CHAPTER 11

Two or more particles in space (unconstrained)

This more advanced chapter concerns the motion of two or more particles in space using $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ in Cartesian coordinates for each particle. The start is the set up of "two-body" type problems. Next are the variety of theorems, especially simplifications involving the center-of-mass, and applications that follow from particle mechanics and the viewing of matter as made up of many pairwise-interacting particles.

Contents

In the previous chapter you saw that once you know the forces on a particle, or how to find those forces given a particle's position, velocity and time, you can easily set up the equations of motion. That is, the linear momentum balance equation for a particle

$$\vec{F} = m\vec{a},$$

with initial conditions, gives a well defined mathematical problem. The solution of this math problem gives the position and velocity of the particle as a function of time. The solution may be hard or impossible to find with pencil and paper, but can usually be found quite directly using numerical integration.

Now we generalize this idea to two, three or more particles. In one model of the universe every system is made of particles and each particle obeys Newton's laws. If we could think of all materials as made of atoms, and of all the atoms moving in deterministic ways governed by Newton's laws and known force laws, and we knew the initial positions and velocities accurately enough, then we could accurately predict the motions of all things for all time. *

To put it in other words, given a simple atomic view of the world and a big computer, we could end a course on dynamics here. You know how to use $\vec{F} = m\vec{a}$ for each atom, so you can simulate anything made of atoms.

Of course there are some serious limitations to this point of view, so before proceeding we list some caveats:

- there are no computers big enough to keep track of the 10^{23} or so atoms needed to describe macroscopic objects or the 10^{79} or so atoms in the universe;
- the laws of interaction between the most fundamental particles are not given by Newton's laws but by quantum chromodynamics, or whatever;
- one feature of the rules of the world, as physicists now understand them, is that they are not deterministic, quantum mechanics says that you *cannot* know the state of the world perfectly;
- the state of the world (the positions and velocities of all the bits is not that well known);
- the solutions of dynamics equations are often unstable in the smallest of errors in the initial conditions propagates into a large error in the predicted motion; and
- massive simulations, even if accurate, are not always the best way to understand how things work.

* *"We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at any given moment knew all of the forces that animate nature and the mutual positions of the beings that compose it, if this intellect were vast enough to submit the data to analysis, could condense into a single formula the movement of the greatest bodies of the universe and that of the lightest atom; for such an intellect nothing could be uncertain and the future just like the past would be present before its eyes."* wrote Marquis Pierre Simon de Laplace (1749-1827). Laplace's supreme intellect, some kind of super data collector and super computer, is sometimes called "Laplace's demon."

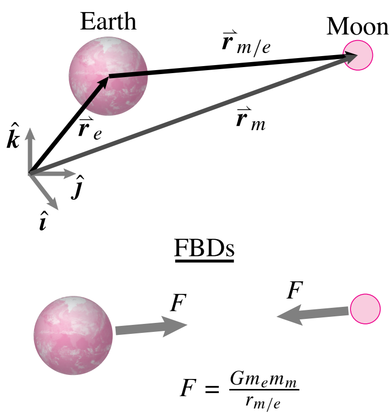

Figure 11.1: The earth and moon. Position is measured relative to some "fixed" point C.
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Despite these limitations, in this chapter we look at the nature of systems of interacting particles. Using this particle model we can, for example, derive some results about angular momentum that turn out to be reliable, despite the questionable microscopic physics. Also, the multi-particle model of the systems is good for intuition and is also useful for modeling machines with many parts and the forms of galaxies.

11.1 Coupled motions of particles in space

Assume you know enough about a system so that you know the forces on each particle if someone tells you the time and the positions and velocities of all the particles. This means you can write the governing equations for the system of particles like this:

$$\begin{aligned} \vec{a}_1 &= \frac{1}{m_1}\vec{F}_1 \\ \vec{a}_2 &= \frac{1}{m_2}\vec{F}_2 \\ \vec{a}_3 &= \frac{1}{m_3}\vec{F}_3 \\ &\text{etc.} \end{aligned} \tag{11.1}$$

where $\vec{F}_1$, $\vec{F}_2$ *etc*. are the total of the forces on the corresponding particles. If the force on each particle comes from air-friction, from springs or dashpots connected here and there, or from gravity interactions with other particles, *etc*., then all the forces on all the particles are known given the positions and velocities of the particles. Thus eqn. (11.1) can be written as a system of first order differential equations in standard form, ready for computer simulation. Given accurate initial conditions and a good computer then the motions of all the particles can be found accurately.

Example: **Coupled motion of the earth and moon in three dimensions.**
Let's neglect the sun and just look at the coupled motions of the earth and moon. They attract each other by the same law of gravity that we used for the sun and earth. The difference between this problem and a "central-force" problem is that we now need to look at the 'absolute' positions of the sun and the moon ($\vec{r}_e$ and $\vec{r}_m$), as well as the 'relative' position $\vec{r}_{m/e} \equiv \vec{r}_m - \vec{r}_e$ (Fig. 11.1).

The linear momentum balance equations are now

$$m_e\ddot{\vec{r}}_e = \frac{-Gm_e m_m \vec{r}_{m/e}}{|\vec{r}_{m/e}|^3} \quad \text{and} \tag{11.2}$$

$$m_m\ddot{\vec{r}}_m = \frac{+Gm_e m_m \vec{r}_{m/e}}{|\vec{r}_{m/e}|^3}, \tag{11.3}$$

which, when broken into x, y, and z components give 6 second order ordinary differential equations. These equations can be written as 12 first order equations by defining a list of 12 z variables: $z_1 = x_e$, $z_2 = \dot{x}_e$, $z_3 = \dot{y}_e$, $z_4 = y_e$, *etc*.

After you can find solutions, using various initial conditions you can check if the computer finds such truths (that is, features of the exact solution of the differential equations) as:

1. that the line between the earth and moon always lies on one fixed plane,

2. the center-of-mass moves at constant speed on a straight line,
3. relative to the center-of-mass both the earth and moon travel on paths that are conic sections (circle, ellipse, parabola, hyperbola or a straight line).
4. the energy of the system is constant,
5. and that the angular momentum of the system about the center-of-mass is a constant.

Linear momentum $\vec{L}$ and its rate of change $\dot{\vec{L}}$

One of our three basic dynamics equations is linear momentum balance:

$$\sum \vec{F} = \dot{\vec{L}}.$$

The first quantity of interest in this section is the linear momentum $\vec{L}$* whose derivative, $\dot{\vec{L}}$, with respect to a Newtonian frame is so important. Linear momentum is a measure of the translational motion of a system.

$$\underbrace{\vec{L}}_{\text{linear momentum}} \equiv \underbrace{\sum m_i \vec{v}_i}_{\substack{\text{summed over} \\ \text{all the mass particles}}} = m_{\text{tot}} \vec{v}_{cm} \tag{11.4}$$

*In Isaac Newton's language: *'The quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjointly'.* In other words, Newton's dynamics equations for a particle were based on the product of $\vec{v}$ and m. This quantity, $m\vec{v}$, is now called $\vec{L}$, the linear momentum of a particle.

Example: **Center of Mass position, velocity, and acceleration**

A particle of mass $m_A = 3\,\text{kg}$ and another point of mass $m_B = 2\,\text{kg}$ have positions, respectively,

$$\vec{r}_A(t) = \left[3\hat{\imath} + 5\left(\frac{t}{\text{s}}\right)\hat{\jmath}\right]\text{m}, \text{ and } \vec{r}_B(t) = \left[6\left(\frac{t^2}{\text{s}^2}\right)\hat{\imath} - 4\hat{\jmath}\right]\text{m}$$

due to forces that we do not discuss here. The position of the center-of-mass of the system of particles, according to equation 2.29 on page 93, is

$$\vec{r}_{cm}(t) = \frac{\overbrace{m_A\vec{r}_A(t) + m_B\vec{r}_B(t)}^{\sum m_i \vec{r}_i}}{\underbrace{(m_A + m_B)}_{m_{\text{tot}}=5\,\text{kg}}}$$

$$\vec{r}_{cm}(t) = \left[\left(\frac{9}{5} + \frac{12}{5}\left(\frac{t^2}{\text{s}^2}\right)\right)\hat{\imath} + \left(3\left(\frac{t}{\text{s}}\right) - \frac{8}{5}\right)\hat{\jmath}\right]\text{m}.$$

11.1 Velocity and acceleration of the center-of-mass of a system of particles

The average position of mass in a system is at a point called the center-of-mass. The position of the center-of-mass is

$$\vec{r}_{cm} = \frac{\sum \vec{r}_i\, m_i}{m_{\text{tot}}}.$$

Multiplying through by m_{tot}, we get

$$\vec{r}_{cm} m_{\text{tot}} = \sum \vec{r}_i\, m_i.$$

By taking the time derivatives of the equation above, we get

$$\vec{v}_{cm} m_{\text{tot}} = \sum \vec{v}_i\, m_i \quad \text{and}$$
$$\vec{a}_{cm} m_{\text{tot}} = \sum \vec{a}_i\, \text{m}_i \quad .$$

for the velocity and acceleration of the center-of-mass. The results above are useful for simplifying various momenta and energy expressions. Note, for example, that

$$\vec{L} = \sum \vec{v}_i\ \text{m}_i = \vec{r}_{cm} m_{\text{tot}}$$
$$\dot{\vec{L}} = \sum \vec{a}_i\ \text{m}_i = \vec{a}_{cm} m_{\text{tot}}.$$

To get the velocity and acceleration of the center-of-mass, we differentiate the position of the center-of-mass once and twice, respectively, to get*

$$\vec{v}_{cm}(t) = \dot{\vec{r}}_{cm}(t) = \left[\frac{24}{5}\left(\frac{t}{s^2}\right)\hat{\imath} + \frac{3}{s}\hat{\jmath}\right] \mathrm{m} = \left[\frac{24}{5}\left(\frac{t}{s}\right)\hat{\imath} + 3\hat{\jmath}\right] \mathrm{m/s}$$

and

$$\vec{a}_{cm}(t) = \dot{\vec{v}}_{cm}(t) = \ddot{\vec{r}}_{cm}(t) = \left[\frac{24}{5}\left(\frac{1}{s^2}\right)\hat{\imath}\right] \mathrm{m} = \left(\frac{24}{5}\right) \mathrm{m/s^2}\hat{\imath}.$$

In this example, the center-of-mass turns out to have constant acceleration in the x-direction.

*That is, particle A travels on the line $x = 3\,\mathrm{m}$ with constant speed $\dot{r}_{A_y} = 5\,\mathrm{m/s}$ and particle B travels on the line $y = -4\,\mathrm{m}$ at changing speed $\dot{r}_{B_x} = 12t\,(\mathrm{m/s^2})$.

The second part of equation 11.4 follows from the definition of the center-of-mass (see box 11.1 on page 527).* The total linear momentum of a system is the same as that of a particle that is located at the center-of-mass and which has mass equal to that of the whole system. The linear momentum is also given by

$$\vec{L} = \frac{d}{dt}\left(m_{\mathrm{tot}}\vec{r}_{cm}\right).$$

*Some books use the symbol $\vec{P}$ for linear momentum. Because $\vec{P}$ is often used to mean force or impulse and P for power we use $\vec{L}$ for linear momentum.

We only consider systems of fixed mass, $\frac{d}{dt}(m_{\mathrm{tot}}) = 0$. Thus, for a fixed mass system, the linear momentum of the system is equal to the total mass of the system times the derivative of the center-of-mass position.

Finally, since the sum defining linear momentum can be grouped any which way (the associative rule of addition) the linear momentum can be found by dividing the system into parts and using the mass of those parts and the center-of-mass motion of those parts. That is, the sum $\sum m_i \vec{v}_i$ can be interpreted as the sum over the center-of-mass velocities and masses of the various subsystems, say the parts of a machine.

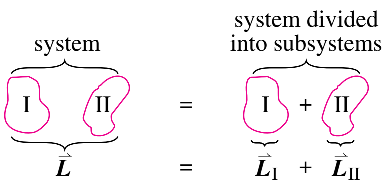

Figure 11.2: System composed of two parts. The momentum of the whole is the sum of the momentums of the two parts.

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Example: **System Momentum**

See figure 11.2 for a schematic example of the total momentum of system being made of the sum of the momenta of its two parts.

The reasoning for this allowed subdivision is similar to that for the center-of-mass in box 2.11 on page 102.

The quantity $\dot{\vec{L}}$ figures a little more directly in our presentation of dynamics than just plain $\vec{L}$ *The rate of change of linear momentum, $\dot{\vec{L}}$, is

*No slight of Sir Isaac is intended.

$$\begin{aligned} \dot{\vec{L}} &= \frac{d}{dt}\vec{L} \\ &= \frac{d}{dt}\sum m_i \vec{v}_i \\ &= m_{\mathrm{tot}}\frac{d\vec{v}_{\mathrm{cm}}}{dt} \\ \dot{\vec{L}} &= m_{\mathrm{tot}}\vec{a}_{\mathrm{cm}} \end{aligned}$$

The last three equations could be thought of as the *definition* of $\dot{\vec{L}}$. That $\dot{\vec{L}}$ turns out to be $\frac{d}{dt}(\vec{L})$ is, then, a derived result. Again, using the definition of center-of-mass, the total rate of change of linear momentum is the same as that of a particle that is located at the center of mass which has mass equal to that of the whole system.

The rate of change of linear momentum is also given by

$$\dot{\vec{L}} = \frac{d}{dt}(m_{\text{tot}}\vec{v}_{\text{cm}}) = \frac{d^2}{dt^2}(m_{\text{tot}}\vec{r}_{\text{cm}}).$$

The momentum $\vec{L}$ and its rate of change $\dot{\vec{L}}$ can be expressed in terms of the total mass of a system and the motion of the center-of-mass. This simplification holds for any system, however complex, and any motion, however contorted and wild.

Angular momentum $\vec{H}$ and its rate of change $\dot{\vec{H}}$

After linear momentum balance, the second basic mechanics principle is angular momentum balance:

$$\sum \vec{M}_{\text{C}} = \dot{\vec{H}}_{\text{C}},$$

where C is any point, preferably one that is fixed in a Newtonian frame.If you choose your point C to be a moving point you may have the confusing problem that the quantity we would like to call $\dot{\vec{H}}_{\text{C}}$ is not the time derivative of $\vec{H}_{\text{C}}$. The first quantity of interest in this sub-section is the angular momentum with respect to some point C, $\vec{H}_{\text{C}}$, whose rate of change $\dot{\vec{H}}_{\text{C}} = d\vec{H}_{\text{C}}/dt$ is so important.

$$\underbrace{\vec{H}_{\text{C}}}_{\text{angular momentum.}} \equiv \underbrace{\sum \vec{r}_{i/C} \times m_i \vec{v}_i}_{\textit{summed over all the mass particles}}$$

A useful theorem about angular momentum is the following (see box 11.2 on page 530), applicable to all systems

$$\vec{H}_{\text{C}} = \overbrace{\vec{r}_{cm/C} \times \vec{v}_{cm} m_{\text{tot}}}^{\text{angular momentum due to center-of-mass motion}} + \overbrace{\sum \underbrace{\vec{r}_{i/cm}}_{\substack{\text{position of } m_i \text{ relative to the center-of-mass} \\ \vec{r}_{i/cm} \equiv \vec{r}_i - \vec{r}_{cm}}} \times \underbrace{\vec{v}_{i/cm}}_{\substack{\text{velocity of } m_i \text{ relative to the center-of-mass} \\ \vec{v}_{i/cm} \equiv \vec{v}_i - \vec{v}_{cm}}} m_i}^{\text{angular momentum relative to the center-of-mass}}. \tag{11.5}$$

A system of particles is shown in figure 11.3. The angular momentum

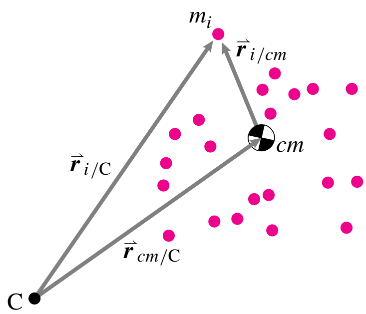

Figure 11.3: A system of particles showing its center-of-mass and the i_{th} particle of mass m_i. The i_{th} particle has position $\vec{r}_{i/cm}$ with respect to the center-of-mass. The center-of-mass has position $\vec{r}_{cm/C}$ with respect to the point C

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of any system is the same as that of a particle at its center-of-mass *plus* the angular momentum associated with motion relative to the center-of-mass.

The angular momentum about point C is a measure of the average rotation rate of the system about point C. Angular momentum is not so intuitive as linear momentum for a number of reasons:

- First, recall that linear momentum is the derivative of the total mass times the center-of-mass position. Unfortunately, in general,
  angular momentum is not the derivative of anything.
- Second, the angular momentum of a given system at a given time depends on the reference point C. So there is not one single quantity that is *the* angular momentum. For different points C_1, C_2, etc., the same system has different angular momentums.
- Finally, calculation of angular momentum involves a vector cross product and many beginning dynamics students are intimidated by vector cross products.

Despite these confusions, the concept of angular momentum allows the solution of many practical problems and eventually becomes somewhat intuitive.

11.2 *THEORY*

Simplifying $\vec{H}_C$ using the center of mass

The definition of angular momentum relative to a point C is

$$\vec{H}_C = \sum \vec{r}_{i/C} \times m_i \vec{v}_i.$$

If we rewrite $\vec{v}_i$ as

$$\vec{v}_i = (\vec{v}_i - \vec{v}_{cm}) + \vec{v}_{cm} = \vec{v}_{i/cm} + \vec{v}_{cm}$$

and

$$\vec{r}_i = (\vec{r}_i - \vec{r}_{cm}) + \vec{r}_{cm} = \vec{r}_{i/cm} + \vec{r}_{cm}$$

then

$$\begin{aligned} \vec{H}_C &= \sum \left(\vec{r}_{cm} + \vec{r}_{i/cm}\right) \times \left[\vec{v}_{cm} + \vec{v}_{i/cm}\right] m_i. \\ &= \sum \vec{r}_{cm} \times \vec{v}_{cm} m_i + \sum \vec{r}_{i/cm} \times \vec{v}_{i/cm} m_i \\ &\quad + \sum \vec{r}_{cm} \times \vec{v}_{i/cm} m_i + \sum \vec{r}_{i/cm} \times \vec{v}_{cm} m_i \\ &= \vec{r}_{cm} \times \vec{v}_{cm} m_{tot} + \sum \vec{r}_{i/cm} \times \vec{v}_{i/cm}\, m_i \\ &\quad + \vec{r}_{cm} \times \underbrace{\left[\sum \vec{v}_{i/cm}\, m_i\right]}_{\vec{0}} + \underbrace{\left[\sum \vec{r}_{i/cm}\, m_i\right]}_{\vec{0}} \times \vec{v}_{cm} \end{aligned}$$

So,

$$\vec{H}_C = \underbrace{\vec{r}_{cm} \times \vec{v}_{cm} m_{tot}}_{\text{contribution of center of mass motion}} + \underbrace{\sum \vec{r}_{i/cm} \times \vec{v}_{i/cm}\, m_i}_{\text{contribution of motion relative to center-of-mass}}.$$

The reason $\sum \vec{r}_{i/cm}\, m_i = \vec{0}$ is somewhat intuitive. It is what you would calculate if you were looking for the center-of-mass relative to the center of mass. More formally,

$$\begin{aligned} \sum \vec{r}_{i/cm}\, m_i &= \sum (\vec{r}_i - \vec{r}_{cm})\, m_i \\ &= \underbrace{\sum \vec{r}_i\, m_i}_{m_{tot} \vec{r}_{cm}} - m_{tot} \vec{r}_{cm} \\ &= \vec{0}. \end{aligned}$$

Similarly, $\sum \vec{v}_{i/cm}\, m_i = \vec{0}$ because it is what you would calculate if you were looking for the velocity of the center-of-mass relative to the center of mass.

The central result of this box is that

angular momentum of any system is that due to motion *of* the center-of-mass *plus* motion *relative to* the center-of-mass.

Actually, it is $\dot{\vec{\boldsymbol{H}}}_\mathrm{C}$ which is the more fundamental quantity. $\dot{\vec{\boldsymbol{H}}}_\mathrm{C}$ is what you use in the equation of motion. You can find $\dot{\vec{\boldsymbol{H}}}_\mathrm{C}$ from $\vec{\boldsymbol{H}}_\mathrm{C}$ as shown in the box on page 532. But, in general,

$$\dot{\vec{\boldsymbol{H}}}_\mathrm{C} \equiv \sum \vec{\boldsymbol{r}}_{i/\mathrm{C}} \times (\mathrm{m}_i \vec{\boldsymbol{a}}_i).$$

A useful theorem about rate of change of angular momentum is the following (see box 11.2 on page 530), applicable to all systems:

rate of change of angular momentum due to center of mass motion

rate of change of angular momentum relative to the center of mass

$$\dot{\vec{\boldsymbol{H}}}_\mathrm{C} = \overbrace{\vec{\boldsymbol{r}}_{cm/C} \times \vec{\boldsymbol{a}}_{cm} m_\mathrm{tot}} + \overbrace{\sum \underbrace{\vec{\boldsymbol{r}}_{i/cm}} \times \underbrace{\vec{\boldsymbol{a}}_{i/cm}} m_i} .$$

$\vec{\boldsymbol{r}}_{i/cm} \equiv \vec{\boldsymbol{r}}_i - \vec{\boldsymbol{r}}_{cm}$

$\vec{\boldsymbol{a}}_{i/cm} \equiv \vec{\boldsymbol{a}}_i - \vec{\boldsymbol{a}}_{cm}$

This expression is completely analogous to equation 11.5 on page 529 and is derived in a manner nearly identical to that shown in box 11.2 on page 530. The rate of change of angular momentum of any system is the same as that of a particle at its center-of-mass *plus* the rate of change of angular momentum associated with motion relative to the center-of-mass. A special point for any system is, as we have mentioned, the center-of-mass. In the above equations for angular momentum we could take C to be a fixed point in space that happens to coincide with the center-of-mass. In this case we would most naturally define $\vec{\boldsymbol{H}}_\mathrm{cm} = \int \vec{\boldsymbol{r}}_{/cm} \times \vec{\boldsymbol{v}}\, dm$ with $\vec{\boldsymbol{v}}$ being the absolute velocity. But we have the following theorem:

$$\vec{\boldsymbol{H}}_\mathrm{cm} = \int \vec{\boldsymbol{r}}_{/cm} \times \vec{\boldsymbol{v}}\, dm = \int \vec{\boldsymbol{r}}_{/cm} \times \vec{\boldsymbol{v}}_{/cm}\, dm$$

where $\vec{\boldsymbol{r}}_{/cm} = \vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_{cm}$ and $\vec{\boldsymbol{v}}_{/cm} = \vec{\boldsymbol{v}} - \vec{\boldsymbol{v}}_{cm}$. Similarly,

$$\dot{\vec{\boldsymbol{H}}}_\mathrm{cm} = \int \vec{\boldsymbol{r}}_{/cm} \times \vec{\boldsymbol{a}}\, dm = \int \vec{\boldsymbol{r}}_{/cm} \times \vec{\boldsymbol{a}}_{/cm}\, dm.$$

with $\vec{\boldsymbol{a}}_{/cm} = \vec{\boldsymbol{a}} - \vec{\boldsymbol{a}}_{cm}$. That is,

> *the angular momentum and rate of change of angular momentum relative to the center-of-mass, defined in terms of the velocity and acceleration* **relative** *to the center-of-mass, are the same as the angular momentum and the rate of change of angular momentum defined in terms of a fixed point in space that coincides with the center-of-mass.*

The angular momentum relative to the center-of-mass $\vec{\boldsymbol{H}}_\mathrm{cm}$ can be calculated with all positions and velocities calculated relative to the center-of-mass. Similarly, the rate of change of angular momentum relative to the center of mass $\dot{\vec{\boldsymbol{H}}}_{\mathrm{cm}}$ can be calculated with all positions and *accelerations* calculated relative to the center-of-mass.

Combining the results above we get the often used result:

$$\sum \vec{\boldsymbol{M}}_{i/cm} = \dot{\vec{\boldsymbol{H}}}_{\mathrm{cm}} \qquad (11.6)$$

This formula is the version of angular momentum balance that many people think of as being basic. In this equation, $\dot{\vec{\boldsymbol{H}}}_{\mathrm{cm}}$ can be found using either the absolute acceleration $\vec{\boldsymbol{a}}$ or the acceleration relative to the center-of-mass, $\vec{\boldsymbol{a}}_{/cm}$. The same $\dot{\vec{\boldsymbol{H}}}_{\mathrm{cm}}$ is found both ways. In this book, we do not give equation 11.6 quite such central status as equations III where the reference point can be any point C not just the center-of-mass.

Kinetic energy E_{K}

The equation of mechanical energy balance (III) is:

$$P = \dot{E}_{\mathrm{K}} + \dot{E}_{\mathrm{P}} + \dot{E}_{int}.$$

For discrete systems, the kinetic energy is calculated as

$$\frac{1}{2}\sum m_i\, v_i^2$$

11.3 Relation between $\frac{d}{dt}\vec{\boldsymbol{H}}_{\mathrm{C}}$ and $\vec{\boldsymbol{H}}_{\mathrm{C}}$

The expression for $\dot{\vec{\boldsymbol{H}}}_{\mathrm{C}}$ follows from that for $\vec{\boldsymbol{H}}_{\mathrm{C}}$ but requires a few steps of algebra to show. Like the rate of change of linear momentum, $\dot{\vec{\boldsymbol{L}}}$, the derivative of $\vec{\boldsymbol{L}}$, the derivative of angular momentum must be taken with respect to a Newtonian frame in order to be useful in momentum balance equations. Note that since we assumed that C is a point fixed in a Newtonian frame that $\frac{d}{dt}\vec{\boldsymbol{r}}_{i/\mathrm{C}} = \vec{\boldsymbol{v}}_{i/\mathrm{C}} = \vec{\boldsymbol{v}}_i$. Starting with the definition of $\dot{\vec{\boldsymbol{H}}}_{\mathrm{C}}$, we can calculate as follows:

$$\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{\mathrm{C}} &= \frac{d}{dt}\vec{\boldsymbol{H}}_{\mathrm{C}} \\
&= \frac{d}{dt}\sum \vec{\boldsymbol{r}}_{i/C} \times (m_i\vec{\boldsymbol{v}}_i) \\
&= \sum \frac{d}{dt}\vec{\boldsymbol{r}}_{i/C} \times (m_i\vec{\boldsymbol{v}}_i) + \vec{\boldsymbol{r}}_{i/C} \times (m_i\frac{d}{dt}\vec{\boldsymbol{v}}_i) \\
&= \sum \overbrace{\underbrace{\vec{\boldsymbol{v}}_i}_{\frac{d}{dt}\vec{\boldsymbol{r}}_{i/C}} \times (m_i\vec{\boldsymbol{v}}_i)}^{\vec{\boldsymbol{0}}} + \vec{\boldsymbol{r}}_{i/C} \times (m_i\frac{d}{dt}\vec{\boldsymbol{v}}_i)
\end{aligned}$$

$$\dot{\vec{\boldsymbol{H}}}_{\mathrm{C}} = \sum \vec{\boldsymbol{r}}_{i/C} \times (m_i\vec{\boldsymbol{a}}_i),$$

We have used the fact that the product rule of differentiation works for cross products between vector-valued functions of time. This final formula, $\dot{\vec{\boldsymbol{H}}}_{\mathrm{C}} = \sum \vec{\boldsymbol{r}}_{i/C} \times (m_i\vec{\boldsymbol{a}}_i)$, or its integral form, $\dot{\vec{\boldsymbol{H}}}_{\mathrm{C}} = \int \vec{\boldsymbol{r}}_{i/C} \times \vec{\boldsymbol{a}}_i\, dm$ are always applicable. They can be simplified in many special cases which we will discuss in this chapter and those that follow.

and its rate of change as

$$\frac{d}{dt}\left[\frac{1}{2}\sum m_i\ v_i^2\right].$$

There is also a general result about the kinetic energy that takes advantage of the center-of-mass. The kinetic energy for any system in any motion can be decomposed into the sum of two terms. One is associated with the motion of the center-of-mass of the system and the other is associated with motion relative to the center-of-mass. Namely,

$$\begin{aligned} E_{\mathrm{K}} &= \underbrace{\frac{1}{2}m_{\mathrm{tot}}v_{cm}^2}_{\text{kinetic energy due to center-of-mass motion}} + \underbrace{\frac{1}{2}\sum m_i v_{i/cm}^2}_{\text{kinetic energy relative to the center-of-mass}}, \\ &= \frac{1}{2}m_{tot}v_{cm}^2 + E_{\mathrm{K}/cm} \end{aligned}$$

where

$$\begin{aligned} E_{\mathrm{K}/cm} &= \frac{1}{2}\sum m_i v_{i/cm}^2 && \text{for discrete systems, and} \\ &= \frac{1}{2}\int (v_{/cm})^2\, dm && \text{for continuous systems.} \end{aligned}$$

The results above can be verified by direct expansion of the basic definitions of E_{K} and the center-of-mass. To repeat,

the kinetic energy of a system is the same as the kinetic energy of a particle with the system's mass at the center-of-mass plus kinetic energy due to motion relative to the center-of-mass.

In this chapter, all particles in the system are assumed to have the same velocity so that they all have the same velocity as the center-of-mass. Thus,

11.4 Using $\vec{H}_{\mathrm{O}}$ and $\dot{\vec{H}}_{\mathrm{O}}$ to find $\vec{H}_{\mathrm{C}}$ and $\dot{\vec{H}}_{\mathrm{C}}$

You can find the angular momentum $\vec{H}_{\mathrm{C}}$ relative to a fixed point C if you know the angular momentum $\vec{H}_{\mathrm{O}}$ relative to some other fixed point O and also know the linear momentum of the system $\vec{L}$ (which does not depend on the reference point). The result is:

$$\vec{H}_{\mathrm{C}} = \vec{H}_{\mathrm{O}} + \vec{r}_{O/C} \times \vec{L}.$$

The formula is similar to the formula for the effective moment of a system of forces that you learned in statics: $\vec{M}_{\mathrm{C}} = \vec{M}_{\mathrm{O}} + \vec{r}_{O/C} \times \vec{F}_{tot}$. Similarly, for the rate of change of angular momentum we have:

$$\dot{\vec{H}}_{\mathrm{C}} = \dot{\vec{H}}_{\mathrm{O}} + \vec{r}_{O/C} \times \dot{\vec{L}}$$

So once you have found $\dot{\vec{L}}$ and also $\dot{\vec{H}}_{\mathrm{O}}$ with respect to some point O you can easily calculate the right hand sides of the momentum balance equations using any point C that you like.

$\vec{\boldsymbol{v}}_{i/cm} = \vec{\mathbf{0}}$ for all particles, and for straight line motion,

$$E_{\mathrm{K}} = \frac{1}{2} m_{\mathrm{tot}} v_{cm}^2.$$

Summary on general results about $\vec{L}$, $\dot{\vec{L}}$, $\vec{H}_{\mathrm{C}}$, $\dot{\vec{H}}_{\mathrm{C}}$, E_{K}, and center-of-mass

$$m_{\mathrm{tot}} \vec{\boldsymbol{r}}_{cm} = \sum \vec{\boldsymbol{r}}_i \; m_i \qquad \text{for all systems}$$

$$m_{\mathrm{tot}} \vec{\boldsymbol{v}}_{cm} = \sum \vec{\boldsymbol{v}}_i \; m_i \qquad \text{for all systems}$$

$$m_{\mathrm{tot}} \vec{\boldsymbol{a}}_{cm} = \sum \vec{\boldsymbol{a}}_i \; m_i \qquad \text{for all systems}$$

$$\vec{\boldsymbol{L}} = \sum m_i \; \vec{\boldsymbol{v}}_i = m_{tot} \vec{\boldsymbol{v}}_{cm} \qquad \text{for all systems}$$

$$\dot{\vec{\boldsymbol{L}}} = \sum m_i \; \vec{\boldsymbol{a}}_i = m_{tot} \vec{\boldsymbol{a}}_{cm} \qquad \text{for all systems}$$

$$\begin{aligned}
\vec{\boldsymbol{H}}_{\mathrm{C}} &= \sum \vec{\boldsymbol{r}}_{i/C} \times (m_i \; \vec{\boldsymbol{v}}_i) && \text{for all systems} \\
&= \vec{\boldsymbol{r}}_{cm/C} \times m_{\mathrm{tot}} \vec{\boldsymbol{v}}_{cm} + \sum \vec{\boldsymbol{r}}_{i/cm} \times (\vec{\boldsymbol{v}}_{i/cm} m_i) && \text{for all systems} \\
&= \vec{\boldsymbol{H}}_{\mathrm{O}} + \vec{\boldsymbol{r}}_{O/C} \times \vec{\boldsymbol{L}} && \text{for all systems} \\
\dot{\vec{\boldsymbol{H}}}_{\mathrm{C}} &= \sum \vec{\boldsymbol{r}}_{i/C} \times (m_i \; \vec{\boldsymbol{a}}_i) && \text{for all systems} \\
&= \vec{\boldsymbol{r}}_{cm/C} \times m_{\mathrm{tot}} \vec{\boldsymbol{a}}_{cm} + \sum \vec{\boldsymbol{r}}_{i/cm} \times (\vec{\boldsymbol{a}}_{i/cm} m_i) && \text{for all systems} \\
&= \dot{\vec{\boldsymbol{H}}}_{\mathrm{O}} + \vec{\boldsymbol{r}}_{O/C} \times \dot{\vec{\boldsymbol{L}}} && \text{for all systems} \\
E_{\mathrm{K}} &= \frac{1}{2} \sum m_i \; v_i^2 && \text{for all systems} \\
&= \frac{1}{2} m_{tot} \; v_{cm}^2 + \frac{1}{2} \sum m_i \; v_{i/cm}^2 && \text{for all systems} \\
\dot{E}_{\mathrm{K}} &= \sum \; m_i \; v_i \; \dot{v}_i && \text{for all systems} \\
&= m_{tot} \; v_{cm} \; \dot{v}_{cm} + \sum m_i \; v_{i/cm} \; \dot{v}_{i/cm} && \text{for all systems}
\end{aligned}$$

SAMPLE 11.1 Location of the center-of-mass. A structure is made up of three point masses, $m_1 = 1\,\text{kg}$, $m_2 = 2\,\text{kg}$ and $m_3 = 3\,\text{kg}$. At the moment of interest, the coordinates of the three masses are (1.25 m, 3 m), (2 m, 2 m), and (0.75 m, 0.5 m), respectively. At the same instant, the velocities of the three masses are $2\,\text{m/s}\hat{\boldsymbol{\imath}}$, $2\,\text{m/s}(\hat{\boldsymbol{\imath}} - 1.5\hat{\boldsymbol{\jmath}})$ and $1\,\text{m/s}\hat{\boldsymbol{\jmath}}$, respectively.

1. Find the coordinates of the center-of-mass of the structure.
2. Find the velocity of the center-of-mass.

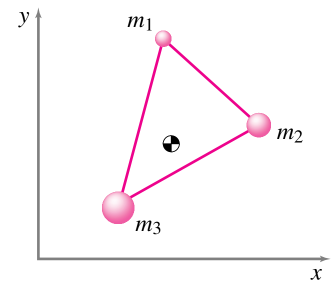


Figure 11.4:

Filename:sfig2-4-2

Solution

1. Let $(\bar{x}, \bar{y})$ be the coordinates of the mass-center. Then from the definition of mass-center

$$
\begin{aligned}
\bar{x} &= \frac{\sum m_i x_i}{\sum m_i} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} \\
&= \frac{1\,\text{kg}\cdot 1.25\,\text{m} + 2\,\text{kg}\cdot 2\,\text{m} + 3\,\text{kg}\cdot 0.75\,\text{m}}{1\,\text{kg} + 2\,\text{kg} + 3\,\text{kg}} \\
&= \frac{7.25\,\cancel{\text{kg}}\cdot\text{m}}{6\,\cancel{\text{kg}}} = 1.25\,\text{m}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\bar{y} &= \frac{\sum m_i y_i}{\sum m_i} \\
&= \frac{1\,\text{kg}\cdot 3\,\text{m} + 2\,\text{kg}\cdot 2\,\text{m} + 3\,\text{kg}\cdot 0.5\,\text{m}}{1\,\text{kg} + 2\,\text{kg} + 3\,\text{kg}} \\
&= \frac{8.55\,\cancel{\text{kg}}\cdot\text{m}}{6\,\cancel{\text{kg}}} = 1.42\,\text{m}.
\end{aligned}
$$

Thus the center-of-mass is located at the coordinates (1.25 m, 1.42 m).

$$\boxed{(1.25\,\text{m},\ 1.42\,\text{m})}$$

2. For a system of particles, the linear momentum

$$
\begin{aligned}
\vec{\boldsymbol{L}} &= \sum m_i \vec{\boldsymbol{v}}_i = m_{\text{tot}} \vec{\boldsymbol{v}}_{cm} \\
\Rightarrow \quad \vec{\boldsymbol{v}}_{cm} &= \frac{\sum m_i \vec{\boldsymbol{v}}_i}{m_{\text{tot}}} \\
&= \frac{1\,\text{kg}\cdot(2\,\text{m/s}\hat{\boldsymbol{\imath}}) + 2\,\text{kg}\cdot(2\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}})\,\text{m/s} + 3\,\text{kg}\cdot(1\,\text{m/s}\hat{\boldsymbol{\jmath}})}{6\,\text{kg}} \\
&= \frac{(6\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}})\,\cancel{\text{kg}}\cdot\text{m/s}}{6\,\cancel{\text{kg}}} \\
&= 1\,\text{m/s}\hat{\boldsymbol{\imath}} + 0.5\,\text{m/s}\hat{\boldsymbol{\jmath}}.
\end{aligned}
$$

$$\boxed{\vec{\boldsymbol{v}}_{cm} = 1\,\text{m/s}\hat{\boldsymbol{\imath}} + 0.5\,\text{m/s}\hat{\boldsymbol{\jmath}}}$$

11.5 *THEORY*
Deriving system momentum balance from the particle equations.

In the front cover you see that we have linear and angular momentum balance equations that apply to arbitrary systems. Another approach to mechanics is to use the equation

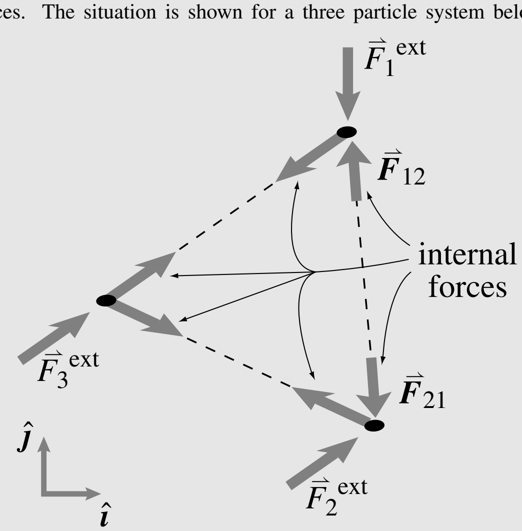
$$\vec{F} = m\vec{a}$$

for every particle in the system and then *derive* the system linear and angular momentum balance equations. This derivation depends on the following assumptions

1. All bodies and systems are composed of point masses.

2. These point masses interact in a pair-wise manner. For every pair of point masses A and B the interaction force is equal and opposite and along the line connecting the point masses.

We then look at any system, which we now assume is a system of point masses, and apply $\vec{F} = m\vec{a}$ to every point mass and add the equations for all point masses in the system. For each point mass we can break the total force into two parts: 1) the interaction forces between the point mass and other point masses in the system, these forces are 'internal' forces ($\vec{F}^{int}$), and 2) the forces acting on the system from the outside, the 'external' forces. The situation is shown for a three particle system below.



System linear momentum balance

Now lets take the equation $\sum \vec{F} = m\vec{a}$ for each particle and add over all the particles.

$$\underbrace{\sum_{\text{all particles}} \left[\sum_{\text{each particle}} \vec{F} \right]}_{\text{The sum of all forces on the system, internal and external}} = \sum_{\text{all particles}} m_i \vec{a}_i$$

Since all the internal forces come in cancelling pairs we can rewrite this equation as:

$$\underbrace{\sum_{\text{all external forces}} \vec{F}^{ext}}_{\text{Only the external forces, the ones acting on the system from the outside.}} = \sum_{\text{all particles}} m_i \vec{a}_i$$

That is, we have derived equation I in the front cover from $\vec{F} = m\vec{a}$ for a point mass by assuming the system is composed of point masses with pair-wise equal and opposite forces.

System angular momentum balance

For any particle we can take the equation

$$\sum_{\text{forces on particle i}} \vec{F} = m_i \vec{a}_i$$

and take the cross product of both sides with the position of the particle relative to some point C:

$$\vec{r}_{i/C} \times \left[\sum_{\text{forces on particle i}} \vec{F} \right] = \vec{r}_{i/C} \times \left[m_i \vec{a}_i \right].$$

Now we can add this equation up over all the particles to get

$$\underbrace{\sum_{particles} \left\{ \vec{r}_{i/C} \times \left[\sum_{on\ particle\ i} \vec{F} \right] \right\}}_{r/C \times \vec{F} \text{ added up for all forces on the system, internal and external}} = \sum_{particles} \left\{ \vec{r}_{i/C} \times \left[m_i \vec{a}_i \right] \right\}.$$

But, by our pair-wise assumption, for every internal force there is an equal and opposite force with the same line of action. So all the internal forces drop out of this sum and we have:

$$\underbrace{\sum_{\text{all external forces}} \vec{r}_{i/C} \times \vec{F}_i^{ext}}_{\text{Only the external forces, the ones acting on the system from the outside.}} = \sum_{\text{all particles}} \vec{r}_{i/C} \times m_i \vec{a}_i .$$

This equation is equation II, the system angular momentum balance equation (assuming we do not allow the application of any pure moments).

The derivations above are classic and are found in essentially all mechanics books. However, some people feel it is fine to take the system linear momentum balance and angular momentum balance equations as postulates and not make the subject of mechanics depend on the unrealistic view of so-simply interacting point masses.

11.6 A preview of rigid body simplifications and advanced kinematics

We have formulas for the motion quantities $\vec{L}$, $\dot{\vec{L}}$, $\vec{H}_C$, and $\dot{\vec{H}}_C$ and E_K in terms of the positions, velocities, and accelerations of all of the mass bits in a system. Most often in this book we deal with the mechanics of *rigid bodies*, objects with negligible deformation. This assumed simplification means that the relative motions of the 10^{23} or so atoms in a body are highly restricted. In fact, if one knows these five vectors:

- $\vec{r}_{cm}$, the position of the center-of-mass,
- $\vec{v}_{cm}$, the velocity of the center-of-mass,
- $\vec{a}_{cm}$, the acceleration of the center-of-mass
- $\vec{\omega}$, the *angular velocity* of the body, and
- $\vec{\alpha}$, the *angular acceleration* of the body,

then one can find the position, velocity, and acceleration of every point on the body in terms of its position relative to the center-of-mass, $\vec{r}_{/cm} = \vec{r} - \vec{r}_{cm}$.

We will save the derivations for later since we have not yet discussed the concepts of angular velocity $\vec{\omega}$ and angular acceleration $\vec{\alpha}$.

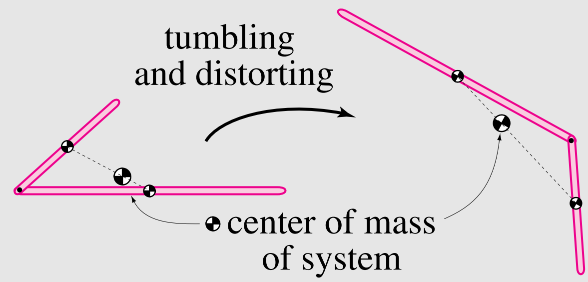
We will also use a new quantity $[I^{cm}]$, the *moment of inertia matrix*. For 2-D problems, $[I^{cm}]$ is just a number. For 3-D problems, $[I^{cm}]$ is a matrix; hence, the square brackets [ ], our notation for a matrix.

As intimidating as these new concepts may appear now, they lead to a *vast* simplification over the alternative — summing over 10^{23} particles or so.

Note that the formulas for linear momentum $\vec{L}$ and rate of change of linear momentum $\dot{\vec{L}}$ do not really look any simpler for a rigid body than the general case.

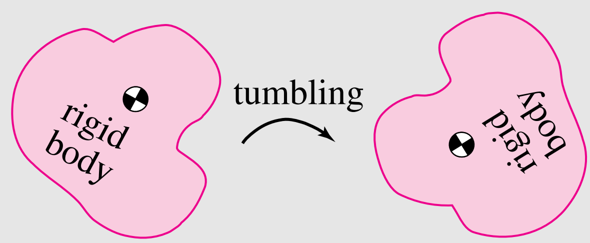
$$\vec{L} = m_{tot}\vec{v}_{cm}$$
$$\dot{\vec{L}} = m_{tot}\vec{a}_{cm}$$

But, they are actually simpler in the following sense. For a general system, when we write $\vec{v}_{cm}$, we are talking about an abstract point that moves in a different way than any point on the system. For example, consider the linked arms below, tumbling in space.

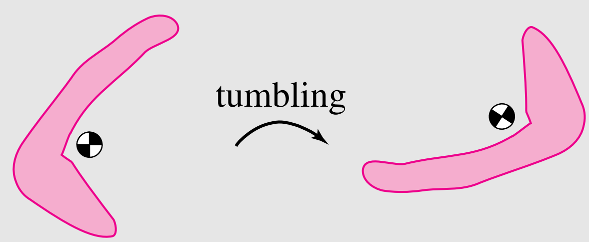


The center-of-mass is not even on any point in the system and, although it represents the average position in the system, it does not move with any point on the system.

On the other hand, for a *rigid body*, the center-of-mass is fixed relative to the body as the body moves,



even if the center-of-mass is not on the body, such as for this 'L-shaped' object.



In this case, the center-of-mass is not literally *on* the body. It is fixed with respect to the body, however. If you were rigidly attached to the body and fixed your gaze on the location of the center of mass, it would not waver in your view as the body, with you attached, tumbled wildly. In this sense the center-of-mass is fixed "on" a rigid body even if not on the body at all.

SAMPLE 11.2 A spring-mass system in space. A spring-mass system consists of two masses, $m_1 = 10\,\mathrm{kg}$ and $m_2 = 1\,\mathrm{kg}$, and a weak spring with stiffness $k = 1\,\mathrm{N/m}$. The spring has zero relaxed length. The system is in 3-D space where there is no gravity. At the moment f observaion, *i.e.*, at $t = 0$, $\vec{r}_1 = \vec{0}$, $\vec{r}_2 = 1\,\mathrm{m}(\hat{\imath} + \hat{\jmath} + \hat{k})$, $\dot{\vec{r}}_1 = \vec{0}$, and $\dot{\vec{r}}_2 = \sqrt{6}\,\mathrm{m/s}(-\hat{\imath} + \hat{\jmath})$. Track the motion of the system for the next 20 seconds. In particular,

1. Plot the trajectory of the two masses in space.
2. Plot the trajectory of the center-of-mass of the system.
3. Plot the trajectory of the two masses as seen by an observer sitting at the center-of-mass.
4. Compute and plot the total energy of the system and show that it remains constant during the entire motion.

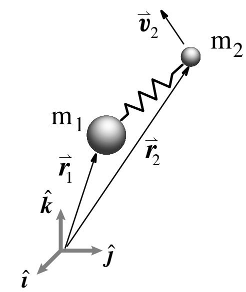


Figure 11.5:

Filename:sfig5-10-hopper

Solution The free-body diagrams of the two masses are shown in Fig. 11.6. The only force acting on each mass is the force due to the spring which is directed along the line joining the two masses. Thus, the system represents a central force problem. From the linear momentum balance of the two masses, we can write the equations of motion as follows.

$$
\begin{aligned}
m_1 \ddot{\vec{r}}_1 &= k(\vec{r}_2 - \vec{r}_1) \\
m_2 \ddot{\vec{r}}_2 &= -k(\vec{r}_2 - \vec{r}_1)
\end{aligned}
$$

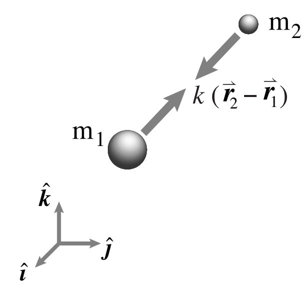


Figure 11.6: Free-body diagram of the two masses.

Filename:sfig5-10-hopper-a

Let $\vec{r}_1 = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$ and $\vec{r}_2 = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$. Substituting above and dotting the two equations with $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, we get

$$
\begin{aligned}
\ddot{x}_1 &= \frac{k}{m_1}(x_2 - x_1); & \ddot{x}_2 &= -\frac{k}{m_2}(x_2 - x_1) \\
\ddot{y}_1 &= \frac{k}{m_1}(y_2 - y_1); & \ddot{y}_2 &= -\frac{k}{m_2}(y_2 - y_1) \\
\ddot{z}_1 &= \frac{k}{m_1}(z_2 - z_1); & \ddot{z}_2 &= -\frac{k}{m_2}(z_2 - z_1)
\end{aligned}
$$

Thus we get six second order coupled linear ODEs as equations of motion.

1. To plot the trajectory of the two masses, we need to solve for $\vec{r}_1(t)$ and $\vec{r}_2(t)$, *i.e.*, for $x_1(t), y_1(t), z_1(t)$, and $x_2(t), y_2(t), z_2(t)$. We can do this by first writing the six second order equations as a set of 12 first order equations and then solving them using a numerical ODE solver. Here is a pseudocode to accomplish this task.

```
ODEs = {x1dot = u1,
        u1dot = k/m1*(x2-x1),
        y1dot = v1,
        v1dot = k/m1*(y2-y1),
        z1dot = w1,
        w1dot = k/m1*(z2-z1),
        x2dot = u2,
        u2dot = -k/m2*(x2-x1),
        y2dot = v2,
        v2dot = -k/m2*(y2-y1),
        z2dot = w2,
        w2dot = -k/m2*(z2-z1) }
IC   = {x1(0)=0, y1(0)=0, z1(0)=0,
        u1(0)=0, v1(0)=0, w1(0)=0,
        x2(0)=1, y2(0)=1, z2(0)=1,
        u2(0)=-sqrt(6), v2(0)=sqrt(6), w2(0)=0}
Set  k=1, m1=10, m2=1
Solve ODEs with IC for t=0 to t=20
Plot {x1,y1,z1} and {x2,y2,z2}
```

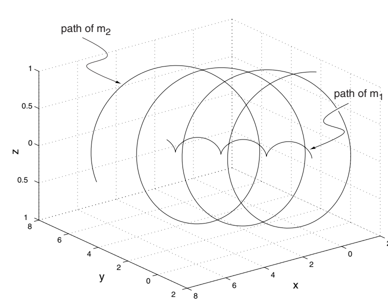


Figure 11.7: 3-D trajectory of m_1 and m_2 plotted from numerical solution of the equations of motion.

Filename:sfig5-10-hopper-b

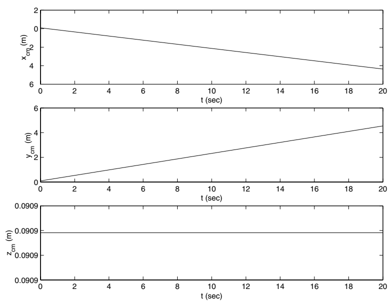


Figure 11.8: The center-of-mass coordinates $x_{\rm cm}(t)$, $y_{\rm cm}(t)$, and $z_{\rm cm}(t)$. The center-of-mass moves on a straight line in a plane parallel to the xy-plane.

Filename:sfig5-10-hopper-c

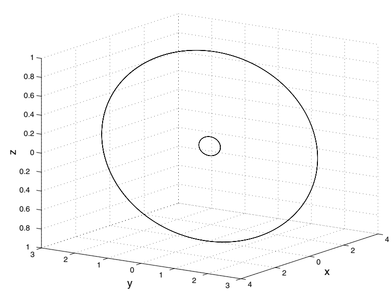


Figure 11.9: The paths of m_1 and m_2 as seen from the center-of-mass. The two masses are on closed orbits with respect to the center-of-mass.

Filename:sfig5-10-hopper-d

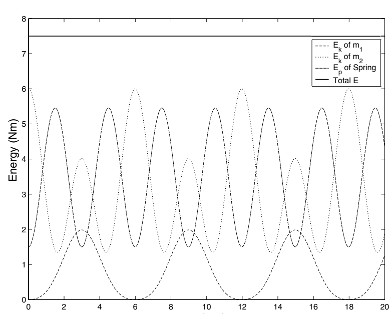


Figure 11.10: The kinetic energy of the two masses and the potential energy of the spring sum up to the constant total energy of the system.

Filename:sfig5-10-hopper-e

The 3-D plot showing the trajectory of the two masses obtained from the numerical solution is shown in Fig. 11.7. From the plot, it seems like the smaller mass goes around the bigger mass as the bigger mass moves on its trajectory.

2. We can find the trajectory of the center-of-mass using the following relationships.

$$x_{\rm cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, \quad y_{\rm cm} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}, \quad z_{\rm cm} = \frac{m_1 z_1 + m_2 z_2}{m_1 + m_2}.$$

Since there is no external force on the system if we consider the two masses and the spring together, the center-of-mass of the system has zero acceleration. Therefore, we expect the center-of-mass to move on a straight path with constant velocity. The center-of-mass coordinates $x_{\rm cm}$, $y_{\rm cm}$, and $z_{\rm cm}$ are plotted against time in Fig. 11.8 which show that the center-of-mass moves on a straight line in a plane parallel to the xy-plane (z is constant). This is expected since the initial velocity of the center of has no z-component:

$$\begin{aligned} \vec{\boldsymbol v}_{\rm cm} &= \frac{m_1 \vec{\boldsymbol v}_1 + m_2 \vec{\boldsymbol v}_2}{m_1 + m_2} \\ &= \frac{m_1 \cdot \vec{\boldsymbol 0} + 1\,\text{kg} \cdot \sqrt(6)\,\text{m/s}(-\hat{\boldsymbol\imath} + \hat{\boldsymbol\jmath})}{10\,\text{kg} + 1\,\text{kg}} \\ &= 0.22\,\text{m/s}(-\hat{\boldsymbol\imath} + \hat{\boldsymbol\jmath}). \end{aligned}$$

3. The trajectory of the two masses with respect to the center-of-mass can be easily obtained by the following relationships.

$$\begin{aligned} x_{1/cm} &= x_1 - x_{\rm cm}, & y_{1/cm} &= y_1 - y_{\rm cm}, & z_{1/cm} &= z_1 - z_{\rm cm} \\ x_{2/cm} &= x_2 - x_{\rm cm}, & y_{2/cm} &= y_2 - y_{\rm cm}, & z_{2/cm} &= z_2 - z_{\rm cm} \end{aligned}$$

The trajectories thus obtained are shown in Fig. 11.8. It is clear that the two masses have closed orbits with respect to the center-of-mass. These closed orbits are actually conic sections as we would expect in a central force problem.

4. We can calculate the kinetic energy of the two masses and the potential energy of the spring at each instant during the motion and add them up to find the total energy.

$$\begin{aligned} (E_k)_{m_1} &= \frac{1}{2} m_1 (u_1^2 + v_1^2 + w_1^2) \\ (E_k)_{m_2} &= \frac{1}{2} m_2 (u_2^2 + v_2^2 + w_2^2) \\ E_p &= \frac{1}{2} k[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2] \\ E_{\rm total} &= (E_k)_{m_1} + (E_k)_{m_2} + E_p \end{aligned}$$

The energies so calculated are plotted in Fig. 11.9. It is clear from the plot that the total energy remains constant during the entire motion.

□

Problems for Chapter 11

Coupled motions for particles in space

11.1 Coupled motions of particles in space

11.1 Linear momentum balance for general systems with multiple interacting parts moving more or less independently reduces to $\vec{F} = m\vec{a}$ *if you interpret the terms correctly.* What does this mean? What is $\vec{F}$? What is m? What is $\vec{a}$?

11.2 A particle of mass $m_1 = 6\,\text{kg}$ and a particle of mass $m_2 = 10\,\text{kg}$ are moving in the xy-plane. At a particular instant of interest, particle 1 has position, velocity, and acceleration $\vec{r}_1 = 3\,\text{m}\hat{\imath} + 2\,\text{m}\hat{\jmath}$, $\vec{v}_1 = -16\,\text{m/s}\hat{\imath} + 6\,\text{m/s}\hat{\jmath}$, and $\vec{a}_1 = 10\,\text{m/s}^2\hat{\imath} - 24\,\text{m/s}^2\hat{\jmath}$, respectively, and particle 2 has position, velocity, and acceleration $\vec{r}_2 = -6\,\text{m}\hat{\imath} - 4\,\text{m}\hat{\jmath}$, $\vec{v}_2 = 8\,\text{m/s}\hat{\imath} + 4\,\text{m/s}\hat{\jmath}$, and $\vec{a}_2 = 5\,\text{m/s}^2\hat{\imath} - 16\,\text{m/s}^2\hat{\jmath}$, respectively.

a) Find the linear momentum $\vec{L}$ and its rate of change $\dot{\vec{L}}$ of each particle at the instant of interest.
b) Find the linear momentum $\vec{L}$ and its rate of change $\dot{\vec{L}}$ of the system of the two particles at the instant of interest.
c) Find the center of mass of the system at the instant of interest.
d) Find the velocity and acceleration of the center of mass.

11.3 A particle of mass $m_1 = 5\,\text{kg}$ and a particle of mass $m_2 = 10\,\text{kg}$ are moving in space. At a particular instant of interest, particle 1 has position, velocity, and acceleration

$$\begin{aligned}\vec{r}_1 &= 1\,\text{m}\hat{\imath} + 1\,\text{m}\hat{\jmath}\\ \vec{v}_1 &= 2\,\text{m/s}\hat{\jmath}\\ \vec{a}_1 &= 3\,\text{m/s}^2\hat{k}\end{aligned}$$

respectively, and particle 2 has position, velocity, and acceleration

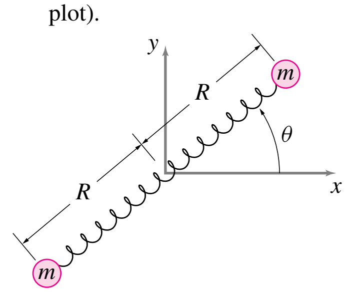
$$\begin{aligned}\vec{r}_2 &= 2\,\text{m}\hat{\imath}\\ \vec{v}_2 &= 1\,\text{m/s}\hat{k}\\ \vec{a}_2 &= 1\,\text{m/s}^2\hat{\jmath}\end{aligned}$$

respectively. For the system of particles at the instant of interest, find its

a) linear momentum $\vec{L}$,
b) rate of change of linear momentum $\dot{\vec{L}}$,
c) angular momentum about the origin $\vec{H}_O$,
d) rate of change of angular momentum about the origin $\dot{\vec{H}}_O$,
e) kinetic energy E_K, and
f) rate of change of kinetic energy.

11.4 Two particles each of mass m are connected by a massless elastic spring of spring constant k and unextended length $2R$. The system slides without friction on a horizontal table, so that no net external forces act.

a) Is the total linear momentum conserved? Justify your answer.
b) Can the center of mass accelerate? Justify your answer.
c) Draw free body diagrams for each mass.
d) Derive the equations of motion for each mass in terms of cartesian coordinates.
e) What are the total kinetic and potential energies of the system?
f) For constant values and initial conditions of your choosing plot the trajectories of the two particles and of the center of mass (on the same plot).



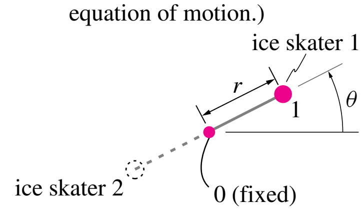
problem 11.4:

Filename:pfigure-blue-65-2

11.5 Two ice skaters whirl around one another. They are connected by a linear elastic cord whose center is stationary in space. We wish to consider the motion of one of the skaters by modeling her as a mass m held by a cord that exerts k Newtons for each meter it is extended from the central position.

a) Draw a free body diagram showing the forces that act on the mass is at an arbitrary position.
b) Write the differential equations that describe the motion.
c) Describe in physical and mathematical terms the nature of the motion for the three cases
   a) $\omega < \sqrt{k/m}$;
   b) $\omega = \sqrt{k/m}$;
   c) $\omega > \sqrt{k/m}$.

(You are not asked to solve the equation of motion.)



problem 11.5:

Filename:pfigure-blue-154-1

11.6 Theory question. If you are given the total mass, the position, the velocity, and the acceleration of the center of mass of a system of particles can you find the angular momentum $\vec{H}_O$ of the system, where O is not at the center of mass? If so, how and why? If not, then give a reason and/or a counter example.

11.7 The equation $(\vec{v}_1' - \vec{v}_2')\cdot\hat{n} = e(\vec{v}_2 - \vec{v}_1)\cdot\hat{n}$ relates relative velocities of two point masses before and after frictionless impact in the normal direction $\hat{n}$ of the impact. If $\vec{v}_1' = v_{1_x}\hat{\imath} + v_{1_y}\hat{\jmath}$, $\vec{v}_2' = -v_0\hat{\imath}$, $e = 0.5$, $\vec{v}_2 = \vec{0}$, $\vec{v}_1 = 2\,\text{ft/s}\hat{\imath} - 5\,\text{ft/s}\hat{\jmath}$, and $\hat{n} = \frac{1}{\sqrt{2}}(\hat{\imath} + \hat{\jmath})$, find the scalar equation relating the velocities in the normal direction.

11.8 Assuming θ, v_0, and e to be known quantities, write the following equations in matrix form set up to solve for v'_{Ax} and v'_{Ay}:

$$\begin{aligned}\sin\theta v'_{Ax} + \cos\theta v'_{Ay} &= e v_0 \cos\theta\\ \cos\theta v'_{Ax} - \sin\theta v'_{Ay} &= v_0 \sin\theta.\end{aligned}$$

11.9 Set up the following equations in matrix form and solve for v_A and v_B, if $v_0 = 2.6\,\text{m/s}$, $e = 0.8$, $m_A = 2\,\text{kg}$, and $m_B = 500\text{g}$:

$$\begin{aligned}m_A v_0 &= m_A v_A + m_B v_B\\ -e v_0 &= v_A - v_B.\end{aligned}$$

11.10 The following three equations are obtained by applying the principle of conservation of linear momentum on some system.

$$m_0 v_0 = 24.0\,\text{m/s}\, m_a - 0.67 m_B v_B - 0.58 m_C v_C$$
$$0 = 36.0\,\text{m/s}\, m_A + 0.33 m_B v_B + 0.3 m_C v_C$$
$$0 = 23.3\,\text{m/s}\, m_A - 0.67 m_B v_B - 0.58 m_C v_C.$$

Assume v_0, v_B, and v_C are the only unknowns. Write the equations in matrix form set up to solve for the unknowns.

11.11 See also problem 11.12. The following three equations are obtained to solve for v'_{Ax}, v'_{Ay}, and v'_{Bx}:

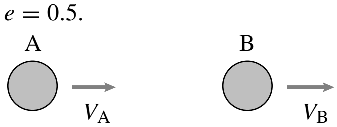
$$(v'_{Bx} - v'_{Ax})\cos\theta = v'_{Ay}\sin\theta - 10\,\text{m/s}$$
$$v'_{Ax}\sin\theta = v'_{Ay}\cos\theta - 36\,\text{m/s}$$
$$m_B v'_{Bx} + m_A v'_{Ax} = (-60\,\text{m/s})\, m_A.$$

Set up these equations in matrix form.

11.12 Solve for the unknowns v'_{A_x}, v'_{A_y}, and v'_{B_x} in problem 11.11 taking $\theta = 50^\circ$, $m_A = 1.5\,\text{m}_B$ and $m_B = 0.8\,\text{kg}$. Use any computer program.

11.13 Using the matrix form of equations in Problem 11.8, solve for v'_{Ax} and v'_{Ay} if $\theta = 20^\circ$ and $v_0 = 5\,\text{ft/s}$.

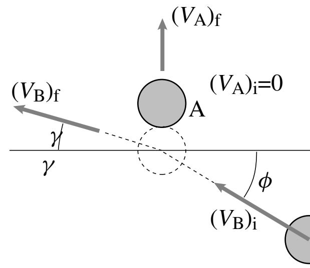
11.14 Two frictionless masses $m_A = 2\,\text{kg}$ and mass $m_B = 5\,\text{kg}$ travel on straight collinear paths with speeds $V_A = 5\,\text{m/s}$ and $V_B = 1\,\text{m/s}$, respectively. The masses collide since $V_A > V_B$. Find the amount of energy lost in the collision assuming normal motion is decoupled from tangential motion. The coefficient of restitution is $e = 0.5$.



problem 11.14:

Filename:Danef94s3q7

11.15 Two frictionless pucks sliding on a plane collide as shown in the figure. Puck A is initially at rest. Given that $(V_B)_i = 1.0\,\text{m/s}$, $(V_A)_i = 0$, and $(V_A)_f = 0.5\,\text{m/s}$, find the approach angle ϕ and rebound angle γ. The coefficient of restitution is $e = 0.9$.



problem 11.15:

Filename:Danef94s2q8

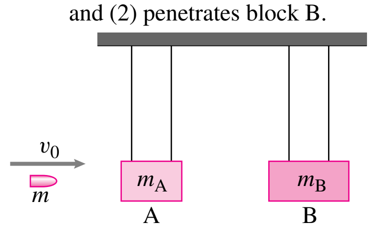
11.16 Reconsider problem 11.15. Given instead that $\gamma = 30°$, $(V_A)_i = 0$, and $(V_A)_f = 0.5\,\text{m/s}$, find the initial velocity of puck B.

11.17 A bullet of mass m with initial speed v_0 is fired in the horizontal direction through block A of mass m_A and becomes embedded in block B of mass m_B. Each block is suspended by thin wires. The bullet causes A and B to start moving with speed of v_A and v_B respectively. Determine

a) the initial speed v_0 of the bullet in terms of v_A and v_B,

b) the velocity of the bullet as it travels from block A to block B,and

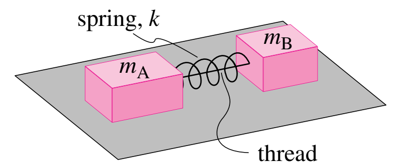
c) the energy loss due to friction as the bullet (1) moves through block A and (2) penetrates block B.



problem 11.17:

Filename:pfigure-blue-23-1

11.18 A massless spring with constant k is held compressed a distance δ from its relaxed length by a thread connecting blocks A and B which are still on a frictionless table. The blocks have mass m_A and m_B, respectively. The thread is suddenly but gently cut, the blocks fly apart and the spring falls to the ground. Find the speed of block A as it slides away.



problem 11.18:

Filename:pfigure-blue-32-2

11.19 Three equal masses, say $m = 1$, are attracted by an inverse-square gravity law with $G = 1$. That is, each mass is attracted to the other by $F = Gm_1m_2/r^2$ where r is the distance between them. Use these unusual and special initial positions:

$$(x1, y1) = (-0.97000436, 0.24308753)$$
$$(x2, y2) = (-x1, -y1)$$
$$(x3, y3) = (0, 0)$$

and initial velocities

$$(vx3, vy3) = (0.93240737, 0.86473146)$$
$$(vx1, vy1) = -(vx3, vy3)/2$$
$$(vx2, vy2) = -(vx3, vy3)/2.$$

Use computer integration to find and plot the motions of the particles.

Answers to *'d problems

2.55) $r_x = \vec{r} \cdot \hat{\imath} = (3\cos\theta + 1.5\sin\theta)$ ft, $r_y = \vec{r} \cdot \hat{\jmath} = (3\sin\theta - 1.5\cos\theta)$ ft.

2.77) No partial credit.

2.78) To get chicken road sin theta.

2.83) $\vec{N}\frac{1000\,\text{N}}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} + \hat{k})$.

2.86) $d = \sqrt{\frac{3}{2}}$.

2.90a) $\hat{\lambda}_{OB} = \frac{1}{\sqrt{50}}(4\hat{\imath} + 3\hat{\jmath} + 5\hat{k})$.

b) $\hat{\lambda}_{OA} = \frac{1}{\sqrt{34}}(3\hat{\jmath} + 5\hat{k})$.

c) $\vec{F}_1 = \frac{5\,\text{N}}{\sqrt{34}}(3\hat{\jmath} + 5\hat{k})$, $\vec{F}_2 = \frac{7\,\text{N}}{\sqrt{50}}(4\hat{\imath} + 3\hat{\jmath} + 5\hat{k})$.

d) $\angle AOB = 34.45$ deg .

e) $F_{1_x} = 0$

f) $\vec{r}_{DO} \times \vec{F}_1 = \left(\frac{100}{\sqrt{34}}\hat{\jmath} - \frac{60}{\sqrt{34}}\hat{k}\right)$ N·m.

g) $M_\lambda = \frac{140}{\sqrt{50}}$ N·m.

h) $M_\lambda = \frac{140}{\sqrt{50}}$ N·m.(same as (7))

2.92a) $\hat{n} = \frac{1}{3}(2\hat{\imath} + 2\hat{\jmath} + \hat{k})$.

b) $d = 1$.

c) $\frac{1}{3}(-2, 19, 11)$.

2.94) $\ell/\sqrt{2}$

2.110) Yes.

2.122a) $\vec{r}_2 = \vec{r}_1 + \vec{F}_1 \times \hat{k}M_1/|\vec{F}_1|^2$, $\vec{F}_2 = \vec{F}_1$.

b) $\vec{r}_2 = \vec{r}_1 + \vec{F}_1 \times \hat{k}M_1/|\vec{F}_1|^2 + c\vec{F}_1$ where c is any real number, $\vec{F}_2 = \vec{F}_1$.

c) $\vec{F}_2 = \vec{0}$ and $\vec{M}_2 = \vec{M}_1$ applied at any point in the plane.

2.123a) $\vec{r}_2 = \vec{r}_1 + \vec{F}_1 \times \vec{M}_1/|\vec{F}_1|^2$, $\vec{F}_2 = \vec{F}_1$, $\vec{M}_2 = \vec{M}_1 \cdot \vec{F}_1\vec{F}_1/|\vec{F}_1|^2$. If $\vec{F}_1 = \vec{0}$ then $\vec{F}_2 = \vec{0}$, $\vec{M}_2 = \vec{M}_1$, and $\vec{r}_2$ is any point at all in space.

b) $\vec{r}_2 = \vec{r}_1 + \vec{F}_1 \times \vec{M}_1 / |\vec{F}_1|^2 + c\vec{F}_1$ where c is any real number, $\vec{F}_2 = \vec{F}_1$, $\vec{M}_2 = \vec{M}_1 \cdot \vec{F}_1 \vec{F}_1 / |\vec{F}_1|^2$. See above for the special case of $\vec{F}_1 = \vec{0}$.

2.124) (0.5 m, −0.4 m)

3.1a) The forces and moments that show on a free body diagram, the *external* forces and moments.

b) The forces and moments that show on a free body diagram, the *external* forces and moments. No "inertial" or "acceleration" forces show.

3.2) You don't.

3.12) Note, no couples show on any of the free body diagrams requested.

4.5) $T_1 = Nmg$, $T_2 = (N-1)mg$, $T_N = (1)mg$, and in general $T_n = (N+1-n)mg$

4.23) (a)$T_{AB} = 30\,\mathrm{N}$, (b) $T_{AB} = \frac{300}{17}\,\mathrm{N}$, (c)$T_{AB} = \frac{5\sqrt{26}}{2}\,\mathrm{N}$

4.59) $\theta \geq \tan^{-1}\left((1-\mu^2)/2\mu\right)$

4.62) For this device to hold, $\mu \geq 1$. (Demanding $\mu \geq 1$ is large for a practical device because typical rock friction has $\mu \approx 0.5$. The too-large number follows from the simplified geometry and numbers chosen for a homework problem.)

4.66) $T_{AB} = \sqrt{10}\mu mg/(3+\mu)$

4.66) Minimum tension if rope slope is μ (instead of $1/3$)

4.68a) $\frac{m}{M} = \frac{R\sin\theta}{R\cos\theta + r} = \frac{2\sin\theta}{1+2\cos\theta}$.

b) $T = mg = 2Mg\frac{\sin\theta}{1+2\cos\theta}$.

c) $\vec{F}_C = Mg\left[-\frac{2\sin\theta}{2\cos\theta+1}\hat{\imath}' + \hat{\jmath}'\right]$ (where $\hat{\imath}'$ and $\hat{\jmath}'$ are aligned with the horizontal and vertical directions)

c) $\tan\phi = \frac{\sin\theta}{2+\cos\theta}$. Needs somewhat involved trigonometry, geometry, and algebra.

d) $\tan\psi = \frac{m}{M} = \frac{2\sin\theta}{1+2\cos\theta}$.

4.69a) $\frac{m}{M} = \frac{R\sin\theta}{R\cos\theta r} = \frac{2\sin\theta}{2\cos\theta - 1}$.

b) $T = mg = 2Mg\frac{\sin\theta}{2\cos\theta-1}$.

c) $\vec{F}_C = \frac{Mg}{1-2\cos\theta}\left[\sin\theta\hat{\imath} + (\cos\theta - 2)\hat{\jmath}\right]$.

4.70a) $\frac{F_1}{F_2} = \frac{R_o + R_i\sin\phi}{R_o - R_i\sin\phi}$

b) For $R_o = 3R_i$ and $\mu = 0.2$, $\frac{F_1}{F_2} \approx 1.14$.

4.75) None are true. The tension is 100 N.

4.90) Maximum overhang when $n \to \infty is\ell$.

4.93) Assuming no side-loads from floor the support from leg AB is 250 N, $T_{AB} = -250\,\mathrm{N}$.

4.94) $T_{IE} = mg/2, T_{CH} = \sqrt{2}mg/2, T_{BH} = -mg/2, A_x = mg/2, A_y = mg/2, A_z = mg$

4.97g) $T_{EH} = 0$ as you can find a number of ways.

4.98a) Use axis EC.

b) Use axis AH.

c) Use $\hat{j}$ axis through B.

d) Use axis DE.

e) Use axis EH.

f) Can't do in one shot.

4.99) $T_{AC} = -\sqrt{2}mg = -1000\sqrt{2}\,\mathrm{N} \approx -1410\,\mathrm{N}$ (the bar is in compression)

4.99) $T_{IP} = 0$

4.99) $T_{KL} = \sqrt{2}mg/6 = \left(1000\sqrt{2}/6\right) N \approx 408\,\mathrm{N}$ (the bar is in tension)

4.101) Hint: With reference to a free body diagram of the robot, use moment balance about axis BC.

5.9) $T_{AC} = -1000\,\mathrm{N}$, (AC is in compression)

5.10) $T_{AB} = 173\,\mathrm{N}$

5.13) 12 of the 15 bars are zero-force members; all but BD, DG, and GJ. The others carry no load but are needed for stability.

5.36) $T_{EB} = -11F/2$

5.36) $T_{HI} = -11bF/2a$

5.36) $T_{JK} = -35bF/2a$, (more than 3 times the compression of HI)

6.1) 1000 N

6.2) 0.08 cm

6.3) 1160 N

6.4) 5 cm

6.5) $k_e = 66.7\,\mathrm{N/cm},\ \delta = 0.75\,\mathrm{cm}$

6.7) $k = 20\,\mathrm{N/cm}$

6.8) Middle spring: $\delta = 1$ cm; side-springs $\delta = 0.5$ cm

6.12) Surprise! This pendulum is in equilibrium for all values of θ.

6.37) 200 N

6.48) $N = (h(w+d)/d\ell)\,F_h$

6.55) Either by looking at part KAP or at part BAQ, if we think of moment balance about A we see that the cutting force has to fight about twice the torque in the gear mechanism as in the ungeared mechanism. For example KAP is aided in its cutting by the torque from the force at G.

6.56) The mechanism multiplies the force at B and C by a factor of 2 compared to having the handle hinged at A. The force at G also gets (a shade less than) this force but with half the lever arm. Together they give a force multiplication of (a shade less than) 2+1=3.

6.57) $F_P = 125\,\mathrm{N}$

6.57) $F_P = 125\,\mathrm{N}$

6.57) For the load at I, $F_P = 75\,\text{N}$. For the load at J, $F_P = 250N$.

6.57) With the welded handle there is just a simple lever and the mechanical advantage comes from the horizontal distance between the load and hinge A. For the 4 bar mechanism the force at C is the applied vertical load, no matter where it is applied. So the lever arm is the horizontal distance from A to C.

6.58) $F_A = 500\,\text{lbf}$

6.59d) reduce the dimension marked "2 inches". The smaller the less the friction needed.

e) As the "2 inch" dimension is reduced to zero, the needed coefficient of friction goes to zero and the forces squeezing the pipe go to infinity. This is bad because it can damage the pipe. It is also bad because a small pipe deformation will cause the hinge on the wrench to snap through, like a so called "toggle mechanism" and thus not grab at all.

6.60) $\vec{\boldsymbol{R}}_A = \vec{\boldsymbol{0}}$

6.60) $T = 200\,\text{lbf}$

6.62) $F_D = \ell_{EC}(\ell_{EH} - d)F/d\ell_{CD}$

6.62) $T_{CC'} = (\ell_{EH}/d - 1)\,(\ell_{EC}/\ell_{CD} + 1)\,F$

6.62) As $d \to 0$, $F_D \to \infty$. Two problems: the amount of motion goes to zero and the assumption of rigidity becomes non-negligibly inaccurate.

6.63) $F_N\,\left(b(a^2+b^2)/a^2\right)\,F = 130F = 1300\,\text{lbf}$

6.63) The mechanism uses three tricks to multiply the force: a lever, a wedge, and a toggle. Each of these multiplies by about 5. Thus the nut-force F_N is on the order of $5^3 = 125$ times as big as F.

7.3) $(117\gamma/2)\,\text{m}^3 = 5.85 * 10^5\,\text{N}$

7.4) Water starts to spill at $h = 3r_{AB} = 3\,\text{m}$.

7.4) Assuming no friction at B, $\vec{\boldsymbol{F}}_A = 2.25 * 10^5\hat{\boldsymbol{\imath}}\,\text{N}$

7.9a) $\rho g \pi r^2 \ell$

b) $-\rho g \pi r^2 (h - \ell)$, note the minus sign, it now takes force to lift the can.

8.14) $F_{Ay} = -500\,\text{N}$, $M_A = -500/3\,\text{N}{\cdot}\text{m}$

8.15) $V(\ell/2) = -w\ell/8$, $M(\ell/2) = w\ell^2/16$, $M_{max} = M(3\ell/8) = 9wl^2/128$

8.17b) [Hint: at every height y the cross sectional area must be big enough to hold the weight plus the wire below that point. From this you can set up and a differential equation for the cross sectional area A as a function of y. Find appropriate initial conditions and solve the equation. Once solved, the volume of wire can be calculated as $V = \int_0^1 0\,\text{mi}A(y)dy$ and the mass as ρV.]

9.11) $x(3\,\text{s}) = 20\,\text{m}$

9.15) (a) $v(3\,\mathrm{s}) = 2\,\mathrm{m/s}$ in each case. (b) $x(3\,\mathrm{s}) = 3\,\mathrm{m}$ for case (a), $x(3\,\mathrm{s}) = 4\,\mathrm{m}$ for case (b).

9.16) $F_s = \frac{\pi}{4} F_T$

9.48) Time span $= 3\pi\sqrt{m/k}/2$

9.51) (a) $m\ddot{x} + kx = F(t)$, (b) $m\ddot{x} + kx = F(t)$, and (c) $m\ddot{y} + 2ky - 2k\ell_0 \frac{y}{\sqrt{\ell_0^2+y^2}} = F(t)$

9.53b) $mg - k(x - \ell_0) = m\ddot{x}$

c) $\ddot{x} + \frac{k}{m}x = g + \frac{k\ell_0}{m}$

e) This solution is the static equilibrium position; i.e., when the mass is hanging at rest, its weight is exactly balanced by the upwards force of the spring at this constant position x.

f) $\ddot{\hat{x}} + \frac{k}{m}\hat{x} = 0$

g) $x(t) = \left[D - \left(\ell_0 + \frac{mg}{k}\right)\right] \cos\sqrt{\frac{k}{m}}\, t + (\ell_0 + \frac{mg}{k})$

h) period=$2\pi\sqrt{\frac{m}{k}}$.

i) If the initial position D is more than $\ell_0 + 2mg/k$, then the spring is in compression for part of the motion. A floppy spring would buckle.

9.55a) period= $\frac{2\pi}{\sqrt{\frac{k}{m}}} = 0.96\,\mathrm{s}$

b) maximum amplitude=0.75 ft

c) period= $2\sqrt{\frac{2h}{g}} + \sqrt{\frac{m}{k}}\left[\pi + 2\tan^{-1}\sqrt{\frac{mg}{2kh}}\right] \approx 1.64\,\mathrm{s}$.

9.56) LHS of Linear Momentum Balance: $\sum \vec{F} = -(kx + b\dot{x})\hat{\imath} + (N - mg)\hat{\jmath}$.

9.70a) Two normal modes.

b) $x_2 = const * x_1 = const * (A\sin(ct) + B\cos(ct))$,where $const = \pm 1$.

c) $\omega_1 = \sqrt{\frac{3k}{m}}$, $\omega_2 = \sqrt{\frac{k}{m}}$.

9.71b) If we start off by assuming that each mass undergoes simple harmonic motion at the same frequency but different amplitudes, we will find that this two-degree-of-freedom system has two natural frequencies. Associated with each natural frequency is a fixed ratio between the amplitudes of each mass. Each mass will undergo simple harmonic motion at one of the two natural frequencies only if the initial displacements of the masses are in the fixed ratio associated with that frequency.

9.73) $\vec{a}_B = \ddot{x}_B\hat{\imath} = \frac{1}{m_B}[-k_4x_B - k_2(x_B - x_A) + c_1(\dot{x}_D - \dot{x}_B) + k_3(x_D - x_B)]\hat{\imath}$.

9.74) $\vec{a}_B = \ddot{x}_B\hat{\imath} = \frac{1}{m_B}[-k_4x_B - c_1(\dot{x}_B - \dot{x}_A) + (k_2 + k_3)(x_D - x_B)]$.

9.77a) $\omega = \sqrt{\frac{2k}{m}}$.

9.81a) One normal mode: [1, 0, 0].

b) The other two normal modes: [0, 1, $\frac{1\pm\sqrt{17}}{4}$].

9.87) $h_{max} = e^2 h$.

10.4a) $\vec{\boldsymbol{v}}(5\,\mathrm{s}) = (30\hat{\boldsymbol{\imath}} + 300\hat{\boldsymbol{\jmath}})\,\mathrm{m/s}$.

b) $\vec{\boldsymbol{a}}(5\,\mathrm{s}) = (6\hat{\boldsymbol{\imath}} + 120\hat{\boldsymbol{\jmath}})\,\mathrm{m/s^2}$.

10.5) $\vec{\boldsymbol{r}}(t) = \left(x_0 + \frac{u_0}{\Omega} - \frac{u_0}{\Omega}\cos(\Omega t)\right)\hat{\boldsymbol{\imath}} + (y_0 + v_0 t)\,\hat{\boldsymbol{\jmath}}$.

10.13) $\vec{\boldsymbol{v}} = 2t\,\mathrm{m/s^2}\hat{\boldsymbol{\imath}} + e^{\frac{t}{s}}\,\mathrm{m/s}\hat{\boldsymbol{\jmath}}$, $\vec{\boldsymbol{a}} = 2\,\mathrm{m/s^2}\hat{\boldsymbol{\imath}} + e^{\frac{t}{s}}\,\mathrm{m/s^2}\,\hat{\boldsymbol{\jmath}}$.

10.48) $T_3 = 13\,\mathrm{N}$

10.61) Equation of motion: $-mg\,\hat{\boldsymbol{\jmath}} - b(\dot{x}^2 + \dot{y}^2)\left(\frac{\dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) = m(\ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}})$.

10.62a) System of equations:

$$\begin{aligned} \dot{x} &= v_x \\ \dot{y} &= v_y \\ \dot{v}_x &= -\frac{b}{m} v_x \sqrt{v_x^2 + v_y^2} \\ \dot{v}_y &= -g - \frac{b}{m} v_y \sqrt{v_x^2 + v_y^2} \end{aligned}$$

11.6) No. You need to know the angular momenta of the particles relative to the center of mass to complete the calculation, information which is not given.

11.17a) $v_0 = \frac{1}{m}(m v_B + m_B v_B + m_A v_A)$.

b) $v_1 = \frac{(m+m_B)}{m} v_B$.

c) (1) $E_{loss} = \frac{1}{2}m\left[v_0^2 - \left(\frac{m+m_B}{m}\right)v_B^2\right] - \frac{1}{2}m_A v_A^2$. $E_{loss} = \frac{1}{2}m\left[\frac{(m+m_B)^2}{m}v_B^2 - (m+m_B)v_B^2\right]$.

11.18) $v_A = \sqrt{\frac{m_B k \delta^2}{m_A^2 + m_B m_A}}$.

11.19) The trajectories should all be the same figure 8.