 \quad \text{and} \quad \left\{\sum \vec{r}_i \times m_i\vec{a}_i\right\} \cdot \hat{\jmath} = 0$$

where $\vec{r}$ is measured relative to any point in the plane. Thus, by linear and angular momentum balance,

$$\sum F_z = 0 \quad \text{and}$$
$$\left\{\sum \vec{r}_i \times \vec{F}_i\right\} \cdot \hat{\imath} = 0 \quad \text{and} \quad \left\{\sum \vec{r}_i \times \vec{F}_i\right\} \cdot \hat{\jmath} = 0$$

so

> A planar object or a symmetric object in planar motion needs no force in the z direction and no moment in the x or y direction to keep it in the plane.

Systems that are symmetric or flat and moving in an approximately planar manner, are thus both accurately and completely modeled with a 2-D analysis. A slight generalization of the result is to any object or collection of objects whose center's of mass are on the plane and each of which is dynamically balanced for rotation about a $\hat{k}$ axis through its center-of-mass.

SAMPLE 14.6 Free planar motion. A rigid rod of length $\ell = 1\,\text{m}$ and mass $m_r = 1\,\text{kg}$, and a rigid square plate of side 1 m and mass $m_p = 10\,\text{kg}$ are launched in motion on a frictionless plane (*e.g.*, an ice hockey rink) with exactly the same initial velocities $\vec{v}_{cm}(0) = 10\,\text{m/s}\hat{\imath} + 1\,\text{m/s}\hat{\jmath}$ and $\vec{\omega}(0) = 1\,\text{rad/s}\hat{k}$. Both the rod and the plate have their center-of-mass at the baseline at $t = 0$.

1. Which of the two is farther from the base line in 3 seconds and which one has undergone more number of revolutions?
2. Find and draw the position of the bar at $t = 1$ sec and at $t = 3$ sec.

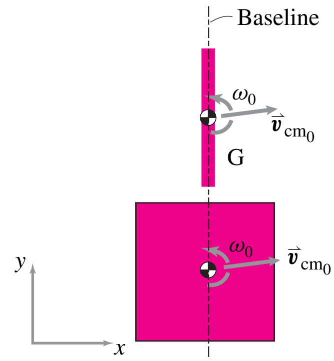


Figure 14.22:
Filename:sfig9-2-rodandplate

Solution

1. The free-body diagram of the rod is shown in Fig. 14.23. There are no forces acting on the rod in the xy-plane. Although there is force of gravity and the normal reaction of the surface acting on the rod, these forces are inconsequential since they act normal to the xy-plane. Therefore, we do not include these forces in our free-body diagram . The linear momentum balance for the rod gives

$$\begin{aligned} \sum \vec{F} &= m_r \vec{a}_{cm} \\ \vec{0} &= m_r \dot{\vec{v}}_{cm} \\ \Rightarrow \quad \vec{v}_{cm} &= \int \vec{0}\, dt = \text{constant} = \vec{v}_{cm0} \\ \Rightarrow \quad \vec{r}_{cm} &= \int \vec{v}_{cm0}\, dt = \vec{r}_{cm0} + \vec{v}_{cm0} t \end{aligned} \tag{14.28}$$

It is clear from the analysis above that in the absence of any applied forces, the position of the body depends only on the initial position and the initial velocity. Since both the plate and the rod start from the same base line with the same initial velocity, they travel the same distance from the base line during any given time period; mass or its geometric distribution play no role in the motion. Thus the center-of-mass of the rod and the plate will be exactly the same distance ($|\vec{r}_{cm}(t) - \vec{r}_{cm0}| = |\vec{v}_{cm0} t|$) at time t. Similarly, the angular momentum balance about the center-of-mass of the rod gives

$$\begin{aligned} \sum \vec{M}_{cm} &= \dot{\vec{H}}_{cm} \\ \vec{0} &= I_{zz}^{cm} \dot{\vec{\omega}} \\ \Rightarrow \quad \vec{\omega} &= \int \vec{0}\, dt = \text{constant} = \vec{\omega}_0 = \dot{\theta}_0 \hat{k} \\ \Rightarrow \quad \theta &= \int \dot{\theta}_0\, dt = \theta_0 + \dot{\theta}_0 t \end{aligned} \tag{14.29}$$

Thus the angular position of the body is also, as expected, independent of the mass and mass distribution of the body, and depends entirely on the initial position and the initial angular velocity. Therefore, both the rod and the plate undergo exactly the same amount of rotation $\left(\theta(t) - \theta_0 = \dot{\theta}_0 t\right)$ during any given time.

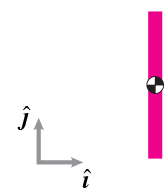


Figure 14.23:
Filename:sfig9-2-rodfbd

2. We can find the position of the rod at $t = 1$ s and $t = 3$ s by substituting these values of t in eqns. (14.28) and 14.29. For convenience, let us assume that $\vec{r}_{cm0} = \vec{0}$. From the initial configuration of the rod, we also know that $\theta_0 = 0$.

$$\begin{aligned} \vec{r}_{cm}(t = 1\,\text{s}) &= \vec{v}_{cm0} \cdot (1\,\text{s}) = (10\,\text{m/s}\hat{\imath} + 1\,\text{m/s}\hat{\jmath}) \cdot (1\,\text{s}) 10\,\text{m}\hat{\imath} + 1\,\text{m}\hat{\jmath} \\ \vec{r}_{cm}(t = 3\,\text{s}) &= \vec{v}_{cm0} \cdot (3\,\text{s}) = 30\,\text{m}\hat{\imath} + 3\,\text{m}\hat{\jmath}. \\ \theta(t = 1\,\text{s}) &= \dot{\theta}_0 \cdot (1\,\text{s}) = (1\,\text{rad/s}) \cdot (1\,\text{s}) = 1\,\text{rad} \\ \theta(t = 3\,\text{s}) &= \dot{\theta}_0 \cdot (3\,\text{s}) = 3\,\text{rad}. \end{aligned}$$

Accordingly, we show the position of the rod in Fig. 14.24.

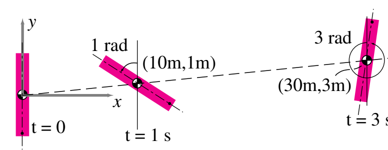


Figure 14.24:
Filename:sfig9-2-rodposition

14.3 *THEORY*

The center-of-mass theorems for 2-D rigid bodies

That all the particles in a system are part of one planar object in planar motion (in that plane) allows highly useful simplification of the expressions for the motion quantities, namely Eqns. 14.15 to 14.19. We can derive these expressions from the center-of-mass theorems of chapter 5. For completeness, we repeat some of those derivations as the start of the derivations here. To save space, we only use the integral ($\int$) forms for the general expressions; the derivations with sums ($\sum$) are similar. In all cases position, velocity, and acceleration are relative to a fixed point in space (that is $\vec{\boldsymbol{r}}$, $\vec{\boldsymbol{v}}$, and $\vec{\boldsymbol{a}}$ mean $\vec{\boldsymbol{r}}_{/0}$, $\vec{\boldsymbol{v}}_{/0}$, and $\vec{\boldsymbol{a}}_{/0}$ respectively).

Linear momentum.

Here we show that to evaluate linear momentum and its rate of change you only need to know the motion of the center of mass.

$$\begin{aligned} \vec{\boldsymbol{L}} &\equiv \int \vec{\boldsymbol{v}}\, dm = \int \frac{d}{dt}\vec{\boldsymbol{r}}\, dm = \frac{d}{dt}\int \vec{\boldsymbol{r}}\, dm = \frac{d}{dt}(m_{\text{tot}}\vec{\boldsymbol{r}}_{\text{cm}}) \\ &= m_{\text{tot}}\frac{d}{dt}\vec{\boldsymbol{r}}_{\text{cm}} = m_{\text{tot}}\vec{\boldsymbol{v}}_{\text{cm}} \end{aligned}$$

By identical reasoning, or by differentiating the expression above with respect to time,

$$\dot{\vec{\boldsymbol{L}}} = m_{\text{tot}}\vec{\boldsymbol{a}}_{\text{cm}}$$

Thus for linear momentum balance one need not pay heed to rotation, only the center-of-mass motion matters.

Angular momentum.

Here we attempt a derivation like the one above but get slightly more complicated results. For simplicity we evaluate angular momentum and its rate of change relative to the origin, but a very similar derivation would hold relative to any fixed point C.

$$\begin{aligned} \vec{\boldsymbol{H}}_{\text{O}} &\equiv \int \vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}\, dm \\ &= \int \left(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_{\text{cm}} + \vec{\boldsymbol{r}}_{\text{cm}}\right) \times \left(\vec{\boldsymbol{v}} - \vec{\boldsymbol{v}}_{\text{cm}} + \vec{\boldsymbol{v}}_{\text{cm}}\right) dm \\ &= \int \left(\vec{\boldsymbol{r}}_{/\text{cm}} + \vec{\boldsymbol{r}}_{\text{cm}}\right) \times \left(\vec{\boldsymbol{v}}_{/\text{cm}} + \vec{\boldsymbol{v}}_{\text{cm}}\right) dm \\ &= \int \vec{\boldsymbol{r}}_{/\text{cm}} \times \vec{\boldsymbol{v}}_{/\text{cm}}\, dm + \int \vec{\boldsymbol{r}}_{\text{cm}} \times \vec{\boldsymbol{v}}_{\text{cm}}\, dm \\ &\quad + \int \vec{\boldsymbol{r}}_{/\text{cm}} \times \vec{\boldsymbol{v}}_{\text{cm}}\, dm + \int \vec{\boldsymbol{r}}_{\text{cm}} \times \vec{\boldsymbol{v}}_{/\text{cm}}\, dm \\ &= \int \vec{\boldsymbol{r}}_{/\text{cm}} \times \vec{\boldsymbol{v}}_{/\text{cm}}\, dm + \vec{\boldsymbol{r}}_{\text{cm}} \times \vec{\boldsymbol{v}}_{\text{cm}} \int dm \\ &\quad + \underbrace{\left(\int \vec{\boldsymbol{r}}_{/\text{cm}}\, dm\right)}_{\vec{\boldsymbol{0}}} \times \vec{\boldsymbol{v}}_{\text{cm}} + \vec{\boldsymbol{r}}_{\text{cm}} \times \underbrace{\left(\int \vec{\boldsymbol{v}}_{/\text{cm}}\, dm\right)}_{\vec{\boldsymbol{0}}} \\ &= \int \vec{\boldsymbol{r}}_{/\text{cm}} \times \vec{\boldsymbol{v}}_{/\text{cm}}\, dm + \vec{\boldsymbol{r}}_{\text{cm}} \times \vec{\boldsymbol{v}}_{\text{cm}} m_{\text{tot}}. \end{aligned}$$

This much is true for any system in any motion. For a rigid object we know about the motions of the parts. Using the center-of-mass as a reference point we know that for all points on the object $\vec{\boldsymbol{v}}_{/\text{cm}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{/\text{cm}}$. Thus we can continue the derivation above, following the same reasoning as was used in chapter 7 for circular motion of rigid bodies:

$$\vec{\boldsymbol{H}}_{\text{O}} = \int \vec{\boldsymbol{r}}_{/\text{cm}} \times \left(\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{/\text{cm}}\right) dm + \vec{\boldsymbol{r}}_{\text{cm}} \times \vec{\boldsymbol{v}}_{\text{cm}} m_{\text{tot}}.$$

Using the identity for the triple cross product (see box **??** on page **??**) or using the geometry of the cross product directly with $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ as in chapters 7 and 8 we get

$$\vec{\boldsymbol{H}}_{\text{O}} = \omega\hat{\boldsymbol{k}} \int r_{/\text{cm}}^2\, dm + \vec{\boldsymbol{r}}_{\text{cm}} \times \vec{\boldsymbol{v}}_{\text{cm}} m_{\text{tot}}.$$

Then defining $I_{zz}^{\text{cm}} \equiv \int r_{/\text{cm}}^2\, dm$ we get the desired result:

$$\vec{\boldsymbol{H}}_{\text{O}} = \vec{\boldsymbol{r}}_{\text{cm}} \times \vec{\boldsymbol{v}}_{\text{cm}} m_{\text{tot}} + I_{zz}^{\text{cm}}\omega\hat{\boldsymbol{k}}.$$

A similar derivation, or differentiation of the result above (and using that $(\frac{d}{dt}\vec{\boldsymbol{r}}) \times \vec{\boldsymbol{v}} = \vec{\boldsymbol{v}} \times \vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}$) gives

$$\dot{\vec{\boldsymbol{H}}}_{\text{O}} = \vec{\boldsymbol{r}}_{\text{cm}} \times \vec{\boldsymbol{a}}_{\text{cm}} m_{\text{tot}} + I_{zz}^{\text{cm}}\dot{\omega}\hat{\boldsymbol{k}}.$$

The results above hold for any reference point, not just the origin of the fixed coordinate system. Thus, relative to a point instantaneously coinciding with the center-of-mass

$$\vec{\boldsymbol{H}}_{\text{cm}} = \underbrace{\vec{\boldsymbol{r}}_{\text{cm/cm}}}_{\vec{\boldsymbol{0}} \times \vec{\boldsymbol{v}}_{\text{cm}} m_{\text{tot}} + I_{zz}^{\text{cm}}\omega\hat{\boldsymbol{k}}} = I_{zz}^{\text{cm}}\omega\hat{\boldsymbol{k}}.$$

and similarly

$$\dot{\vec{\boldsymbol{H}}}_{\text{cm}} = I_{zz}^{\text{cm}}\dot{\omega}\hat{\boldsymbol{k}}.$$

Kinetic energy.

Unsurprisingly the expression for kinetic energy and it's rate of change are also simplified by derivations very similar to those above. Skipping the details (or leaving them as an exercise for the peppy reader):

$$\begin{aligned} E_{\text{K}} &\equiv \int \frac{1}{2}\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}}\, dm \\ &= \frac{1}{2}m_{\text{tot}}v_{cm}^2 + \frac{1}{2}I_{zz}^{\text{cm}}\omega^2 \end{aligned}$$

and

$$\begin{aligned} \dot{E}_{\text{K}} &\equiv \frac{d}{dt}E_{\text{K}} \\ &= m_{\text{tot}}v\dot{v} + I_{zz}^{\text{cm}}\omega\dot{\omega}. \end{aligned}$$

SAMPLE 14.7 A passive rigid diver. An experimental model of a rigid diver is to be launched from a diving board that is 3 m above the water level. Say that the initial velocity of the center-of-mass and the initial angular velocity of the diver can be controlled at launch. The diver is launched into the dive in almost vertical position, and it is required to be as vertical as possible at the very end of the dive (which is marked by the position of the diver's center-of-mass at 1 m above the water level). If the initial vertical velocity of the diver's center-of-mass is 3 m/s, find the required initial angular velocity for the vertical entry of the diver into the water.

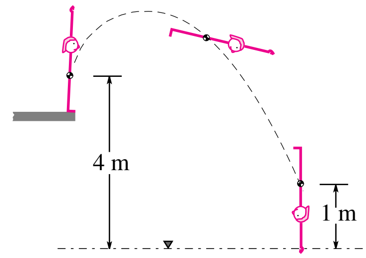


Figure 14.25:

Filename:sfig9-2-diver

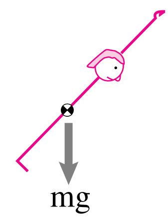


Figure 14.26:

Filename:sfig9-2-diver-a

Solution Once the diver leaves the diving board, it is in free flight under gravity, *i.e.*, the only force acting on it is the force due to gravity. The free-body diagram of the diver is shown in Fig. 14.26. The linear momentum balance for the diver gives

$$\begin{aligned}
\sum \vec{F} &= m\vec{a}_{\text{cm}} \\
-mg\hat{j} &= m\ddot{y}\hat{j} \\
\Rightarrow \quad \ddot{y} &= -g \\
\sum \vec{M}_{\text{cm}} &= \dot{\vec{H}}_{\text{cm}} \\
\vec{0} &= I_{zz}^{\text{cm}}\ddot{\theta}\hat{k} \\
\Rightarrow \quad \ddot{\theta} &= 0.
\end{aligned}$$

From these equations of motion, it is clear that the linear and the angular motions of the diver are uncoupled. We can easily solve the equations of motion to get

$$\begin{aligned}
y(t) &= y_0 + \dot{y}_0 - \frac{1}{2}gt^2 \\
\theta(t) &= \theta_0 + \dot{\theta}_0 t.
\end{aligned}$$

We need to find the initial angular speed $\dot{\theta}_0$ such that $\theta = \pi$ when $y = 1$ m (the center-of-mass position at the water entry). From the expression for $\theta(t)$, we get, $\dot{\theta}_0 = \pi/t$. Thus we need to find the value of t at the instant of water entry. We can find t from the expression for $y(t)$ since we know that $y = 1$ m at that instant, and that $y_0 = 3$ m and $\dot{y}_0 = 3$ m/s. We have,

$$\begin{aligned}
y &= y_0 + \dot{y}_0 - \frac{1}{2}gt^2 \\
\Rightarrow \quad t &= \frac{\dot{y}_0 \pm \sqrt{\dot{y}_0^2 + 2g(y_0 - y)}}{g} \\
&= \frac{3\,\text{m/s} \pm \sqrt{(3\,\text{m/s})^2 + 2\cdot 9.8\,\text{m/s}^2\cdot(3\,\text{m} - 1\,\text{m})}}{9.8\,\text{m/s}^2} \\
&= 1.15 \text{ or } -0.53\,\text{s}.
\end{aligned}$$

We reject the negative value of time as meaningless in this context. Thus it takes the diver 1.15 s to complete the dive. Since, the diver must rotate by π during this time, we have

$$\dot{\theta}_0 = \pi/t = \pi/(1.15\,\text{s}) = 2.73\,\text{rad/s}.$$

$$\boxed{\dot{\theta}_0 = 2.73\,\text{rad/s}}$$

14.4 *THEORY*

The work of a moving force and of a couple

The work of a force acting on a object from state one to state two is

$$W_{12} = \int_{t_1}^{t_2} P dt.$$

But sometimes we like to think not of the time integral of the power, but of the path integral of the moving force. So we rearrange this integral as follows.

$$\begin{aligned} W_{12} &= \int_{t_1}^{t_2} P dt \\ &= \int_{t_1}^{t_2} \vec{F} \cdot \underbrace{\vec{v} dt}_{d\vec{r}} \\ &= \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} \end{aligned} \qquad (14.24)$$

The validity of equation 14.24 depends on the force acting on the same material point of the moving object as it moves from position 1 to position 2; i.e., the force moves with the object. If the material point of force application changes with time, eqn. (14.24) is senseless and should be replaced with the following more generally applicable equation:

$$W_{12} \equiv \int_{t_1}^{t_2} P \, dt = \int_{t_1}^{t_2} \vec{F} \cdot \vec{v} dt \qquad (14.25)$$

where $\vec{v}$ is the velocity of the material point at the instantaneous location of the applied force.

Hand drags on a passing train: a subtlety

There is a subtle distinction between eqn. (14.24) and eqn. (14.25). As an example think of standing still and dragging your hand on a passing train. Your hand slows down the train with the force

$$\vec{F}_{\text{hand on train}}.$$

It might seem that the work of the hand on the train is zero because your hand doesn't move; work is force times distance and the distance is zero and eqn. (14.24) superficially evaluates to

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = 0.$$

But we have violated the condition for the validity of eqn. (14.24): the force be applied to a fixed material point as time progresses. Whereas on the train your hand smears a whole line of material points.

Clearly your hand does slow the train, so it must do (negative) work on the train, as eqn. (14.25) correctly shows because

$$P_{\text{force on train}} = \vec{F}_{\text{hand on train}} \cdot \vec{v}_{\text{train}} \neq 0.$$

The power of the hand force on the train is the force on the train dotted with the velocity of the train (*not* with the velocity of your hand. Thus, your hand does negative work on the train. eqn. (14.25) applies to the train and eqn. (14.24) does not.

On the other hand (so to speak) if one looks at the power of the force on the hand we have:

$$\vec{F}_{\text{train on hand}} = -\vec{F}_{\text{hand on train}}$$

while the velocity of the hand is zero so

$$P_{\text{force on hand}} = \vec{F}_{\text{train on hand}} \cdot \vec{v}_{\text{hand}} = 0.$$

So the train does *no* work on your hand since while your hand does (negative) work on the train. The difference, of course, is mechanical energy lost to heat.

Work of an applied torque

By thinking of an applied torque as really a distribution of forces, the work of an applied torque is the sum of the contributions of the applied forces. If a collection of forces equivalent to a torque is applied to one rigid object the power of these forces turns out to be $\vec{M} \cdot \vec{\omega}$. At a given angular velocity a bigger torque applies more power. And a given torque applies more power to a faster spinning object.

Here's a quick derivation for a collection of forces $\vec{F}_i$ that add to zero acting at points with positions $\vec{r}_i$ relative to a reference point on the object o′.

$$\begin{aligned} P &= \sum \vec{F}_i \cdot \vec{v}_i \\ &= \sum \vec{F}_i \cdot \left(\vec{v}_{o'} + \vec{\omega} \times \vec{r}_{i/o'}\right) \\ &= \vec{v}_{o'} \cdot \underbrace{\sum \vec{F}_i}_{\vec{0}} + \sum \vec{F}_i \cdot \left(\vec{\omega} \times \vec{r}_{i/o'}\right) \\ &= \sum \vec{\omega} \cdot \left(\vec{r}_{i/o'} \times \vec{F}_i\right) \\ &= \vec{\omega} \cdot \sum \vec{r}_{i/o'} \times \vec{F}_i \\ &= \vec{\omega} \cdot \vec{M}_{o'} \end{aligned} \qquad (14.26)$$

Work of a general force distribution

A general force distribution has, by reasoning close to that above, a power of:

$$P = \vec{F}_{\text{tot}} \cdot \vec{v}_{o'} + \vec{\omega} \cdot \vec{M}_{o'}. \qquad (14.27)$$

For a given force system applied to a given object in a given motion any point o′ can be used. The terms in the formula above will depend on o′, but the sum does not. Besides the center-of-mass, another convenient locations for o′ is a fixed hinge, in which case the applied force has no power.

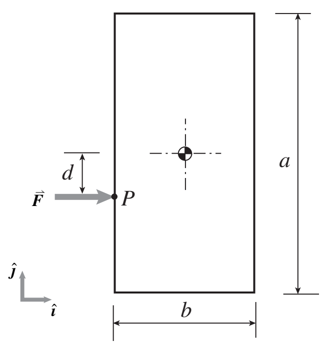


Figure 14.27:

Filename:sfig9-tumblingplate1

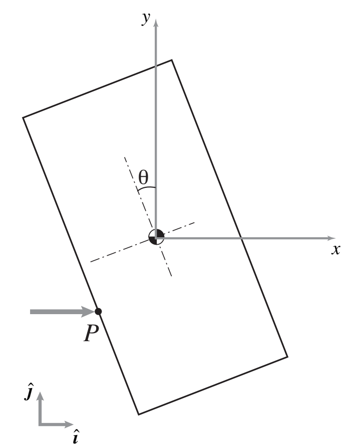


Figure 14.28: Free body diagram of the plate at some instant t when the longitudinal axis of the plate makes an angle $\theta(t)$ with the fixed vertical axis.

Filename:sfig9-tumblingplate1a

SAMPLE 14.8 A plate tumbling in space. A rectangular plate of mass $m = 0.5\,\text{kg}$, $I_{zz}^{cm} = 2.08 \times 10^{-3}\,\text{kg}\cdot\text{m}^2$, and dimensions $a = 200\,\text{mm}$ and $b = 100\,\text{mm}$ is pushed by a force $\vec{F} = 0.5\,\text{N}\hat{\imath}$, acting at $d = 30\,\text{mm}$ away from the mass-center, as shown in the figure. Assume that the force remains constant in magnitude and direction but remains attached to the material point P of the plate. There is no gravity.

1. Find the initial acceleration of the mass-center.
2. Find the initial angular acceleration of the plate.
3. Write the equations of motion of the plate (for both linear and angular motion).

Solution The only force acting on the plate is the applied force $\vec{F}$. Thus, Fig. 14.27 is also the free-body diagram of the plate at the start of motion.

1. From the linear momentum balance we get,

$$\sum \vec{F} = m\vec{a}_{cm}$$
$$\Rightarrow \quad \vec{a}_{cm} = \frac{\sum \vec{F}}{m} = \frac{0.5\,\text{N}\hat{\imath}}{0.5\,\text{kg}} = 1\,\text{m/s}^2\hat{\imath}.$$

which is the initial acceleration of the mass-center.

$$\boxed{\vec{a}_{cm} = 1\,\text{m/s}^2\hat{\imath}}$$

2. From the angular momentum balance about the mass-center, we get

$$\vec{M}_{cm} = \dot{\vec{H}}_{cm}$$
$$Fd\hat{k} = I_{zz}^{cm}\dot{\vec{\omega}}$$
$$\Rightarrow \quad \dot{\vec{\omega}} = \frac{Fd}{I_{zz}^{cm}}\hat{k} = \frac{0.5\,\text{N}\cdot 0.03\,\text{m}}{2.08\,\text{kg}\cdot\,\text{m}^2} = 7.2\,\text{rad/s}^2\hat{k}$$

which is the initial angular acceleration of the plate.

$$\boxed{\dot{\vec{\omega}} = 7.2\,\text{rad/s}^2\hat{k}}$$

3. To find the equations of motion, we can use the linear momentum balance and the angular momentum balance as we have done above. So, why aren't the equations obtained above for the linear acceleration, $\vec{a}_{cm} = F/m\hat{\imath}$, and the angular acceleration, $\dot{\vec{\omega}} = Fd/I_{zz}^{cm}\hat{k}$, qualified to be called equations of motion? Because, they are not valid for a general configuration of the plate during its motion. The expressions for the accelerations are valid only in the initial configuration (and hence those are initial accelerations).

   Let us first draw a free-body diagram of the plate in a general configuration during its motion (see Fig. **??**). Assume the center-of-mass to be displaced by $x\hat{\imath}$ and $y\hat{\jmath}$, and the longitudinal axis of the plate to be rotated by $\theta\hat{k}$ with respect to the vertical (inertial y-axis). The applied force remains horizontal and attached to the material point P, as stated in the problem. The linear momentum balance gives

$$\sum \vec{F} = m\vec{a}_{cm} \quad \Rightarrow \quad \vec{a}_{cm} = \frac{\sum \vec{F}}{m}$$
$$\text{or} \quad \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} = \frac{F}{m}\hat{\imath} \quad \Rightarrow \quad \ddot{x} = \frac{F}{m}, \quad \ddot{y} = 0.$$

   Since F/m is constant, the equations of motion of the center-of-mass indicate that the acceleration is constant and that the mass-center moves in the x-direction.

Similarly, we now use angular momentum balance to determine the rotation (angular motion) of the plate. The angular momentum balance about the mass-center give

$$\begin{aligned}\vec{M}_{cm} &= \dot{\vec{H}}_{cm} \\ \vec{r}_{P/cm} \times \vec{F} &= I_{zz}^{cm}\ddot{\theta}\hat{k}.\end{aligned}$$

Now,

$$\begin{aligned}\vec{r}_{P/cm} &= -r[\cos(\theta+\alpha)\hat{\imath} + \sin(\theta+\alpha)\hat{\jmath}] \\ \vec{F} &= F\hat{\imath} \\ \Rightarrow \quad \vec{r}_{P/cm} \times \vec{F} &= Fr\sin(\theta+\alpha)\hat{k}.\end{aligned}$$

Thus,

$$\ddot{\theta} = \frac{Fr}{I_{zz}^{cm}}\sin(\theta+\alpha)$$

where $r = \sqrt{d^2 + (b/2)^2}$ and $\alpha = \tan^{-1}(2d/b)$.

Thus, we have got the equations of motion for both the linear and the angular motion.

$$\boxed{\ddot{x} = \frac{F}{m}, \quad \ddot{y} = 0, \quad \ddot{\theta} = \frac{Fr}{I_{zz}^{cm}}\sin(\theta+\alpha)}$$

4. The equations of linear motion of the plate are very simple and we can solve them at once to get

$$\begin{aligned}x(t) &= x_0 + \dot{x}_0 t + \frac{1}{2}\frac{F}{m}t^2 \\ y(t) &= y_0 + \dot{y}_0 t.\end{aligned}$$

If the plate starts from rest ($\dot{x}_0 = 0,\ \dot{y}_0 = 0$) with the center-of-mass at the origin ($x_0 = 0,\ y_0 = 0$), then we have

$$x(t) = \frac{F}{2m}t^2, \quad \text{and} \quad y(t) = 0.$$

Thus the center-of-mass moves along the x-axis with acceleration F/m.

The equation of angular motion of the plate is, however, not so simple. In fact, it is a nonlinear ODE. It is very difficult to get an analytical solution of this equation. However, we can solve it numerically using, say, a Runge-Kutta ODE solver:

```
ODEs = {thetadot = w, wdot = (F*r/Icm)*sin(theta+a)}
IC   = {theta(0) = 0, w(0) = 0}
Set  F=.5, d=0.03; b=0.1; Icm=2.08e-03
compute r = sqrt(d^2+.25*b^2),   a = atan(2*d/b)
Solve ODEs with IC for t=0 to t=10
Plot theta(t)
```

The plot obtained from this calculation is shown in Fig. 14.30.

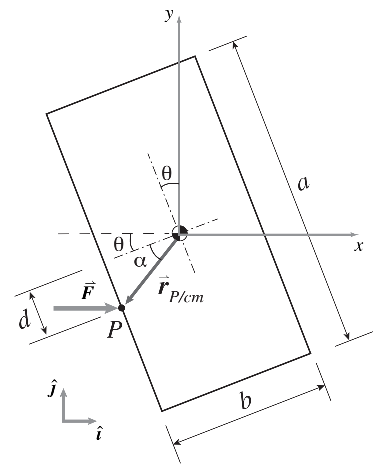


Figure 14.29: Geometry of the plate at the instant t when the longitudinal axis of the plate makes an angle $\theta(t)$ with the fixed y-axis. The position of point P is $\vec{r}_{P/cm}$ which makes a fixed angle $\alpha (= \tan^{-1}\frac{d}{b/2})$ with the transverse axis of the plate. This angle is shown here merely for ease of calculation.

Filename:sfig9-tumblingplate1b

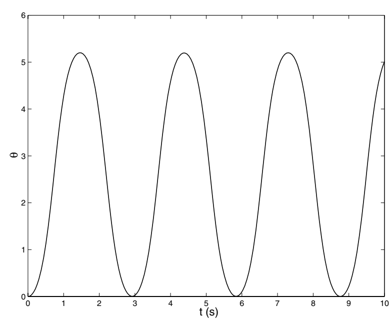


Figure 14.30:

Filename:sfig9-2-odesoln

SAMPLE 14.9 Impulse-momentum. Consider the plate problem of Sample 14.8 (page 722) again. Assume that the plate is at rest at $t = 0$ in the vertical upright position and that the force acts on the plate for 2 seconds.

1. Find the velocity of the center-of-mass of the plate at the end of 2 seconds.
2. Can you also find the angular velocity of the plate at the end of 2 seconds?

Solution

1. Since we are interested in finding the velocity at a particular instant t, given the velocity at another instant $t = 0$, we can use the impulse-momentum equations to find the desired velocity.

$$\begin{aligned}
\vec{\boldsymbol{L}}_2 - \vec{\boldsymbol{L}}_1 &= \int_{t_1}^{t_2} \sum \vec{\boldsymbol{F}}\, dt \\
m\,\vec{\boldsymbol{v}}_{\text{cm}}(t) - m\,\vec{\boldsymbol{v}}_{\text{cm}}(0) &= \int_0^t \vec{\boldsymbol{F}}\, dt \\
\Rightarrow \quad \vec{\boldsymbol{v}}_{\text{cm}}(t) &= \vec{\boldsymbol{v}}_{\text{cm}}(0) + \frac{1}{m}\int_0^2 \vec{\boldsymbol{F}}\, dt \\
&= \vec{\boldsymbol{0}} + \frac{1}{0.5\,\text{kg}}\int_0^2 (0.5\,\text{N}\hat{\boldsymbol{\imath}})\, dt \\
&= 2\,\text{m/s}\hat{\boldsymbol{\imath}}.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{v}}_{\text{cm}}(2\,\text{s}) = 2\,\text{m/s}\hat{\boldsymbol{\imath}}}$$

2. Now, let us try to find the angular velocity the same way, using angular impulse-momentum relation. We have,

$$\begin{aligned}
(\vec{\boldsymbol{H}}_{\text{cm}})_2 - (\vec{\boldsymbol{H}}_{\text{cm}})_1 &= \int_{t_1}^{t_2} \sum \vec{\boldsymbol{M}}_{\text{cm}}\, dt \\
I_{zz}^{\text{cm}}\vec{\boldsymbol{\omega}}(t) - I_{zz}^{\text{cm}}\vec{\boldsymbol{\omega}}(0) &= \int_0^t \sum \vec{\boldsymbol{M}}_{\text{cm}}\, dt \\
\Rightarrow \quad \vec{\boldsymbol{\omega}}(t) &= \vec{\boldsymbol{\omega}}(0) + \frac{1}{I_{zz}^{\text{cm}}}\int_0^t \sum \vec{\boldsymbol{M}}_{\text{cm}}\, dt \\
&= \vec{\boldsymbol{\omega}}(0) + \frac{1}{I_{zz}^{\text{cm}}}\int_0^t (\vec{\boldsymbol{r}}_{\text{P/cm}} \times \vec{\boldsymbol{F}})\, dt \\
&= \vec{\boldsymbol{0}} + \frac{1}{I_{zz}^{\text{cm}}}\int_0^t (Fr\sin(\theta+\alpha)\hat{\boldsymbol{k}})\, dt \\
&= \frac{Fr}{I_{zz}^{\text{cm}}}\left(\int_0^t \sin(\theta+\alpha)\, dt\right)\hat{\boldsymbol{k}}.
\end{aligned}$$

Now, we are in trouble; how do we evaluate the integral? In the integrand, we have θ which is an implicit function of t. Unless we know how θ depends on t we cannot evaluate the integral. To find $\theta(t)$ we have to solve the equation of angular motion we derived in the previous sample. However, we were not able to solve for $\theta(t)$ analytically, we had to resort to numerical solution. Thus, it is not possible to evaluate the integral above and, therefore, we cannot find the angular velocity of the plate at the end of 2 seconds using impulse-momentum equations. We could, however, find the desired velocity easily from the numerical solution.

14.3 Kinematics of rolling and sliding

Pure rolling in 2-D

In this section, we would like to add to the vocabulary of special motions by considering *pure rolling*. Most commonly, one discusses pure rolling of round objects on flat ground, like wheels and balls, and rolling of round things on other round things like gears and cams.

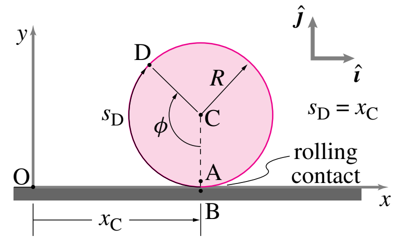


Figure 14.31: Pure rolling of a round wheel on a level support.
Filename:tfigure7-2D-pure-rolling

2-D rolling of a round wheel on level ground

The simplest case, the no-slip rolling of a round wheel, is an instructive starting point. First, we define the geometric and kinematic variables as shown in Fig. 14.31. For convenience, we pick a point D which was at $x_D = 0$ at the start of rolling, when $x_C = 0$. The key to the kinematics is that:

The arc length traversed on the wheel is the distance traveled by the wheel center.

That is,

$$\begin{aligned} x_C &= s_D \\ &= R\phi \\ \Rightarrow v_C = \dot{x}_C &= R\dot{\phi} \\ \Rightarrow a_C = \dot{v}_C = \ddot{x}_C &= R\ddot{\phi} \end{aligned}$$

So the rolling condition amounts to the following set of restrictions on the position of C, $\vec{r}_C$, and the rotations of the wheel ϕ:

$$\vec{r}_C = R\phi\hat{\imath} + R\hat{\jmath}, \quad \vec{v}_C = R\dot{\phi}\hat{\imath}, \quad \vec{a}_C = R\ddot{\phi}\hat{\imath}, \quad \vec{\omega} = -\dot{\phi}\hat{k}, \text{ and } \quad \vec{\alpha} = \dot{\vec{\omega}} = -\ddot{\phi}\hat{k}.$$

If we want to track the motion of a particular point, say D, we could do so by using the following parametric formula:

$$\begin{aligned} \vec{r}_D &= \vec{r}_C + \vec{r}_{D/C} \\ &= R(\phi\hat{\imath} + \hat{\jmath}) + R(-\sin\phi\hat{\imath} - \cos\phi\hat{\jmath}) \\ &= R\left[(\phi - \sin\phi)\hat{\imath} + (1 - \cos\phi)\hat{\jmath})\right] \\ \Rightarrow \vec{v}_D &= R\left[(\dot{\phi}(1 - \cos\phi)\hat{\imath} + \dot{\phi}\sin\phi\hat{\jmath})\right] \qquad (14.30) \\ \Rightarrow \vec{a}_D &= R\dot{\phi}^2(\sin\phi\hat{\imath} + \cos\phi\hat{\jmath}). \end{aligned}$$

assuming $\dot{\phi}$= constant

Note that if $\phi = 0$ or 2π or 4π, etc., then the point D is on the ground and eqn. (14.30) correctly gives that

$$\vec{v}_D = R\left[\dot{\phi}(\underbrace{1 - \overbrace{\cos(2n\pi)}^{1})}_{0}\right]\hat{\imath} + \dot{\phi}\underbrace{\sin(2n\pi)}_{0}\hat{\jmath} = \vec{0}.$$

Instantaneous Kinematics

Instead of tracking the wheel from its start, we could analyze the kinematics at the instant of interest. Here, we make the observation that the wheel rolls without slip. Therefore, the point on the wheel touching the ground has no velocity relative to the ground.

$$\underbrace{\vec{v}_A}_{\text{Velocity of point on the wheel touching the ground}} = \underbrace{\vec{v}_B}_{\text{Velocity of ground} = \vec{0}} \tag{14.31}$$

Now, we know how to calculate the velocity of points on a rigid body. So,

$$\vec{v}_A = \vec{v}_C + \vec{v}_{A/C},$$

where, since A and C are on the same rigid body (Fig. 14.31), we have from eqn. (13.36) that

$$\vec{v}_{A/C} = \vec{\omega} \times \vec{r}_{A/C}.$$

Putting this equation together with eqn. (14.31), we get

$$\begin{aligned} \vec{v}_A &= \vec{v}_B \\ \Rightarrow \underbrace{\vec{v}_C}_{v_C\hat{\imath}} + \underbrace{\vec{\omega}}_{\omega\hat{k}} \times \underbrace{\vec{r}_{A/C}}_{-R\hat{\jmath}} &= \vec{0} \\ \Rightarrow v_C\hat{\imath} + \omega R\hat{\imath} &= \vec{0} \end{aligned}$$

$$\Rightarrow \qquad v_C = -\omega R. \tag{14.32}$$

We use $\vec{v}_C = v\hat{\imath}$ since the center of the wheel goes neither up nor down. Note that if you measure the angle by ϕ, like we did before, then $\vec{\omega} = -\dot{\phi}\hat{k}$ so that positive rotation rate is in the counter-clockwise direction. Thus, $v_C = -\omega R = -(-\dot{\phi})R = \dot{\phi}R$.

Since there is always *some* point of the wheel touching the ground, we know that $v_C = -\omega R$ for all time. Therefore,

$$\vec{a}_C = \dot{v}_C\hat{\imath} = -\dot{\omega}R\hat{\imath}.$$

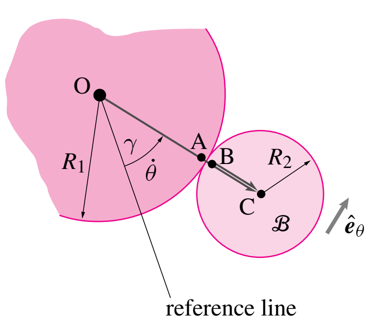


Figure 14.32:
Filename:tfigure7-rolling-on-another

Rolling of round objects on round surfaces

For round objects rolling on or in another round object, the analysis is similar to that for rolling on a flat surface. A common application is the so-called epicyclic, hypo-cyclic, or planetary gears (See Box 14.5 on planetary gears on page 728). Referring to Fig. 14.32, we can calculate the velocity of C with respect to a fixed frame two ways and compare:

$$\begin{aligned}
\vec{v}_C &= \vec{v}_B + \vec{v}_{C/B} \\
\vec{v}_C &= \overbrace{\underbrace{\vec{v}_A}_{\vec{0}} + \underbrace{\vec{v}_{B/A}}_{\vec{0}}}^{\vec{v}_B} + \vec{v}_{C/B}. \\
\dot{\theta}(R_1 + R_2)\hat{e}_\theta &= \omega_\mathcal{B} R_2 \hat{e}_\theta \\
\Rightarrow \omega_\mathcal{B} &= \frac{\dot{\theta}(R_1 + R_2)}{R_2} = \dot{\theta}(1 + \frac{R_1}{R_2}).
\end{aligned}$$

Example: **Two quarters.**
The formula above can be tested in the case of $R_1 = R_2$ by using two quarters or two dimes on a table. Roll one quarter, call it $\mathcal{B}$, around another quarter pressed fast to the table. You will see that as the rolling quarter $\mathcal{B}$ travels around the stationary quarter one time, it makes two full revolutions. That is, the orientation of $\mathcal{B}$ changes twice as fast as the angle of the line from the center of the stationary quarter to its center. Or, in the language of the calculation above, $\omega_\mathcal{B} = 2\dot{\theta}$.

Sliding

Although wheels and balls are known for rolling, they do sometimes slide such as when a car screeches at fast acceleration or sudden braking or when a bowling ball is released on the lane.

The *sliding velocity* is the velocity of the material point on the wheel (or ball) relative to its contacting substrate. In the case of pure rolling, the sliding velocity is zero. In the case of a ball or wheel moving against a stationary support surface, whether round or curved, the sliding velocity is

$$\vec{v}_{\text{sliding}} = \vec{v}_{\text{circle center}} + \vec{\omega} \times \vec{r}_{\text{contact/center}} \tag{14.34}$$

Example: **Bowling ball**
The velocity of the point on the bowling ball instantaneously in contact with the alley (ground) is $\vec{v}_\text{C} = v_\text{G}\hat{\imath} + \omega\hat{k} \times \vec{r}_{\text{C/G}} = (v_\text{G} + \omega R)\hat{\imath}$. So unless $\omega = -v_\text{G}/R$ the ball is sliding.

Note that, if sliding, the friction force on the ball opposes the slip of the ball and tends to accelerate the balls rotation towards rolling. That is, for example, if the ball is not rotating the sliding velocity is $v_G\hat{\imath}$, the friction force is in the $-\hat{\imath}$ direction and angular momentum balance about the center-of-mass implies $\dot{\omega} < 0$ and a counter-clockwise rotational acceleration. No matter what the initial velocity or rotational rate the ball will eventually roll.

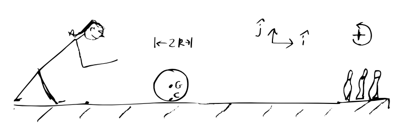
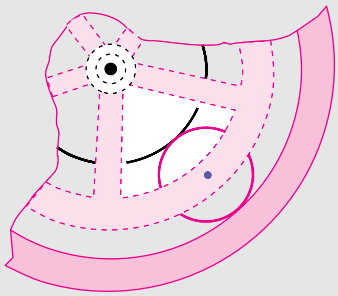


Figure 14.33: The bowling ball is sliding so long as $v_G \neq -\omega R$

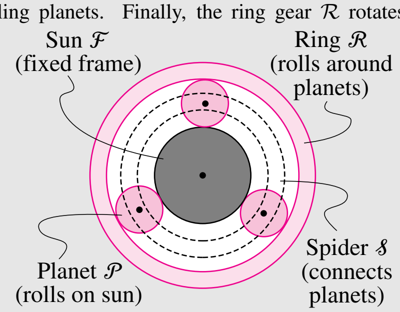
Filename:tfigure-bowlball

14.5 The Sturmey-Archer hub

In 1903, the year the Wright Brothers first flew powered airplanes, the Sturmey-Archer company patented the internal-hub three-speed bicycle transmission. This marvel of engineering was sold on the best bikes until finicky but fast racing bicycles using derailleurs started to push them out of the market in the 1960's. Now, a hundred years later, internal bicycle hubs (now made by Shimano and Sachs) are having something of a revival, particularly in Europe. These internal-hub transmissions utilize a system called *planetary gears*, gears which roll around other gears. See the figure below.



In order to understand this gear system, we need to understand its kinematics—the motion of its parts. Referring to figure above, the central 'sun' gear $\mathcal{F}$ is stationary, at least we treat it as stationary in this discussion since it is fixed to the bike frame, so it is fixed in body $\mathcal{F}$. The 'planet' gears roll around the sun gear. Let's call one of these planets $\mathcal{P}$. The spider $\mathcal{S}$ connects the centers of the rolling planets. Finally, the ring gear $\mathcal{R}$ rotates around the sun.



The gear transmission steps up the angular velocity when the spider $\mathcal{S}$ is driven and ring $\mathcal{R}$, which moves faster, is connected to the wheel. The transmission steps down the angular velocity when the ring gear is driven and the slower spider is connected to the wheel. The third 'speed' in the three-speed gear transmission is direct drive (the wheel is driven directly).

What are the 'gear ratios' in the planetary gear system? The 'trick' is to recognize that for rolling contact that the contacting points have the same velocity, $\vec{v}_A = \vec{v}_B$ and $\vec{v}_D = \vec{v}_E$. Let's define some terms.

$\vec{\omega}_{\mathcal{S}} = \omega_{\mathcal{S}}\hat{k}$		angular velocity of the spider
$\vec{\omega}_{\mathcal{P}} = \omega_{\mathcal{P}}\hat{k}$		angular velocity of the planet
$\vec{\omega}_{\mathcal{R}} = \omega_{\mathcal{R}}\hat{k}$		angular velocity of the ring

Now, we can find the relation of these angular velocities as follows.

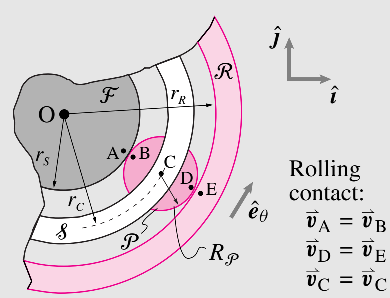
Look at the velocity of point C in two ways. First,

$$\begin{aligned}
\overbrace{\vec{v}_C}^{\text{A point on the spider}} &= \overbrace{\vec{v}_C}^{\text{A point on the planetary gear}} \\
\Rightarrow \vec{\omega}_{\mathcal{S}} \times \vec{r}_C &= \underbrace{\vec{v}_B}_{\vec{0}} + \vec{\omega}_{\mathcal{P}} \times \vec{r}_{C/B} \\
\Rightarrow \omega_{\mathcal{S}} r_C &= \omega_{\mathcal{P}} R_P \\
\Rightarrow \omega_{\mathcal{P}} &= \frac{r_C}{R_P}\omega_{\mathcal{S}} \qquad (14.33)
\end{aligned}$$

Next, let's look at point D and E:

$$\begin{aligned}
\vec{v}_D &= \vec{v}_E \\
\vec{v}_A + \vec{v}_{D/A} &= \vec{\omega}_{\mathcal{R}} \times \vec{r}_R \\
\vec{0} + \vec{\omega}_{\mathcal{P}} \times \vec{r}_{D/A} &= \omega_{\mathcal{R}}\hat{k} \times \vec{r}_R \\
\omega_{\mathcal{P}}(2R_P)\hat{e}_\theta &= \omega_{\mathcal{R}} r_R \hat{e}_\theta \\
\Rightarrow \omega_{\mathcal{R}} &= \frac{2R_P}{r_R}\underbrace{\omega_{\mathcal{P}}}_{\omega_{\mathcal{P}} = \frac{r_C}{R_P}\omega_{\mathcal{S}}}
\end{aligned}$$

$$\begin{aligned}
\omega_{\mathcal{R}} &= \frac{2R_P}{r_R}\overbrace{\frac{r_C}{R_P}}^{r_C = r_S + R_P}\omega_{\mathcal{S}} \\
&= \frac{2(r_S + R_P)}{\underbrace{r_R}_{r_R = r_S + 2R_P}}\omega_{\mathcal{S}} \\
\Rightarrow \frac{\omega_{\mathcal{R}}}{\omega_{\mathcal{S}}} &= 2\frac{1 + \frac{R_P}{r_S}}{1 + \frac{2R_P}{r_S}} = \text{angular velocity step-up.}
\end{aligned}$$



Typically, the gears have radius ratio of $\frac{R_P}{r_S} = \frac{3}{2}$ which gives a gear ratio of $\frac{5}{4}$. Thus, the ratio of the highest gear to the lowest gear on a Sturmey-Archer hub is $\frac{5}{4}/\frac{4}{5} = \frac{25}{16} = 1.5625$. You might compare this ratio to that of a modern mountain bike, with eighteen or twenty-one gears, where the ratio of the highest gear to the lowest is about 4:1.

SAMPLE 14.10 Falling ladder: The ends of a ladder of length $L = 3\,\text{m}$ slip along the frictionless wall and floor shown in Figure 14.34. At the instant shown, when $\theta = 60^\circ$, the angular speed $\dot{\theta} = 1.15\,\text{rad/s}$ and the angular acceleration $\ddot{\theta} = 2.5\,\text{rad/s}^2$. Find the absolute velocity and acceleration of end B of the ladder.

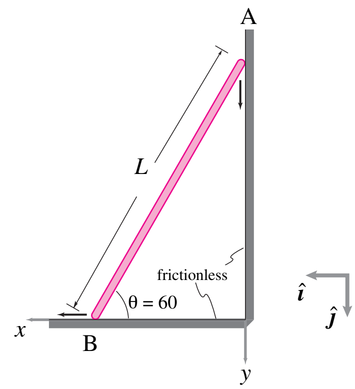


Figure 14.34:
Filename:sfig7-2-2

Solution Since the ladder is falling, it is rotating clockwise. From the given information:

$$\begin{aligned}\vec{\omega} &= \dot{\theta}\hat{k} = -1.15\,\text{rad/s}\hat{k}\\ \dot{\vec{\omega}} &= \ddot{\theta}\hat{k} = -2.5\,\text{rad/s}^2\hat{k}.\end{aligned}$$

We need to find $\vec{v}_B$, the absolute velocity of end B, and $\vec{a}_B$, the absolute acceleration of end B.

Since the end A slides along the wall and end the B slides along the floor, we know the directions of $\vec{v}_A$, $\vec{v}_B$, $\vec{a}_A$ and $\vec{a}_B$.

Let $\vec{v}_A = v_A\hat{j}$, $\vec{a}_A = a_A\hat{j}$, $\vec{v}_B = v_B\hat{\imath}$ and $\vec{a}_B = a_B\hat{\imath}$ where the scalar quantities v_A, a_A, v_B and a_B are unknown.

$$\begin{aligned}\text{Now,}\quad \vec{v}_A &= \vec{v}_B + \vec{v}_{A/B} = \vec{v}_B + \vec{\omega}\times\vec{r}_{A/B}\\ \text{or}\quad v_A\hat{j} &= v_B\hat{\imath} + \dot{\theta}\hat{k}\times\underbrace{L(-\cos\theta\hat{\imath} - \sin\theta\hat{j})}_{\vec{r}_{A/B}}\\ &= (v_B + \dot{\theta}L\sin\theta)\hat{\imath} - \dot{\theta}L\cos\theta\hat{j}.\end{aligned}$$

Dotting both sides of the equation with $\hat{\imath}$, we get:

$$\begin{aligned}v_A\underbrace{\hat{j}\cdot\hat{\imath}}_{0} &= (v_B + \dot{\theta}L\sin\theta)\underbrace{\hat{\imath}\cdot\hat{\imath}}_{1} + \dot{\theta}L\cos\theta\underbrace{\hat{j}\cdot\hat{\imath}}_{0}\\ \Rightarrow\quad 0 &= v_B + \dot{\theta}L\sin\theta\\ \Rightarrow\quad v_B &= -\dot{\theta}L\sin\theta = -(-1.15\,\text{rad/s})\cdot 3\,\text{m}\cdot\frac{\sqrt{3}}{2}\\ &= 2.99\,\text{m/s}.\end{aligned}$$

$$\boxed{\vec{v}_B = 2.9\,\text{m/s}\hat{\imath}}$$

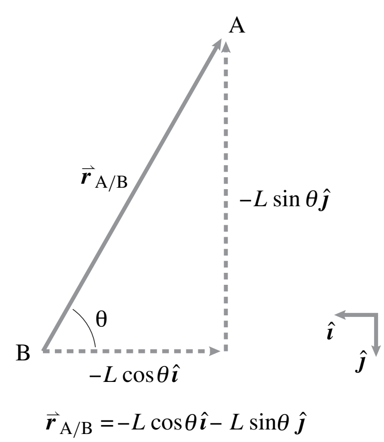


Figure 14.35:
Filename:sfig7-2-2b

Similarly,

$$\begin{aligned}\vec{a}_A &= \vec{a}_B + \dot{\vec{\omega}}\times\vec{r}_{A/B} + \overbrace{\vec{\omega}\times(\vec{\omega}\times\vec{r}_{A/B})}^{-\omega^2\vec{r}_{A/B}}\\ a_A\hat{j} &= a_B\hat{\imath} + \ddot{\theta}\hat{k}\times L(-\cos\theta\hat{\imath} - \sin\theta\hat{j}) - \dot{\theta}^2L(-\cos\theta\hat{\imath} - \sin\theta\hat{j})\\ &= (a_B + \ddot{\theta}L\sin\theta + \dot{\theta}^2L\cos\theta)\hat{\imath} + (-\ddot{\theta}L\cos\theta + \dot{\theta}^2L\sin\theta)\hat{j}.\end{aligned}$$

Dotting both sides of this equation with $\hat{\imath}$ (as we did for velocity) we get:

$$\begin{aligned}0 &= a_B + \ddot{\theta}L\sin\theta + \dot{\theta}^2L\cos\theta\\ \Rightarrow\quad a_B &= -\ddot{\theta}L\sin\theta - \dot{\theta}^2L\cos\theta\\ &= -(-2.5\,\text{rad/s}^2\cdot 3\,\text{m}\cdot\frac{\sqrt{3}}{2}) - (-1.15\,\text{rad/s})^2\cdot 3\,\text{m}\cdot\frac{1}{2}\\ &= 4.51\,\text{m/s}^2.\end{aligned}$$

$$\boxed{\vec{a}_B = 4.51\,\text{m/s}^2}$$

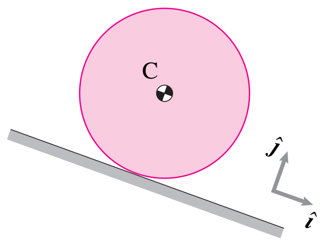


Figure 14.36:

Filename:sfig9-rolling-may00

SAMPLE 14.11 A cylinder of diameter 500 mm rolls down an inclined plane with uniform acceleration (of the center-of-mass) $a = 0.1\,\text{m/s}^2$. At an instant t_0, the mass-center has speed $v_0 = 0.5\,\text{m/s}$.

1. Find the angular speed ω and the angular acceleration $\dot{\omega}$ at t_0.
2. How many revolutions does the cylinder make in the next 2 seconds?
3. What is the distance travelled by the center-of-mass in those 2 seconds?

Solution This problem is about simple kinematic calculations. We are given the velocity, $\dot{x}$, and the acceleration, $\ddot{x}$, of the center-of-mass. We are supposed to find angular velocity ω, angular acceleration $\dot{\omega}$, angular displacement θ in 2 seconds, and the corresponding linear distance x along the incline. The radius of the cylinder $R = \text{diameter}/2 = 0.25\,\text{m}$.

1. From the kinematics of pure rolling,

$$\begin{aligned} \omega &= \frac{\dot{x}}{R} = \frac{0.5\,\text{m/s}}{0.25\,\text{m}} = 2\,\text{rad/s}, \\ \dot{\omega} &= \frac{\ddot{x}}{R} = \frac{0.1\,\text{m/s}^2}{0.25\,\text{m}} = 0.4\,\text{rad/s}^2. \end{aligned}$$

$$\boxed{\omega = 2\,\text{rad/s}, \quad \dot{\omega} = 0.4\,\text{rad/s}^2}$$

2. We can find the number of revolutions the cylinder makes in 2 seconds by solving for the angular displacement θ in this time period. Since,

$$\ddot{\theta} \equiv \dot{\omega} = \text{constant},$$

we integrate this equation twice and substitute the initial conditions, $\dot{\theta}(t = 0) = \omega = 2\,\text{rad/s}$ and $\theta(t = 0) = 0$, to get

$$\begin{aligned} \theta(t) &= \omega t + \frac{1}{2}\dot{\omega}t^2 \\ \Rightarrow \quad \theta(t = 2\,\text{s}) &= (2\,\text{rad/s}) \cdot (2\,\text{s}) + \frac{1}{2}(0.4\,\text{rad/s}) \cdot (4\,\text{s}^2) \\ &= 4.8\,\text{rad} = \frac{4.8}{2\pi}\,\text{rev} = 0.76\,\text{rev}. \end{aligned}$$

$$\boxed{\theta = 0.76\,\text{rev}}$$

3. Now that we know the angular displacement θ, the distance travelled by the mass-center is the arc-length corresponding to θ, *i.e.*,

$$x = R\theta = (0.25\,\text{m}) \cdot (4.8) = 1.2\,\text{m}.$$

$$\boxed{x = 1.2\,\text{m}}$$

Note that we could have found the distance travelled by the mass-center by integrating the equation $\ddot{x} = 0.1\,\text{m/s}^2$ twice.

SAMPLE 14.12 Condition of pure rolling. A cylinder of radius $R = 20\,\text{cm}$ rolls on a flat surface with absolute angular speed $\omega = 12\,\text{rad/s}$ under the conditions shown in the figure (In cases (ii) and (iii), you may think of the 'flat surface' as a conveyor belt). In each case,

1. Write the condition for pure rolling.
2. Find the velocity of the center C of the cylinder.

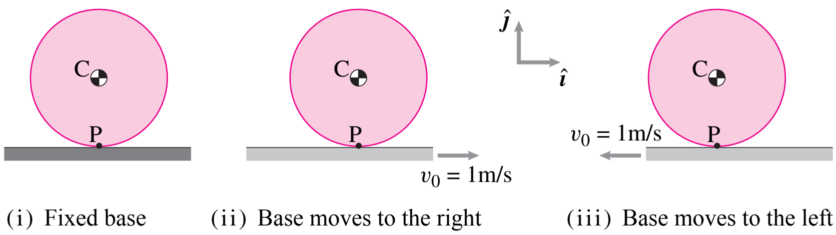


Figure 14.37:

Filename:sfig7-rolling1

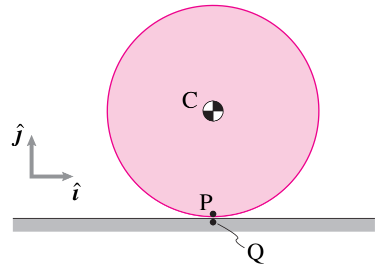


Figure 14.38: The cylinder rolls on the flat surface. Instantaneously, point P on the cylinder is in contact with point Q on the flat surface. For pure rolling, points P and Q must have the same velocity.

Filename:sfig7-rolling1a

Solution At any instant during rolling, the cylinder makes a point-contact with the flat surface. Let the point of instantaneous contact on the cylinder be P, and let the corresponding point on the flat surface be Q. The condition of pure rolling, in each case, is $\vec{v}_P = \vec{v}_Q$, that is, there is no relative motion between the two contacting points (a relative motion will imply slip). Now, we analyze each case.

Case(i) In this case, the bottom surface is fixed. Therefore,

1. The condition of pure rolling is: $\vec{v}_P = \vec{v}_Q = \vec{0}$.
2. Velocity of the center:

$$\begin{aligned}\vec{v}_C &= \vec{v}_P + \vec{\omega} \times \vec{r}_{C/P} = \vec{0} + (-\omega\hat{k}) \times R\hat{\jmath} \\ &= \omega R\hat{\imath} = (12\,\text{rad/s}) \cdot (0.2\,\text{m})\hat{\imath} = 2.4\,\text{m/s}\hat{\imath}.\end{aligned}$$

Case(ii) In this case, the bottom surface moves with velocity $\vec{v} = 1\,\text{m/s}\hat{\imath}$. Therefore, $\vec{v}_Q = 1\,\text{m/s}\hat{\imath}$. Thus,

1. The condition of pure rolling is: $\vec{v}_P = \vec{v}_Q = 1\,\text{m/s}\hat{\imath}$.
2. Velocity of the center:

$$\begin{aligned}\vec{v}_C &= \vec{v}_P + \vec{\omega} \times \vec{r}_{C/P} = v_0\hat{\imath} + \omega R\hat{\imath} \\ &= 1\,\text{m/s}\hat{\imath} + 2.4\,\text{m/s}\hat{\imath} = 3.4\,\text{m/s}\hat{\imath}.\end{aligned}$$

Case(iii) In this case, the bottom surface moves with velocity $\vec{v} = -1\,\text{m/s}\hat{\imath}$. Therefore, $\vec{v}_Q = -1\,\text{m/s}\hat{\imath}$. Thus,

1. The condition of pure rolling is: $\vec{v}_P = \vec{v}_Q = -1\,\text{m/s}\hat{\imath}$.
2. Velocity of the center:

$$\begin{aligned}\vec{v}_C &= \vec{v}_P + \vec{\omega} \times \vec{r}_{C/P} = -v_0\hat{\imath} + \omega R\hat{\imath} \\ &= -1\,\text{m/s}\hat{\imath} + 2.4\,\text{m/s}\hat{\imath} = 1.4\,\text{m/s}\hat{\imath}.\end{aligned}$$

(a) :	(i) $\vec{v}_P = \vec{0}$,	(ii) $\vec{v}_P = 1\,\text{m/s}\hat{\imath}$,	(iii) $\vec{v}_P = -1\,\text{m/s}\hat{\imath}$,
(b) :	(i) $\vec{v}_C = 2.4\,\text{m/s}\hat{\imath}$,	(ii) $\vec{v}_C = 3.4\,\text{m/s}\hat{\imath}$,	(iii) $\vec{v}_C = 1.4\,\text{m/s}\hat{\imath}$

SAMPLE 14.13 Motion of a point on a disk rolling inside a cylinder. A uniform disk of radius r rolls without slipping with constant angular speed ω inside a fixed cylinder of radius R. A point P is marked on the disk at a distance ℓ ($\ell < r$) from the center of the disk. at a general time t during rolling, find

1. the position of point P,
2. the velocity of point P, and
3. the acceleration of point P

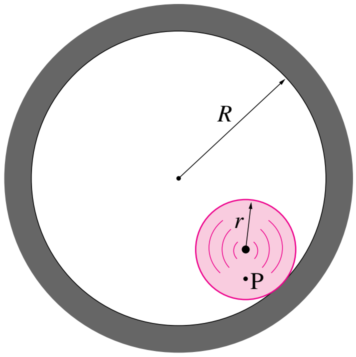


Figure 14.39: A uniform disk of radius r rolls without slipping inside a fixed cylinder.

Filename:sfig6-5-3

Solution Let the disk be vertically below the center of the cylinder at $t = 0\,\mathrm{s}$ such that point P is vertically above the center of the disk (Fig. 14.40). At this instant, Q is the point of contact between the disk and the cylinder. Let the disk roll for time t such that at instant t the line joining the two centers (line OC) makes an angle ϕ with its vertical position at $t = 0\,\mathrm{s}$. Since the disk has rolled for time t at a constant angular speed ω, point P has rotated counter-clockwise by an angle $\theta = \omega t$ from its original vertical position P'.

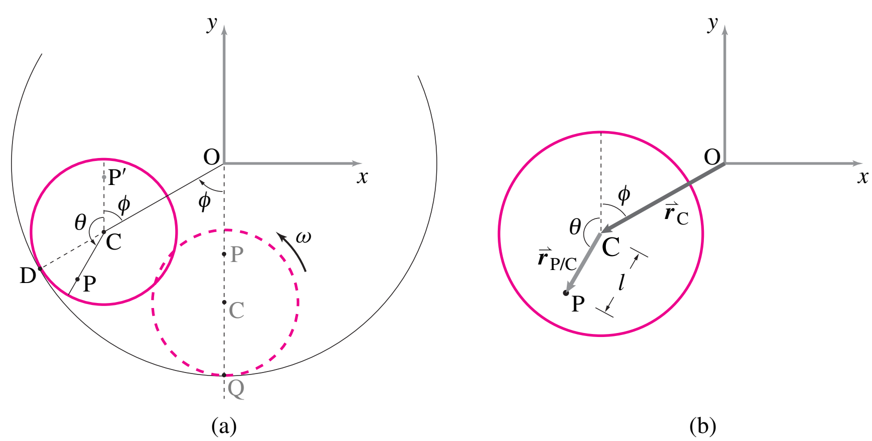


Figure 14.40: Geometry of motion: keeping track of point P while the disk rolls for time t, rotating by angle $\theta = \omega t$ inside the cylinder.

Filename:sfig6-5-3a

1. **Position of point P:** From Fig. 14.40(b) we can write

$$\vec{r}_P = \vec{r}_C + \vec{r}_{P/C} = (R-r)\hat{\boldsymbol{\lambda}}_{OC} + \ell\hat{\boldsymbol{\lambda}}_{CP}$$

where

$$\begin{aligned}\hat{\boldsymbol{\lambda}}_{OC} &= \text{a unit vector along OC } = -\sin\phi\,\hat{\imath} - \cos\phi\,\hat{\jmath},\\ \hat{\boldsymbol{\lambda}}_{CP} &= \text{a unit vector along CP } = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}.\end{aligned}$$

Thus,

$$\vec{r}_P = [-(R-r)\sin\phi - \ell\sin\theta]\hat{\imath} + [-(R-r)\cos\phi + \ell\cos\theta]\hat{\jmath}.$$

We have thus obtained an expression for the position vector of point P as a function of ϕ and θ. Since we also want to find velocity and acceleration of point P, it will be nice to express $\vec{r}_P$ as a function of t. As noted above, $\theta = \omega t$; but how do we find ϕ as a function of t? Note that the center of the disk C is going around point O in circles with angular velocity $-\dot{\phi}\hat{\boldsymbol{k}}$. The disk, however, is rotating with angular velocity $\vec{\omega} = \omega\hat{\boldsymbol{k}}$

about the instantaneous center of rotation, point D. Therefore, we can calculate the velocity of point C in two ways:

$$\begin{aligned}
\vec{\boldsymbol{v}}_C &= \vec{\boldsymbol{v}}_C \\
\text{or} \quad \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{C/D} &= -\dot{\phi}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{C/O} \\
\text{or} \quad \omega\hat{\boldsymbol{k}} \times r(-\hat{\boldsymbol{\lambda}}_{OC}) &= -\dot{\phi}\hat{\boldsymbol{k}} \times (R-r)\hat{\boldsymbol{\lambda}}_{OC} \\
\text{or} \quad -\omega r(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}_{OC}) &= -\dot{\phi}(R-r)(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}_{OC}) \\
\Rightarrow \quad \frac{r}{R-r}\omega &= \dot{\phi}.
\end{aligned}$$

Integrating the last expression with respect to time, we obtain

$$\phi = \frac{r}{R-r}\omega t.$$

Let

$$q = \frac{r}{R-r},$$

then, the position vector of point P may now be written as

$$\vec{\boldsymbol{r}}_P = [-(R-r)\sin(q\omega t) - \ell\sin(\omega t)]\hat{\boldsymbol{\imath}} + [-(R-r)\cos(q\omega t) + \ell\cos(\omega t)]\hat{\boldsymbol{\jmath}}. \quad (14.35)$$

2. **Velocity of point P:** Differentiating Eqn. (14.35) once with respect to time we get

$$\vec{\boldsymbol{v}}_P = -\omega[(R-r)q\cos(q\omega t) + \ell\cos(\omega t)]\hat{\boldsymbol{\imath}} + \omega[(R-r)q\sin(q\omega t) - \ell\sin(\omega t)]\hat{\boldsymbol{\jmath}}.$$

Substituting $(R-r)q = r$ in $\vec{\boldsymbol{v}}_P$ we get

$$\vec{\boldsymbol{v}}_P = -\omega r[\{\cos(q\omega t) + \frac{\ell}{r}\cos(\omega t)\}\hat{\boldsymbol{\imath}} - \{\sin(q\omega t) - \frac{\ell}{r}\sin(\omega t)\}\hat{\boldsymbol{\jmath}}]. \quad (14.36)$$

3. **Acceleration of point P:** Differentiating Eqn. (14.36) once with respect to time we get

$$\vec{\boldsymbol{a}}_P = -\omega^2 r[-\{q\sin(q\omega t) + \frac{\ell}{r}\sin(\omega t)\}\hat{\boldsymbol{\imath}} - \{q\cos(q\omega t) - \frac{\ell}{r}\cos(\omega t)\}\hat{\boldsymbol{\jmath}}]. \quad (14.37)$$

$$\boxed{\begin{aligned}
\vec{\boldsymbol{r}}_P &= [-(R-r)\sin(q\omega t) - \ell\sin(\omega t)]\hat{\boldsymbol{\imath}} + [-(R-r)\cos(q\omega t) + \ell\cos(\omega t)]\hat{\boldsymbol{\jmath}} \\
\vec{\boldsymbol{v}}_P &= -\omega r[\{\cos(q\omega t) + \tfrac{\ell}{r}\cos(\omega t)\}\hat{\boldsymbol{\imath}} - \{\sin(q\omega t) - \tfrac{\ell}{r}\sin(\omega t)\}\hat{\boldsymbol{\jmath}}] \\
\vec{\boldsymbol{a}}_P &= -\omega^2 r[-\{q\sin(q\omega t) + \tfrac{\ell}{r}\sin(\omega t)\}\hat{\boldsymbol{\imath}} - \{q\cos(q\omega t) - \tfrac{\ell}{r}\cos(\omega t)\}\hat{\boldsymbol{\jmath}}]
\end{aligned}}$$

SAMPLE 14.14 The rolling disk: instantaneous kinematics. For the rolling disk in Sample 14.13, let $R = 4\,\text{ft}$, $r = 1\,\text{ft}$ and point P be on the rim of the disk. Assume that at $t = 0$, the center of the disk is vertically below the center of the cylinder and point P is on the vertical line joining the two centers. If the disk is rolling at a constant speed $\omega = \pi$ rad/s, find

1. the position of point P and center C at $t = 1\,\text{s}$, $3\,\text{s}$, and $5.25\,\text{s}$,
2. the velocity of point P and center C at those instants, and
3. the acceleration of point P and center C at the same instants as above.

Draw the position of the disk at the three instants and show the velocities and accelerations found above.

Solution The general expressions for position, velocity, and acceleration of point P obtained in Sample 14.13 can be used to find the position, velocity, and acceleration of any point on the disk by substituting an appropriate value of ℓ in equations (14.35), (14.36), and (14.37). Since $R = 4r$,

$$q = \frac{r}{R-r} = \frac{1}{3}.$$

Now, point P is on the rim of the disk and point C is the center of the disk. Therefore,

$$\begin{aligned} \text{for point P:} \quad \ell &= r, \\ \text{for point C:} \quad \ell &= 0. \end{aligned}$$

Substituting these values for ℓ, and $q = 1/3$ in equations (14.35), (14.36), and (14.37) we get the following.

1. **Position:**

$$\begin{aligned} \vec{r}_C &= -3r\left[\sin\left(\frac{\omega t}{3}\right)\hat{\imath} + \cos\left(\frac{\omega t}{3}\right)\hat{\jmath}\right], \\ \vec{r}_P &= \vec{r}_C + r\left[-sin\,(\omega t)\,\hat{\imath} + \cos\,(\omega t)\,\hat{\jmath}\right]. \end{aligned}$$

2. **Velocity:**

$$\begin{aligned} \vec{v}_C &= -\omega r\left[\cos\left(\frac{\omega t}{3}\right)\hat{\imath} - \sin\left(\frac{\omega t}{3}\right)\hat{\jmath}\right], \\ \vec{v}_P &= -\omega r\left[\left\{\cos\left(\frac{\omega t}{3}\right) + \cos\,(\omega t)\right\}\hat{\imath} - \left\{\sin\left(\frac{\omega t}{3}\right) - \sin\,(\omega t)\right\}\hat{\jmath}\right]. \end{aligned}$$

3. **Acceleration:**

$$\begin{aligned} \vec{a}_C &= \frac{\omega^2 r}{3}\left[\sin\left(\frac{\omega t}{3}\right)\hat{\imath} + \cos\left(\frac{\omega t}{3}\right)\hat{\jmath}\right], \\ \vec{a}_P &= \omega^2 r\left[\left\{\frac{1}{3}\sin\left(\frac{\omega t}{3}\right) + \sin\,(\omega t)\right\}\hat{\imath} + \left\{\frac{1}{3}\cos\left(\frac{\omega t}{3}\right) - \cos\,(\omega t)\right\}\hat{\jmath}\right]. \end{aligned}$$

We can now use these expressions to find the position, velocity, and acceleration of the two points at the instants of interest by substituting $r = 1\,\text{ft}$, $\omega = \pi$ rad/s, and appropriate values of t. These values are shown in Table 14.1.

The velocity and acceleration of the two points are shown in Figures 14.41(a) and (b) respectively.

It is worthwhile to check the directions of velocities and the accelerations by thinking about the velocity and acceleration of point P as a vector sum of the velocity (same for acceleration) of the center of the disk and the velocity (same for acceleration) of point P with respect to the center of the disk. Since the motions involved are circular motions at constant rate, a visual inspection of the velocities and the accelerations is not very difficult. Try it.

t	1 s	3 s	5.25 s
$\vec{r}_C$ (ft)	$3(-\frac{\sqrt{3}}{2}\hat{\imath} - \frac{1}{2}\hat{\jmath})$	$3\hat{\jmath}$	$3(\frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath})$
$\vec{r}_P$ (ft)	$\vec{r}_C - \hat{\jmath}$	$\vec{r}_C - \hat{\jmath}$	$4(\frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath})$
$\vec{v}_C$ (ft/s)	$\pi(-\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath})$	$\pi\hat{\imath}$	$\pi(-\frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath})$
$\vec{v}_P$ (ft/s)	$\pi(\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath})$	$2\pi\hat{\imath}$	$\vec{0}$
$\vec{a}_C$ (ft/s^2)	$\frac{\pi^2}{3}(\frac{\sqrt{3}}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath})$	$-\frac{\pi^2}{3}\hat{\jmath}$	$\frac{\pi^2}{3}(-\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath})$
$\vec{v}_P$ (ft/s^2)	$11.86(.24\hat{\imath} + .97\hat{\jmath})$	$\frac{2\pi^2}{3}\hat{\jmath}$	$13.16(-\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath})$

Table 14.1: Position, velocity, and acceleration of point P and point C

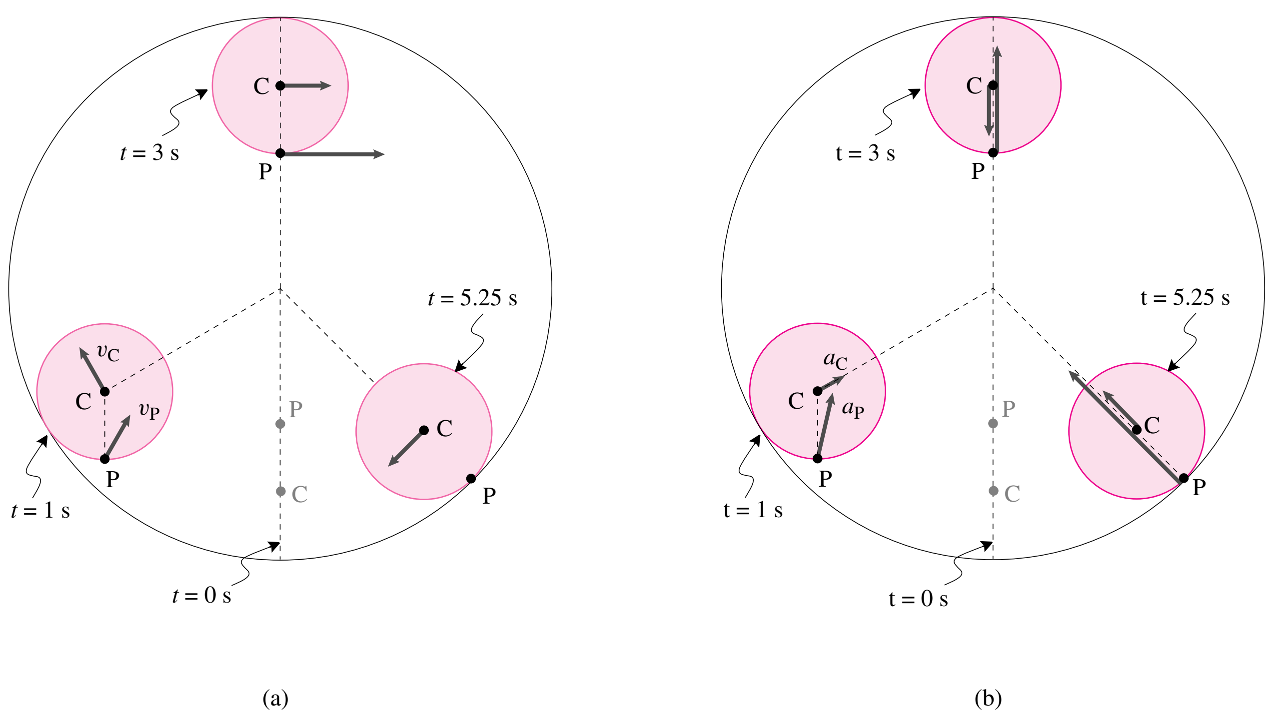


Figure 14.41: (a) Velocity and (b) Acceleration of points P and C at $t = 1$ s, 3 s, and 5.25 s.

Filename:sfig6-5-4a

SAMPLE 14.15 The rolling disk: path of a point on the disk. For the rolling disk in Sample 14.13, take $\omega = \pi$ rad/s. Draw the path of a point on the rim of the disk for one complete revolution of the center of the disk around the cylinder for the following conditions:

1. $R = 8r$,
2. $R = 4r$, and
3. $R = 2r$.

Solution In Sample 14.13, we obtained a general expression for the position of a point on the disk as a function of time. By computing the position of the point for various values of time t up to the time required to go around the cylinder for one complete cycle, we can draw the path of the point. For the various given conditions, the variable that changes in Eqn. (14.35) is q. We can write a computer program to generate the path of any point on the disk for a given set of R and r. Here is a pseudocode to generate the required path on a computer according to Eqn. (14.35).

A pseudocode to plot the path of a point on the disk:

```
(pseudo-code) program rollingdisk
%--------------------------------------------------------------
% This code plots the path of any point on a disk of radius
% 'r' rolling with speed 'w' inside a cylinder of radius 'R'.
% The point of interest is distance 'l' away from the center of
% the disk. The coordinates x and y of the specified point P are
% calculated according to the relation mentioned above.
%---------------------------------------------------------------
phi = pi/50*[1,2,3,...,100]    % make a vector phi from 0 to 2*pi
x = R*cos(phi)                 % create points on the outer cylinder
y = R*sin(phi)
plot y vs x                    % plot the outer cylinder
hold this plot                 % hold to overlay plots of paths
q = r/(R-r)                    % calculate q.
T = 2*pi/(q*w)                 % calculate time T for going around-
                               % the cylinder once at speed 'w'.
t = T/100*[1,2,3, ..., 100]    % make a time vector t from 0 to T-
                               % taking 101 points.

rcx = -(R-r)*(sin(q*w*t))      % find the x coordinates of pt. C.
rcy = -(R-r)*(cos(q*w*t))      % find the y coordinates of pt. C.
rpx = rcx-l*sin(w*t)           % find the x coordinates of pt. P.
rpy = rcy + l*cos(q*t)         % find the y coordinates of pt. P.
plot rpy vs rpx                % plot the path of P and the path
plot rcy vs rcx                % of C. For path of C
```

Once coded, we can use this program to plot the paths of both the center and the point P on the rim of the disk for the three given situations. Note that for any point on the rim of the disk $l = r$ (see Fig 14.40).

1. Let $R = 4$ units. Then $r = 0.5$ for $R = 8r$. To plot the required path, we run our program `rollingdisk` with desired input,

```
R = 4
r = 0.5
w = pi
l = 0.5
execute rollingdisk
```

The plot generated is shown in Fig.14.42 with a few graphic elements added for illustrative purposes.

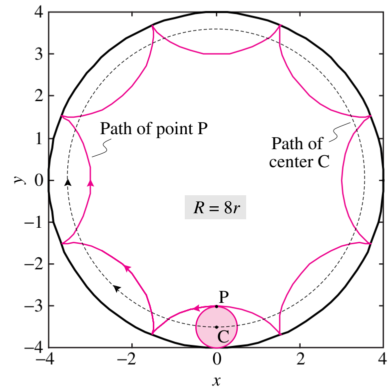


Figure 14.42: Path of point P and the center C of the disk for $R = 8r$.

Filename:sfig6-5-5a

2. Similarly, for $R = 4r$ we type:

```
R = 4
r = 1
w = pi
l = 1
execute rollingdisk
```

to plot the desired paths. The plot generated in this case is shown in Fig.14.43

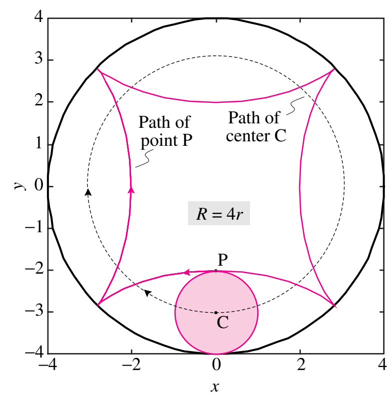


Figure 14.43: Path of point P and the center C of the disk for $R = 4r$.

Filename:sfig6-5-5b

3. The last one is the most interesting case. The plot obtained in this case by typing:

```
R = 4
r = 2
w = pi
l = 2
execute rollingdisk
```

is shown in Fig.14.44. Point P just travels on a straight line! In fact, every point on the rim of the disk goes back and forth on a straight line. Most people find this motion odd at first sight. You can roughly verify the result by cutting a whole twice the diameter of a coin (say a US quarter or dime) in a piece of cardboard and rolling the coin around inside while watching a marked point on the perimeter.

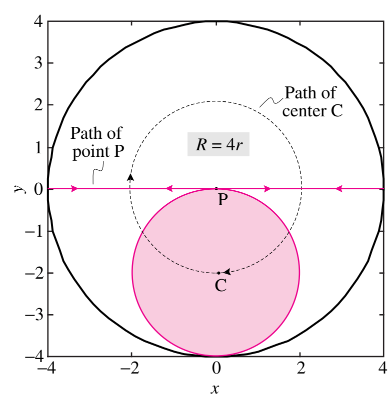


Figure 14.44: Path of point P and the center C of the disk for $R = 2r$.

Filename:sfig6-5-5c

A curiosity. We just discovered something simple about the path of a point on the edge of a circle rolling in another circle that is twice as big. The edge point moves in a straight line. In contrast one might think about the motion of the center G of a *straight* line segment that *slides* against two *straight* walls as in sample 14.23. A problem couldn't be more different. Naturally the path of point G is a circle (as you can check physically by looking at the middle of a ruler as you hold it as you sliding against a wall-floor corner).

14.4 Mechanics of contact

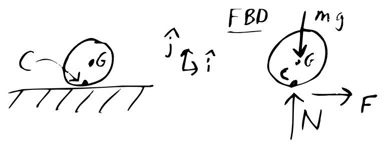


Figure 14.45: A ball rolls or slides on level ground.

Filename:tfigure-levelrolling

Mechanics of contacting bodies: rolling and sliding A typical machine part has forces that come from contact with other parts. In fact, with the major exception of gravity, most of the forces that act on bodies of engineering interest come from contact. Many of the forces you have drawn in free body diagrams have been contact forces: The force of the ground on an ideal wheel, of an axle on a bearing, etc.

We'd now like to consider some mechanics problems that involve sliding or rolling contact. Once you understand the kinematics from the previous section, there is nothing new in the mechanics. As always, the mechanics is linear momentum balance, angular momentum balance and energy balance. Because we are considering single rigid bodies in 2D the expressions for the motion quantities are especially simple (as you can look up in Table I at the back of the book): $\dot{\vec{L}} = m_{\text{tot}}\vec{a}_{\text{cm}}$, $\dot{\vec{H}}_{\text{C}} = \vec{r}_{\text{cm/C}} \times (m_{\text{tot}}\vec{a}_{\text{cm}}) + I\dot{\omega}\hat{k}$ (where $I = I_{zz}^{\text{cm}}$), and $E_{\text{K}} = m_{\text{tot}}v_{cm}^2/2 + I\omega^2/2$.

The key to success, as usual, is the drawing of appropriate free body diagrams (see Chapter 3 pages 88-91 and Chapter 6 pages 328-9). The two cases one needs to consider as possible are rolling, where the contact point has no relative velocity and the tangential reaction force is unknown but less than μN, and sliding where the relative velocity could be anything and the tangential reaction force is usually assumed to have a magnitude of μN but oppose the relative motion.

For friction forces in rolling refer to chapter 2 on free body diagrams. Note that in pure rolling contact, the contact force does no work because the material point of contact has no velocity. However, when there is sliding mechanical energy is dissipated. The rate of loss of kinetic and potential energy is

$$\text{Rate of frictional dissipation} = P_{diss} = F_{\text{friction}} \cdot v_{\text{slip}} \qquad (14.38)$$

where v_{slip} is the relative velocity of the contacting slipping points. If either the friction force (ideal lubrication) or sliding velocity (no slip) is zero there is no dissipation. Work-energy relations and impulse-momentum relations are useful to solve some problems both with and without slip.

As for various problems throughout the text, it is often a savings of calculation to use angular momentum balance (or moment balance in statics) relative to a point where there are unknown reaction forces. For rolling and slipping problems this often means making use of contact points.

Example: **Pure rolling on level ground**
A ball or wheel rolling on level ground, with no air friction etc, rolls at constant speed (see Fig. 14.45). This is most directly deduced from angular momentum balance about the contact point C:

$$\begin{array}{rcccl} \vec{M}_{\text{C}} = \dot{\vec{H}}_{\text{C}} & \Rightarrow & \vec{r}_{\text{G/C}} \times -mg\hat{\jmath} & = & \vec{r}_{\text{G/C}} \times m\vec{a}_{\text{G}} + \dot{\omega}I_{zz}^{\text{cm}}\hat{k} \\ & \Rightarrow & \vec{0} & = & R\hat{\jmath} \times (-m\dot{\omega}R\hat{\imath}) + \dot{\omega}I_{zz}^{\text{cm}}\hat{k} \\ \text{dotting with } \hat{k} & \Rightarrow & \dot{\omega} = 0 & \Rightarrow & \omega = \text{constant.} \end{array}$$

Because for rolling $v_G = -\omega R$ we thus have that v_G is a constant. [The result can also be obtained by combining angular momentum balance about the center-of-mass with linear momentum balance.]

Finally, linear momentum balance gives the reaction force at C to be $\vec{F} = mg\hat{\jmath}$. So,

assuming point contact, there is no rolling resistance.

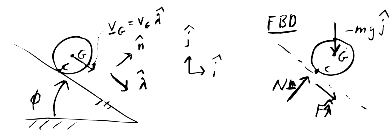


Figure 14.46: A ball rolls or slides down a slope.

Filename:tfigure-sloperolling

Example: **Bowling ball with initial sliding**

A bowling ball is released with an initial speed of v_0 and no rotation rate. What is it's subsequent motion? To start with, the motion is incompatible with rolling, the bottom of the ball is sliding to the right. So there is a frictional force which opposes motion and $F = -\mu N$ (see Fig. 14.45). Linear and angular momentum balance give:

$$\begin{array}{lll} \text{LMB:} & \Rightarrow & \left\{-F\hat{\imath} + N\hat{\jmath} - mg\hat{\jmath} = ma\hat{\imath}\right\} \\ \{\}\cdot\hat{\jmath} & \Rightarrow & N = mg \\ \{\}\cdot\hat{\imath} & \Rightarrow & a = -\mu g \\ \text{AMB}_{/\text{G}}: & \Rightarrow & -R\mu mg = I_{zz}^{\text{cm}}\dot{\omega} \end{array}$$

$$\Rightarrow \quad v = v_0 - \mu g t \quad \text{and} \quad \omega = -\mu R m g t / I_{zz}^{\text{cm}}$$

Thus the forward speed of the ball decreases linearly with time while the counter-clockwise angular velocity decreases linearly with time.

This solution is only appropriate so long as there is rightward slip, $v_G > -\omega R$. Just like for a sliding block, there is no impetus for reversal, and the block switches to pure rolling when

$$v = -\omega R \Rightarrow v_0 - \mu g t = -\left(-\mu R m g t / I_{zz}^{\text{cm}}\right) R \Rightarrow t = \frac{v_0}{\mu g \left(1 + \frac{mR^2}{I_{zz}^{\text{cm}}}\right)}.$$

Note that the energy lost during sliding is less than μmg times the distance the center of the ball moves during slip.

Example: **Ball rolling down hill.**

Assuming rolling we can find the acceleration of a ball as it rolls downhill (see Fig. 14.46). We start out with the kinematic observations that $\vec{a}_\text{G} = a_\text{G}\hat{\boldsymbol{\lambda}}$, that $R\omega = -v_\text{G}$ and that $R\dot{\omega} = -a_\text{G}$. Angular momentum balance about the stationary point on the ground instantaneously coinciding with the contact point gives

$$\begin{array}{lll} \text{AMB}_{/\text{C}} & \Rightarrow & \vec{r}_{\text{G/C}} \times (-mg\hat{\jmath}) = \vec{r}_{\text{G/C}} \times m\vec{a}_\text{G} + I_{zz}^{\text{cm}}\dot{\omega}\hat{k} \\ & & \left\{-R\sin\phi mg\hat{k} = (R\hat{n}) \times (ma_\text{G}\hat{\boldsymbol{\lambda}}) + I_{zz}^{\text{cm}}\dot{\omega}\hat{k}\right\} \\ \{\}\cdot\hat{k} & \Rightarrow & -Rmg\sin\phi = -Rma_\text{G} - I_{zz}^{\text{cm}}a_\text{G}/R \\ & \Rightarrow & a_\text{G} = \frac{g\sin\phi}{1 + I_{zz}^{\text{cm}}/(mR^2)}. \end{array}$$

Which is less than the acceleration of a block sliding on a ramp without friction: $a = g\sin\phi$ (unless the mass of the rolling ball is concentrated at the center with $I_{zz}^{\text{cm}} = 0$). Note that a very small ball rolls just as slowly. In the limit as the ball radius goes to zero the behavior does not approach that of a point mass that slides; the rolling remains significant.

Example: **Ball rolling down hill: energy approach**

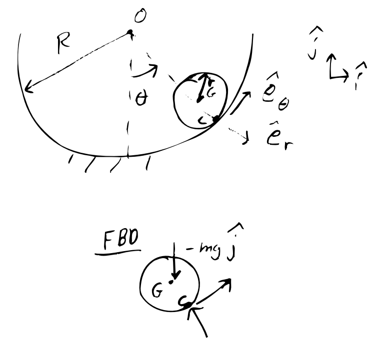


Figure 14.47: A ball rolls in a round cross-section bowl.

Filename:tfigure-ballinbowl

We can find the acceleration of the rolling ball using power balance or conservation of energy. For example

$$\begin{aligned}
0 = \tfrac{d}{dt}E_{\text{T}} \Rightarrow \qquad 0 &= \dot{E_{\text{K}}} + \dot{E_{\text{P}}} \\
&= \tfrac{d}{dt}\left(mv^2/2 + I_{zz}^{\text{cm}}\omega^2/2\right) + \tfrac{d}{dt}(mgy) \\
&= mv\dot{v} + I_{zz}^{\text{cm}}\omega\dot{\omega} + mg\dot{y} \\
&= mv\dot{v} + I_{zz}^{\text{cm}}(v/R)\dot{v}/R - mg(\sin\phi)v \\
\text{assuming } v \neq 0 \Rightarrow \qquad 0 &= (m + I_{zz}^{\text{cm}}/R^2)\dot{v} - mg\sin\phi \\
\Rightarrow \qquad \dot{v} &= \tfrac{g\sin\phi}{1+I_{zz}^{\text{cm}}/(mR^2)}
\end{aligned}$$

as before.

Example: **Does the ball slide?**

How big is the coefficient of friction μ needed to prevent slip for a ball rolling down a hill? Use linear momentum balance to find the normal and frictional components of the contact force, using the rolling example above.

$$\begin{aligned}
\text{AMB } (\vec{\boldsymbol{F}}_{\text{tot}} = m\vec{\boldsymbol{a}}_{\text{G}}) \qquad &\Rightarrow \quad \left\{N\hat{\boldsymbol{n}} + F\hat{\boldsymbol{\lambda}} - mg\hat{\boldsymbol{\jmath}} = ma_{\text{G}}\hat{\boldsymbol{\lambda}}\right\} \\
\{\}\cdot\hat{\boldsymbol{n}} \qquad &\Rightarrow \quad N = mg\cos\phi \\
\{\}\cdot\hat{\boldsymbol{\lambda}} \qquad &\Rightarrow \quad F + mg\sin\phi = m\tfrac{g\sin\phi}{1+I_{zz}^{\text{cm}}/(mR^2)} \\
&\qquad\quad F = \tfrac{-mg\sin\phi}{1+mR^2/I_{zz}^{\text{cm}}} \\
\text{Critical condition:} \qquad &\Rightarrow \quad \mu = \tfrac{|F|}{N} = \tfrac{\tan\phi}{1+mR^2/I_{zz}^{\text{cm}}}
\end{aligned}$$

If I_{zz}^{cm} is very small (the mass concentrated near the center of the ball) then small friction is needed to prevent rolling. For a uniform rubber ball on pavement (with $\mu \approx 1$ and $I_{zz}^{\text{cm}} \approx 2mR^2/5$) the steepest slope for rolling without slip is a steep $\phi = \tan^{-1}(7/2) \approx 74^{\circ}$. A metal hoop on the other hand (with $\mu \approx .3$ and $I_{zz}^{\text{cm}} \approx mR^2$) will only roll without slip for slopes less than about $\phi = \tan^{-1}(.6) \approx 31^{\circ}$.

Example: **Oscillations of a ball in a bowl.**

A round ball can oscillate back and forth in the bottom of a circular cross section bowl or pipe (see Fig. 14.47). Similarly, a cylindrical object can roll inside a pipe. What is the period of oscillation? Start with angular momentum balance about the contact point

$$\begin{aligned}
\vec{\boldsymbol{r}}_{\text{G/C}} \times (-mg\hat{\boldsymbol{\jmath}}) &= \vec{\boldsymbol{r}}_{\text{G/C}} \times m\vec{\boldsymbol{a}}_{\text{G}} + I_{zz}^{\text{cm}}\dot{\omega}\hat{\boldsymbol{k}} \\
rmg\sin\theta\hat{\boldsymbol{k}} &= -r\hat{\boldsymbol{e}}_r \times \left(m\left((R-r)\ddot{\theta}\hat{\boldsymbol{e}}_\theta - (R-r)\dot{\theta}^2\hat{\boldsymbol{e}}_r\right)\right) \\
&\quad + I_{zz}^{\text{cm}}\dot{\omega}\hat{\boldsymbol{k}}.
\end{aligned}$$

Evaluating the cross products (using that $\hat{\boldsymbol{e}}_r \times \hat{\boldsymbol{e}}_t = \hat{\boldsymbol{k}}$) and using the kinematics from the previous section (that $(R-r)\dot{\theta} = -r\omega$) and dotting the left and right sides with $\hat{\boldsymbol{k}}$ gives

$$(R-r)\ddot{\theta} = \frac{g\sin\theta}{1 + I_{zz}^{\text{cm}}/mr^2},$$

the tangential acceleration is the same as would have been predicted by putting the ball on a constant slope of $-\theta$. Using the small angle approximation that $\sin\theta = \theta$ the equation can be rearranged as a standard harmonic oscillator equation

$$\ddot{\theta} + \left(\frac{g}{(R-r)(1 + I_{zz}^{\text{cm}}/mr^2)}\right)\theta,$$

If all the ball's mass were concentrated in its middle (so $I_{zz}^{\text{cm}} = 0$) this is naturally the same as for a simple pendulum with length $R - r$. For any parameter values the period of small oscillation is

$$T = 2\pi\sqrt{\frac{(R-r)(1 + I_{zz}^{\text{cm}}/mr^2)}{g}}.$$

For a marble, ball bearing, or AAA battery in a sideways glass (with $R - r \approx 2\,\mathrm{cm} = .04\,\mathrm{m}$, $I_{zz}^{\mathrm{cm}}/mr^2 \approx 2/5$ and $g \approx 10\,\mathrm{m/s^2}$) this gives about one oscillation every half second. See page **??** for the energy approach to this problem.

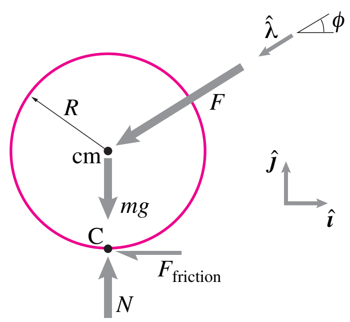


Figure 14.48: FBD of a wheel with mass m. Force F is applied by the axle.

Filename:tfigure2-wheel-mass-lhs

SAMPLE 14.16 A rolling wheel with non-negligible mass. Consider the wheel with mass m shown in figure 14.48. The wheel rolls to the left without slipping. The free-body diagram of the wheel is shown here again. Write the equation of motion of the wheel.

Solution We can write the equation of motion of the wheel in terms of either the center-of-mass position x or the angular displacement of the wheel θ. Since in pure rolling, these two variables share a simple relationship ($x = R\theta$), we can easily get the equation of motion in terms of x if we have the equation in terms of θ and vice versa. Let $\vec{\omega} = \omega\hat{k}$ and $\dot{\vec{\omega}} = \dot{\omega}\hat{k}$.

Since all the forces are shown in the free body diagram, we can readily write the angular momentum balance for the wheel. We choose the point of contact C as our reference point for the angular momentum balance (because the gravity force, $-mg\hat{j}$, the friction force $-F_{friction}\hat{\imath}$, and the normal reaction of the ground $N\hat{j}$, all pass through the contact point C and therefore, produce no moment about this point). We have

$$\sum \vec{M}_C = \dot{\vec{H}}_C$$

where

$$\begin{aligned}
\sum \vec{M}_C &= \overbrace{\vec{r}_{cm/C}}^{R\hat{j}} \times (F\hat{\lambda}) \\
&= R\hat{j} \times F(-\cos\phi\hat{\imath} - \sin\phi\hat{j}) \\
&= FR\cos\phi\hat{k}
\end{aligned}$$

and

$$\begin{aligned}
\dot{\vec{H}}_C &= \vec{r}_{cm/C} \times m\vec{a}_{cm} + I_{zz}^{cm}\dot{\vec{\omega}} \\
&= R\hat{j} \times m \underbrace{\ddot{x}}_{-\dot{\omega}R} \hat{\imath} + I_{zz}^{cm}\dot{\omega}\hat{k} \\
&= m\dot{\omega}R^2\hat{k} + I_{zz}^{cm}\dot{\omega}\hat{k} \\
&= (I_{zz}^{cm} + mR^2)\dot{\omega}\hat{k}.
\end{aligned}$$

Thus,

$$\begin{aligned}
FR\cos\phi\hat{k} &= (I_{zz}^{cm} + mR^2)\dot{\omega}\hat{k} \\
\Rightarrow \quad \dot{\omega} \equiv \ddot{\theta} &= \frac{FR\cos\phi}{I_{zz}^{cm} + mR^2}
\end{aligned}$$

which is the equation of motion we are looking for. Note that we can easily substitute $\ddot{\theta} = -\ddot{x}/R$ in the equation of motion above to get the equation of motion in terms of the center-of-mass displacement x as

$$\ddot{x} = -\frac{FR^2\cos\phi}{I_{zz}^{cm} + mR^2}.$$

$$\boxed{\ddot{\theta} = \frac{FR\cos\phi}{I_{zz}^{cm} + mR^2}}$$

Comments: We could have, of course, used linear momentum balance with angular momentum balance about the center-of-mass to derive the equation of motion. Note, however, that the linear momentum balance will essentially give two scalar equations in the x and y directions involving all forces shown in the free-body diagram . The angular momentum balance , on the other hand, gets rid of some of them. Depending on which forces are known, we may or may not need to use all the three scalar equations. In the final equation of motion, we must have only one unknown.

SAMPLE 14.17 Energy and power of a rolling wheel. A wheel of diameter 2 ft and mass 20 lbm rolls without slipping on a horizontal surface. The kinetic energy of the wheel is 1700 ft· lbf. Assume the wheel to be a thin, uniform disk.

1. Find the rate of rotation of the wheel.
2. Find the average power required to bring the wheel to a complete stop in 5 s.

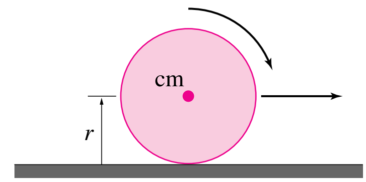


Figure 14.49:
Filename:sfig7-4-1a

Solution

1. Let ω be the rate of rotation of the wheel. Since the wheel rotates without slip, its center-of-mass moves with speed $v_{\text{cm}} = \omega r$. The wheel has both translational and rotational kinetic energy. The total kinetic energy is

$$\begin{aligned} E_K &= \frac{1}{2} m v_{\text{cm}}^2 + \frac{1}{2} I^{\text{cm}} \omega^2 \\ &= \frac{1}{2} m \omega^2 r^2 + \frac{1}{2} I^{\text{cm}} \omega^2 \\ &= \frac{1}{2} (mr^2 + \underbrace{I^{\text{cm}}}_{\frac{1}{2}mr^2}) \omega^2 \\ &= \frac{3}{4} m r^2 \omega^2 \\ \Rightarrow \quad \omega^2 &= \frac{4E_K}{3mr^2} \\ &= \frac{4 \times 1700\,\text{ft}\cdot\text{lbf}}{3 \times 20\,\text{lbm}\cdot 1\,\text{ft}^2} \\ &= \frac{4 \times 1700 \times 32.2\,\text{lbm}\cdot\text{ft}/\text{s}^2}{3 \times 20\,\text{lbm}\cdot\text{ft}} \\ &= 3649.33 \frac{1}{\text{s}^2} \\ \Rightarrow \quad \omega &= 60.4\,\text{rad/s}. \end{aligned}$$

$$\boxed{\omega = 60.4\,\text{rad/s}}$$

Note: This rotational speed, by the way, is extremely high. At this speed the center-of-mass moves at 60.4 ft/s!

2. Power is the rate of work done on a body or the rate of change of kinetic energy. Here we are given the initial kinetic energy, the final kinetic energy (zero) and the time to achieve the final state. Therefore, the average power is,

$$\begin{aligned} P &= \frac{E_{K1} - E_{K2}}{\Delta t} \\ &= \frac{1700\,\text{ft}\cdot\text{lbf} - 0}{5\,\text{s}} = 340\,\text{ft}\cdot\text{lbf/s} \\ &= 340\,\text{ft}\cdot\text{lbf/s} \cdot \frac{1\,\text{hp}}{550\,\text{ft}\cdot\text{lbf/s}} \\ &= 0.62\,\text{hp} \end{aligned}$$

$$\boxed{P = 0.62\,\text{hp}}$$

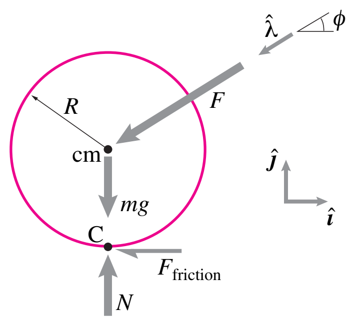


Figure 14.50: FBD of a rolling wheel.

Filename:tfigure2-wheel-mass-lhs-en

SAMPLE 14.18 Equation of motion of a rolling wheel from energy balance. Consider the wheel with mass m from figure 14.50. The free-body diagram of the wheel is shown here again. Derive the equation of motion of the wheel using energy balance.

Solution From energy balance, we have

$$P = \dot{E}_{\rm K}$$

where

$$\begin{aligned}
P &= \sum \vec{F}_i \cdot \vec{v}_i \\
&= -F_{\rm friction}\hat{\imath} \cdot \overbrace{\vec{v}_C}^{\vec{0}} + N\hat{\jmath} \cdot \overbrace{\vec{v}_C}^{\vec{0}} - mg\hat{\jmath} \cdot \overbrace{\vec{v}_{cm}}^{v\hat{\imath}} + F\hat{\lambda} \cdot \overbrace{\vec{v}_{cm}}^{v\hat{\imath}} \\
&= -mgv\underbrace{(\hat{\imath} \cdot \hat{\jmath})}_{0} + Fv(\underbrace{\hat{\lambda} \cdot \hat{\imath}}_{-\cos\phi}) \\
&= -Fv\cos\phi
\end{aligned}$$

and

$$\begin{aligned}
\dot{E}_{\rm K} &= \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_{zz}^{\rm cm}\overbrace{\dot{\omega}^2}^{(\dot{x}/R)^2}\right) \\
&= \frac{1}{2}\frac{d}{dt}[(m + I_{zz}^{\rm cm}/R^2)\dot{x}^2] \\
&= (m + I_{zz}^{\rm cm}/R^2)\dot{x}\ddot{x}.
\end{aligned}$$

Thus,

$$\begin{aligned}
-Fv\cos\phi &= (m + I_{zz}^{\rm cm}/R^2)\dot{x}\ddot{x} \\
\text{or} \quad -F\not{\dot{x}}\cos\phi &= (m + I_{zz}^{\rm cm}/R^2)\not{\dot{x}}\ddot{x} \\
\Rightarrow \quad \ddot{x} &= -\frac{F\cos\phi}{m + I_{zz}^{\rm cm}/R^2}.
\end{aligned}$$

We can also write the equation of motion in terms of θ by replacing $\ddot{x}$ with $\ddot{\theta}R$ giving,

$$\ddot{\theta} = \frac{FR\cos\phi}{m + I_{zz}^{\rm cm}/R^2}.$$

$$\boxed{\ddot{x} = -\frac{F\cos\phi}{m+I_{zz}^{\rm cm}/R^2}}$$

Comments: In the equations above (for calculating P), we have set $\vec{v}_C = \vec{0}$ because in pure rolling, the instantaneous velocity of the contact point is zero. Note that the force due to gravity is normal to the direction of the velocity of the center-of-mass. So, the only power supplied to the wheel is due to the force $F\hat{\lambda}$ acting at the center-of-mass.

SAMPLE 14.19 Equation of motion of a rolling disk on an incline. A uniform circular disk of mass $m = 1\,\mathrm{kg}$ and radius $R = 0.4\,\mathrm{m}$ rolls down an inclined shown in the figure. Write the equation of motion of the disk assuming pure rolling, and find the distance travelled by the center-of-mass in 2 s.

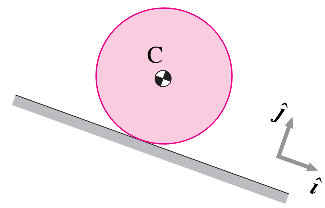


Figure 14.51:
Filename:sfig9-rolling-incline1

Solution The free-body diagram of the disk is shown in Fig. 14.52. In addition to the base unit vectors $\hat{\imath}$ and $\hat{\jmath}$, let us use unit vectors $\hat{\boldsymbol{\lambda}}$ and $\hat{\boldsymbol{n}}$ along the plane and perpendicular to the plane, respectively, to express various vectors. We can write the equation of motion using linear momentum balance or angular momentum balance. However, note that if we use linear momentum balance we have two unknown forces in the equation. On the other hand, if we use angular momentum balance about the contact point C, these forces do not show up in the equation. So, let us use angular momentum balance about point C:

$$\sum \vec{\boldsymbol{M}}_C = \dot{\vec{\boldsymbol{H}}}_C$$

where

$$\begin{aligned}\sum \vec{\boldsymbol{M}}_C &= \vec{\boldsymbol{r}}_{O/C} \times m\vec{\boldsymbol{g}} = R\hat{\boldsymbol{n}} \times (-mg\hat{\boldsymbol{\jmath}}) \\ &= -Rmg\sin\alpha\hat{\boldsymbol{k}}\end{aligned}$$

and

$$\begin{aligned}\dot{\vec{\boldsymbol{H}}}_C &= -I_{zz}^{cm}\dot{\omega}\hat{\boldsymbol{k}} + \overbrace{\vec{\boldsymbol{r}}_{O/C}}^{R\hat{\boldsymbol{n}}} \times m\overbrace{\vec{\boldsymbol{a}}_{cm}}^{R\dot{\omega}\hat{\boldsymbol{\lambda}}} \\ &= -I_{zz}^{cm}\dot{\omega}\hat{\boldsymbol{k}} + mR^2\dot{\omega}(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\lambda}}) \\ &= -(I_{zz}^{cm} + mR^2)\dot{\omega}\hat{\boldsymbol{k}}.\end{aligned}$$

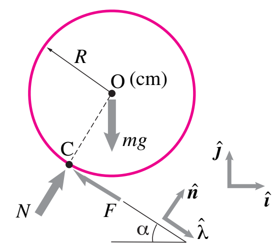


Figure 14.52:
Filename:sfig9-rolling-incline1a

Thus,

$$\begin{aligned}-Rmg\sin\alpha\hat{\boldsymbol{k}} &= -(I_{zz}^{cm} + mR^2)\dot{\omega}\hat{\boldsymbol{k}} \\ \Rightarrow \quad \dot{\omega} &= \frac{g\sin\alpha}{R[1 + I_{zz}^{cm}/(mR^2)]}.\end{aligned}$$

$$\boxed{\dot{\omega} = \frac{g\sin\alpha}{R[1+I_{zz}^{cm}/(mR^2)]}}$$

Note that in the above equation of motion, the right hand side is constant. So, we can solve the equation for ω and θ by simply integrating this equation and substituting the initial conditions $\omega(t = 0) = 0$ and $\theta(t = 0) = 0$. Let us write the equation of motion as $\dot{\omega} = \beta$ where $\beta = g\sin\alpha/R(1 + I_{zz}^{cm}/mR^2)$. Then,

$$\begin{aligned}\omega &\equiv \dot{\theta} = \beta t + C_1 \\ \theta &= \frac{1}{2}\beta t^2 + C_1 t + C_2.\end{aligned}$$

Substituting the given initial conditions $\dot{\theta}(0) = 0$ and $\theta(0) = 0$, we get $C_1 = 0$ and $C_2 = 0$, which implies that $\theta = \frac{1}{2}\beta t^2$. Now, in pure rolling, $x = R\theta$. Therefore,

$$\begin{aligned}x(t) &= R\theta(t) = \frac{1}{2}\beta t^2 = \not{R} \cdot \frac{1}{2}\frac{g\sin\alpha}{\not{R}(1 + I_{zz}^{cm}/mR^2)}t^2 \\ &= \frac{1}{2}\frac{g\sin\alpha}{1 + \frac{\frac{1}{2}mR^2}{mR^2}}t^2 = \frac{1}{3}(g\sin\alpha)t^2 \\ x(2\,\mathrm{s}) &= \frac{1}{3} \cdot 9.8\,\mathrm{m/s^2} \cdot \sin(30^\circ) \cdot (2\,\mathrm{s})^2 = 6.53\,\mathrm{m}.\end{aligned}$$

$$\boxed{x(2\,\mathrm{s}) = 6.53\,\mathrm{m}}$$

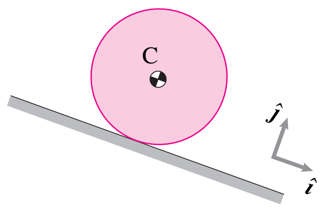


Figure 14.53:
Filename:sfig9-rolling-incline2

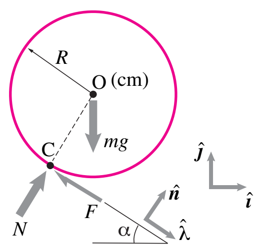


Figure 14.54:
Filename:sfig9-rolling-incline2a

SAMPLE 14.20 Using Work and energy in pure rolling. Consider the disk of Sample 14.19 rolling down the incline again. Suppose the disk starts rolling from rest. Find the speed of the center-of-mass when the disk is 2 m down the inclined plane.

Solution We are given that the disk rolls down, starting with zero initial velocity. We are to find the speed of the center-of-mass after it has travelled 2 malong the incline. We can, of course, solve this problem using equation of motion, by first solving for the time t the disk takes to travel the given distance and then evaluating the expression for speed $\omega(t)$ or $x(t)$ at that t. However, it is usually easier to use work energy principle whenever positions are specified at two instants, speed is specified at one of those instants, and speed is to be found at the other instant. This is because we can, presumably, compute the work done on the system in travelling the specified distance and relate it to the change in kinetic energy of the system between the two instants. In the problem given here, let ω_1 and ω_2 be the initial and final (after rolling down by $d = 2\,\text{m}$) angular speeds of the disk, respectively. We know that in rolling, the kinetic energy is given by

$$E_K = \frac{1}{2}m \overbrace{v_{cm}^2}^{(\omega R)^2} + \frac{1}{2}I_{zz}^{cm}\omega^2 = \frac{1}{2}(mR^2 + I_{zz}^{cm})\omega^2.$$

Therefore,

$$\Delta E_K = E_{K2} - E_{K1} = \frac{1}{2}(mR^2 + I_{zz}^{cm})(\omega_2^2 - \omega_1^2). \tag{14.39}$$

Now, let us calculate the work done by all the forces acting on the disk during the displacement of he mass-center by d along the plane. Note that in ideal rolling, the contact forces do no work. Therefore, the work done on the disk is only due to the gravitational force:

$$W = (-mg\hat{\jmath}) \cdot (d\hat{\lambda}) = -mgd(\overbrace{\hat{\jmath} \cdot \hat{\lambda}}^{-\sin\alpha}) = mgd\sin\alpha. \tag{14.40}$$

From work-energy principle (integral form of power balance, $P = \dot{E}_K$), we know that $W = \Delta E_K$. Therefore, from eqn. (14.39) and eqn. (14.40), we get

$$\begin{aligned} mgd\sin\alpha &= \frac{1}{2}(mR^2 + I_{zz}^{cm})(\omega_2^2 - \omega_1^2) \\ \Rightarrow \quad \omega_2^2 &= \omega_1^2 + \frac{2mgd\sin\alpha}{mR^2 + I_{zz}^{cm}} = \omega_1^2 + \frac{2gd\sin\alpha}{R^2\left(1 + \frac{I_{zz}^{cm}}{mR^2}\right)} \\ &= \omega_1^2 + \frac{4gd\sin\alpha}{3R^2}. \end{aligned}$$

Substituting the values of g, d, α, R, etc., and setting $\omega_1 = 0$, we get

$$\begin{aligned} \omega_2^2 &= \frac{4 \cdot (9.8\,\text{m/s}^2) \cdot (2\,\text{m}) \cdot (\sin(30^\circ))}{3 \cdot (0.4\,\text{m})^2} = 81.67/s^2 \\ \Rightarrow \quad \omega_2 &= 9.04\,\text{rad/s}. \end{aligned}$$

The corresponding speed of the center-of-mass is

$$v_{cm} = \omega_2 R = 9.04\,\text{rad/s} \cdot 0.4\,\text{m} = 3.61\,\text{m/s}.$$

$$\boxed{v_{cm} = 3.61\,\text{m/s}}$$

SAMPLE 14.21 Impulse and momentum calculations in pure rolling. Consider the disk of Sample 14.19 rolling down the incline again. Find an expression for the rolling speed (ω) of the disk after a finite time Δt, given the initial rolling speed ω_1.

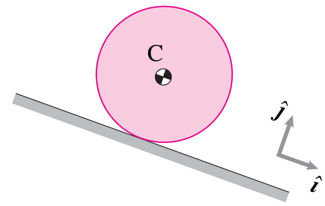


Figure 14.55:

Filename:sfig9-rolling-incline3

Solution Once again, this problem can be solved by integrating the equation of motion (as done in Sample 14.19). However, we will solve this problem here using impulse-momentum relationship. Note that we need the speed of the disk ω_2, after a finite time Δt, given the initial speed ω_1. Since the forces acting on the disk do not change during this time (assuming pure rolling), it is easy to calculate impulse and then relate it to the change in the momenta of the disk between the two instants. Now, from the linear impulse momentum relationship, $\sum \vec{F}\cdot\Delta t = \vec{L}_2 - \vec{L}_1$, we have

$$(-F\hat{\lambda} + N\hat{\imath} - mg\hat{\jmath})\Delta t = m(v_2 - v_1)\hat{\lambda}. \quad (14.41)$$

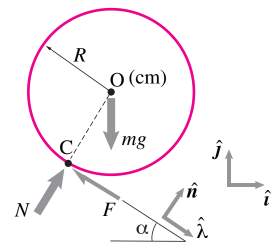


Figure 14.56:

Filename:sfig9-rolling-incline3a

Dotting eqn. (14.41) with $\hat{\lambda}$ gives

$$\begin{aligned}(-F - mg(\underbrace{\hat{\jmath}\cdot\hat{\lambda}}_{-\sin\alpha}))\Delta t &= m(v_2 - v_1)\\ (-F + mg\sin\alpha)\Delta t &= mR(\omega_2 - \omega_1). \quad (14.42)\end{aligned}$$

Similarly, the angular impulse-momentum relationship about the mass-center, $\vec{M}_O\Delta t = (\vec{H}_O)_2 - (\vec{H}_O)_1$, gives

$$\begin{aligned}(-FR\hat{k})\Delta t &= -I_{zz}^{cm}(\omega_2 - \omega_1)\hat{k}\\ \Rightarrow \quad FR\Delta t &= I_{zz}^{cm}(\omega_2 - \omega_1). \quad (14.43)\end{aligned}$$

Note that the other forces (N and mg) do not produce any moment about the mass-center as they pass through this point. We can now eliminate the unknown force F from eqn. (14.42) and eqn. (14.43) by multiplying eqn. (14.42) with R and adding to eqn. (14.43):

$$\begin{aligned}mgR\sin\alpha\Delta t &= (I_{zz}^{cm} + mR^2)(\omega_2 - \omega_1)\\ \text{or} \quad g\sin\alpha\Delta t &= R\left(1 + \frac{I_{zz}^{cm}}{mR^2}\right)(\omega_2 - \omega_1)\\ \Rightarrow \quad \omega_2 &= \omega_1 + \frac{g\sin\alpha}{R\left(1 + \frac{I_{zz}^{cm}}{mR^2}\right)}\Delta t.\end{aligned}$$

$$\boxed{\omega_2 = \omega_1 + \frac{g\sin\alpha}{R\left(1+\frac{I_{zz}^{cm}}{mR^2}\right)}\Delta t}$$

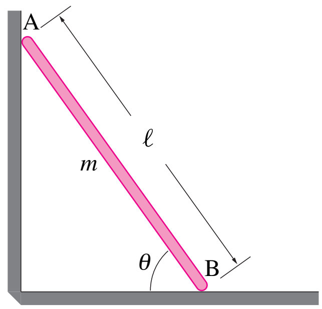


Figure 14.57: A ladder, modeled as a uniform rod of mass m and length ℓ, falls from a rest position at $\theta = \theta_o$ ($< \pi/2$) such that its ends slide along frictionless vertical and horizontal surfaces.

Filename:sfig7-3-1

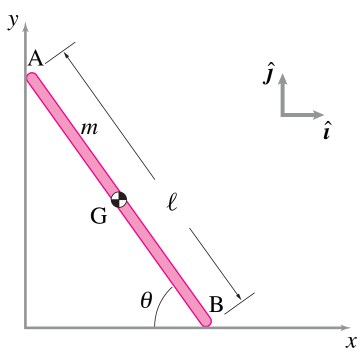


Figure 14.58:

Filename:sfig7-3-1a

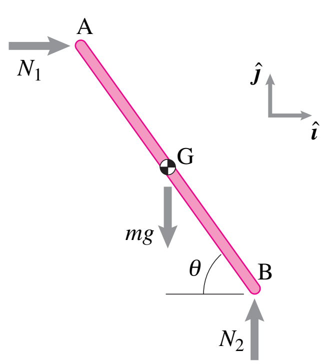


Figure 14.59: The free-body diagram of the ladder.

Filename:sfig7-3-1b

SAMPLE 14.22 Falling ladder. A ladder AB, modeled as a uniform rigid rod of mass m and length ℓ, rests against frictionless horizontal and vertical surfaces. The ladder is released from rest at $\theta = \theta_o$ ($\theta_o < \pi/2$). Assume the motion to be planar (in the vertical plane).

1. As the ladder falls, what is the path of the center-of-mass of the ladder?
2. Find the equation of motion (e.g., a differential equation in terms of θ and its time derivatives) for the ladder.
3. How does the angular speed ω ($= \dot{\theta}$) depend on θ?

Solution Since the ladder is modeled by a uniform rod AB, its center-of-mass is at G, half way between the two ends. As the ladder slides down, the end A moves down along the vertical wall and the end B moves out along the floor. Note that it is a single degree of freedom system as angle θ (a single variable) is sufficient to determine the position of every point on the ladder at any instant of time.

1. **Path of the center-of-mass:** Let the origin of our x-y coordinate system be the intersection of the two surfaces on which the ends of the ladder slide (see Fig. 14.58). The position vector of the center-of-mass G may be written as

$$\begin{aligned} \vec{r}_G &= \vec{r}_B + \vec{r}_{G/B} \\ &= \ell\cos\theta\hat{\imath} + \frac{\ell}{2}(-\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) \\ &= \frac{\ell}{2}(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}). \end{aligned} \tag{14.44}$$

Thus the coordinates of the center-of-mass are

$$x_G = \frac{\ell}{2}\cos\theta \quad \text{and} \quad y_G = \frac{\ell}{2}\sin\theta,$$

from which we get

$$x_G^2 + y_G^2 = \frac{\ell^2}{4}$$

which is the equation of a circle of radius $\frac{\ell}{2}$. Therefore, the center-of-mass of the ladder follows a circular path of radius $\frac{\ell}{2}$ centered at the origin. Of course, the center-of-mass traverses only that part of the circle which lies between its initial position at $\theta = \theta_o$ and the final position at $\theta = 0$.

2. **Equation of motion:** The free-body diagram of the ladder is shown in Fig. 14.59. Since there is no friction, the only forces acting at the end points A and B are the normal reactions from the contacting surfaces. Now, writing the the linear momentum balance ($\sum \vec{F} = m\vec{a}$) for the ladder we get

$$N_1\hat{\imath} + (N_2 - mg)\hat{\jmath} = m\vec{a}_G = m\ddot{\vec{r}}_G.$$

Differentiating eqn. (14.44) twice we get $\ddot{\vec{r}}_G$ as

$$\ddot{\vec{r}}_G = \frac{\ell}{2}[(-\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta)\hat{\imath} + (\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\hat{\jmath}].$$

Substituting this expression in the linear momentum balance equation above and dotting both sides of the equation by $\hat{\imath}$ and then by $\hat{\jmath}$ we get

$$\begin{aligned} N_1 &= -\frac{1}{2}m\ell(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) \\ N_2 &= \frac{1}{2}m\ell(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + mg. \end{aligned}$$

Next, we write the angular momentum balance for the ladder about its center-of-mass, $\sum \vec{\boldsymbol{M}}_{/G} = \dot{\vec{\boldsymbol{H}}}_{/G}$, where

$$
\begin{aligned}
\sum \vec{\boldsymbol{M}}_{/G} &= \left(-N_1 \frac{\ell}{2}\sin\theta + N_2 \frac{\ell}{2}\cos\theta\right)\hat{\boldsymbol{k}} \\
&= \frac{1}{2}m\ell(\ddot{\theta}\,\sin\theta + \dot{\theta}^2\cos\theta)\frac{\ell}{2}\sin\theta\hat{\boldsymbol{k}} \\
&\quad + \left[\frac{1}{2}m\ell(\ddot{\theta}\,\cos\theta - \dot{\theta}^2\sin\theta) + mg\right]\frac{\ell}{2}\cos\theta\hat{\boldsymbol{k}} \\
&= \left(\frac{1}{4}m\ell^2\ddot{\theta} + \frac{1}{2}mg\ell\cos\theta\right)\hat{\boldsymbol{k}}
\end{aligned}
$$

and

$$
\dot{\vec{\boldsymbol{H}}}_{/G} = I_{zz/G}\dot{\vec{\boldsymbol{\omega}}} = \frac{1}{12}m\ell^2\ddot{\theta}(-\hat{\boldsymbol{k}}),
$$

where $\dot{\vec{\boldsymbol{\omega}}} = \ddot{\theta}(-\hat{\boldsymbol{k}})$ because θ is measured positive in the clockwise direction $(-\hat{\boldsymbol{k}})$. Now, equating the two quantities $\sum \vec{\boldsymbol{M}}_{/G} = \dot{\vec{\boldsymbol{H}}}_{/G}$ and dotting both sides with $\hat{\boldsymbol{k}}$ we get

$$
\begin{aligned}
\frac{1}{4}\not{m}\ell^2\ddot{\theta} + \frac{1}{2}\not{m}g\ell\cos\theta &= -\frac{1}{12}\not{m}\ell^2\ddot{\theta} \\
\text{or} \qquad (\frac{1}{12} + \frac{1}{4})\ell^2\ddot{\theta} &= -\frac{1}{2}g\ell\cos\theta \\
\text{or} \qquad \ddot{\theta} &= -\frac{3g}{2\ell}\cos\theta
\end{aligned}
\tag{14.45}
$$

which is the required equation of motion. Unfortunately, it is a nonlinear equation which does not have a nice closed form solution for $\theta(t)$.

3. **Angular Speed of the ladder:** To solve for the angular speed $\omega\ (=\dot{\theta})$ as a function of θ we need to express eqn. (14.45) in terms of $\omega,\ \theta$, and derivatives of ω with respect to θ. Now,

$$
\ddot{\theta} = \dot{\omega} = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\cdot\frac{d\theta}{dt} = \omega\,\frac{d\omega}{d\theta}.
$$

Substituting in eqn. (14.45) and integrating both sides from the initial rest position to an arbitrary position θ we get

$$
\begin{aligned}
\int_0^{\omega}\omega\,d\omega &= -\int_{\theta_0}^{\theta}\frac{3g}{2\ell}\cos\theta\,d\theta \\
\Rightarrow \qquad \frac{1}{2}\omega^2 &= -\frac{3g}{2\ell}(\sin\theta - \sin\theta_0) \\
\Rightarrow \qquad \omega &= \pm\sqrt{\frac{3g}{\ell}(\sin\theta_0 - \sin\theta)}.
\end{aligned}
$$

Since end B is sliding to the right, θ is decreasing; hence it is the negative sign in front of the square root which gives the correct answer, *i.e.*,

$$
\vec{\boldsymbol{\omega}} = \dot{\theta}(-\hat{\boldsymbol{k}}) = -\sqrt{\frac{3g}{\ell}(\sin\theta_0 - \sin\theta)}\;\hat{\boldsymbol{k}}.
$$

SAMPLE 14.23 The falling ladder again. Consider the falling ladder of Sample 14.10 again. The mass of the ladder is m and the length is ℓ. The ladder is released from rest at $\theta = 80^\circ$.

1. At the instant when $\theta = 45^\circ$, find the speed of the center-of-mass of the ladder using energy.
2. Derive the equation of motion of the ladder using work-energy balance.

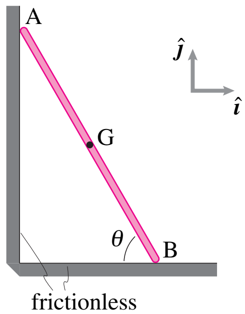


Figure 14.60:

Filename:sfig7-4-3

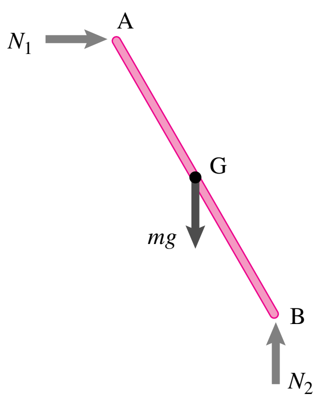


Figure 14.61:

Filename:sfig7-4-3a

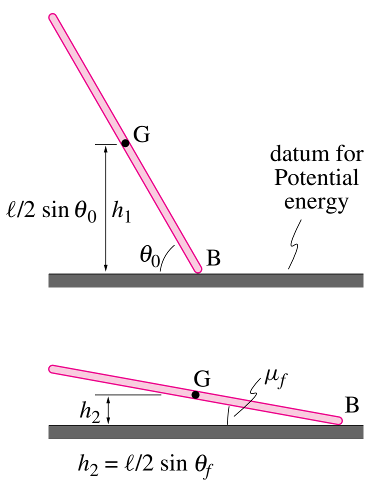


Figure 14.62:

Filename:sfig7-4-3b

Solution

1. Since there is no friction, there is no loss of energy between the two states: $\theta_0 = 80^\circ$ and $\theta_f = 45^\circ$. The only external forces on the ladder are N_1, N_2, and mg as shown in the free body diagram. Since the displacements of points A and B are perpendicular to the normal reactions of the walls, N_1 and N_2, respectively, no work is done by these forces on the ladder. The only force that does work is the force due to gravity. But this force is conservative. Therefore, the conservation of energy holds between any two states of the ladder during its fall.

Let E_1 and E_2 be the total energy of the ladder at θ_0 and θ_f, respectively. Then

$$E_1 = E_2 \quad \text{(conservation of energy).}$$

$$\begin{aligned} \text{Now} \quad E_1 &= \underbrace{E_{K_1}}_{\text{K.E.}} + \underbrace{E_{P_1}}_{\text{P.E.}} \\ &= 0 + mgh_1 \\ &= mg\frac{\ell}{2}\sin\theta_0 \\ \text{and} \quad E_2 &= E_{K_2} + E_{P_2} = \underbrace{\frac{1}{2}mv_G^2 + \frac{1}{2}I_{zz}^G\omega^2}_{E_{K_2}} + mgh_2. \end{aligned}$$

Equating E_1 and E_2 we get

$$\begin{aligned} \not{m}g\frac{\ell}{\not{2}}(\sin\theta_0 - \sin\theta_f) &= \frac{1}{\not{2}}(\not{m}v_G^2 + \underbrace{\frac{1}{12}\not{m}\ell^2}_{I_{zz}^G}\omega^2) \\ \text{or} \quad g\ell(\sin\theta_0 - \sin\theta_f) &= v_G^2 + \frac{1}{12}\ell^2\omega^2. \end{aligned} \tag{14.46}$$

Clearly, we cannot find v_G from this equation alone because the equation contains another unknown, ω. So we need to find another equation which relates v_G and ω. To find this equation we turn to kinematics. Note that

$$\begin{aligned} \vec{r}_G &= \frac{\ell}{2}(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) \\ \Rightarrow \quad \vec{v}_G &= \dot{\vec{r}}_G = \frac{\ell}{2}(-\sin\theta\cdot\dot{\theta}\hat{\imath} + \cos\theta\cdot\dot{\theta}\hat{\jmath}) \\ \Rightarrow \quad v_G &= |\vec{v}_G| = \sqrt{\frac{\ell^2}{4}(\cos^2\theta + \sin^2\theta)\,\dot{\theta}^2} \\ &= \frac{\ell}{2}\dot{\theta} = \frac{\ell}{2}\omega \\ \Rightarrow \quad \omega &= \frac{2v_G}{\ell}. \end{aligned}$$

Substituting the expression for ω in eqn. (14.46) we get

$$\begin{aligned} g\ell(\sin\theta_0 - \sin\theta_f) &= v_G^2 + \frac{1}{12}\ell^2 \cdot \frac{4v_G^2}{\ell^2} \\ &= \frac{4}{3}v_G^2 \\ \Rightarrow \quad v_G &= \sqrt{\frac{3g\ell}{4}(\sin\theta_0 - \sin\theta_f)} \\ &= 0.46\sqrt{g\ell}. \end{aligned}$$

$$\boxed{v_G = 0.46\sqrt{g\ell}}$$

2. Equation of motion: Since the ladder is a single degree of freedom system, we can use the power equation to derive the equation of motion:

$$P = \dot{E}_K.$$

For the ladder, the only force that does work is mg. This force acts on the center-of-mass G. Therefore,

$$\begin{aligned} P &= \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}} = -mg\hat{\boldsymbol{\jmath}} \cdot \vec{\boldsymbol{v}}_G \\ &= -mg\hat{\boldsymbol{\jmath}} \cdot \left[\frac{\ell}{2}(-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}})\,\dot{\theta}\right] \\ &= -mg\frac{\ell}{2}\dot{\theta}\cos\theta. \end{aligned}$$

Now, the rate of change of kinetic energy is

$$\begin{aligned} \dot{E}_K &= \frac{d}{dt}\left(\frac{1}{2}mv_G^2 + \frac{1}{2}I_{zz}^G\omega^2\right) \\ &= \frac{d}{dt}\left(\frac{1}{2}m\frac{\ell^2\omega^2}{4} + \frac{1}{2}\frac{m\ell^2}{12}\omega^2\right) \\ &= \frac{m\ell^2}{4}\omega\dot{\omega} + \frac{m\ell^2}{12}\omega\dot{\omega} \\ &= \frac{m\ell^2}{3}\omega\dot{\omega} \equiv \frac{m\ell^2}{3}\dot{\theta}\ddot{\theta} \quad (\text{since } \omega = \dot{\theta} \text{ and } \dot{\omega} = \ddot{\theta}). \end{aligned}$$

Now equating P and $\dot{E}_K$ we get

$$\begin{aligned} \frac{\not{m}\ell^2}{3}\not{\dot{\theta}}\ddot{\theta} &= -\not{m}g\frac{\ell}{2}\not{\dot{\theta}}\cos\theta \\ \Rightarrow \quad \ddot{\theta} &= -\frac{3g}{2\ell}\cos\theta \end{aligned}$$

which is the same expression as obtained in Sample 14.22 (b).

$$\boxed{\ddot{\theta} = -\frac{3g}{2\ell}\cos\theta}$$

Note: To do this problem we have assumed that the upper end of the ladder stays in contact with the wall as it slides down. One might wonder if this is a consistent assumption. Does this assumption correspond to the non-physical assumption that the wall is capable of pulling on the ladder? Or in other words, if a real ladder was sliding against a slippery wall and floor would it lose contact? The answer is yes. One way of finding when contact would be lost is to calculate the normal reaction N_1 and finding out at what value of θ it passes through zero. It turns out that N_1 is zero at about $\theta = 41°$.

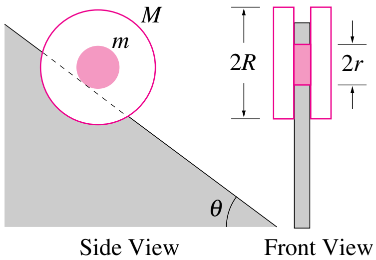


Figure 14.63: A composite wheel made of three uniform disks rolls down an inclined wedge without slipping.

Filename:sfig7-3-3

SAMPLE 14.24 Rolling on an inclined plane. A wheel is made up of three uniform disks— the center disk of mass $m = 1\,\text{kg}$, radius $r = 10\,\text{cm}$ and two identical outer disks of mass $M = 2\,\text{kg}$ each and radius R. The wheel rolls down an inclined wedge without slipping. The angle of inclination of the wedge with horizontal is $\theta = 30^\circ$. The radius of the bigger disks is to be selected such that the linear acceleration of the wheel center does not exceed 0.2g. Find the radius R of the bigger disks.

Solution Since a bound is prescribed on the linear acceleration of the wheel and the radius of the bigger disks is to be selected to satisfy this bound, we need to find an expression for the acceleration of the wheel (hopefully) in terms of the radius R.

The free-body diagram of the wheel is shown in Fig. 14.64. In addition to the weight $(m+2M)g$ of the wheel and the normal reaction N of the wedge surface there is an unknown force of friction F_f acting on the wheel at point C. This friction force is necessary for the condition of rolling motion. You must realize, however, that $F_f \neq \mu N$ because there is neither slipping nor a condition of impending slipping. Thus the magnitude of F_f is not known yet.

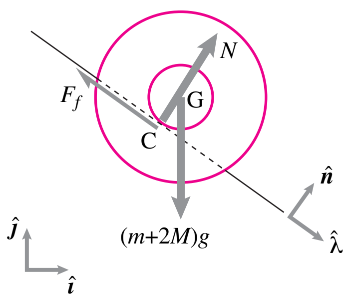


Figure 14.64: Free body diagram of the wheel.

Filename:sfig7-3-3a

Let the acceleration of the center-of-mass of the wheel be

$$\vec{\boldsymbol{a}}_G = a_G \hat{\boldsymbol{\lambda}}$$

and the angular acceleration of the wheel be

$$\dot{\vec{\boldsymbol{\omega}}} = -\dot{\omega}\hat{\boldsymbol{k}}.$$

We assumed $\dot{\vec{\boldsymbol{\omega}}}$ to be in the *negative* $\hat{\boldsymbol{k}}$ direction. But, if this assumption is wrong, we will get a negative value for $\dot{\omega}$.

Now we write the equation of linear momentum balance for the wheel:

$$\begin{aligned} \sum \vec{\boldsymbol{F}} &= m_{\text{total}}\, \vec{\boldsymbol{a}}_{\text{cm}} \\ -(m+2M)g\hat{\boldsymbol{\jmath}} + N\hat{\boldsymbol{n}} - F_f\hat{\boldsymbol{\lambda}} &= (m+2M)a_G\hat{\boldsymbol{\lambda}} \end{aligned}$$

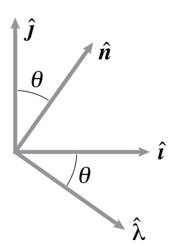


Figure 14.65: Geometry of unit vectors. This diagram can be used to find various dot and cross products between any two unit vectors. For example, $\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\jmath}} = \sin\theta\hat{\boldsymbol{k}}$.

Filename:sfig7-3-3b

This 2-D vector equation gives (at the most) two independent scalar equations. But we have three unknowns: N, F_f, and a_G. Thus we do not have enough equations to solve for the unknowns including the quantity of interest a_G. So, we now write the equation of angular momentum balance for the wheel about the point of contact C (using $\vec{\boldsymbol{r}}_{G/C} = r\hat{\boldsymbol{n}}$):

$$\sum \vec{\boldsymbol{M}}_{\text{C}} = \dot{\vec{\boldsymbol{H}}}_{\text{C}}$$

where

$$\begin{aligned} \vec{\boldsymbol{M}}_{\text{C}} &= \vec{\boldsymbol{r}}_{G/C} \times (m+2M)g(-\hat{\boldsymbol{\jmath}}) \\ &= r\hat{\boldsymbol{n}} \times (m+2M)g(-\hat{\boldsymbol{\jmath}}) \\ &= -(m+2M)gr\sin\theta\hat{\boldsymbol{k}} \quad \text{(see Fig. 14.65)} \end{aligned}$$

and

$$\begin{aligned} \dot{\vec{\boldsymbol{H}}}_{\text{C}} &= I_{zz}^{G}\,\dot{\vec{\boldsymbol{\omega}}} + \vec{\boldsymbol{r}}_{G/C} \times m_{\text{total}}\,\vec{\boldsymbol{a}}_G \\ &= I_{zz}^{G}(-\dot{\omega}\hat{\boldsymbol{k}}) - m_{\text{total}}\,\dot{\omega}r^2\hat{\boldsymbol{k}} \\ &= (I_{zz}^{G} + m_{\text{total}}r^2)(-\dot{\omega}\hat{\boldsymbol{k}}) \\ &= \left[(\frac{1}{2}mr^2 + 2\cdot\frac{1}{2}MR^2) + \overbrace{(m+2M)}^{m_{\text{total}}}r^2\right](-\dot{\omega}\hat{\boldsymbol{k}}) \\ &= -\left[\frac{3}{2}mr^2 + M(R^2+2r^2)\right]\dot{\omega}\hat{\boldsymbol{k}}. \end{aligned}$$

Thus,

$$\begin{aligned} -(m+2M)gr\sin\theta\hat{\boldsymbol{k}} &= -\left[\frac{3}{2}mr^2 + M(R^2+2r^2)\right]\dot{\omega}\hat{\boldsymbol{k}} \\ \Rightarrow \quad \dot{\omega} &= \frac{(m+2M)gr\sin\theta}{\frac{3}{2}mr^2 + M(R^2+2r^2)}. \end{aligned} \tag{14.47}$$

Now we need to relate $\dot{\omega}$ to a_G. From the kinematics of rolling,

$$a_G = \dot{\omega}r.$$

Therefore, from Eqn. (14.47) we get

$$a_G = \frac{(m+2M)gr^2\sin\theta}{\frac{3}{2}mr^2 + M(R^2+2r^2)}.$$

Now we can solve for R in terms of a_G:

$$\begin{aligned} \frac{3}{2}mr^2 + M(R^2+2r^2) &= \frac{(m+2M)gr^2\sin\theta}{a_G} \\ \Rightarrow \quad M(R^2+2r^2) &= \frac{(m+2M)g}{a_G}r^2\sin\theta - \frac{3}{2}mr^2 \\ \Rightarrow \quad R^2 &= \frac{(m+2M)g}{Ma_G}r^2\sin\theta - \frac{3m}{2M}r^2 - 2r^2. \end{aligned}$$

Since we require $a_G \leq 0.2g$ we get

$$\begin{aligned} R^2 &\geq \left(\frac{(m+2M)g}{M\cdot 0.2g}\sin\theta - \frac{3m}{2M} - 2\right)r^2 \\ &\geq \left(\frac{5\,\text{kg}}{0.4\,\text{kg}}\cdot\frac{1}{2} - \frac{3\,\text{kg}}{4\,\text{kg}} - 2\right)(0.1\,\text{m})^2 \\ &\geq 0.035\,\text{m}^2 \\ \Rightarrow \quad R &\geq 0.187\,\text{m}. \end{aligned}$$

Thus the outer disks of radius 20 cm will do the job.

$$\boxed{R \geq 18.7\,\text{cm}}$$

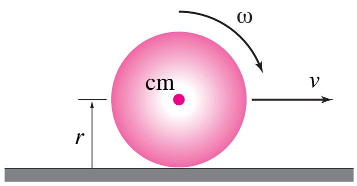


Figure 14.66:

Filename:sfig9-rollandslide-ball1

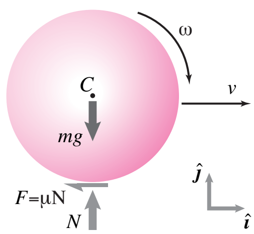


Figure 14.67: Free-body diagram of the ball during sliding.

Filename:sfig9-rollandslide-ball1a

SAMPLE 14.25 Which one starts rolling first — a marble or a bowling ball? A marble and a bowling ball, made of the same material, are launched on a horizontal platform with the same initial velocity, say v_0. The initial velocity is large enough so that both start out sliding. Towards the end of their motion, both have pure rolling motion. If the radius of the bowling ball is 16 times that of the marble, find the instant, for each ball, when the sliding motion changes to rolling motion.

Solution Let us consider one ball, say the bowling ball, first. Let the radius of the ball be r and mass m. The ball starts with center-of-mass velocity $\vec{v}_o = v_0\hat{\imath}$. The ball starts out sliding. During the sliding motion, the force of friction acting on the ball must equal μN (see the FBD). The friction force creates a torque about the mass-center which, in turn, starts the rolling motion of the ball. However, rolling and sliding coexist for a while, till the speed of the mass-center slows down enough to satisfy the pure rolling condition, $v = \omega r$. Let the instant of transition from the mixed motion to pure rolling be t^*. From linear momentum balance , we have

$$
\begin{aligned}
& & m\dot{v}\hat{\imath} &= -\mu N\hat{\imath} + (N - mg)\hat{\jmath} && (14.48)\\
\text{eqn. (14.48)}\cdot\hat{\jmath} &\Rightarrow & N &= mg \\
\text{eqn. (14.48)}\cdot\hat{\imath} &\Rightarrow & m\dot{v} &= -\mu N = -\mu mg \\
&\Rightarrow & \dot{v} &= -\mu g \\
&\Rightarrow & v &= v_0 - \mu g t. && (14.49)
\end{aligned}
$$

Similarly, from angular momentum balance about the mass-center, we get

$$
\begin{aligned}
-I_{zz}^{cm}\dot{\omega}\hat{k} &= -\mu N r\hat{k} = -\mu mgr\hat{k} \\
\Rightarrow \quad \dot{\omega} &= \frac{\mu mgr}{I_{zz}^{cm}} \\
\Rightarrow \quad \omega &= \underbrace{\omega_0}_{0} + \frac{\mu mgr}{I_{zz}^{cm}}t. && (14.50)
\end{aligned}
$$

At the instant of transition from mixed rolling and sliding to pure rolling, *i.e.*, at $t = t^*$, $v = \omega r$. Therefore, from eqn. (14.49) and eqn. (14.50), we get

$$
\begin{aligned}
v_0 - \mu g t^* &= \frac{\mu mgr^2}{I_{zz}^{cm}}t^* \\
\Rightarrow \quad v_0 &= \mu g t^*(1 + \frac{mr^2}{I_{zz}^{cm}}) \\
\Rightarrow \quad t^* &= \frac{v_0}{\mu g(1 + \frac{mr^2}{I_{zz}^{cm}})}.
\end{aligned}
$$

Now, for a sphere, $I_{zz}^{cm} = \frac{2}{5}mr^2$. Therefore,

$$
t^* = \frac{v_0}{\mu g(1 + \frac{mr^2}{\frac{2}{5}mr^2})} = \frac{2v_0}{7\mu g}.
$$

Note that the expression for t^* is independent of mass and radius of the ball! Therefore, the bowling ball and the marble are going to change their mixed motion to pure rolling at exactly the same instant. This is not an intuitive result.

$t^* = \frac{2v_0}{7\mu g}$ for both.

SAMPLE 14.26 Transition from a mix of sliding and rolling to pure rolling, using impulse-momentum. Consider the problem in Sample 14.25 again: A ball of radius $r = 10\,\text{cm}$ and mass $m = 1\,\text{kg}$ is launched horizontally with initial velocity $v_0 = 5\,\text{m/s}$ on a surface with coefficient of friction $\mu = 0.12$. The ball starts sliding, rolls and slides simultaneously for a while, and then rolls without sliding. Find the time it takes to start pure rolling.

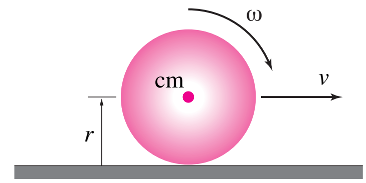


Figure 14.68:

Filename:sfig9-rollandslide-ball2

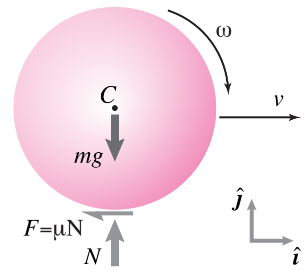


Figure 14.69: Free-body diagram of the ball during sliding.

Filename:sfig9-rollandslide-ball2a

Solution Let us denote the time of transition from mixed motion (rolling and sliding) to pure rolling by t^*. At $t = 0$, we know that $v_{\text{cm}} = v_0 = 5\,\text{m/s}$, and $\omega_0 = 0$. We also know that at $t = t^*$, $v_{\text{cm}} = v_{t^*} = \omega_{t^*} r$, where r is the radius of the ball. We do not know t^* and v_{t^*}. However, we are considering a finite time event (during t^*) and the forces acting on the ball during this duration are known. Recall that impulse momentum equations involve the net force on the body, the time of impulse, and momenta of the body at the two instants. Momenta calculations involve velocities. Therefore, we should be able to use impulse-momentum equations here and find the desired unknowns. From linear impulse-momentum, we have

$$\begin{aligned}\left(\sum \vec{F}\right) t^* &= m v_{t^*}\hat{\imath} - m v_0 \hat{\imath} \\ (-\mu N \hat{\imath} + (N - mg)\hat{\jmath})\, t^* &= m(v_{t^*} - v_0)\hat{\imath}.\end{aligned}$$

Dotting the above equation with $\hat{\jmath}$ and $\hat{\imath}$, respectively, we get

$$\begin{aligned} N &= mg \\ -\mu \underbrace{N}_{mg} t^* &= m(v_{t^*} - v_0) \\ \Rightarrow \quad -\mu g t^* &= v_{t^*} - v_0. \end{aligned} \tag{14.51}$$

Similarly, from angular impulse-momentum relation about the mass-center, we get

$$\begin{aligned} \sum \vec{M}_{\text{cm}} t^* &= (\vec{H}_{\text{cm}})_{t^*} - (\vec{H}_{\text{cm}})_0 \\ (-\mu N r \hat{k}) t^* &= (I_{zz}^{\text{cm}} \omega_{t^*} - I_{zz}^{\text{cm}} \underbrace{\omega_0}_{0})(-\hat{k}) \\ \text{or} \qquad -\mu m g r t^* &= -I_{zz}^{\text{cm}} \omega_{t^*} \\ \Rightarrow \quad \omega_{t^*} &= \mu m g r t^* / I_{zz}^{\text{cm}} \\ \Rightarrow \quad v_{t^*} &\equiv \omega_{t^*} r = \mu m g r^2 t^* / I_{zz}^{\text{cm}}. \end{aligned}$$

Substituting this expression for v_{t^*} in eqn. (14.51), we get

$$\begin{aligned} -\mu g t^* &= \mu m g r^2 t^* / I_{zz}^{\text{cm}} - v_0 \\ \Rightarrow \quad t^* &= \frac{v_0}{\mu g (1 + \frac{mr^2}{I_{zz}^{\text{cm}}})} \end{aligned}$$

which is, of course, the same expression we obtained for t^* in Sample 14.25. Again, noting that $I_{zz}^{\text{cm}} = \frac{2}{5} m r^2$ for a sphere, we calculate the time of transition as

$$t^* = \frac{2 v_0}{7 \mu g} = \frac{2 \cdot (5\,\text{m/s})}{7 \cdot (0.2) \cdot (9.8\,\text{m/s}^2)} = 0.73\,\text{s}.$$

$$\boxed{t^* = 0.73\,\text{s}}$$

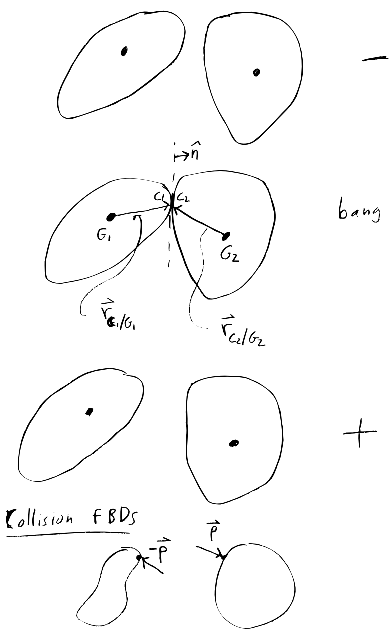


Figure 14.70: Two bodies collide at point C. The only non-negligible collision impulse is $\vec{\boldsymbol{P}}$ acting on body 2 (and $-\vec{\boldsymbol{P}}$ on body 1) at point C. The material points on the contacting bodies are C_1 and C_2. The outward normal to body 1 at C_1 is $\hat{\boldsymbol{n}}$.

Filename:tfigure-2Dcollision

14.5 Collision mechanics

Now we extend the concepts from 1D collisions (section 9.5 starting on page 449).

2D collisions

For collisions between rigid bodies with more general motions before and after the collisions we depend on the three ideas from the start of this section, namely that

I. Collision forces are big,
II. Collisions are quick, and
III. The laws of mechanics apply during the collision.

There are two extra assumptions that are needed in simple analysis:

IV. Collision forces are few. For a given rigid body there is one, or at most two non-negligible collision forces. This is the real import of idea (I) above. Because collision forces are big most other forces can be neglected.
V. The collision force(s) act at a well defined point which does not move during the collision.

Based on these assumptions one then uses linear and angular momentum balance in their time-integrated form.

Example: **Two bodies in space**

Two bodies collide at point C. The impulse acting on body 2 is $\vec{\boldsymbol{P}} = \int \vec{\boldsymbol{F}}_{\text{coll}}\, dt$. If the mass and inertia properties of both bodies is known, as are the velocities and rotation rates before the collision we have the following linear and angular momentum balance equations for the two bodies:

$$\begin{aligned} -\vec{\boldsymbol{P}} &= m_1 \left(\vec{\boldsymbol{v}}^{+}_{G1} - \vec{\boldsymbol{v}}^{-}_{G1}\right) \\ \vec{\boldsymbol{P}} &= m_2 \left(\vec{\boldsymbol{v}}^{+}_{G2} - \vec{\boldsymbol{v}}^{-}_{G2}\right) \\ \vec{\boldsymbol{r}}_{C/G1} \times (-\vec{\boldsymbol{P}}) &= I^{cm1}_{zz} \left(\omega^{+}_{1} - \omega^{-}_{1}\right) \hat{\boldsymbol{k}} \\ \vec{\boldsymbol{r}}_{C/G2} \times \vec{\boldsymbol{P}} &= I^{cm2}_{zz} \left(\omega^{+}_{2} - \omega^{-}_{2}\right) \hat{\boldsymbol{k}}. \end{aligned} \qquad (14.52)$$

These make up 6 scalar equations (2 for each momentum equation, 1 for each angular momentum equation). There are 8 scalar unknowns: $\vec{\boldsymbol{v}}^{+}_{G1}$ (2), $\vec{\boldsymbol{v}}^{+}_{G2}$ (2), ω^{+}_{1} (1), ω^{+}_{2} (1), and $\vec{\boldsymbol{P}}$ (2). Thus the motion after the collision cannot be determined.

[Note that we could write linear and angular momentum balance for the system, but this would only give equations which could be obtained by adding and subtracting combinations of the equations above. That is, the equations so obtained would not be independent.]

So, as for 1-D collisions, momentum balance is not enough to determine the outcome of the collision. Eqns. 14.52 aren't enough. A thousand different models and assumptions could be added to make the system solvable. But there are only two cases that are non-controversial and also relatively simple: 1) sticking collisions, and 2) frictionless collisions.

Sticking collisions

A ‘perfectly-plastic’ sticking collision is one where the relative velocities of the two contacting points are assumed to go suddenly to zero. That is

$$\vec{v}^{+}_{C1} = \vec{v}^{+}_{C2}$$

Writing $\vec{v}^{+}_{C1} = \vec{v}^{+}_{G1} + (\omega^{+}_{1}\hat{k}) \times \vec{r}_{C/G1}$ and similarly for $\vec{v}_{C2}$ thus adds a vector equation (2 scalar equations) to the equation set 14.52. This gives 8 equations in 8 unknowns.

A little cleverness can reduce the problem to one of solving only 4 equations in 4 unknowns. Linear momentum balance for the system, angular momentum balance for the system and angular momentum balance for object 2 make up 4 scalar equations. None of these equations includes the impulse $\vec{P}$. Because the system moves as if hinged at C_1 after the collision, the state of motion after the system is fully characterized by $\vec{v}^{+}_{G1}$, ω^{+}_{1}, and ω^{+}_{2}. Thus we have 4 equations in 4 unknowns.

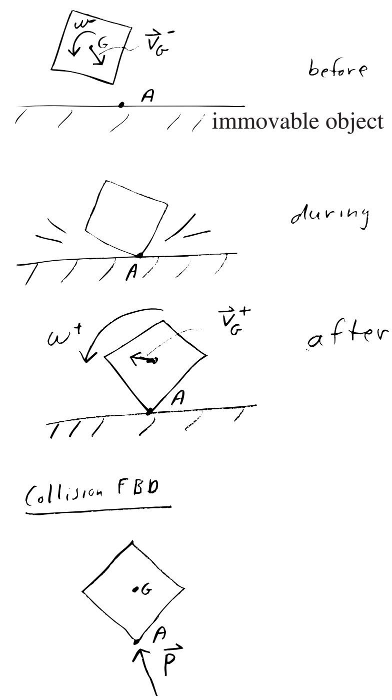


Figure 14.71: Sticking collision with an immovable object. The box sticks at A and then rotates about A. Angular momentum about point A is conserved in the collision.

Filename:tfigure-collimmovable

Example: **One body is hugely massive: collision with an immovable object**
If body 2, say, is huge compared to body 1 then it can be taken to be immovable and collision problems can be solved by only considering body 1 (see Fig. 14.71). In the case of a sticking collision the full state of the system after the collision is determined by ω^{+}_{1}. This can be found from the single scalar equation obtained from angular momentum balance about the collision point.

$$\begin{aligned} \vec{H}^{-}_{A} &= \vec{H}^{+}_{A} \\ \vec{r}_{G/A} \times m\vec{v}^{-}_{G} + I^{cm}_{zz}\omega^{-}\hat{k} &= \vec{r}_{G/A} \times m\vec{v}^{+}_{G} + I^{cm}_{zz}\omega^{+}\hat{k} \end{aligned}$$

Because the state of the system before the collision is assumed known (the left “-” side of the equation, and because the post-collision (+) state is a rotation about A, this equation is one scalar equation in the one unknown ω^{+}. Note that $\vec{H}^{+}_{A}$ could also be evaluated as $\vec{H}^{+}_{A} = \omega^{+} I^{A}_{zz}\hat{k}$. So one way of expressing the post-collision state is as

$$\omega^{+} = \frac{\left(\vec{r}_{G/A} \times m\vec{v}^{-}_{G} + I^{cm}_{zz}\omega^{-}\hat{k}\right)\cdot\hat{k}}{I^{A}_{zz}} \quad \text{and} \quad \vec{v}^{+}_{G} = \omega^{+}\hat{k} \times \vec{r}_{G/A}.$$

Note also that the same $\vec{r}_{G/A}$ is used on the right and left sides of the equation because only the velocity and not the position is assumed to jump during the collision.

The collision impulse $\vec{P}$ can then be found from linear momentum balance as

$$\vec{P} = m\left(\vec{v}^{+}_{G} - \vec{v}^{-}_{G}\right).$$

Sticking collisions are used as models of projectiles hitting targets, of robot and animal limbs making contact with the ground, of monkeys and acrobats grabbing hand holds, and of some particularly dead and frictional collisions between solids (such as when a car trips on a curb).

Frictionless collisions

The second special case is that of a frictionless collision. Here we add two assumptions:

1. There is no friction so $\vec{P} = P\hat{n}$. The number of unknowns is thus reduces from 8 to 7.
2. There is a coefficient of (normal) restitution e.

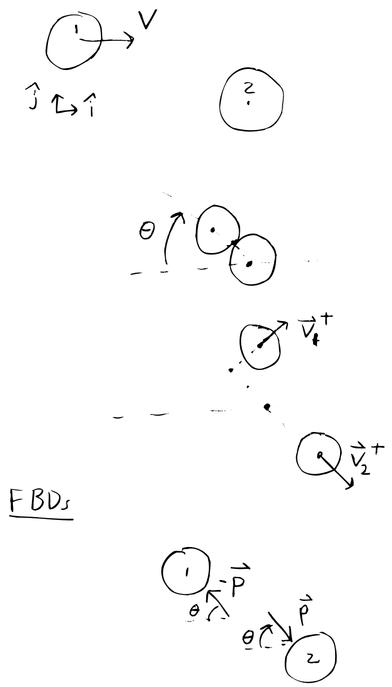


Figure 14.72: Frictionless collision between two identical round objects. Ball one is initially moving to the right, ball 2 is initially stationary. The impulse of ball 1 *on* ball 2 is $\vec{\boldsymbol{P}}$.

Filename:tfigure-poolballs

The normal restitution coefficient is taken as a property of the colliding bodies. It is a given number with $0 < e < 1$ with this defining equation:

$$(\vec{\boldsymbol{v}}_{C2}^{+} - \vec{\boldsymbol{v}}_{C1}^{+}) \cdot \hat{\boldsymbol{n}} = -e(\vec{\boldsymbol{v}}_{C2}^{-} - \vec{\boldsymbol{v}}_{C1}^{-}) \cdot \hat{\boldsymbol{n}}.$$

This says that the normal part of the relative velocity of the contacting points reverses sign and its magnitude is attenuated by e. This adds a scalar equation to the set Eqns. 14.52 thus giving 7 scalar equations (4 momentum, 2 angular momentum, 1 restitution) for 7 unknowns (4 velocity components, 2 angular velocities and the normal impulse).

The most popular application of the frictionless collision model is for billiard or pool balls, or carrom pucks. These things have relatively small coefficients of friction.

We state without proof that a frictionless collision with $e = 1$ conserves energy.

Example: **Pool balls**

Assume one ball approaches the other with initial velocity $\vec{\boldsymbol{v}}_{G1}^{+} = v\hat{\boldsymbol{\imath}}$ and has an elastic frictionless collision with the other ball at a collision angle of θ as shown in Fig. 14.72. Defining $\hat{\boldsymbol{n}} \equiv \cos\theta\hat{\boldsymbol{\imath}} - \sin\theta\hat{\boldsymbol{\jmath}}$ we have that $\vec{\boldsymbol{P}} = P\hat{\boldsymbol{n}}$. To determine the outcome of the equation we have the angular momentum balance equations (about the center-of-mass) which trivially tell us that

$$\omega_1^+ = \omega_2^+ = 0$$

because the balls start with no spin and the frictionless collision impulses $\vec{\boldsymbol{P}} = P\hat{\boldsymbol{n}}$ and $-\vec{\boldsymbol{P}} = -P\hat{\boldsymbol{n}}$ have no moment about the center-of-mass. Linear momentum balance for each of the balls

$$\begin{aligned} -P\hat{\boldsymbol{n}} &= m\vec{\boldsymbol{v}}_{G1}^{+} - mv\hat{\boldsymbol{\imath}} \\ P\hat{\boldsymbol{n}} &= m\vec{\boldsymbol{v}}_{G2}^{+} - \vec{\boldsymbol{0}} \end{aligned}$$

gives 4 scalar equations which are supplemented by the restitution equation (using $e = 1$)

$$\begin{aligned} &\left(\Delta\vec{\boldsymbol{v}}^{+}\right) \cdot \hat{\boldsymbol{n}} = -e\left(\Delta\vec{\boldsymbol{v}}^{-}\right) \cdot \hat{\boldsymbol{n}} \\ \Rightarrow\ & -v\cos\theta = \vec{\boldsymbol{v}}_{G2}^{+} \cdot \hat{\boldsymbol{n}} - \vec{\boldsymbol{v}}_{G1}^{+} \cdot \hat{\boldsymbol{n}} \end{aligned}$$

which together make 5 scalar equations in the 5 scalar unknowns $\vec{\boldsymbol{v}}_{G1}^{+}$, $\vec{\boldsymbol{v}}_{G2}^{+}$, and P (each vector has 2 unknown components). These have the solution

$$\begin{aligned} \vec{\boldsymbol{v}}_{G1}^{+} &= v\sin\theta(\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}), \\ \vec{\boldsymbol{v}}_{G1}^{+} &= v\cos\theta(\cos\theta\hat{\boldsymbol{\imath}} - \sin\theta\hat{\boldsymbol{\jmath}}), \quad \text{and} \\ P &= mv\cos\theta. \end{aligned}$$

The solution can be checked by plugging back into the momentum and restitution equations. Also, as promised, this $e = 1$ solution conserves kinetic energy. The solution has the interesting property that the outgoing trajectories of the two balls are orthogonal for all θ but $\theta = 0$ in which case ball 1 comes to rest in the collision. [The solution can be found graphically by looking for two outgoing vectors which add to the original velocity of mass 1, where the sum of the squares of the outgoing speeds must add to the square of the incoming speed.]

Frictional collisions

For a collision with friction, but not so much that total sticking is accurate, the modeling is complex and subtle. As of this writing there are no standard acceptable ways of dealing with such situations. Commercial simulation packages should be used for such with skeptical caution. They are generally defective in that they can predict only a limited range of phenomena and/or they can create energy even with innocent input parameters.

Why is it hard to find a good collision law

Ideally one would like a rule to determine how bodies move after a collision from how they move before the collision. Such a rule would be called a collision law or a constitutive relation for collisions. That accurate collision laws are rare at best might be surmised from the basic problem that the phrase *rigid body collisions* is in some sense a contradiction in terms, an oxymoron. The force generated in the contact comes from material deformation, and deformation is just what we generally try to neglect when doing rigid body mechanics.

There is a temptation to say that one wants to continue to neglect deformation during the collision, but for in an infinitesimal contact region. And some collision laws are formulated with this approach. Even then, there are no reliable models for the deformation in that small region, and such laws are doomed to inaccuracy in situations where the deformation is not so limited.

For complex shaped bodies touching at various points that are generally not known *a priori*, no collision law is reliably accurate.

SAMPLE 14.27 The vector equation $m_1\vec{v}_1 + \mathrm{m}_2\vec{v}_2 = m_1\vec{v}_1^{\,+} + \mathrm{m}_2\vec{v}_2^{\,+}$ expresses the conservation of linear momentum of two masses. Suppose $\vec{v}_1 = \vec{0}$, $\vec{v}_2 = -v_0\hat{\jmath}$, $\vec{v}_1^{\,+} = v_1^+\hat{\imath}$ and $\vec{v}_2^{\,+} = v_{2_t}^+\hat{e}_t + v_{2_n}^+\hat{e}_n$, where $\hat{e}_t = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$ and $\hat{e}_n = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}$.

1. Obtain two independent scalar equations from the momentum equation corresponding to projections in the $\hat{e}_n$ and $\hat{e}_t$ directions.
2. Assume that you are given another equation $v'_{2_t} = -v_0\sin\theta$. Set up a matrix equation to solve for v_1^+, $v_{2_t}^+$, and $v_{2_n}^+$ from the three equations.

Solution

1. The given equation of conservation of linear momentum is

$$m_1\underbrace{\vec{v}_1}_{\mathbf{0}} + \mathrm{m}_2\vec{v}_2 = m_1\vec{v}_1^{\,+} + \mathrm{m}_2\vec{v}_2^{\,+}$$

$$\text{or} \qquad -m_2v_0\hat{\jmath} = m_1v_1^+\hat{\imath} + m_2(v_{2_t}^+\hat{e}_t + v_{2_n}^+\hat{e}_n). \tag{14.53}$$

Dotting both sides of eqn. (14.53) with $\hat{e}_n$ gives

$$-m_2v_0\overbrace{(\hat{e}_n\cdot\hat{\jmath})}^{\cos\theta} = m_1v_1^+\overbrace{(\hat{e}_n\cdot\hat{\imath})}^{-\sin\theta} + m_2v_{2_t}^+\overbrace{(\hat{e}_n\cdot\hat{e}_t)}^{0} + m_2v_{2_n}^+\overbrace{(\hat{e}_n\cdot\hat{e}_n)}^{1}$$

$$\text{or} \qquad -m_2v_0\cos\theta = -m_1v_1^+\sin\theta + m_2v_{2_n}^+. \tag{14.54}$$

Dotting both sides of eqn. (14.53) with $\hat{e}_t$ gives

$$-m_2v_0\overbrace{(\hat{e}_t\cdot\hat{\jmath})}^{\sin\theta} = m_1v_1^+\overbrace{(\hat{e}_t\cdot\hat{\imath})}^{\cos\theta} + m_2v_{2_t}^+\overbrace{(\hat{e}_t\cdot\hat{e}_t)}^{1} + m_2v_{2_n}^+\overbrace{(\hat{e}_t\cdot\hat{e}_n)}^{0}$$

$$\text{or} \qquad -m_2v_0\sin\theta = m_1v_1^+\cos\theta + m_2v_{2_t}^+. \tag{14.55}$$

$$\boxed{-m_2v_0\cos\theta = -m_1v_1^+\sin\theta + m_2v_{2_n}^+, \quad -m_2v_0\sin\theta = m_1v_1^+\cos\theta + m_2v_{2_t}^+}$$

2. Now, we rearrange eqn. (14.54) and 14.55 along with the third given equation, $v'_{2_t} = -v_0\sin\theta$, so that all unknowns are on the left hand side and the known quantities are on the right hand side of the equal sign. These equations, in matrix form, are as follows.

$$\begin{bmatrix} -m_1\sin\theta & 0 & m_2 \\ -m_1\cos\theta & m_2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} v_1^+ \\ v_{2_t}^+ \\ v_{2_n}^+ \end{Bmatrix} = \begin{Bmatrix} -m_2v_0\cos\theta \\ -m_2v_0\sin\theta \\ -v_0\sin\theta \end{Bmatrix}.$$

This equation can be easily solved on a computer for the unknowns.

SAMPLE 14.28 Cueing a billiard ball. A billiard ball is cued by striking it horizontally at a distance $d = 10\,\text{mm}$ above the center of the ball. The ball has mass $m = 0.2\,\text{kg}$ and radius $r = 30\,\text{mm}$. Immediately after the strike, the center-of-mass of the ball moves with linear speed $v = 1\,\text{m/s}$. Find the angular speed of the ball immediately after the strike. Ignore friction between the ball and the table during the strike.

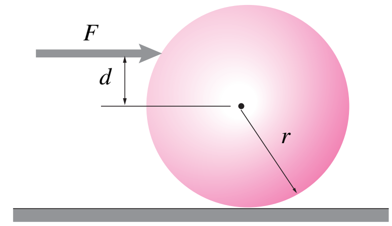


Figure 14.73:
Filename:sfig7-3-DH1

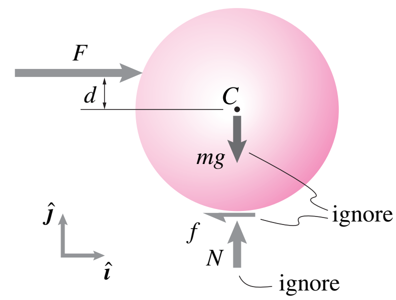


Figure 14.74: FBD of the ball during the strike. The nonimpulsive forces mg, N, and f can be ignored in comparison to the strike force F.
Filename:sfig7-3-DH2

Solution Let the force imparted during the strike be F. Since the ball is cued by giving a blow with the cue, F is an impulsive force. Impulsive forces, such as F, are in general so large that all non-impulsive forces are negligible in comparison during the time such forces act. Therefore, we can ignore all other forces (mg, N, f) acting on the ball from its free body diagram during the strike.

Now, from the linear momentum balance of the ball we get

$$F\hat{\imath} = \dot{\vec{L}} \quad \text{or} \quad (F\hat{\imath})dt = d\vec{L} \quad \Rightarrow \quad \int (F\hat{\imath})dt = \vec{L}_2 - \vec{L}_1$$

where $L_2 - L_1 = \Delta\vec{L}$ is the net change in the linear momentum of the ball during the strike. Since the ball is at rest before the strike, $\vec{L}_1 = m \underbrace{\vec{v}_1}_{0} = \vec{0}$. Immediately after the strike, $\vec{v} = v\hat{\imath} = 1\,\text{m/s}$.

$$\text{Thus} \quad \vec{L}_2 = m\vec{v} = 0.2\,\text{kg}\cdot 1\,\text{m/s}\hat{\imath} = 0.2\,\text{N}\cdot\text{s}\hat{\imath}.$$

$$\text{Hence} \quad \int (F\hat{\imath})dt = 0.2\,\text{N}\cdot\text{s}\hat{\imath} \quad \text{or} \quad \int F dt = 0.2\,\text{N}\cdot\text{s}. \tag{14.56}$$

To find the angular speed we apply the angular momentum balance. Let ω be the angular speed immediately after the strike and $\vec{\omega} = \omega\hat{k}$. Now,

$$\sum \vec{M}_{\text{cm}} = \dot{\vec{H}}_{\text{cm}} \quad \Rightarrow \quad \int \sum \vec{M}_{\text{cm}}\,dt = \int d\vec{H}_{\text{cm}} = (\vec{H}_{\text{cm}})_2 - (\vec{H}_{\text{cm}})_1.$$

$$\begin{aligned}
\text{Since} \quad \vec{H}_{\text{cm}} &= I_{cm}^{zz}\vec{\omega} \quad \text{and just before the strike, } \vec{\omega} = \vec{0}, \\
(\vec{H}_{\text{cm}})_1 &\equiv \text{angular momentum just before the strike } = \vec{0} \\
(\vec{H}_{\text{cm}})_2 &\equiv \text{angular momentum just after the strike } = I_{cm}^{zz}\omega\hat{k}, \\
\int \sum \vec{M}_{\text{cm}}\,dt &= I_{cm}^{zz}\omega\hat{k} = \frac{2}{5}mr^2\omega\hat{k} \text{ (since for a sphere, } I_{cm}^{zz} = \tfrac{2}{5}mr^2).
\end{aligned}$$

$$\begin{aligned}
\text{But} \quad & \sum \vec{M}_{\text{cm}} = -Fd\hat{k}, \\
\text{therefore} \quad & -\int (Fd)dt\hat{k} = \frac{2}{5}mr^2\omega\hat{k} \\
\text{or} \quad & -\underbrace{d}_{\text{constant}} \int F dt = \frac{2}{5}mr^2\omega \quad \Rightarrow \quad \omega = -\frac{5d}{2mr^2}\int F dt.
\end{aligned}$$

Substituting the given values and $\int F dt = 0.2\,\text{N}\cdot\text{s}$ from equation 14.56 we get

$$\omega = -\frac{5(0.01\,\text{m})}{2\cdot 0.2\,\text{kg}\cdot(0.03\,\text{m})^2}\cdot 0.2\,\text{N}\cdot\text{s} = -27.78\,\text{rad/s}.$$

The negative value makes sense because the ball will spin clockwise after the strike, but we assumed that ω was anticlockwise.

$$\boxed{\omega = -27.78\,\text{rad/s}.}$$

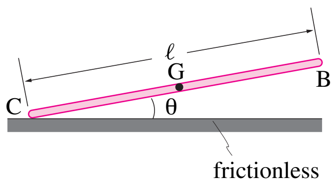


Figure 14.75:

Filename:sfig9-5-fallingbar

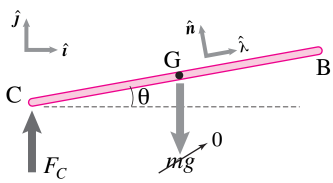


Figure 14.76: The free-body diagram of the bar during collision. The impulsive force at the point of impact C is so large that the force of gravity can be completely ignored in comparison.

Filename:sfig9-5-fallingbar-a

SAMPLE 14.29 Falling stick. A uniform bar of length ℓ and mass m falls on the ground at an angle θ as shown in the figure. Just before impact at point C, the entire bar has the same velocity v directed vertically downwards. Assume that the collision at C is plastic, *i.e.*, end C of the bar gets stuck to the ground upon impact.

1. Find the angular velocity of the bar just after impact.
2. Assuming θ to be small, find the velocity of end B of the bar just after impact.

Solution We are given that the impact at point C is plastic. That is, end C of the bar has zero velocity after impact. Thus end C gets stuck to the ground. Then we expect the rod to rotate about point C as rest of the bar moves (perhaps faster) to touch the ground. The free-body diagram of the bar is shown in Fig. 14.76 during the impact at point C. Note that we can ignore the force of gravity in comparison to the large impulsive force F_c due to impact at C.

1. Now, if we carry out angular momentum balance about point C, there will be no net moment acting on the bar, and therefore, angular momentum about the impact point C is conserved. Distinguishing the kinematic quantities before and after impact with superscripts '-' and '+', respectively, we get from the conservation of angular momentum about point C,

$$\begin{aligned}\vec{\boldsymbol{H}}_C^- &= \vec{\boldsymbol{H}}_C^+ \\ I_{zz}^{\text{cm}}\vec{\boldsymbol{\omega}}^- + \vec{r}_{\text{G/C}} \times m\vec{\boldsymbol{v}}_G^- &= I_{zz}^{\text{cm}}\vec{\boldsymbol{\omega}}^+ + \vec{r}_{\text{G/C}} \times m\vec{\boldsymbol{v}}_G^+.\end{aligned}$$

Now, we know that $\vec{\boldsymbol{\omega}}^- = \vec{\boldsymbol{0}}$ since every point on the bar has the same vertical velocity $\vec{\boldsymbol{v}} = -v\hat{\boldsymbol{\jmath}}$, and that just after impact, $\vec{\boldsymbol{v}}_G^+ = \vec{\boldsymbol{\omega}}^+ \times \vec{r}_{\text{G/C}}$ where we can take $\vec{\boldsymbol{\omega}}^+ = \omega(-\hat{\boldsymbol{k}})$. Thus, *

$$\begin{aligned}\vec{\boldsymbol{H}}_C^- &= \vec{r}_{\text{G/C}} \times m\vec{\boldsymbol{v}}_G^- = (\ell/2)\hat{\boldsymbol{\lambda}} \times mv(-\hat{\boldsymbol{\jmath}}) \\ &= -\frac{mv\ell}{2}\cos\theta\hat{\boldsymbol{k}} \qquad (\text{since } \hat{\boldsymbol{\lambda}} = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) \\ \vec{\boldsymbol{H}}_C^+ &= I_{zz}^{\text{cm}}\vec{\boldsymbol{\omega}}^+ + \vec{r}_{\text{G/C}} \times m(\vec{\boldsymbol{\omega}}^+ \times \vec{r}_{\text{G/C}}) \\ &= -I_{zz}^{\text{cm}}\omega\hat{\boldsymbol{k}} + (\ell/2)\hat{\boldsymbol{\lambda}} \times m\underbrace{(-\omega\hat{\boldsymbol{k}} \times \ell/2\hat{\boldsymbol{\lambda}})}_{\omega\ell/2(-\hat{\boldsymbol{n}})} \\ &= -\frac{1}{12}m\ell^2\omega\hat{\boldsymbol{k}} - \frac{1}{4}m\ell^2\omega\hat{\boldsymbol{k}} = -\frac{1}{3}m\ell^2\omega\hat{\boldsymbol{k}}.\end{aligned}$$

Now, equating $\vec{\boldsymbol{H}}_C^-$ and $\vec{\boldsymbol{H}}_C^+$ we get

$$\omega = \frac{3v}{2\ell}\cos\theta, \quad \Rightarrow \quad \vec{\boldsymbol{\omega}} = -\frac{3v}{2\ell}\cos\theta\hat{\boldsymbol{k}}.$$

$$\boxed{\vec{\boldsymbol{\omega}} = -\tfrac{3v}{2\ell}\cos\theta\hat{\boldsymbol{k}}}$$

2. The velocity of the end B is now easily found using $\vec{\boldsymbol{v}}_{\text{B}} = \vec{\boldsymbol{v}}_{\text{C}} + \vec{\boldsymbol{v}}_{\text{B/C}} = \vec{\boldsymbol{v}}_{\text{B/C}}$ and $\vec{\boldsymbol{v}}_{\text{B/C}} = \vec{\boldsymbol{\omega}} \times \vec{r}_{\text{B/C}}$. Thus,

$$\begin{aligned}\vec{\boldsymbol{v}}_{\text{B/C}} &= \vec{\boldsymbol{\omega}} \times \vec{r}_{\text{B/C}} = -\omega\hat{\boldsymbol{k}} \times \ell\hat{\boldsymbol{\lambda}} \\ &= -\omega\ell\hat{\boldsymbol{n}} = -\frac{3v}{2}\cos\theta(-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}})\end{aligned}$$

but, for small θ, $\cos\theta \approx 1$, and $\sin\theta \approx 0$. Therefore, $\vec{\boldsymbol{v}}_{\text{B/C}} = -\frac{3v}{2}\hat{\boldsymbol{\jmath}}$. Thus, end B of the bar speeds up by one and a half times its original speed due to the plastic impact at C.

$$\boxed{\vec{\boldsymbol{v}}_{\text{B/C}} = -(3/2)v\hat{\boldsymbol{\jmath}}}$$

* Since C is a fixed point for the motion of the bar after impact, we could calculate $\vec{\boldsymbol{H}}_C^+$ as follows.

$$\vec{\boldsymbol{H}}_C^+ = I_{zz}^C\vec{\boldsymbol{\omega}} = \underbrace{\frac{1}{3}m\ell^2}_{I_{zz}^C}\omega(-\hat{\boldsymbol{k}}).$$

SAMPLE 14.30 tipping box. A box of mass $m = 20\,\text{kg}$ and dimensions $2a = 1\,\text{m}$ and $2b = 0.4\,\text{m}$ moves along a horizontal surface with uniform speed $v = 1\,\text{m/s}$. Suddenly, it bumps into an obstacle at A. Assume that the impact is plastic and point A is at the lowest level of the box. Determine if the box can tip over following the impact. If not, what is the maximum v the box can have so that it does not tip over after the impact.

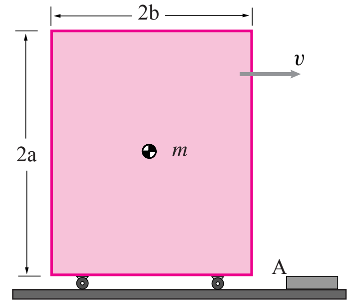


Figure 14.77:

Filename:sfig9-5-tipping

Solution Whether the box can tip or not depends on whether it gets sufficient initial angular speed just after collision to overcome the restoring moment due to gravity about the point of rotation A. So, first we need to find the angular velocity of the box immediately following the collision. The free-body diagram of the box during collision is shown in Fig. 14.78. There is an impulse $\vec{P}$ acting at the point of impact. If we carry out the angular momentum balance about point A, we see that the impulse at A produces no moment impulse about A, and therefore, the angular momentum about point A has to be conserved. That is, $\vec{H}_A^{\,+} = \vec{H}_A^{\,-}$. Now,

$$\vec{H}_A^{\,-} = \vec{r}_{G/A} \times m\vec{v}_G^{\,-} = (-b\hat{\imath} + a\hat{\jmath}) \times mv\hat{\imath} = -mav\hat{k}$$

Let the box have angular velocity $\vec{\omega}^{\,+} = \omega\hat{k}$ just after impact. Then,

$$\begin{aligned} \vec{H}_A^{\,+} &= I_{zz}^{cm}\vec{\omega}^{\,+} + \vec{r}_{G/A} \times m\vec{v}_G^{\,+} = I_{zz}^{cm}\omega\hat{k} + r\hat{\lambda} \times m(\omega\hat{k} \times r\hat{\lambda}) \\ &= I_{zz}^{cm}\omega\hat{k} + mr^2\omega\hat{k} = \frac{1}{12}(4a^2 + 4b^2)m\omega\hat{k} + m(a^2 + b^2)\omega\hat{k} \\ &= \frac{4}{3}(a^2 + b^2)m\omega\hat{k}. \end{aligned}$$

Now equating the two momenta, we get

$$\omega = -\frac{3a}{4(a^2+b^2)}v \quad \Rightarrow \quad \vec{\omega}^{\,+} = -\frac{3a}{4(a^2+b^2)}v\hat{k}.$$

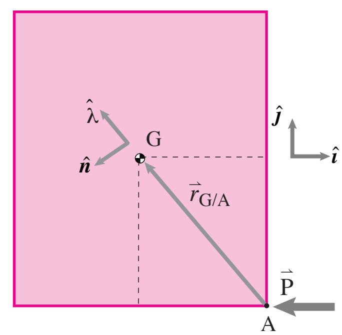


Figure 14.78: The free-body diagram of the box during collision.

Filename:sfig9-5-tipping-a

Thus we know the angular velocity immediately after impact. Now let us find out if it is enough to get over the hill, so to speak. We need to find the equation of motion of the box for the motion that follows the impact. Once the impact is over (in a few milliseconds), the usual forces show up on the free-body diagram (see Fig. 14.79).

We can find the equation of subsequent motion by carrying out angular momentum balance about point A (the box rotates about this point), $\sum \vec{M}_A = \dot{\vec{H}}_A$.

$$\begin{aligned} \vec{r}_{G/A} \times mg(-\hat{\jmath}) &= I_{zz}^{A}\dot{\omega}\hat{k} \\ \Rightarrow \quad \dot{\omega} &= \frac{mgb}{I_{zz}^{A}} = \frac{3gb}{4(a^2+b^2)}. \end{aligned}$$

Thus the angular acceleration (due to the restoring moment of the weight of the box) is counterclockwise and constant. Therefore, we can use $\omega^2 = \omega_0^2 + 2\dot{\omega}\Delta\theta$ to find if the box can make it to the tipping position (the center-of-mass on the vertical line through A). Let us take θ to be positive in the clockwise direction (direction of tipping). Then $\dot{\omega}$ is negative. Starting from the position of impact, the box must rotate by $\Delta\theta = \tan^{-1}(b/a)$ in order to tip over. In this position, we must have $\omega \geq 0$.

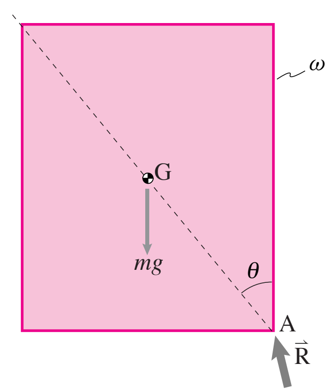


Figure 14.79: The free-body diagram of the box just after the collision is over.

Filename:sfig9-5-tipping-b

$$\omega^2 = \omega_0^2 - 2\dot{\omega}\Delta\theta \geq 0 \quad \Rightarrow \quad \omega_0^2 \geq 2\dot{\omega}\Delta\theta \quad \Rightarrow \quad v^2 \geq \frac{24bg(a^2+b^2)}{9a^2}\Delta\theta.$$

Substituting the given numerical values for a, b, and $g = 9.8\,\text{m/s}^2$, we get

$$v \geq 1.52\,\text{m/s}^2.$$

Thus the given initial speed of the box, $v = 1\,\text{m/s}$, is not enough for tipping over.

$$\boxed{v \geq 1.52\,\text{m/s}^2}$$

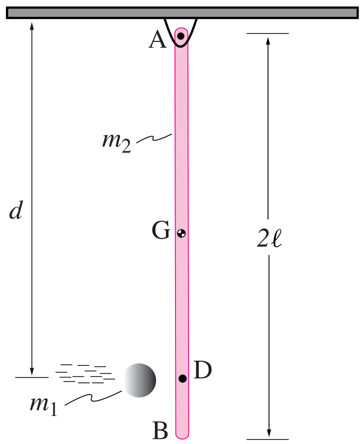


Figure 14.80:

Filename:sfig9-5-ballbar

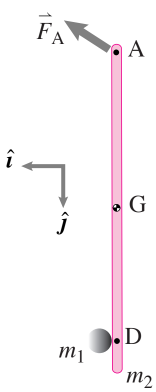


Figure 14.81: The free-body diagram of the ball and the bar together during collision. The impulsive force at the point of impact is internal to the system and hence, does not show on the free-body diagram.

Filename:sfig9-5-ballbar-a

SAMPLE 14.31 Ball hits the bat. A uniform bar of mass $m_2 = 1\,\text{kg}$ and length $2\ell = 1\,\text{m}$ hangs vertically from a hinge at A. A ball of mass $m_1 = 0.25\,\text{kg}$ comes and hits the bar horizontally at point D with speed $v = 5\,\text{m/s}$. The point of impact D is located at $d = 0.75\,\text{m}$ from the hinge point A. Assume that the collision between the ball and the bar is plastic.

1. Find the velocity of point D on the bar immediately after impact.
2. Find the impulse on the bar at D due to the impact.
3. Find and plot the impulsive reaction at the hinge point A as a function of d, the distance of the point of impact from the hinge point. What is the value of d which makes the impulse at A to be zero?

Solution The free-body diagram of the ball and the bar as a single system is shown in Fig. 14.81 during impact. There is only one external impulsive force $\vec{F}_A$ acting at the hinge point A. We take the ball and the bar together here so that the impulsive force acting between the ball and the bar becomes internal to the system and we are left with only one external force at A. Then, the angular momentum balance about point A yields $\dot{\vec{H}}_A = \vec{0}$ since there is no net moment about A. Thus the angular momentum about A is conserved during the impact.

1. Let us distinguish the kinematic quantities just before impact and immediately after impact with superscripts '-' and '+', respectively. Then, from the conservation of angular momentum about point A, we get $\vec{H}_A^- = \vec{H}_A^+$. Now,

$$\begin{aligned}
\vec{H}_A^- &= (\vec{H}_A^-)_{\text{ball}} + (\vec{H}_A^-)_{\text{bar}} \\
&= \vec{r}_{D/A} \times m_1 \vec{v}^- + I_{zz}^A \vec{\omega}^- \\
&= d\hat{\jmath} \times m_1 v(-\hat{\imath}) + \vec{0} = m_1 d v \hat{k}.
\end{aligned}$$

Similarly,

$$\vec{H}_A^+ = \vec{r}_{D/A} \times m_1 \vec{v}^+ + I_{zz}^A \vec{\omega}^+$$

but, $\vec{v}^+ = \vec{\omega}^+ \times \vec{r}_{D/A} = -\omega d\hat{\imath}$, where $\vec{\omega}^+ = \omega\hat{k}$ (let). Hence,

$$\begin{aligned}
\vec{H}_A^+ &= d\hat{\jmath} \times m_1(-\omega d\hat{\imath}) + \frac{1}{3} m_2 (2\ell)^2 \omega \hat{k} \\
&= (m_1 d^2 + \frac{4}{3} m_2 \ell^2)\omega\hat{k}.
\end{aligned}$$

Equating the two momenta, we get

$$\begin{aligned}
\omega &= \frac{m_1 d v}{m_1 d^2 + (4/3) m_2 \ell^2} \\
&= \frac{v}{d\left(1 + \frac{4}{3}\frac{m_2}{m_1}\left(\frac{\ell}{d}\right)^2\right)} \\
\Rightarrow \quad \vec{v}_D &= \vec{\omega}^+ \times \vec{r}_{D/A} = \omega d(-\hat{\imath}) \\
&= -\frac{v}{1 + \frac{4}{3}\frac{m_2}{m_1}\left(\frac{\ell}{d}\right)^2}\hat{\imath}.
\end{aligned}$$

Now, substituting the given numerical values, $v = 5\,\text{m/s}$, $m_1 = 0.25\,\text{kg}$, $m_2 = 1\,\text{kg}$, $\ell = 0.5\,\text{m}$, and $d = 0.75\,\text{m}$, we get $\vec{v}_D = -2.08\,\text{m/s}\hat{\imath}$

$$\boxed{\vec{v}_D = -2.08\,\text{m/s}\hat{\imath}}$$

2. To find the impulse at D due to the impact, we can consider either the ball or the bar separately, and find the impulse by evaluating the change in the linear momentum of the body. Let us consider the ball since it has only one impulse acting on it. The

free-body diagram of the ball during impact is shown in Fig. 14.82. From the linear impulse-momentum relationship we get,

$$\begin{aligned} \vec{\boldsymbol{P}}_D = \int \vec{\boldsymbol{F}}_\mathrm{D}\,dt &= \vec{\boldsymbol{L}}^+ - \vec{\boldsymbol{L}}^- = m_1(\vec{\boldsymbol{v}}^+ - \vec{\boldsymbol{v}}^-) \\ &= m_1\left(-\frac{v}{1+\frac{4}{3}\frac{m_2}{m_1}\left(\frac{\ell}{d}\right)^2}\hat{\boldsymbol{\imath}} + v\hat{\boldsymbol{\imath}}\right) \\ &= m_1 v\left(1 - \frac{1}{1+\frac{4}{3}\frac{m_2}{m_1}\left(\frac{\ell}{d}\right)^2}\right)\hat{\boldsymbol{\imath}}. \end{aligned}$$

Substituting the given numerical values, we get $\vec{\boldsymbol{P}}_D = 0.73\,\mathrm{kg\cdot m/s}\hat{\boldsymbol{\imath}}$. The impulse on the bar is equal and opposite. Therefore, the impulse on the bar is $-\vec{\boldsymbol{P}}_D = -0.73\,\mathrm{kg\cdot m/s}\hat{\boldsymbol{\imath}}$.

$$\boxed{\text{Impulse at D} = -0.73\,\mathrm{kg\cdot m/s}\hat{\boldsymbol{\imath}}}$$

3. Now that we know the impulse at D, we can easily find the impulse at A by applying impulse-momentum relationship to the bar. Since, the bar is stationary just before impact, its initial momentum is zero. Thus, for the bar,

$$\int(\vec{\boldsymbol{F}}_\mathrm{A} - \vec{\boldsymbol{F}}_\mathrm{D})dt = \vec{\boldsymbol{L}}^+ - \vec{\boldsymbol{L}}^- = \vec{\boldsymbol{L}}^+ = m_2\vec{\boldsymbol{v}}^+_\mathrm{cm}.$$

Denoting the impulse at A with $\vec{\boldsymbol{P}}_A$, the mass ratio m_2/m_1 by m_r, and the length ratio ℓ/d byh q, and noting that $\vec{\boldsymbol{v}}^+_\mathrm{cm} = \omega\hat{\boldsymbol{k}} \times \ell\hat{\boldsymbol{\jmath}} = -\omega\ell\hat{\boldsymbol{\imath}}$, we get

$$\begin{aligned} \vec{\boldsymbol{P}}_A &\equiv \int \vec{\boldsymbol{F}}_\mathrm{A}\,dt = \int \vec{\boldsymbol{F}}_\mathrm{D}\,dt + m_2(-\omega\ell\hat{\boldsymbol{\imath}}) \\ &= m_1 v\left(1 - \frac{1}{1+\frac{4}{3}m_r q^2}\right)\hat{\boldsymbol{\imath}} - m_2\ell\frac{v}{d\left(1+\frac{4}{3}m_r q^2\right)}\hat{\boldsymbol{\imath}} \\ &= m_1 v\left(\frac{\frac{4}{3}m_r q^2}{1+\frac{4}{3}m_r q^2}\right)\hat{\boldsymbol{\imath}} - m_2 v\left(\frac{q}{1+\frac{4}{3}m_r q^2}\right)\hat{\boldsymbol{\imath}} \\ &= \frac{(4/3)m_2 q^2 - m_2 q}{1+\frac{4}{3}m_r q^2}v\hat{\boldsymbol{\imath}} = \frac{q(4q-3)}{3\left(1+\frac{4}{3}m_r q^2\right)}m_2 v\hat{\boldsymbol{\imath}}. \end{aligned}$$

Now, we are ready to graph the impulse at A as a function of $q \equiv \ell/d$. However, note that a better quantity to graph will be $P_A/(m_1 v)$, that is, the nondimensional impulse at A, normalized with respect to the initial linear momentum $m_1 v$ of the ball. The plot is shown in Fig. **??**. It is clear from the plot, as well as from the expression for $\vec{\boldsymbol{P}}_A$, that the impulse at A is zero when $q = 3/4$ or $d = 4\ell/3 = 2/3(2\ell)$, that is, when the ball strikes at two thirds the length of the bar. Note that this location of the impact point is independent of the mass ratio m_r.

$$\boxed{d = 2/3(2\ell) \text{ for } \vec{\boldsymbol{P}}_A = \vec{\boldsymbol{0}}}$$

Comment: This particular point of impact D (when $d = 2/3(2\ell)$) which induces no impulse at the support point A is called the *center of percussion*. If you imagine the bar to be a bat or a racquet and point A to be the location of your grip, then hitting a ball at D gives you an impulse-free shot. In sports, point D is called a *sweet spot*.

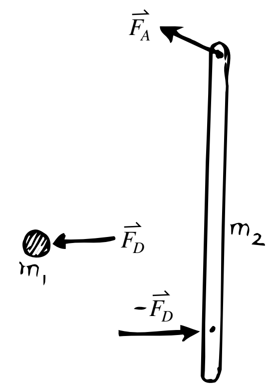


Figure 14.82: Separate free-body diagrams of the ball and the bar during collision.

Filename:sfig9-5-ballbar-b

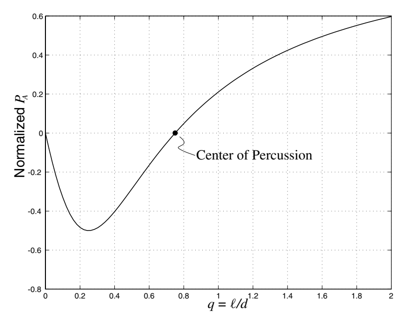


Figure 14.83: Plot of normalized impulse at A as a function of $q = \ell/d$.

Filename:sweetspot

SAMPLE 14.32 Flying dish and the solar panel. A uniform rectangular plate of dimensions $2a = 2\,\mathrm{m}$ and $2b = 1\,\mathrm{m}$ and mass $m_P = 2\,\mathrm{kg}$ drifts in space at a uniform speed $v_p = 10\,\mathrm{m/s}$ (in a local Newtonian reference frame) in the direction shown in the figure. Another circular disk of radius $R = 0.25\,\mathrm{m}$ and mass $m_D = 1\,\mathrm{kg}$ is heading towards the plate at a linear speed $v_D = 1\,\mathrm{m/s}$ directed normal to the facing edge of the plate. In addition, the disk is spinning at $\omega_D = 5\,\mathrm{rad/s}$ in the clockwise direction. The plate and the disk collide at point A of the plate, located at $d = 0.8\,\mathrm{m}$ from the center of the long edge. Assume that the collision is frictionless and purely elastic. Find the linear and angular velocities of the plate and the disk immediately after the collision.

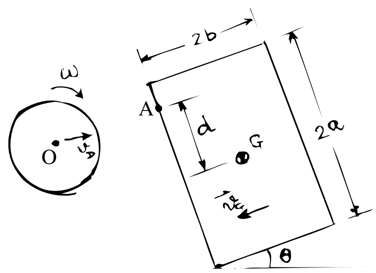


Figure 14.84:

Filename:sfig9-5-diskplate

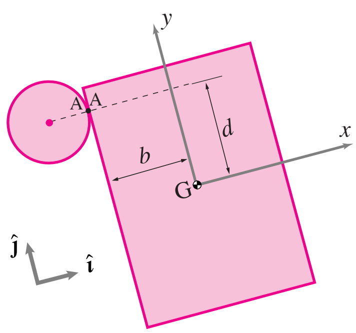


Figure 14.85: The free-body diagram of the disk and the plate together during collision. The impulsive force at the point of impact is internal to the system and hence, does not show on the free-body diagram .

Filename:sfig9-5-diskplate-a

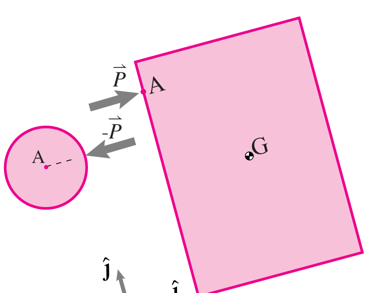


Figure 14.86: Separate free-body diagrams of the disk and the plate during collision.

Filename:sfig9-5-diskplate-b

Solution To find the linear as well as the angular velocities of the disk and the plate, we will have to use linear and angular momentum-impulse relations. In total, we have 7 scalar unknowns here — 4 for linear velocities of the disk and the plate (each velocity has two components), 2 for the two angular velocities, and 1 for the collision impulse. Naturally, we need 7 independent equations. We have 6 independent equations from the linear and angular impulse-momentum balance for the two bodies (3 each). We need one more equation. That equation is the relationship between the normal components of the relative velocities of approach and departure with the coefficient of restitution e (=1 for elastic collision). Thus we have enough equations. Let us set up all the required equations. We can then solve the equations using a computer.

The free-body diagrams of the disk and the plate together and the two separately are shown in Fig. 14.85 and 14.86, respectively. Using an xy coordinate system oriented as shown in Fig. 14.85, we can write

$$\begin{array}{lrcl}
\text{LMB for disk:} & m_D(\vec{v}_D^+ - \vec{v}_D^-) & = & -P\hat{\imath} \\
\text{LMB for plate:} & m_P(\vec{v}_P^+ - \vec{v}_P^-) & = & P\hat{\imath} \\
\text{AMB for disk:} & I_D^{\mathrm{cm}}(\vec{\omega}_D^+ - \vec{\omega}_D^-) & = & \vec{0} \\
\text{AMB for plate:} & I_P^{\mathrm{cm}}(\vec{\omega}_P^+ - \vec{\omega}_P^-) & = & \vec{r}_{\mathrm{A/G}} \times P\hat{\imath} \\
\text{kinematics:} & \hat{\imath}\cdot\{\vec{v}_{A_D}^+ - \vec{v}_{A_P}^+ & = & e(\vec{v}_{A_P}^- - \vec{v}_{A_D}^-)\}
\end{array}$$

where, in the last equation $\vec{v}_{A_D}$ and $\vec{v}_{A_P}$ refer to the velocities of the material points located at A on the disk and on the plate, respectively. Other linear velocities in the equations above refer to the velocities at the center-of-mass of the corresponding bodies. We are given that $\vec{v}_D^- = v_D\hat{\imath}$, $\vec{v}_P^- = -v_P\hat{\imath}$, $\vec{\omega}_D^- = -\Omega_D\hat{k}$, and $\vec{\omega}_P^- = \vec{0}$. Let us assume that $\vec{\omega}_D^+ = \omega_D\hat{k}$, $\vec{\omega}_P^+ = \omega_P\hat{k}$, $vv_D^+ = v_{D_x}^+\hat{\imath} + v_{D_y}^+\hat{\jmath}$, and similarly, $vv_P^+ = v_{P_x}^+\hat{\imath} + v_{P_y}^+\hat{\jmath}$. Then,

$$\begin{array}{rcl}
\vec{v}_{A_D}^- & = & \vec{v}_D^- + \vec{\omega}_D^- \times \vec{r}_{\mathrm{A/O}} = v_D\hat{\imath} - \omega_D R\hat{\jmath} \\
\vec{v}_{A_D}^+ & = & \vec{v}_D^+ + \vec{\omega}_D^+ \times \vec{r}_{\mathrm{A/O}} = v_{D_x}^+\hat{\imath} + (v_{D_y}^+ + \omega_D^+ R)\hat{\jmath} \\
\vec{v}_{A_P}^- & = & \vec{v}_P^- = -v_P\hat{\imath} \\
\vec{v}_{A_P}^+ & = & \vec{v}_P^+ + \vec{\omega}_P^+ \times \vec{r}_{\mathrm{A/G}} = (v_{P_x}^+ - \omega_P^+ d)\hat{\imath} + (v_{P_y}^+ - \omega_P^+ d)\hat{\jmath}.
\end{array}$$

Substituting these quantities in the kinematics equation above and dotting with the normal direction at A, $\hat{\imath}$, we get

$$v_{D_x}^+ - v_{P_x}^+ + \omega_P^+ d = \underbrace{e}_{1}(-v_P - v_D) = -v_P - v_D. \qquad (14.57)$$

Now, let us extract the scalar equations from the impulse-momentum equations for the disk and the plate by dotting with appropriate unit vectors.

Dotting LMB for the disk with $\hat{\imath}$ and $\hat{\jmath}$, respectively, we get

$$m_D(v^+_{D_x} - v_D) = -P \tag{14.58}$$

$$m_D v^+_{D_y} = 0. \tag{14.59}$$

Dotting LMB for the plate with $\hat{\imath}$ and $\hat{\jmath}$, respectively, we get

$$m_P(v^+_{P_x} - v_P) = P \tag{14.60}$$

$$m_P v^+_{P_y} = 0. \tag{14.61}$$

Dotting AMB for the disk and the plate with $\hat{k}$, we get

$$I^{cm}_D(\omega^+_D - \omega_D) = 0 \tag{14.62}$$

$$I^{cm}_P \omega^+_P = Pd. \tag{14.63}$$

We have all the equations we need. Let us rearrange these equations in a matrix form, taking the known quantities to the right and putting all unknowns to the left side. We then, write eqns. (14.58)–(14.63), and then eqn. (14.57) as

$$\begin{bmatrix} m_D & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & m_D & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_P & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & m_P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I^{cm}_D & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I^{cm}_P & -d \\ 1 & 0 & -1 & 0 & 0 & d & 0 \end{bmatrix} \begin{Bmatrix} v^+_{D_x} \\ v^+_{D_y} \\ v^+_{P_x} \\ v^+_{P_y} \\ \omega^+_D \\ \omega^+_P \\ P \end{Bmatrix} = \begin{Bmatrix} m_D v_D \\ 0 \\ m_P v_P \\ 0 \\ I^{cm}_D \omega_D \\ 0 \\ -v_P - v_D \end{Bmatrix}.$$

Substituting the given numerical values for the masses and the pre-collision velocities, and the moments of inertia, $I^{cm}_D = (1/2)m_D R^2$ and $I^{cm}_P = (1/12)m_P(4a^2 + 4b^2)$, and then solving the matrix equation on a computer, we get,

$$\begin{aligned} \vec{v}^+_D &= 0.34\,\text{m/s}\hat{\imath}, \quad \vec{v}^+_P = -9.67\,\text{m/s}\hat{\imath} \\ \vec{\omega}^+_D &= -5\,\text{rad/s}\hat{k}, \quad \vec{\omega}^+_P = -1.26\,\text{rad/s}\hat{k} \\ P &= -0.66\,\text{kg}\cdot\text{m/s}. \end{aligned}$$

You can easily check that the results obtained satisfy the conservation of linear momentum for the plate and the disk taken together as one system.

$$\boxed{\vec{v}^+_D = 0.34\,\text{m/s}\hat{\imath},\ \vec{v}^+_P = -9.67\,\text{m/s}\hat{\imath},\ \vec{\omega}^+_D = -5\,\text{rad/s}\hat{k},\ \vec{\omega}^+_P = -1.26\,\text{rad/s}\hat{k}}$$

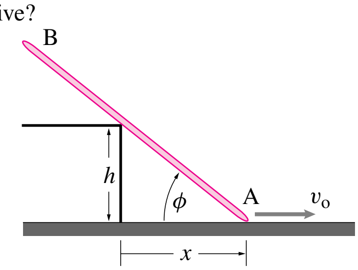
Comments: In this particular problem, the equations are simple enough to be solved by hand. For example, eqns. (14.59), (14.61), and (14.62) are trivial to solve and immediately give, $v^+_{D_y} = 0$, $v^+_{P_y} = 0$, and $\omega^+_D = \omega^D = 5\,\text{rad/s}$. Rest of the equations can be solved by usual eliminations and substitutions, etc. However, it is important to learn how to set up these equations in matrix form so that no matter how complicated the equations are, they can be easily solved on a computer. What really counts is do you have 7 linear independent equations for the 7 unknowns. If you do, you are home.

Problems for Chapter 14

General planar motion of a single rigid body

14.1 Kinematics of planar rigid-body motion

14.1 The slender rod AB rests against the step of height h, while end "A" is moved along the ground at a constant velocity v_o. Find $\dot{\phi}$ and $\ddot{\phi}$ in terms of x, h, and v_o. Is $\dot{\phi}$ positive or negative? Is $\ddot{\phi}$ positive or negative?



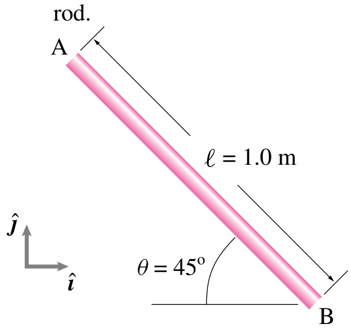
problem 14.1:

Filename:pfigure-blue-98-1

14.2 A ten foot ladder is leaning between a floor and a wall. The top of the ladder is sliding down the wall at one foot per second. (The foot is simultaneously sliding out on the floor). When the ladder makes a 45 degree angle with the vertical what is the speed of the midpoint of the ladder?

14.3 A uniform rigid rod AB of length $\ell = 1$ m rotates at a constant angular speed ω about an unknown fixed point. At the instant shown, the velocities of the two ends of the rod are $\vec{v}_A = -1\,\text{m/s}\hat{\imath}$ and $\vec{v}_B = 1\,\text{m/s}\hat{\jmath}$.

a) Find the angular velocity of the rod.
b) Find the center of rotation of the rod.

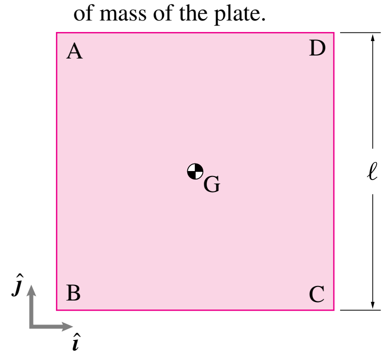


problem 14.3:

Filename:pfigure4-3-rp6

14.4 A square plate ABCD rotates at a constant angular speed about an unknown point in its plane. At the instant shown, the velocities of the two corner points A and D are $\vec{v}_A = -2\,\text{ft/s}(\hat{\imath} + \hat{\jmath})$ and $\vec{v}_D = -(2\,\text{ft/s})\hat{\imath}$, respectively.

a) Find the center of rotation of the plate.
b) Find the acceleration of the center of mass of the plate.

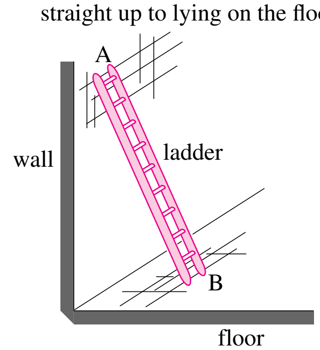


problem 14.4:

Filename:pfigure4-3-rp7

14.5 Consider the motion of a rigid ladder which can slide on a wall and on the floor as shown in the figure. The point A on the ladder moves parallel to the wall. The point B moves parallel to the floor. Yet, at a given instant, both have velocities that are consistent with the ladder rotating about some special point, the center of rotation (COR). Define appropriate dimensions for the problem.

a) Find the COR for the ladder when it is at some given position (and moving, of course). Hint, if a point is A is 'going in circles' about another point C, that other point C must be in the direction perpendicular to the motion of A.
b) As the ladder moves, the COR changes with time. What is the set of points on the plane that are the COR's for the ladder as it falls from straight up to lying on the floor?



problem 14.5:

Filename:pfigure-blue-96-1

14.6 A car driver on a very boring highway is carefully monitoring her speed. Over a one hour period, the car travels on a curve with constant radius of curvature, $\rho = 25$ mi, and its speed increases uniformly from 50 mph to 60 mph. What is the acceleration of the of the car half-way through this one hour period, in path coordinates?

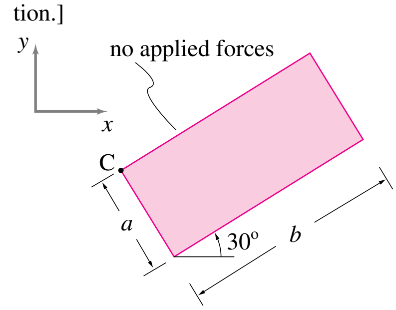
14.7 Find expressions for $\hat{e}_t$, a_t, a_n, $\hat{e}_n$, and the radius of curvature ρ, at any position (or time) on the given particle paths for

a) problem 10.13,
b) problem 10.15,
c) problem **??**,
d) problem 10.17,
e) problem 10.16, and
f) problem 10.14.

14.8 A particle travels at non-constant speed on an elliptical path given by $y^2 = b^2(1 - \frac{x^2}{a^2})$. Carefully sketch the ellipse for particular values of a and b. For various positions of the particle on the path, sketch the position vector $\vec{r}(t)$; the polar coordinate basis vectors $\hat{e}_r$ and $\hat{e}_\theta$; and the path coordinate basis vectors $\hat{e}_n$ and $\hat{e}_t$. At what points on the path are $\hat{e}_r$ and $\hat{e}_n$ parallel(or $\hat{e}_\theta$ and $\hat{e}_t$ parallel)?

14.2 General planar mechanics of a rigid body

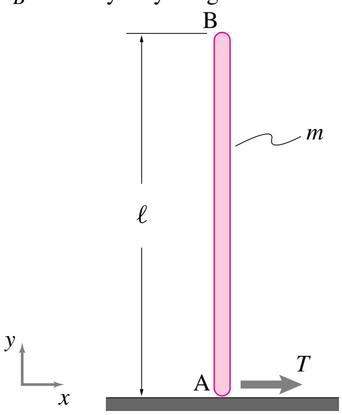
14.9 The uniform rectangle of width $a = 1$ m, length $b = 2$ m, and mass $m = 1$ kg in the figure is sliding on the xy-plane with no friction. At the moment in question, point C is at $x_C = 3$ m and $y_C = 2$ m. The linear momentum is $\vec{L} = 4\hat{\imath} + 3\hat{\jmath}$ (kg·m/s) and the angular momentum about the center of mass is $\vec{H}_{cm} = 5\hat{k}$(kg·m/s^2). Find the acceleration of any point on the body that you choose. (Mark it.) [Hint: You have been given some redundant information.]



problem 14.9:

Filename:pfigure-blue-101-1

14.10 The vertical pole AB of mass m and length ℓ is initially, *at rest* on a frictionless surface. A tension T is suddenly applied at A. What is $\ddot{x}_{cm}$? What is $\ddot{\theta}_{AB}$? What is $\ddot{x}_B$? Gravity may be ignored.



problem 14.10:

Filename:pfigure-blue-95-1

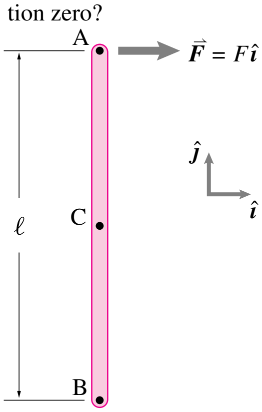
14.11 Force on a stick in space. 2-D . No gravity. A uniform thin stick with length ℓ and mass m is, at the instant of interest, parallel to the y axis and has no velocity and no angular velocity. The force $\vec{F} = F\hat{\imath}$ with $F > 0$ is suddenly applied at point A. The questions below concern the instant after the force $\vec{F}$ is applied.

a) What is the acceleration of point C, the center of mass?

b) What is the angular acceleration of the stick?

c) What is the acceleration of the point A?

d) (relatively harder) What additional force would have to be applied to point B to make point B's acceleration zero?



problem 14.11:

Filename:pfigure-s94h10p2

14.12 A uniform thin rod of length ℓ and mass m stands vertically, with one end resting on a frictionless surface and the other held by someone's hand. The rod is released from rest, displaced slightly from the vertical. No forces are applied during the release. There is gravity.

a) Find the path of the center of mass.

b) Find the force of the floor on the end of the rod just before the rod is horizontal.

14.13 A uniform disk, with mass center labeled as point G, is sitting motionless on the frictionless xy plane. A massless peg is attached to a point on its perimeter. This disk has radius of 1 m and mass of 10 kg. A constant force of $F = 1000\,\text{N}\hat{\imath}$ is applied to the peg for .0001 s (one ten-thousandth of a second).

a) What is the velocity of the center of mass of the center of the disk after the force is applied?

b) Assuming that the idealizations named in the problem statement are exact is your answer to (a) exact or approximate?

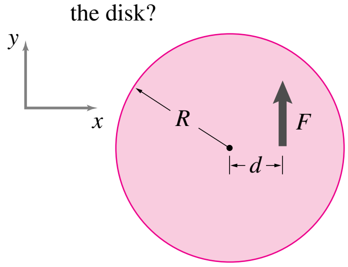
c) What is the angular velocity of the disk after the force is applied?

d) Assuming that the idealizations named in the problem statement are exact is your answer to (c) exact or approximate?

14.14 A uniform thin flat disc is floating in space. It has radius R and mass m. A force F is applied to it a distance d from the center in the y direction. Treat this problem as two-dimensional.

a) What is the acceleration of the center of the disc?

b) What is the angular acceleration of the disk?



problem 14.14:

Filename:pfigure-blue-102-1

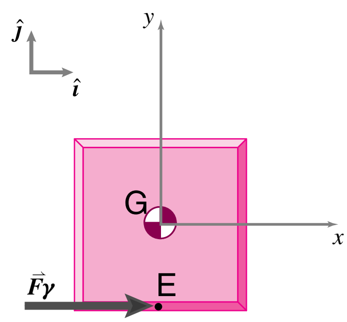
14.15 A uniform 1kg plate that is one meter on a side is initially at rest in the position shown. A constant force $\vec{F} = 1\,\text{N}\hat{\imath}$ is applied at $t = 0$ and maintained henceforth. If you need to calculate any quantity that you don't know, but can't do the calculation to find it, assume that the value is given.

a) Find the position of G as a function of time (the answer should have numbers and units).

b) Find a differential equation, and initial conditions, that when solved would give θ as a function of time. θ is the counterclockwise rotation of the plate from the configuration shown.

c) Write computer commands that would generate a drawing of the outline of the plate at $t = 1\,\text{s}$. You can use hand calculations or the computer for as many of the intermediate commands as you like. Hand work and sketches should be provided as needed to justify or explain the computer work.

d) Run your code and show clear output with labeled plots. Mark output by hand to clarify any points.



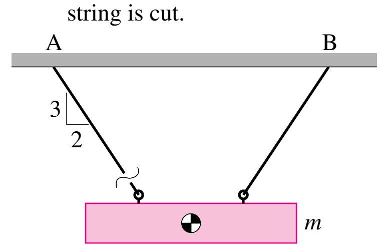
problem 14.15:

Filename:S02p2p2flyingplate

14.16 A uniform rectangular metal beam of mass m hangs symmetrically by two strings as shown in the figure.

a) Draw a free-body diagram of the beam and evaluate $\sum \vec{F}$.

b) Repeat (a) immediately after the left string is cut.

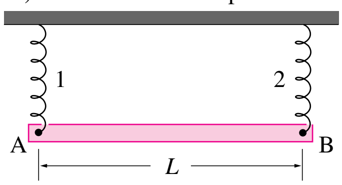


problem 14.16:

Filename:pfig2-2-rp8

14.17 A uniform slender bar AB of mass m is suspended from two springs (each of spring constant K) as shown. If spring 2 breaks, determine at that instant

a) the angular acceleration of the bar,
b) the acceleration of point A, and
c) the acceleration of point B.

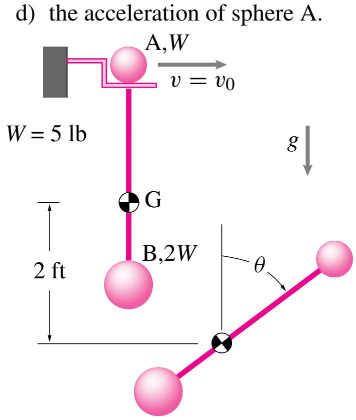


problem 14.17:

Filename:pfigure-blue-50-2

14.18 Two small spheres A and B are connected by a rigid rod of length $\ell = 1.0$ ft and negligible mass. The assembly is hung from a hook, as shown. Sphere A is struck, suddenly breaking its contact with the hook and giving it a horizontal velocity $v_0 = 3.0$ ft/s which sends the assembly into free fall. Determine the angular momentum of the assembly about its mass center at point G immediately after A is hit. After the center of mass has fallen two feet, determine:

a) the angle θ through which the rod has rotated,
b) the velocity of sphere A,
c) the total kinetic energy of the assembly of spheres A and B and the rod, and
d) the acceleration of sphere A.



problem 14.18:

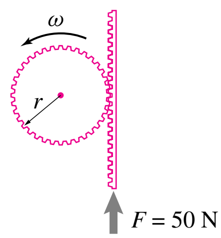
Filename:pfigure-blue-105-1

The next several problems concern Work, power and energy **14.19** Verify that the expressions for work done by a force F, $W = F\Delta S$, and by a moment M, $W = M\Delta\theta$, are dimensionally correct if ΔS and $\Delta\theta$ are linear and angular displacements respectively.

14.20 A uniform disc of mass m and radius r rotates with angular velocity $\omega\hat{\boldsymbol{k}}$. Its center of mass translates with velocity $\vec{\boldsymbol{v}} = \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}}$ in the xy-plane. What is the total kinetic energy of the disk?

14.21 Calculate the energy stored in a spring using the expression $E_P = \frac{1}{2}k\delta^2$ if the spring is compressed by 100 mm and the spring constant is 100 N/m.

14.22 In a rack and pinion system, the rack is acted upon by a constant force $F = 50$ N and has speed $v = 2$ m/s in the direction of the force. Find the input power to the system.

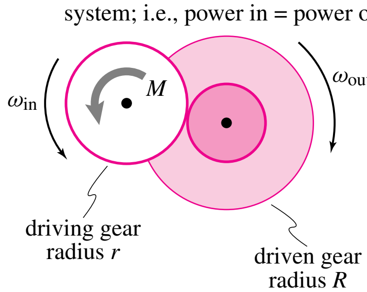


problem 14.22:

Filename:pfig2-3-rp6

14.23 The driving gear in a compound gear train rotates at constant speed ω_0. The driving torque is M_{in}. If the driven gear rotates at a constant speed ω_{out}, find:

a) the input power to the system, and
b) the output torque of the system assuming there is no power loss in the system; i.e., power in = power out.

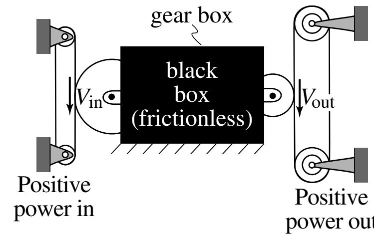


problem 14.23:

Filename:pfig2-3-rp7

14.24 An elaborate frictionless gear box has an input and output roller with $V_{in} = const$. Assuming that $V_{out} = 7V_{in}$ and the force between the left belt and roller is $F_{in} = 3$ lb:

a) What is F_{out} (draw a picture defining the signs of F_{in} and F_{out})?.
b) Is F_{out} greater or less than the F_{in}? (Assume $F_{in} > 0$.) Why?



problem 14.24:

Filename:pfigure-blue-138-1

14.3 Kinematics of rolling and sliding

14.25 A stone in a wheel. A round wheel rolls to the right. At time $t = 0$ it picks up a stone the road. The stone is stuck in the edge of the wheel. You want to know the direction of the rock's motion just before and after it next hits the ground. Here are some candidate answers:

- When the stone approaches the ground its motion is tangent to the ground.
- The stone approaches the ground at angle x (you name it).
- When the stone approaches the ground its motion is perpendicular to the ground.
- The stone approaches the ground at various angles depending on the following conditions(...you list the conditions.)

Although you could address this question analytically, you are to try to get a clear answer by looking at computer generated plots. In particular, you are to plot the pebble's path for a small interval of time near when the stone next touches the ground. You should pick the parameters that make your case for an answer the strongest. You may make more than one plot.

Here are some steps to follow:

a) Assuming the wheel has radius R_w and the pebble is a distance R_p from the center (not necessarily equal to R_w). The pebble is directly below the center of the wheel at time $t = 0$. The wheel spins at constant clockwise rate ω. The x-axis is on the ground and $x(t = 0) = 0$. The wheel rolls without slipping. Using a clear well labeled drawing (use a compass and ruler or a computer drawing program), show that

$$x(t) = \omega t R_w - R_p \sin(\omega t)$$

$$y(t) = R_w - R_p \cos(\omega t)$$

b) Using this relation, write a program to make a plot of the path of the pebble as the wheel makes a little more than one revolution. Also show the outline of the wheel and the pebble itself at some intermediate time of interest. [Use any software and computer that pleases you.]

c) Change whatever you need to change to make a good plot of the pebble's path for a small amount of time as the pebble approaches and leaves the road. Also show the wheel and the pebble at some time in this interval.

d) In this configuration the pebble moves a *very* small distance in a small time so your axes need to be scaled. But make sure your x- and y- axes have the same scale so that the path of the pebble and the outline of the wheel will not be distorted.

e) How does your computer output buttress your claim that the pebble approaches and leaves the ground at the angles you claim?

f) Think of something about the pebble in the wheel that was not explicitly asked in this problem and explain it using the computer, and/or hand calculation and/or a drawing.

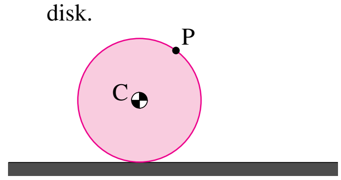
14.26 A uniform disk of radius r rolls at a constant rate without slip. A small ball of mass m is attached to the outside edge of the disk. What is the force required to hold the disk in place when the mass is above the center of the disk?

14.27 Rolling at constant rate. A round disk rolls on the ground at constant rate. It rolls $1\frac{1}{4}$ revolutions over the time of interest.

a) **Particle paths.** Accurately plot the paths of three points: the center of the disk C, a point on the outer edge that is initially on the ground, and a point that is initially half way between the former two points. [Hint: Write a parametric equation for the position of the points. First find a relation between ω and v_C. Then note that the position of a point is the position of the center plus the position of the point relative to the center.] Draw the paths on the computer, make sure x and y scales are the same.

b) **Velocity of points.** Find the velocity of the points at a few instants in the motion: after $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, and 1 revolution. Draw the velocity vector (by hand) on your plot. Draw the direction accurately and draw the lengths of the vectors in proportion to their magnitude. You can find the velocity by differentiating the position vector or by using relative motion formulas appropriately. Draw the disk at its position after one quarter revolution. Note that the velocity of the points is perpendicular to the line connecting the points to the ground contact.

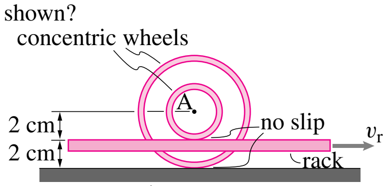
c) **Acceleration of points.** Do the same as above but for acceleration. Note that the acceleration of the points is parallel to the line connecting the points to the center of the disk.



problem 14.27:

Filename:pfigure-s94h11p2

14.28 The concentric wheels are welded to each other and roll without slip on the rack and stationary support. The rack moves to the right at $v_r = 1\,\text{m/s}$. What is the velocity of point A in the middle of the wheels shown?



problem 14.28:

Filename:pfigure-blue-118-1

14.29 Questions (a) - (e) refer to the cylinders in the configuration shown figure. Question (f) is closely related. Answer the questions in terms of the given quantities (and any other quantities you define if needed).

a) What is the speed (magnitude of velocity) of point c?

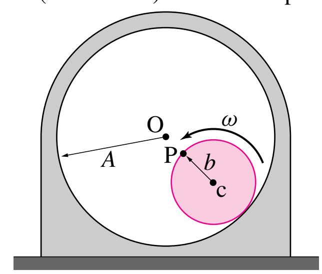
b) What is the speed of point P?

c) What is the magnitude of the acceleration of point c?

d) What is the magnitude of the acceleration of point P?

e) What is the radius of curvature of the path of the particle P at the point of interest?

f) In the special case of $A = 2b$ what is the curve which particle P traces (for all time)? Sketch the path.



problem 14.29: A little cylinder (with outer radius b and center at point c) rolls without slipping inside a bigger fixed cylinder (with inner radius A and center at point O). The *absolute* angular velocity of the little cylinder ω is *constant*. P is attached to the outside edge of the little cylinder. At the instant of interest, P is on the line between O and c.

Filename:pfigure-blue-50-1

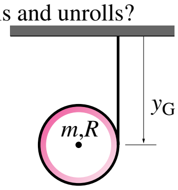
14.4 Mechanics of contacting bodies: rolling and sliding

14.30 A uniform disc of mass m and radius r rolls without slip at constant rate. What is the total kinetic energy of the disk?

14.31 A non-uniform disc of mass m and radius r rolls without slip at constant rate. The center of mass is located at a distance $\frac{r}{2}$ from the center of the disc. What is the total kinetic energy of the disc when the center of mass is directly above the center of the disc?

14.32 Falling hoop. A bicycle rim (no spokes, tube, tire, or hub) is idealized as a hoop with mass m and radius R. G is at the center of the hoop. An inextensible string is wrapped around the hoop and attached to the ceiling. The hoop is released from rest at the position shown at $t = 0$.

a) Find y_G at a later time t in terms of any or all of m, R, g, and t.
b) Does G move sideways as the hoop falls and unrolls?

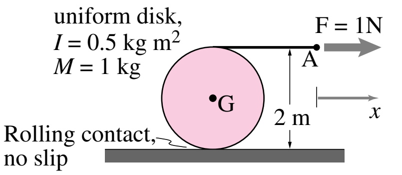


problem 14.32:

Filename:p-s96-p3-2

14.33 A uniform disk with radius R and mass m has a string wrapped around it. The string is pulled with a force F. The disk rolls without slipping.

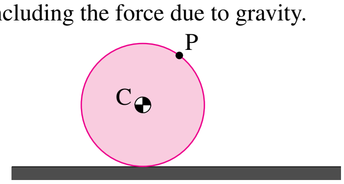
a) What is the angular acceleration of the disk, $\vec{\boldsymbol{\alpha}}_{Disk} = -\ddot{\theta}\hat{\boldsymbol{k}}$? Make any reasonable assumptions you need that are consistent with the figure information and the laws of mechanics. State your assumptions.
b) Find the acceleration of the point A in the figure.



problem 14.33:

Filename:pfigure-blue-43-1

14.34 If a pebble is stuck to the edge of the wheel in problem 14.27, what is the maximum speed of the pebble during the motion? When is the force on the pebble from the wheel maximum? Draw a good FBD including the force due to gravity.

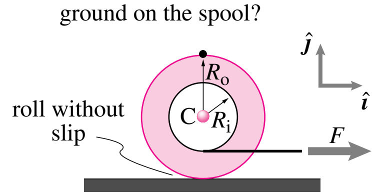


problem 14.34:

Filename:pfigure-s94h11p2-a

14.35 Spool Rolling without Slip and Pulled by a Cord. The light-weight spool is nearly empty but a lead ball with mass m has been placed at its center. A force F is applied in the horizontal direction to the cord wound around the wheel. Dimensions are as marked. Coordinate directions are as marked.

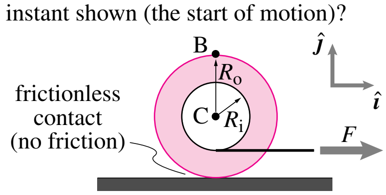
a) What is the acceleration of the center of the spool?
b) What is the horizontal force of the ground on the spool?



problem 14.35:

Filename:pfigure-s94h11p5

14.36 A film spool is placed on a very slippery table. Assume that the film and reel (together) have mass distributed the same as a uniform disk of radius R_i. What, in terms of R_i, R_o, m, g, $\hat{\imath}$, $\hat{\jmath}$, and F are the accelerations of points C and B at the instant shown (the start of motion)?



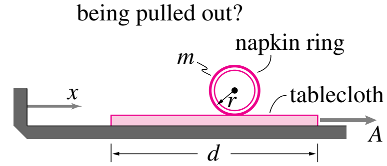
problem 14.36:

Filename:p-f96-f-4

14.37 Again, Spool Rolling without Slip and Pulled by a Cord. Reconsider the spool from problem 14.35. This time, a force F is applied vertically to the cord wound around the wheel. In this case, what is the acceleration of the center of the spool? Is it possible to pull the cord at some angle between horizontal and vertical so that the angular acceleration of the spool or the acceleration of the center of mass is zero? If so, find the angle in terms of R_i, R_0, m, and F.

14.38 A napkin ring lies on a thick velvet tablecloth. The thin ring (of mass m, radius r) rolls without slip as a mischievous child pulls the tablecloth (mass M) out with acceleration A. The ring starts at the right end ($x = d$). You can make a reasonable physical model of this situation with an empty soda can and a piece of paper on a flat table.

a) What is the ring's acceleration as the tablecloth is being withdrawn?
b) How far has the tablecloth moved to the right from its starting point $x = 0$ when the ring rolls off its left-hand end?
c) Clearly describe the subsequent motion of the ring. Which way does it end up rolling at what speed?
d) Would your answer to the previous question be different if the ring slipped on the cloth as the cloth was being pulled out?

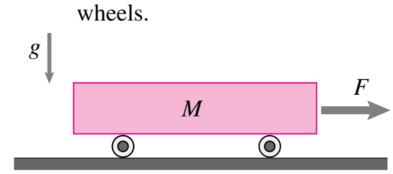


problem 14.38:

Filename:pfigure-blue-51-1

14.39 A block of mass M is supported by two rollers (uniform cylinders) each of mass m and radius r. They roll without slip on the block and the ground. A force F is applied in the horizontal direction to the right, as shown in the figure. Given F, m, r, and M, find:

a) the acceleration of the block,
b) the acceleration of the center of mass of this block/roller system,
c) the reaction at the wheel bases,
d) the force of the right wheel on the block,
e) the acceleration of the wheel centers, and
f) the angular acceleration of the wheels.



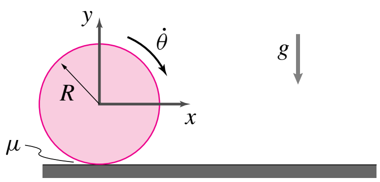
problem 14.39:

Filename:pfigure-blue-108-2

14.40 Dropped spinning disk. 2-D . A uniform disk of radius R and mass m is gently dropped onto a surface and doesn't bounce. When it is released it is spinning clockwise at the rate $\dot{\theta}_0$. The disk skids for a while and then is eventually rolling.

a) What is the speed of the center of the disk when the disk is eventually rolling (answer in terms of g, μ, R, $\dot{\theta}_0$, and m)?

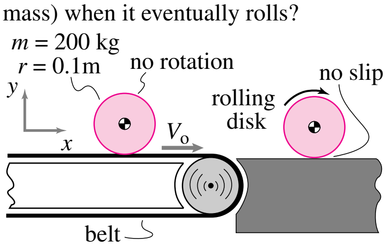
b) In the transition from slipping to rolling, energy is lost to friction. How does the amount lost depend on the coefficient of friction (and other parameters)? How does this loss make or not make sense in the limit as $\mu \to 0$ and the dissipation rate $\to$ zero?



problem 14.40:

Filename:pfigure-s95q10

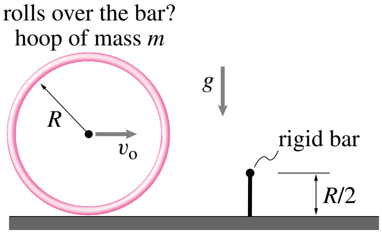
14.41 Disk on a conveyor belt. A uniform metal cylinder with mass of 200 kg is carried on a conveyer belt which moves at $V_0 = 3\,\mathrm{m/s}$. The disk is not rotating when on the belt. The disk is delivered to a flat hard platform where it slides for a while and ends up rolling. How fast is it moving (i.e. what is the speed of the center of mass) when it eventually rolls?



problem 14.41:

Filename:pfigure-s94h11p3

14.42 A rigid hoop with radius R and mass m is rolling without slip so that its center has translational speed v_o. It then hits a narrow bar with height $R/2$. When the hoop hits the bar suddenly it sticks and doesn't slide. It does hinge freely about the bar, however. The gravitational constant is g. How big is v_o if the hoop just barely rolls over the bar?



problem 14.42:

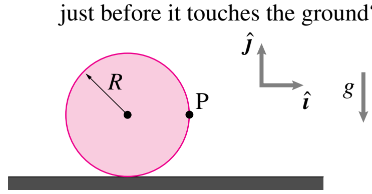
Filename:pfigure-blue-87-5

14.43 2-D rolling of an unbalanced wheel. A wheel, modeled as massless, has a point mass (mass $= m$) at its perimeter. The wheel is released from rest at the position shown. The wheel makes contact with coefficient of friction μ.

a) What is the acceleration of the point P just after the wheel is released if $\mu = 0$?

b) What is the acceleration of the point P just after the wheel is released if $\mu = 2$?

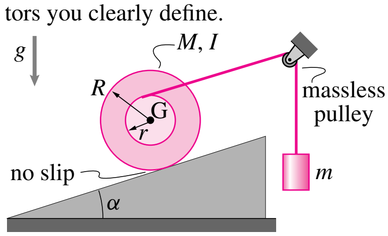
c) Assuming the wheel rolls without slip (no-slip requires, by the way, that the friction be high: $\mu = \infty$) what is the velocity of the point P just before it touches the ground?



problem 14.43:

Filename:pfigure-s94f1p1

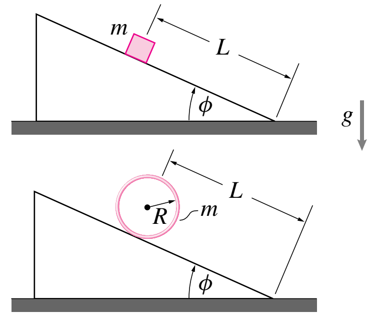
14.44 Spool and mass. A reel of mass M and moment of inertia $I_{zz}^{cm} = I$ rolls without slipping upwards on an incline with slope-angle α. It is pulled up by a string attached to mass m as shown. Find the acceleration of point G in terms of some or all of M, m, I, R, r, α, g and any base vectors you clearly define.



problem 14.44:

Filename:s97f4

14.45 Two objects are released on two identical ramps. One is a sliding block (no friction), the other a rolling hoop (no slip). Both have the same mass, m, are in the same gravity field and have the same distance to travel. It takes the sliding mass 1 s to reach the bottom of the ramp. How long does it take the hoop? [Useful formula: "$s = \frac{1}{2}at^2$"]



problem 14.45:

Filename:pfigure-blue-51-2

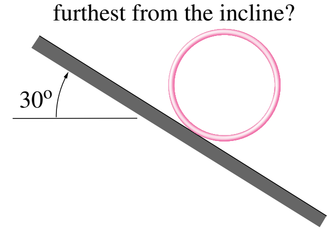
14.46 The hoop is rolled down an incline that is 30° from horizontal. It does not slip. It does not fall over sideways. It is let go from rest at t = 0.

a) At $t = 0^+$ what is the acceleration of the hoop center of mass?

b) At $t = 0^+$ what is the acceleration of the point on the hoop that is on the incline?

c) At $t = 0^+$ what is the acceleration of the point on the hoop that is furthest from the incline?

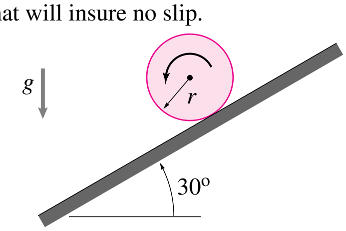
d) After the hoop has descended 2 vertical meters (and traveled an appropriate distance down the incline) what is the acceleration of the point on the hoop that is (at that instant) furthest from the incline?



problem 14.46:

Filename:pfigure-blue-47-1

14.47 A uniform cylinder of mass m and radius r rolls down an incline *without slip*, as shown below. Determine: (a) the angular acceleration α of the disk; (b) the minimum value of the coefficient of friction μ that will insure no slip.



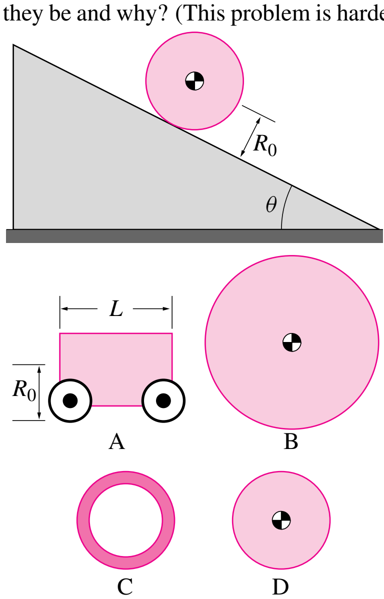
problem 14.47:

Filename:pfigure-blue-49-3

14.48 Race of rollers. A uniform disk with mass M_0 and radius R_0 is allowed to roll down the frictionless but quite slip-resistant ($\mu = 1$) 30° ramp shown. It is raced against four other objects (A, B, C and D), one at a time. Who wins the races, or are there ties? First try to construct any plausible reasoning. Good answers will be based, at least in part, on careful use of equations of mechanics.

a) Block A has the same mass and has center of mass a distance R_0 from the ramp. It rolls on massless wheels with frictionless bearings.
b) Uniform disk B has the same mass ($M_B = M_0$) but twice the radius ($R_B = 2R_0$).
c) Hollow pipe C has the same mass ($M_C = M_0$) and the same radius ($R_C = R_0$).
d) Uniform disk D has the same radius ($R_D = R_0$) but twice the mass ($M_D = 2M_0$).

Can you find a round object which will roll as fast as the block slides? How about a massless cylinder with a point mass in its center? Can you find an object which will go slower than the slowest or faster than the fastest of these objects? What would they be and why? (This problem is harder.)



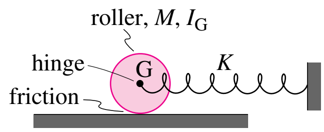
problem 14.48:

Filename:pfigure-s94h11p4

14.49 A roller of mass M and polar moment of inertia about the center of mass I_G is connected to a spring of stiffness K by a frictionless hinge as shown in the figure. Consider two kinds of friction between the roller and the surface it moves on:

1. Perfect slipping (no friction), and
2. Perfect rolling (infinite friction).

a) What is the period of oscillation in the first case?
b) What is the period of oscillation in the second case?

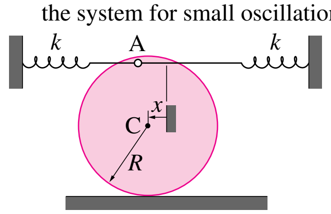


problem 14.49:

Filename:pfigure-blue-141-2

14.50 A uniform cylinder of mass m and radius R rolls back and forth without slipping through small amplitudes (i.e., the springs attached at point A on the rim act linearly and the vertical change in the height of point A is negligible). The springs, which act both in compression and tension, are unextended when A is directly over C.

a) Determine the differential equation of motion for the cylinder's center.
b) Calculate the natural frequency of the system for small oscillations.

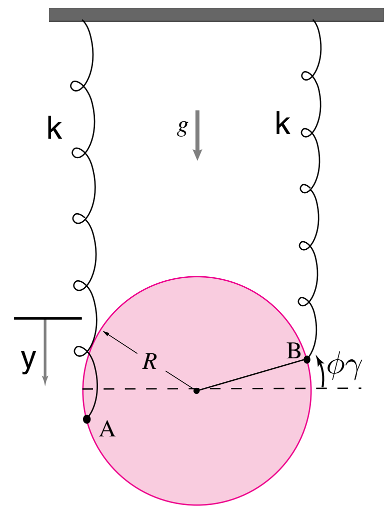


problem 14.50:

Filename:pfigure-blue-152-1

14.51 Hanging disk, 2-D. A uniform thin disk of radius R and mass m hangs in a gravity field g from a pair of massless springs each with constant k. In the static equilibrium configuration the springs are vertical and attached to points A and B on the right and left edges of the disk. In the equilibrium configuration the springs carry the weight, the disk counter-clockwise rotation is $\phi = 0$, and the downwards vertical deflection is $y = 0$. Assume throughout that the center of the disk only moves up and down, and that ϕ is small so that the springs may be regarded as vertical when calculating their stretch ($\sin\phi \approx \phi$ and $\cos\phi \approx 1$).

a) Find $\ddot{\phi}$ and $\ddot{y}$ in terms of some or all of ϕ, $\dot{\phi}$, y, $\dot{y}$, k, m, R, and g.
b) Find the natural frequencies of vibration in terms of some or all of k, m, R, and g.



problem 14.51:

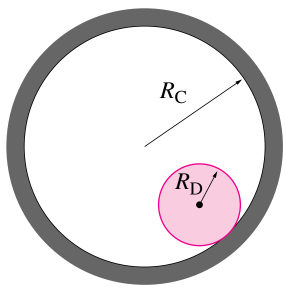
Filename:pfigure-disktwosprings

14.52 A disk rolls in a cylinder. For all of the problems below, the disk rolls without slip and s back and forth due to gravity.

a) **Sketch.** Draw a neat sketch of the disk in the cylinder. The sketch should show all variables, coordinates and dimension used in the problem.
b) **FBD.** Draw a free body diagram of the disk.
c) **Momentum balance.** Write the equations of linear and angular momentum balance for the disk. Use the point on the cylinder which touches the disk for the angular momentum balance equation. Leave as unknown in these equations variables which you do not know.
d) **Kinematics.** The disk rolling in the cylinder is a *one*-degree-of-freedom system. That is, the values of only *one* coordinate and its derivatives are enough to determine the positions, velocities and accelerations of all points. The angle that the line from the center of the cylinder to the center of the disk makes from the vertical can be used as such a variable. Find all of the velocities and accelerations needed in the momentum balance equation in terms of this variable and it's derivative. [Hint: you'll need to think about the rolling contact in order to do this part.]
e) **Equation of motion.** Write the angular momentum balance equation as a single second order differential equation.

f) **Simple pendulum?** Does this equation reduce to the equation for a pendulum with a point mass and length equal to the radius of the cylinder, when the disk radius gets arbitrarily small? Why, or why not?

g) **How many?** How many parts can one simple question be divided into?

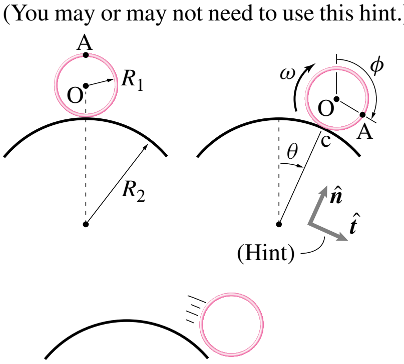


problem 14.52: A disk rolls without slip inside a bigger cylinder.

Filename:h12-3

14.53 A uniform hoop of radius R_1 and mass m rolls from rest down a semi-circular track of radius R_2 as shown. Assume that no slipping occurs. At what angle θ does the hoop leave the track and what is its angular velocity ω and the linear velocity $\vec{v}$ of its center of mass at that instant? If the hoop slides down the track without friction, so that it does not rotate, will it leave at a smaller or larger angle θ than if it rolls without slip (as above)? Give a qualitative argument to justify your answer.

HINT: Here is a geometric relationship between angle ϕ hoop turns through and angle θ subtended by its center when no slipping occurs: $\phi = [(R_1 + R_2)/R_2]\theta$. (You may or may not need to use this hint.)

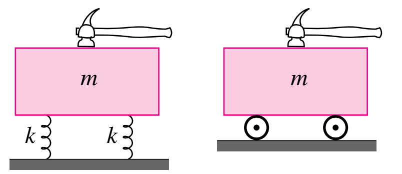


problem 14.53:

Filename:pfigure-blue-46-1

14.5 Collisions

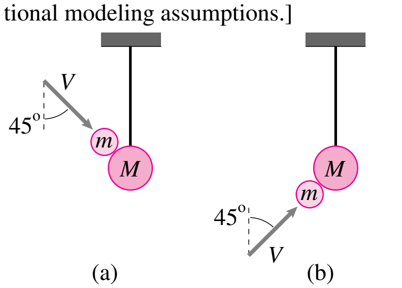
14.54 The two blocks shown in the figure are identical except that one rests on two springs while the other one sits on two massless wheels. Draw free-body diagrams of each mass as each is struck by a hammer. Here we are interested in the free-body diagrams only during collision. Therefore, ignore all forces that are much smaller than the *impulsive* forces. State in words why the forces you choose to show should not be ignored during the collision.



problem 14.54:

Filename:pfig2-1-rp9

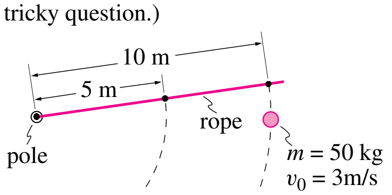
14.55 These problems concerns two colliding masses. In the first case In (a) the smaller mass hits the hanging mass from above at an angle 45° with the vertical. In (b) second case the smaller mass hits the hanging mass from below at the same angle. Assuming perfectly elastic impact between the balls, find the velocity of the hanging mass just after the collision. [Note, these problems are *not* well posed and can only be solved if you make additional modeling assumptions.]



problem 14.55:

Filename:pfig2-2-rp7

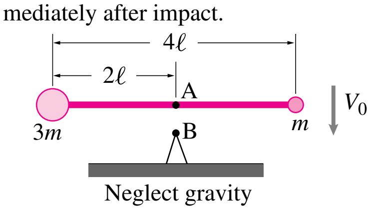
14.56 A narrow pole is in the middle of a pond with a 10 m rope tied to it. A frictionless ice skater of mass 50 kg and speed 3 m/s grabs the rope. The rope slowly wraps around the pole. What is the speed of the skater when the rope is 5 m long? (A tricky question.)



problem 14.56:

Filename:pfigure-blue-49-1

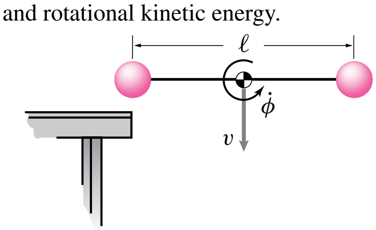
14.57 The masses m and $3m$ are joined by a light-weight bar of length 4ℓ. If point A in the center of the bar strikes fixed point B vertically with velocity V_0, and is not permitted to rebound, find $\dot{\theta}$ of the system immediately after impact.



problem 14.57: Neglect gravity!

Filename:pfigure-blue-81-1

14.58 Two equal masses each of mass m are joined by a massless rigid rod of length ℓ. The assembly strikes the edge of a table as shown in the figure, when the center of mass is moving downward with a linear velocity v and the system is rotating with angular velocity $\dot{\phi}$ in the counter-clockwise sense. The impact is 'elastic'. Find the immediate subsequent motion of the system, assuming that no energy is lost during the impact and that there is no gravity. Show that there is an interchange of translational and rotational kinetic energy.



problem 14.58:

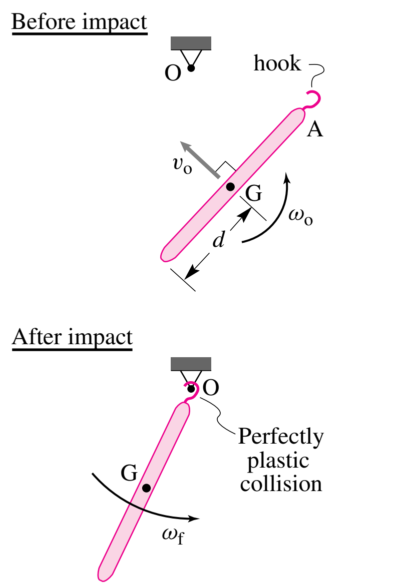
Filename:pfigure-blue-137-2

14.59 In the *absence of gravity*, a thin rod of mass m and length ℓ is initially tumbling with constant angular speed ω_o, in the counter-clockwise direction, while its mass center has constant speed v_o, directed as shown below. The end A then makes a perfectly plastic collision with a rigid peg O (via a hook). The velocity of the mass center happens to be perpendicular to the rod just before impact.

a) What is the angular speed ω_f immediately after impact?

b) What is the angular speed 10 seconds after impact? Why?

c) What is the loss in energy in the plastic collision?



problem 14.59:

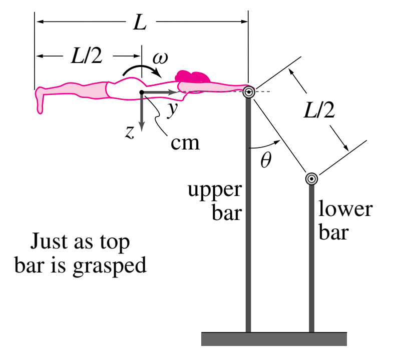
Filename:pfigure-blue-78-2

14.60 A gymnast of mass m and extended height L is performing on the uneven parallel bars. About the x, y, z axes which pass through her center of mass, her radii of gyration are k_x, k_y, and k_z, respectively. Just before she grasps the top bar, her fully extended body is horizontal and rotating with angular rate ω; her center of mass is then stationary. Neglect any friction between the bar and her hands and assume that she remains rigid throughout the entire stunt.

a) What is the gymnast's rotation rate just *after* she grasps the bar? State clearly any approximations/assumptions that you make.

b) Calculate the linear speed with which her hips (CM) strike the lower bar. State all assumptions/approximations.

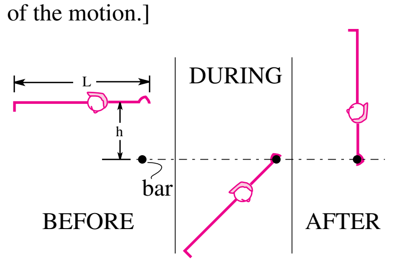
c) Describe (in words, no equations please) her motion immediately after her hips strike the lower bar if she releases her hands just prior to this impact.



problem 14.60:

Filename:pfigure-blue-130-2

14.61 An acrobat modelled as a rigid body. An acrobat is modeled as a uniform rigid mass m of length l. The acrobat falls without rotation in the position shown from height h where she was stationary. She then grabs a bar with a firm but slippery grip. What is h so that after the subsequent motion the acrobat ends up in a stationary handstand? [Hints: Note what quantities are preserved in what parts of the motion.]



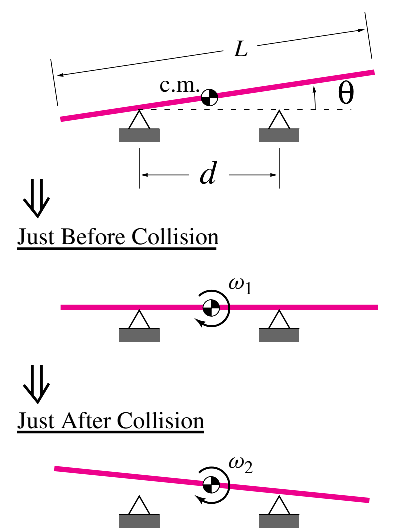
problem 14.61:

Filename:pfigure-s94h10p4

14.62 A crude see-saw is built with two supports separated by distance d about which a rigid plank (mass m, length L) can pivot smoothly. The plank is placed symmetrically, so that its center of mass is midway between the supports when the plan is at rest.

a) While the left end is resting on the left support, the right end of the plank is lifted to an angle θ and released. At what angular velocity ω_1 will the plank strike the right hand support?

b) Following the impact, the left end of the plank can pivot purely about the right end if d/L is properly chosen and the right end does not bounce. Find ω_2 under these circumstances.



problem 14.62:

Filename:pfigure-blue-81-2

14.63 Baseball bat. In order to convey the ideas without making the calculation to complicated, some of the simplifying assumptions here are highly approximate. Assume that a bat is a uniform rigid stick with length L and mass m_s. The motion of the bat is a pivoting about one end held firmly in place with hands that rotate but do not move. The swinging of the bat occurs by the application of a constant torque M_s at the hands over an angle of $\theta = \pi/2$ until the point of impact with the ball. The ball has mass m_b and arrives perpendicular to the bat at an absolute speed v_b at a point a distance ℓ from the hands. The collision between the bat and the ball is completely elastic.

a) To maximize the speed v_{hit} of the hit ball *How heavy should a baseball bat be? Where should the ball hit the bat?* Here are some hints for one way to approach the problem.

- Find the angular velocity of the bat just before collision by drawing a FBD of the bat etc.
- Find the total energy of the ball and bat system just before the collision.
- Draw a FBD of the ball and of the bat during the collision (with this model there is an impulse at the hands on the bat). Call the magnitude of the impulse of the ball on the bat (and vice versa) $\int F dt$.
- Use various momentum equations to find

the angular velocity of the bat and velocity of the ball just after the collision in terms of $\int F dt$ and other quantities above. Use these to find the energy of the system just after collision.

- Solve for the value of $\int F dt$ that conserves energy. As a check you should see if this also predicts that the relative separation speed of the ball and bat (at the impact point) is the same as the relative approach speed (it should be).
- You now know can calculate v_{hit} in terms of m_b, m_s, M_s, L, ℓ, and v_b.
- Find the maximum of the above expression by varying m_s and ℓ. Pick numbers for the fixed quantities if you like.

b) Can you explain in words what is wrong with a bat that is too light or too heavy?

c) Which aspects of the model above do you think lead to the biggest errors in predicting what a real ball player should pick for a bat and place on the bat to hit the ball?

d) Describe as clearly as possible a different model of a baseball swing that you think would give a more accurate prediction. (You need not do the calculation).