CHAPTER 10

Particle dynamics in space (unconstrained)

This chapter is about the vector equation $\vec{F} = m\vec{a}$ for one particle. Concepts and applications include ballistics and planetary motion. The differential equations of motion are set-up in cartesian coordinates and integrated either numerically, or for special simple cases, by hand. Constraints, forces from ropes, rods, chains floors, rails and guides that can only be found once one knows the acceleration, are not considered.

Contents

The previous chapter was about particles that move in a straight line. Now we will consider particles that move in more complicated ways. More specifically, in this chapter we will consider the curving motion of a single particle using cartesian coordinates. We will be able to calculate the path of a hit baseball (perhaps taking account of air friction), a satellite, or a bungy jumper.

The key tool is, in Newton's words,

> *"Any change of motion is proportional to the force that acts, and it is made in the direction of the straight line in which that force is acting."*

Realizing that the quantification of motion is the product of mass and velocity, and that the rate of change of velocity is acceleration, in modern language we could rephrase Newton's as:

> *'the net force on a particle is its mass times its acceleration.'*

Informally we think 'force causes motion in the direction of the force'. Then, thinking more carefully we fill in the details that in this context 'motion' means acceleration and that the amount of force needed for a given acceleration is also proportional to the mass.

You can also think of $\vec{F} = m\vec{a}$ as a special case of the more general principle of linear momentum balance (LMB) for a system, where the system of interest is just a single particle. If we start with the general form of LMB given in the front cover, and discussed in general terms in chapter 1, we get:

$$\begin{aligned} \sum \vec{F}_\mathrm{i} &= \dot{\vec{L}} && \text{Linear momentum balance for any system} \\ &= \sum m_i \vec{a}_\mathrm{i} && \text{for a system of particles} \\ &= m\vec{a} && \text{for one particle} \end{aligned}$$

If we define $\vec{F}$ to be the net force on the particle ($\vec{F} = \sum \vec{F}_\mathrm{i}$) then linear momentum balance becomes 'Newton's second law',

$$\vec{F} = m\vec{a}. \qquad (10.1)$$

Does force cause acceleration or is it the other way around? Whether force causes acceleration or acceleration necessitates force,

Figure 10.1: Some small blobs of water fly in nearly (neglecting air friction) parabolic arcs. This fountain is in the Detroit airport.
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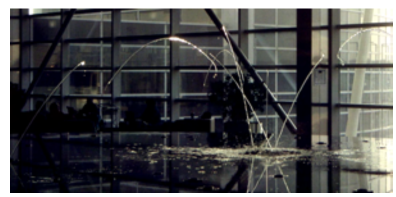

Figure 10.2: Small streams start to show the arcs in a still photo.
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Figure 10.3: A full parabola shows.
Filename:FountainDetroit5041

Figure 10.4: The water runs in continuous streams. Some fancy valve work (under the visible part of the fountain) is required to get such a laminar stream that holds together for the whole flight. This water draws its own graph of the trajectory of its particles. You can do the same kind of thing with a squirt gun next to a blackboard.
Filename:FountainDetroit5040

the issue of *causality*, is a philosophical question of no import. All that shows up in the math, and in any problem solution, is that when there is a net force there is acceleration of mass, and when there is acceleration of mass there is a net force. When a car crashes into a pole there is a big force and a big deceleration of the car. You could think of the force on the bumper as causing the car to slow down rapidly. Or you could think of the rapid car deceleration as necessitating a force. It is only a matter of personal taste because in both cases the same *eqn*. (10.1) applies. Equations don't have a 'cause' side and a 'result' side (If $A = B$ does A cause B or does B cause A?).

Acceleration is the second derivative of position

What is acceleration? If $\vec{r}(t)$ is the position of a particle relative to some origin, the particle's acceleration is

$$\vec{a} \equiv \ddot{\vec{r}}.$$

As for scalars, one or two dots over a vector is a short hand notation for the first or second time derivative. In the next section we'll explain how to take the derivative of a vector. As explained in box 10.1 the vector diferentiation has to be done using an appropriate coordinate system.

10.1 Dynamics of a particle in space

Time derivative of a vector: position, velocity and acceleration

From here to the end of the book most of our calculations will involve vector-valued functions of time. For example, the vectors linear momentum $\vec{\boldsymbol{L}}$ and angular momentum $\vec{\boldsymbol{H}}$ have a central place in mechanics. Evaluating them depends, in turn, on understanding the relation between position $\vec{\boldsymbol{r}}$, and its rate of change, called velocity $\vec{\boldsymbol{v}}$. We also need to know the relation between velocity $\vec{\boldsymbol{v}}$ and its rate of change, the acceleration $\vec{\boldsymbol{a}}$.

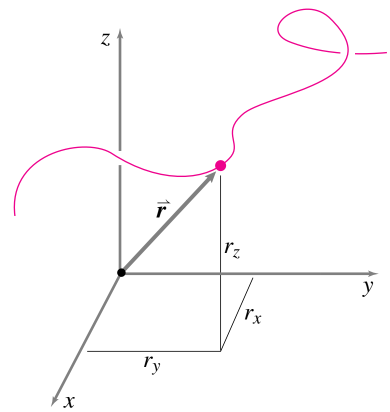

Figure 10.5: The path of a particle and its position at one time projected onto Cartesian coordinates. The path, or trajectory, of the particle is the sequence of locations of the tip of the $\vec{\boldsymbol{r}}$ vector, if the vector is drawn with its tail at the origin.

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What do we mean by the rate of change of a vector? The rate of change of any quantity, including a vector, is the ratio of the change of that quantity to the amount of time that passes, for very small amounts of time. *

$$\text{rate of change of any } (thing) \equiv \frac{\text{amount } thing \text{ changes}}{\text{amount of time for that change}}$$

* As you should remember from Calculus, these words really describe the average rate of change over the time interval. Only in the mathematical limit, as the time interval approaches zero, is the ratio of "amount of change over the time interval" not just approximately, but exactly, the instantaneous rate of change.

The notation for the rate of change of a vector $\vec{\boldsymbol{r}}$ is

$$\frac{d\vec{\boldsymbol{r}}}{dt}.$$

Or, in the short hand 'dot' notation invented by Newton for just this purpose, $\vec{\boldsymbol{v}} = \dot{\vec{\boldsymbol{r}}}$. The definition of the derivative $\frac{d\vec{\boldsymbol{r}}}{dt}$ or $\dot{\vec{\boldsymbol{r}}}$ is the same as for any*thing* else,

$$\frac{d\vec{\boldsymbol{r}}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{\boldsymbol{r}}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{\boldsymbol{r}}(t + \Delta t) - \vec{\boldsymbol{r}}(t)}{\Delta t}$$

where the top of the fraction is the change in $\vec{\boldsymbol{r}}$ and the bottom is the change in t. Underlying most dynamics calculations are derivatives of $\vec{\boldsymbol{r}}(t)$. But we also sometimes need to take derivatives of linear momentum $\vec{\boldsymbol{L}}$, angular momentum $\vec{\boldsymbol{H}}$ and some other quantities (e.g., the angular velocity $\vec{\omega}$ of a rigid object). But all of these quantities somehow depend derivatives of $\vec{\boldsymbol{r}}(t)$.

Cartesian coordinates

A simple way to think about vector derivatives is with cartesian coordinates. A moving point has a location $\vec{\boldsymbol{r}}$, relative to the origin of a 'good' (i.e., $\mathcal{N}$ewtonion) reference frame as shown in figure 10.5, which can be written as:

$$\vec{\boldsymbol{r}} = r_x\hat{\imath} + r_y\hat{\jmath} + r_z\hat{\boldsymbol{k}} \qquad \text{or} \qquad \vec{\boldsymbol{r}} = x\hat{\imath} + y\hat{\jmath} + z\hat{\boldsymbol{k}}.$$

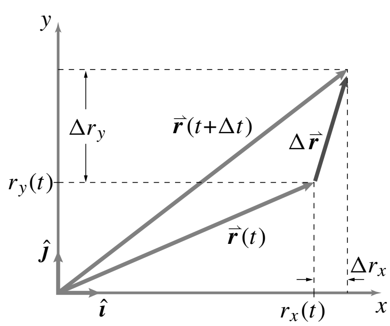

Figure 10.6: Change of position in Δt broken into components in 2-D . $\Delta\vec{r}$ is $\vec{r}(t+\Delta t)-\vec{r}(t)$. $\Delta\vec{r}$ has components Δr_x and Δr_y. So $\Delta\vec{r} = \Delta r_x\hat{\imath} + \Delta r_y\hat{\jmath}$. In the limit as Δt goes to zero, $\dot{\vec{r}}$ is the ratio of $\Delta\vec{r}$ to Δt.

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So velocity is the derivative of $\vec{r}$. Since the base vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are constant, differentiation to get velocity and acceleration is simple:

$$\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k} \qquad \text{and} \qquad \vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k}.$$

The idea is illustrated in $Fig.$ 10.6. Let's take

$$\vec{r} = r_x\hat{\imath} + r_y\hat{\jmath} \quad \text{or} \quad \vec{r}(t) = r_x(t)\hat{\imath} + r_y(t)\hat{\jmath}.$$

We can apply the definition of derivative and find

$$\begin{aligned}
\dot{\vec{r}}(t) &= \lim_{\Delta t\to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} \\
&= \frac{\overbrace{\left(r_x(t+\Delta t)\hat{\imath} + r_y(t+\Delta t)\hat{\jmath}\right)}^{\vec{r}(t+\Delta t)} - \overbrace{\left(r_x(t)\hat{\imath} + r_y(t)\hat{\jmath}\right)}^{\vec{r}(t)}}{\Delta t} \\
&= \frac{r_x(t+\Delta t) - r_x(t)}{\Delta t}\hat{\imath} + \frac{r_y(t+\Delta t) - r_y(t)}{\Delta t}\hat{\jmath} \\
&= \dot{r}_x(t)\hat{\imath} + \dot{r}_y(t)\hat{\jmath}
\end{aligned}$$

thus showing the palatable result that

10.1 *THEORY*
Newton's laws are accurate in a Newtonian reference frame

Acceleration is calculated from position using a particular coordinate system. For our purposes here, a coordinate system is also a *reference frame.* The calculation of acceleration of a particle depends on how the coordinate system itself is moving. So the simple equation

$$\vec{F} = m\vec{a}$$

has as many different interpretations as there are differently moving coordinate systems (and there are an infinite number of those). In each different coordinate system, the coordinates of a given particle are different from the coordinates in another system. And the calculated accelerations are also different. Sir Isaac Newton was sitting on earth contemplating position relative to the ground at his feet when he noticed that his second law accurately described things like falling apples. So the equation $\vec{F} = m\vec{a}$ is valid using coordinate systems that are fixed to the earth. Well, not quite. Isaac noticed that the motion of the planets around the sun only followed his law if the acceleration was calculated using a coordinate system that was still relative to 'the fixed stars.' With a fixed-star coordinate system you calculate slightly (about 0.25%) different accelerations for things like falling apples than you do using a coordinate system that is stuck to the earth. And nowadays when astrophysicists try to figure out how the laws of mechanics explain the shapes of spiral galaxies, they realize that none of the so-called 'fixed stars' are so totally fixed. They need even more care to pick a coordinate system where $eqn.$ (10.1) is accurate.

Despite all this confusion, it is generally agreed that no matter where you are there exists some coordinate system for which Newton's laws are incredibly accurate.

Further, once you know one 'good' coordinate system you know many others. Any system which translates (has no relative rotation) with constant velocity relative to a 'good' system is also a 'good' system. Why? Because the difference between the accelerations measured in the two frames is the relative acceleration of the frames, which is zero. Mechanics is the same on a constant velocity train or plane as on a stationary plane or train. Any reference frame in which Newton's laws are accurate is called a $\mathcal{N}$*ewtonian reference frame.* Sometimes people also call such a frame a $\mathcal{F}$*ixed frame*, as in 'fixed to the earth' or 'fixed to the stars'. But a Newtonian frame could also be 'fixed' to a constant velocity train or plane.

For most engineering purposes a coordinate system attached to the ground under your feet is a good approximation to a $\mathcal{N}$ewtonian frame. Fortunately. Or else apples would fall differently. Imagine Newton's apple having fallen on some crazy curved path leaving Newton confounded and the subject of mechanics still a mystery. The fall of apples, both in Newton's day and now, is well predicted using Newton's laws and treating the ground as a $\mathcal{N}$ewtonian frame. However, if you are interested in trajectory control of satellites, you need to use something more like the 'fixed stars' as your (even more accurate) Newtonian reference frame in order to make accurate predictions using Newton's laws.

the components of the velocity vector are the time derivatives of the components of the position vector①.

① **Beware.** Later in the book we will use base vectors that change in time, such as polar coordinate base vectors, path basis vectors, or basis vectors attached to a rotating frame. For these vectors the components of the vector's derivative will *not* be the derivatives of its components. See box 10.2.

Figure 10.7 shows a particle P's path, its position at a sequence of times. The position vector $\vec{r}_{P/O}$ is the arrow from the origin to a point on the curve, a different point on the curve at each instant of time. The velocity $\vec{v}$ at time t is the rate of change of position at that time, $\vec{v} \equiv \dot{\vec{r}}$.

Example: **Given position as a function of time, find the velocity.**
Given that the position of a point is:

$$\vec{r}(t) = C_1 \cos(\omega\ t)\hat{\imath} + C_2 \sin(\omega\ t)\hat{\jmath}$$

with C_1, C_2 and ω given constants what is the velocity (a vector) at a given time t?

First we note that the components of $\vec{r}(t)$ have been given implicitly as

$$r_x(t) = C_1 \cos(\omega\ t) \qquad \text{and} \qquad r_y(t) = C_2 \sin(\omega\ t).$$

Then we find the velocity by differentiating each of the components with respect to time and re-assembling as a vector to get

$$\vec{v}(t) = \dot{\vec{r}} = -C_1\omega \sin(\omega\ t)\hat{\imath} + C_2\omega \cos(\omega\ t)\hat{\jmath}$$

Now we evaluate this expression with the given values of C_1, C_2, ω and t .

Are position, velocity and acceleration all parallel? Sometimes this is a right intuition. For example, after some time has passed the change in position is exactly the average velocity. And the change in velocity is exactly the average acceleration. So in the long run, if something accelerates in some more-or-less constant direction then the position will change in that same direction. But actually, at any instant in time, position, velocity and acceleration are basically unrelated.

Example: **Position, velocity and acceleration can be mutually orthogonal.**
Here is a motion where, at least at one instant in time, the position, velocity, and acceleration are mutually orthogonal as in $Fig.$ 10.8. For example, look at the path in $Fig.$ 10.9. At the point where the path intersects the y axis the position relative to the origin is in the $\hat{\jmath}$ direction, the velocity is tangent to the path in the $\hat{\imath}$ direction and the acceleration is at least partially up, in the $\hat{k}$ direction. Working this out with equations, if we take the position as a function of time to be

$$\vec{r}(t) = A\hat{\jmath} - Bt\hat{\imath} + Ct^2\hat{k}$$

we can calculate the velocity and acceleration by differentiation as

$$\vec{v} = \frac{d\vec{r}}{dt} \quad = \quad -B\hat{\imath} + 2Ct\hat{k}, \quad \vec{a} = \frac{d\vec{v}}{dt} \quad = \quad 2C\hat{k}.$$

So, at $t = 0$,

$$\vec{r} = A\hat{\jmath}, \quad \vec{v} = -B\hat{\imath}, \qquad \text{and} \qquad \vec{a} = 2C\hat{k}.$$

The dot products between $\vec{r}$, $\vec{v}$ and $\vec{a}$ are: $\vec{r}\cdot\vec{v} = 0$, $\vec{v}\cdot\vec{a} = 0$, and $\vec{r}\cdot\vec{a} = 0$, so these vectors are mutually orthogonal at the instant marked. (**Aside:** Why is there a $-B$ in this example? Answer: no reason, we could have used $+B$ just as well.)

In constant rate circular motion position (relative to the circle's center) and velocity remain perpendicular for all time, and so do velocity and

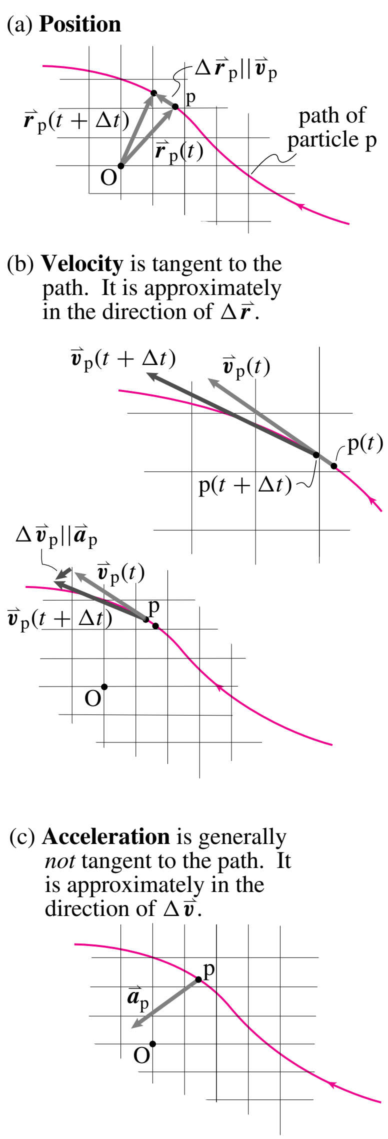

Figure 10.7: A particle moving on a curve. (a) shows the position vector is an arrow from the origin to the point on the curve. On the position curve the particle is shown at two times: t and $t + \Delta t$. The velocity at time t is roughly parallel to the difference between these two positions. The velocity is then shown at these two times in (b). The acceleration is roughly parallel to the difference between these two velocities. In (c) the acceleration is drawn on the path roughly parallel to the difference in velocities.

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acceleration. However, the directions of position, velocity and acceleration are not arbitrary. For example, there is no motion where position, velocity and acceleration are exactly mutually orthogonal for an extended time. Imagine a slender circular cone. If position is measured relative to the apex of the cone then constant-rate circular motion about the base of the cone *almost* has position, velocity and acceleration mutually orthogonal for all time. But position and acceleration are only exactly orthogonal in the limit as the cone becomes infinitely slender.

The product rule of differentiation

We know three ways to multiply vectors: multiplying a vector by a scalar, taking the dot product of two vectors, and taking the cross product of two vectors (please review Chapter 2). Because all of these quantities might be functions of time we need to know how to differentiate products. It's simple. All three kinds of vector multiplication obey 'the product rule' that you learned in freshmen calculus.

$$\begin{aligned}\frac{d}{dt}(a\vec{A}) &= \dot{a}\vec{A} + a\dot{\vec{A}}\\ \frac{d}{dt}(\vec{A}\cdot\vec{B}) &= \dot{\vec{A}}\cdot\vec{B} + \vec{A}\cdot\dot{\vec{B}}\\ \frac{d}{dt}(\vec{A}\times\vec{B}) &= \dot{\vec{A}}\times\vec{B} + \vec{A}\times\dot{\vec{B}}.\end{aligned}$$

The proofs of these identities is the same as the proof used for scalar multiplication, it follows from the definition of derivative (above) and the elimination of terms with Δ^2 as negligible compared to terms with just Δ in the limit $\Delta \to 0$..

Example: **Derivative of a vector of constant length.**

Assume a vector $\vec{C}$ has constant length so

$$|\vec{C}| = \text{constant} \qquad \text{and} \qquad |\vec{C}|^2 = \text{constant}$$

so, differentiating and using the product rule, working from left to right:

$$\frac{d}{dt}|\vec{C}|^2 = \frac{d}{dt}(\vec{C}\cdot\vec{C}) = \vec{C}\cdot\dot{\vec{C}} + \dot{\vec{C}}\cdot\vec{C} = 2\vec{C}\cdot\dot{\vec{C}} = 0.$$

Because $\vec{C}\cdot\dot{\vec{C}} = 0$ we then know that $\vec{C} \perp \dot{\vec{C}}$. That is, for any vector $\vec{C}$ that has constant length, its rate of change is perpendicular to itself. This is a useful fact to remember about time-varying constant-magnitude vectors, especially time-varying unit vectors.

To make this more intuitive, imagine a dog on a taught fixed-length leash anchored to the ground. The length of the leash is the magnitude $|\vec{C}|$ of the position vector $\vec{C}$, from ground-to-neck, and is constant. So our result is obvious, the neck can only move with a velocity $\dot{\vec{C}}$ that is tangent to the circle that the neck moves on because the tangent of a circle is orthogonal to the radius.

In 3D, space-dogs on taught leashes can only move tangent to the sphere they are stuck on $|\vec{C}| = \text{constant} \quad \Rightarrow \quad \vec{C}\cdot\dot{\vec{C}} = 0 \quad \Rightarrow \quad \vec{C} \perp \dot{\vec{C}}$. And, intuitively again, all tangents to the surface of a sphere are orthogonal to the radius of the sphere at that point.

Dynamics in space

Isaac Newton wondered how the planets move around the sun. By applying his equation $\vec{F} = m\vec{a}$, his law of gravitation, his calculus, and his inimitable geometric reasoning, he learned a lot about the moon and the planets. After you learn the material in this section you will know enough to reproduce many of Newton's calculations. You won't need to be a Newton-like genius to solve Newton's differential equations. You can solve them on a computer. And you can use the same computer approach to find motions that Newton could never find, say the trajectory of projectile with a realistic model of air friction. In this chapter, the the basic recipe is this: *

$$\text{Write} \quad \vec{F} = m\vec{a} \quad \text{and solve the equations.}$$

In some sense it's that simple.

A sure-fire recipe. Here's how to find the motion of a particle:

1. Draw a free body diagram of the particle,
2. Find the forces on the particle in terms of its position, velocity and time. External forces (external forces might come, for example, from a spring, dashpot, gravity, or air friction),
3. Write the linear momentum balance equation for the particle (translation: write $\vec{F} = m\vec{a}$).
4. Break the vector equation into components to make 2 or 3 2nd order scalar ODEs, in 2 or 3 dimensions, respectively.
5. Write the 2 or 3 2nd order ODEs in first order form. You now have 4 or 6 first order ordinary differential equations (for a 2 or 3 dimensional problem, respectively).
6. Write these first order equations in standard form, with all the time derivatives on the left hand side.
7. Feed these equations to the computer, substituting values for the various parameters and appropriate initial conditions.
8. Plot some aspect(s) of the solution and
   a) Use the solution to help you find errors in your formulation, and
   b) Interpret the solution so that it makes sense to you and increases your understanding of the system of study.

Instantaneous dynamics. Some problems are even easier, problems of the *instantaneous dynamics* type. They use the equations of dynamics but do not track the motion over time.

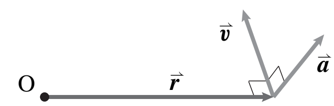

Figure 10.8: It is possible for position, velocity and acceleration to be mutually orthogonal.

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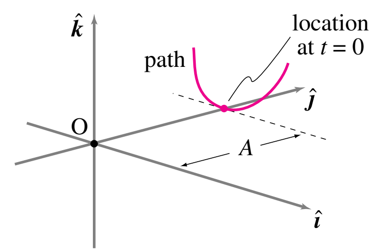

Figure 10.9: A path where, at the marked dot, the position (relative to the origin), velocity and acceleration can be mutually orthogonal.

Filename:sfig3-1-three-orthos-a

* Eventually you may gain the math skills to shortcut this brute-force numerical approach, at least for some simple problems. But for most problems, even math geniuses use the numerical approach here.

Example: **Knowing the forces find the acceleration.**
Say you know the forces on a particle at some instant in time, say $\vec{F}_1$ and $\vec{F}_2$, and you just want to know the acceleration at that instant. The answer is given directly by linear momentum balance as

$$\sum \vec{F}_i = m\vec{a} \quad \Rightarrow \quad \vec{a} = \frac{\vec{F}_1 + \vec{F}_2}{m}$$

Sometimes this 'instantaneous' dynamics, with the motion given and the forces to be determined, is called *'inverse dynamics'*. The inside back cover of the book compares the solution methods for instantaneous dynamics to those where differential equations need be solved.

Analytic solution. Some problems involving motion are simple and you can determine almost all you want to know with pencil and paper. You can bypass the whole computer recipe above.

Example: **Parabolic trajectory of a projectile**
If we assume a constant gravitational field, neglect air drag, and take the y direction as up the only force acting on a projectile is $\vec{F} = mg\,\hat{\jmath}$. Thus the "equations of motion" (linear momentum balance) are

$$-mg\,\hat{\jmath} = m\vec{a}.$$

Taking the dot product of this equation with $\hat{\imath}$ and $\hat{\jmath}$ (equivalent to taking the x and y components) we get the following two differential equations,

$$\ddot{x} = 0 \quad \text{and} \quad \ddot{y} = -g$$

which are decoupled and have the general solution

$$\vec{r} = (A + Bt)\hat{\imath} + (C + Dt - gt^2/2)\hat{\jmath}$$

which is a parametric description of all possible trajectories. By making plots or by using simple algebra you could convince yourself that these trajectories are parabolas for all possible A, B, C, and D. That is, neglecting air drag, the predicted trajectory of a thrown ball is a parabola.

Some other special problems turn out to be easy, although you might not recognize such problems at first glance.

Example: **Mass tethered by a zero-length spring**
Imagine a massless spring whose unstretched length is zero (See page 329 in section 6.1 for a discussion of zero length springs). Assume one end is connected to a pivot at the origin and the other to a particle. Neglect gravity and air drag. The force on the mass is thus proportional to its distance from the pivot and the spring constant and pointed towards the origin: $\vec{F} = -k\vec{r}$. Thus linear momentum balance yields

$$-k\vec{r} = m\vec{a}.$$

Breaking into components we get

$$\ddot{x} = (-k/m)x \quad \text{and} \quad \ddot{y} = (-k/m)y.$$

Thus the motion can be thought of as two independent harmonic oscillators, one in the x direction and one in the y direction. The general solution is

$$\vec{r} = \left(A\cos\sqrt{\frac{k}{m}}\,t + B\sin\sqrt{\frac{k}{m}}\,t\right)\hat{\imath} + \left(C\cos\sqrt{\frac{k}{m}}\,t + D\sin\sqrt{\frac{k}{m}}\,t\right)\hat{\jmath}$$

which is always an ellipse (special cases of which are a circle and a straight line).

Even with gravity and springs together some (only some) problems are easy.

Example: **Mass hanging from a zero-length spring**
If gravity in the $-\hat{\jmath}$ direction is included in the above problem the solution is only changed by the addition of a constant to be:

$$\vec{r}_{\text{with gravity}} = \vec{r}_{\text{previous example}} - (mg/k)\hat{\jmath}$$

Analytic methods sometimes just can't do the job. Some problems are hard and can't be solved without a computer.

Example: **Trajectory with quadratic air drag.**
For motions of things you can see with your bare eyes moving in air, the drag force is roughly proportional to the speed squared and opposes the motion. Thus the total force on a particle is $\vec{F} = -mg\,\hat{\jmath} - Cv^2(\vec{v}/v)$, where $\vec{v}/v$ is a unit vector in the direction of motion. So linear momentum balance gives

$$-mg\,\hat{\jmath} - Cv\vec{v} = m\vec{a}.$$

If we dot this equation with $\hat{\imath}$ and $\hat{\jmath}$ we get

$$\ddot{x} = -(C/m)\left(\sqrt{\dot{x}^2+\dot{y}^2}\right)\dot{x} \quad \text{and} \quad \ddot{y} = -(C/m)\left(\sqrt{\dot{x}^2+\dot{y}^2}\right)\dot{y} - g.$$

These are two coupled second order equations that are probably not solvable with pencil and paper. But they are easily put in the form of a set of four first order equations for direct numerical solution.

On the edge. Some problems are within the reach of advanced analytic methods, but might be more-easily solved with a computer.

Example: **Path of the earth around the sun.**
Assume the sun is big and unmovable with mass M and the earth has mass m. Take the origin to be at the sun. The force on the earth is $\vec{F} = -(mMG/r^2)(\vec{r}/r)$ where $\vec{r}/r$ is a unit vector pointing from the sun to the earth. So linear momentum balance gives

$$-\frac{mMG\vec{r}}{r^3} = m\vec{a}.$$

This equation *can* be solved with pencil and paper, Newton did it but many of us find it too tricky (see box **??** on page **??**). On the other hand the equations of motion for planetary trajectories are easily broken into components and then into a set of 4 ODEs which can be easily solved on the computer. Either by pencil and paper, or by investigation of numerical solutions, you will find that all solutions are conic sections (straight lines, parabolas, hyperbolas, and ellipses). The special case of circular motion is not far from what the earth does around the sun, what the moon does around the earth, and what most artificial satellites do around the earth.

Summary

If, given the time, the particle's position and the particles velocity, you know the force on a particle, then you know $\vec{F}(t, \vec{r}, \vec{v})$*. That means you can write $\vec{F} = m\vec{a}$ as

$$\vec{a} = \vec{F}(t, \vec{r}, \vec{v})/m$$

where $\vec{F}$ is known. This can, in turn, be written as two vector first order equations

$$\begin{aligned} \dot{\vec{r}} &= \vec{v} \\ \dot{\vec{v}} &= \vec{F}(t, \vec{r}, \vec{v})/m. \end{aligned} \qquad (10.2)$$

* Typically you would know this because an applied force would vary in time in a known way (the dependence on t), gravity and spring forces would vary with position in a known way (dependence on r), and you would know forces due to various friction (dependence on $\vec{v}$).

which are equivalent, written out long hand, to the 6 first order equations

$$\begin{aligned}\dot{x} &= v_x \\ \dot{y} &= v_y \\ \dot{z} &= v_z \\ \dot{v}_x &= F_x(t,x,y,z,v_x,v_y,v_z)/m \\ \dot{v}_y &= F_y(t,x,y,z,v_x,v_y,v_z)/m \\ \dot{v}_z &= F_z(t,x,y,z,v_x,v_y,v_z)/m.\end{aligned} \tag{10.3}$$

Given the position and velocity at some starting time, these equations can be integrated, sometimes by hand but generally on the computer, to give position and velocity as a function of time.

Example: **Simple ballistics.**

This is an example that can be solved with pencil and paper. A computer is not needed. It is the classic from high school and freshman physics. A particle has only one force on it, gravity. A free body diagram is shown in ***Fig.* 10.10**. Linear momentum balance gives

$$\begin{aligned}\vec{F} &= m\vec{a} \\ -mg\,\hat{\jmath} &= m\vec{a} \\ \vec{a} &= -g\,\hat{\jmath}\end{aligned}$$

So

$$\begin{aligned}\dot{x} &= v_x \\ \dot{y} &= v_y \\ \dot{v}_x &= 0 \\ \dot{v}_y &= -g\end{aligned}$$

Integrating the last two of these equations and plugging the result into the first two we get:

$$\begin{aligned}\vec{v} &= v_{0x}\hat{\imath} + (v_{0y} - gt)\hat{\jmath} \\ \vec{r} &= (x_0 + v_{0x}t)\hat{\imath} + (y_0 + v_{0y}t - gt^2/2)\hat{\jmath}\end{aligned}$$

This solution is plotted various ways in ***Fig.* 10.22** on page 568.

More complicated examples are given in the samples on the following pages.

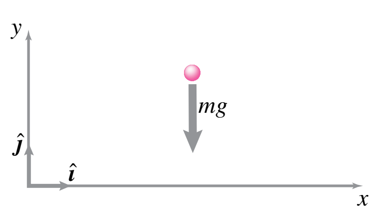

Figure 10.10: Free body diagram of a particle in flight, neglecting air friction.

Filename:tfig10-ballistics-FBD

SAMPLE 10.1 Velocity and acceleration from derivative of position: The position vector of a particle is given as a functions time:

$$\vec{r}(t) = (C_1 + C_2 t + C_3 t^2)\hat{\imath} + C_4 t\,\hat{\jmath}$$

where $C_1 = 1\,\text{m}, C_2 = 3\,\text{m/s}, C_3 = 1\,\text{m/s}^2$, and $C_4 = 2\,\text{m/s}$.

1. Find the position, velocity, and acceleration of the particle at $t = 2\,\text{s}$.
2. Find the change in the position of the particle between $t = 2\,\text{s}$ and $t = 3\,\text{s}$.

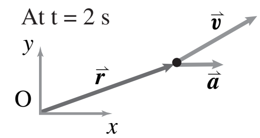

Figure 10.11:

Filename:sfig10-1-kinvectors1

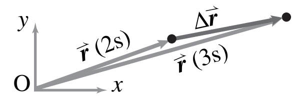

Figure 10.12:

Filename:sfig10-1-kinvectors2

Solution We are given,

$$\vec{r} = (C_1 + C_2 t + C_3 t^2)\hat{\imath} + C_4 t\,\hat{\jmath}.$$

Therefore,

$$\begin{aligned}\vec{v} &\equiv \dot{\vec{r}} = \frac{d\vec{r}}{dt} = (C_2 + 2C_3 t)\hat{\imath} + C_4\hat{\jmath}\\ \vec{a} &\equiv \ddot{\vec{r}} = \frac{d^2\vec{r}}{dt^2} = 2C_3\hat{\imath}.\end{aligned}$$

1. Substituting the given values of the constants and $t = 2\,\text{s}$ in the equations above we get,

$$\begin{aligned}\vec{r}(t = 2\,\text{s}) &= (1\,\text{m} + 3\frac{\text{m}}{\text{s}}\cdot 2\,\text{s} + 1\frac{\text{m}}{\text{s}^2}\cdot 4\,\text{s}^2)\hat{\imath} + (2\frac{\text{m}}{\text{s}}\cdot 2\,\text{s})\hat{\jmath}\\ &= 11\,\text{m}\hat{\imath} + 4\,\text{m}\hat{\jmath}\\ \vec{v}(t = 2\,\text{s}) &= (3\frac{\text{m}}{\text{s}} + 2\cdot 1\frac{\text{m}}{\text{s}^2}\cdot 2\,\text{s})\hat{\imath} + (2\frac{\text{m}}{\text{s}})\hat{\jmath}\\ &= 7\,\text{m/s}\hat{\imath} + 2\,\text{m/s}\hat{\jmath}\\ \vec{a}(t = 2\,\text{s}) &= (2\cdot 1\frac{\text{m}}{\text{s}^2})\hat{\imath} = 2\,\text{m/s}^2\hat{\imath}.\end{aligned}$$

$$\boxed{\vec{r} = (11\hat{\imath} + 4\hat{\jmath})\,\text{m},\quad \vec{v} = (7\hat{\imath} + 2\hat{\jmath})\,\text{m/s},\quad \vec{a} = 2\,\text{m/s}^2\hat{\imath}}$$

2. The change in the position of the particle between the two time instants is,

$$\Delta\vec{r} = \vec{r}(t = 3\,\text{s}) - \vec{r}(t = 2\,\text{s}).$$

We already have $\vec{r}$ at $t = 2\,\text{s}$. We need to calculate $\vec{r}$ at $t = 3\,\text{s}$.

$$\begin{aligned}\vec{r}(t = 3\,\text{s}) &= (1\,\text{m} + 3\frac{\text{m}}{\text{s}}\cdot 3\,\text{s} + 1\frac{\text{m}}{\text{s}^2}\cdot 9\,\text{s}^2)\hat{\imath} + (2\frac{\text{m}}{\text{s}}\cdot 3\,\text{s})\hat{\jmath}\\ &= 19\,\text{m}\hat{\imath} + 6\,\text{m}\hat{\jmath}.\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta\vec{r} &= (19\,\text{m}\hat{\imath} + 6\,\text{m}\hat{\jmath}) - (11\,\text{m}\hat{\imath} + 4\,\text{m}\hat{\jmath})\\ &= 8\,\text{m}\hat{\imath} + 2\,\text{m}\hat{\jmath}.\end{aligned}$$

$$\boxed{\Delta\vec{r} = 8\,\text{m}\hat{\imath} + 2\,\text{m}\hat{\jmath}}$$

10.2 *THEORY*

The rate of change of a vector depends on reference frame

The time derivative of a vector can be found by differentiating each of its components. This calculation depended on having a reference frame, an imaginary piece of big graph paper, and a corresponding set of base (or basis) vectors, say $\hat{\imath}$, $\hat{\jmath}$ and $\hat{\boldsymbol{k}}$. But there can be more than one piece of imaginary graph paper. You could be holding one, Jo another, and Tanya a third. Each could be moving their graph paper around and on each paper the same given vector would change in a different way.

As noted earlier, but for the special case of one frame moving at constant velocity (without rotation) with respect to another, the rate of change of a given vector is different if calculated in different reference frames.

For mechanics we have to differentiate vectors with respect to a Newtonian frame.

Because most often we use the "fixed" ground under us as a practical approximation of a Newtonian frame, we label a Newtonian frame with a curly script $\mathcal{F}$, for fixed. So, when being fussy about notation we will sometimes write

$$^{\mathcal{F}}\dot{\vec{\boldsymbol{r}}}_{\mathrm{B/O}} = \text{The velocity of point B as calculated in frame } \mathcal{F}.$$

Non-Newtonian frames

Even though the laws of mechanics are not valid in non-Newtonian frames, non-Newtonian frames are useful help with the understanding of the motion of and forces on systems composed of objects with complex relative motion. So eventually we need to understand frames that accelerate and rotate with respect to each other and with reference to Newtonian frames. Such non-Newtonian frames will be discussed in later chapters.

SAMPLE 10.2 Velocity and acceleration from position on a helix. Given that the position of a particle is

$$\vec{r} = A\cos(\lambda t)\hat{\imath} + B\sin(\lambda t)\hat{\jmath} + Ct\hat{k},$$

with A, B, C, and λ constants, find

1. the velocity as a function of time,
2. the acceleration as a function of time,
3. a condition under which the acceleration vector is normal to the velocity vector.

Solution

1. The velocity:

$$\begin{aligned} \vec{v} &= \frac{d\vec{r}}{dt} \\ &= \frac{d}{dt}[A\cos(\lambda t)\hat{\imath} + B\sin(\lambda t)\hat{\jmath} + Ct\hat{k}] \\ &= -A\lambda\sin(\lambda t)\hat{\imath} + B\lambda\cos(\lambda t)\hat{\jmath} + C\hat{k}. \end{aligned}$$

$$\boxed{\vec{v} = -A\lambda\sin(\lambda t)\hat{\imath} + B\lambda\cos(\lambda t)\hat{\jmath} + C\hat{k}}$$

2. The acceleration:

$$\begin{aligned} \vec{a} &= \frac{d\vec{v}}{dt} \\ &= \frac{d}{dt}[-A\lambda\sin(\lambda t)\hat{\imath} + B\lambda\cos(\lambda t)\hat{\jmath} \\ &= -A\lambda^2\cos(\lambda t)\hat{\imath} - B\lambda^2\sin(\lambda t)\hat{\jmath} \end{aligned}$$

$$\boxed{\vec{a} = -A\lambda^2\cos(\lambda t)\hat{\imath} - B\lambda^2\sin(\lambda t)\hat{\jmath}}$$

3. The velocity vector and the acceleration vector will be orthogonal to each other if $\vec{v}\cdot\vec{a} = 0$. Taking the dot product of the two vectors, we find,

$$\begin{aligned} \vec{v}\cdot\vec{a} &= (-A\lambda\sin(\lambda t)\hat{\imath} + B\lambda\cos(\lambda t)\hat{\jmath} + C\hat{k})\cdot(-A\lambda^2\cos(\lambda t)\hat{\imath} - B\lambda^2\sin(\lambda t)\hat{\jmath}) \\ &= A^2\lambda^3\sin(\lambda t)\cos(\lambda t) - B^2\lambda^3\sin(\lambda t)\cos(\lambda t) \\ &= (A^2 - B^2)\lambda^3\sin(\lambda t)\cos(\lambda t). \end{aligned}$$

Now, this dot product must be zero for all t if $\vec{a}$ is normal to $\vec{v}$. This is indeed the case if $A = B$. Thus, the condition for orthogonality o $\vec{v}$ and $\vec{a}$ is $A = B$.

$$\boxed{A = B \quad \Rightarrow \quad \vec{v}\cdot\vec{a} = 0}$$

Note: The path is an elliptical helix with axis in the z direction. The z-component of velocity is constant so the acceleration is entirely in the xy plane. In fact, the acceleration vector points from the particle towards the axis of the helix.

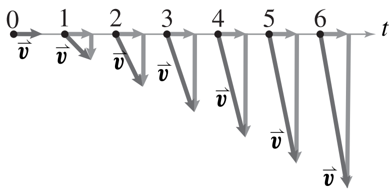

Figure 10.13: The given velocity vector at different time instants ($t = 0$, 1 2, etc.). Note that the x-component of the velocity remains constant (v_0) while the y-component grows linearly with time ($-gt$).

Filename:sfig10-1-4-vel

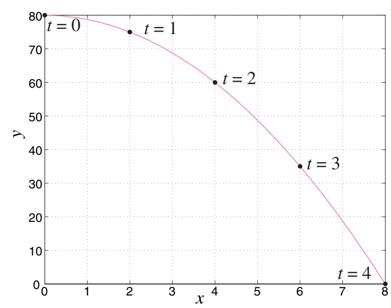

Figure 10.14: Path of the particle plotted by computing the coordinates $x(t)$ and $y(t)$ at 20 equal time intervals between $t = 0$ to 4 seconds. Five positions are marked on the path corresponding to $t = 0$, 1, 2, 3, and 4 seconds.

Filename:sfig10-1-4-path

SAMPLE 10.3 Position from velocity. Assume the expression for velocity $\vec{v}$ of a particle is given: $\vec{v} = v_0\hat{\imath} - gt\hat{\jmath}$. Find the expressions for the x and y coordinates of the particle at a general time t, if the initial coordinates at $t = 0$ are (x_0, y_0). Plot the path of the particle taking $x0 = 0$, $y_0 = 80\,\text{m}$, $v_0 = 2\,\text{m/s}$, $g = 10\,\text{m/s}^2$, and $t = 1\ldots 4\,\text{s}$.

Solution The position vector of the particle at any time t is

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}.$$

We are given that

$$\vec{r}(t=0) = x_0\hat{\imath} + y_0\hat{\jmath}.$$

Now

$$\begin{aligned} \vec{v} \equiv \frac{d\vec{r}}{dt} &= v_0\hat{\imath} - gt\hat{\jmath} \\ \text{or} \qquad dx\hat{\imath} + dy\hat{\jmath} &= (v_0\hat{\imath} - gt\hat{\jmath})\,dt. \end{aligned}$$

Integrating both sides of the equation with appropriate limits, we get

$$\begin{aligned} \int_{x_0\hat{\imath}+y_0\hat{\jmath}}^{x\hat{\imath}+y\hat{\jmath}} (dx\hat{\imath} + dy\hat{\jmath}) &= \int_0^t (v_0\hat{\imath} - gt\hat{\jmath})dt \\ \int_{x_0}^{x} dx\hat{\imath} + \int_{y_0}^{y} dy\hat{\jmath} &= v_0\hat{\imath}\int_0^t dt - g\hat{\jmath}\int_0^t t\,dt \\ (x - x_0)\hat{\imath} + (y - y_0)\hat{\jmath} &= v_0 t\hat{\imath} - \frac{1}{2}gt^2\hat{\jmath} \\ x\hat{\imath} + y\hat{\jmath} &= (x_0 + v_0 t)\hat{\imath} + (y_0 - \frac{1}{2}gt^2)\hat{\jmath}. \end{aligned}$$

Therefore,

$$\vec{r}(t) = (x_0 + v_0 t)\hat{\imath} + (y_0 - \frac{1}{2}gt^2)\hat{\jmath}$$

and the (x, y) coordinates are

$$\begin{aligned} x(t) &= x_0 + v_0 t \\ y(t) &= y_0 - \frac{1}{2}gt^2. \end{aligned}$$

$$\boxed{(x_0 + v_0 t,\ y_0 - \tfrac{1}{2}gt^2)}$$

Plugging in $x0 = 0$, $y_0 = 80\,\text{m}$, $v_0 = 2\,\text{m/s}$, $g = 10\,\text{m/s}^2$, and taking 20 points between $t = 0$ to $t = 4$, we compute the values of x and y and plot them to get the path of the particle. The plot is shown in $Fig.$ 10.14 with a few intermediate positions marked on the path.

Comments: From the x and y coordinates, it is possible to get the equation of the path of the particle by eliminating the time from the two equations. From the expression for $x(t)$, we get $t = (x - x_0)/v_0$. Substituting this expression for t in the equation for $y(t)$, we get,

$$y - y_0 = \frac{g}{2v_0^2}(x - x_0)^2$$

which is the equation of the path. From this equation it should be clear that the path is parabolic. It is easier to see this if you shift the origin to (x_0, y_0) and use the new coordinates $\hat{x} = x - x_0$ and $\hat{y} = y - y_0$. Then, in terms of the new coordinates, the path becomes,

$$\hat{y} = \frac{g}{2v_0^2}\hat{x}^2.$$

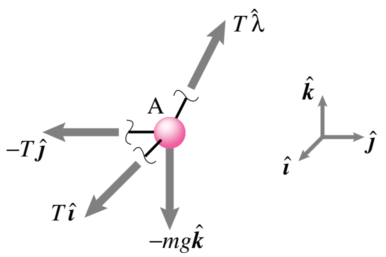

Figure 10.16: FBD of the ball

Filename:sfig2-1-10b

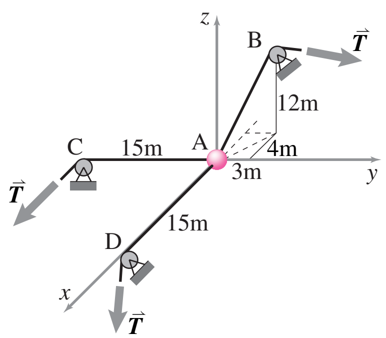

Figure 10.15: A ball in 3-D

Filename:sfig2-1-10a

SAMPLE 10.4 Acceleration of a point mass in 3-D. A ball of mass $m = 13\,\text{kg}$ is being pulled by three strings as shown in Fig. 10.15. The tension in each string is $T = 13\,\text{N}$. Find the acceleration of the ball.

Solution The forces acting on the body are shown in the free-body diagram in Fig.10.16. From geometry:

$$\begin{aligned}\hat{\lambda} &= \frac{\vec{\boldsymbol{r}}_{AB}}{|\vec{\boldsymbol{r}}_{AB}|} = \frac{-4\hat{\imath} + 3\hat{\jmath} + 12\hat{\boldsymbol{k}}}{\sqrt{4^2 + 3^2 + 12^2}} \\ &= \frac{-4\hat{\imath} + 3\hat{\jmath} + 12\hat{\boldsymbol{k}}}{13}.\end{aligned}$$

The balance of linear momentum for the ball gives

$$\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}} \tag{10.4}$$

where

$$\begin{aligned}\sum \vec{\boldsymbol{F}} &= T\hat{\imath} - T\hat{\jmath} + T\hat{\lambda} - mg\hat{\boldsymbol{k}} \\ &= T\left(\hat{\imath} - \hat{\jmath} + \frac{-4\hat{\imath} + 3\hat{\jmath} + 12\hat{\boldsymbol{k}}}{13}\right) - mg\hat{\boldsymbol{k}} \\ &= \frac{T}{13}(9\hat{\imath} + 10\hat{\jmath} + 12\hat{\boldsymbol{k}}) - mg\hat{\boldsymbol{k}}.\end{aligned}$$

Substituting $\sum \vec{\boldsymbol{F}}$ in eqn. (10.4):

$$\vec{\boldsymbol{a}} = \frac{T}{13m}(9\hat{\imath} + 10\hat{\jmath} + 12\hat{\boldsymbol{k}}) - g\hat{\boldsymbol{k}}.$$

Now plugging in the given values: $T = 13\,\text{N}$, $m = 13\,\text{kg}$, and $g = 10\,\text{m/s}^2$, we get

$$\begin{aligned}\vec{\boldsymbol{a}} &= \frac{\cancel{13}\,\text{N}}{\cancel{13}\cdot 13\,\text{kg}}(9\hat{\imath} - 10\hat{\jmath} + 12\hat{\boldsymbol{k}}) - 10\,\text{m/s}^2\hat{\boldsymbol{k}} \\ &= (0.69\hat{\imath} - 0.77\hat{\jmath} - 9.08\hat{\boldsymbol{k}})\,\text{m/s}^2.\end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}} = (0.69\hat{\imath} - 0.77\hat{\jmath} - 9.08\hat{\boldsymbol{k}})\,\text{m/s}^2}$$

SAMPLE 10.5 Projectile motion with air drag. A projectile is fired into the air at an initial angle θ_0 and with initial speed v_0. The air resistance to the motion is proportional to the square of the speed of the projectile. Take the constant of proportionality to be k. Find the equations of motion of the projectile in the horizontal and vertical directions assuming the air resistance to be in the opposite direction of the velocity.

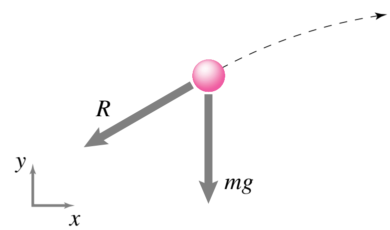

Figure 10.17: FBD of the projectile.

Filename:sfig6-4-DH1

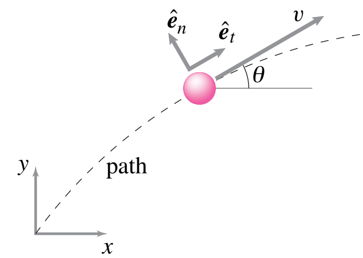

Figure 10.18:

Filename:sfig6-4-DH2

Solution The free body diagram of the projectile is shown in the figure at some constant t during motion. At the instant shown, let the velocity of the projectile be $\vec{v} = v\hat{e}_t$ where

$$\hat{e}_t = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}.$$

Then the force due to air resistance is

$$\vec{R} = -kv^2\hat{e}_t.$$

Now applying the linear momentum balance on the projectile, we get

$$\begin{aligned} \vec{R} + m\vec{g} &= m\vec{a} \\ \text{or} \qquad -kv^2\hat{e}_t - mg\,\hat{\jmath} &= m\overbrace{(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath})}^{\vec{a}}. \end{aligned} \qquad (10.5)$$

Noting that $v = |\vec{v}| = |\dot{x}\hat{\imath} + \dot{y}\hat{\jmath}| = \sqrt{\dot{x}^2 + \dot{y}^2}$, and dotting both sides of eqn. (10.5) with $\hat{\imath}$ and $\hat{\jmath}$ we get

$$\begin{aligned} -k(\dot{x}^2 + \dot{y}^2)\cdot(\hat{e}_t\cdot\hat{\imath}) &= m\ddot{x} \\ -k(\dot{x}^2 + \dot{y}^2)\cdot(\hat{e}_t\cdot\hat{\jmath}) - mg &= m\ddot{y}. \end{aligned}$$

Rearranging terms and carrying out the dot products, we get

$$\begin{aligned} \ddot{x} &= -\frac{k}{m}(\dot{x}^2 + \dot{y}^2)\cos\theta \\ \ddot{y} &= -g - \frac{k}{m}(\dot{x}^2 + \dot{y}^2)\sin\theta. \end{aligned}$$

Note that θ changes with time. We can express θ in terms of $\dot{x}$ and $\dot{y}$ because θ is the slope of the trajectory:

$$\theta = \tan^{-1}\frac{dy}{dx} = \tan^{-1}\frac{dy/dt}{dx/dt} = \tan^{-1}\frac{\dot{x}}{\dot{y}} \qquad (\text{i.e., } \tan\theta = \frac{\dot{y}}{\dot{x}})$$

$$\Rightarrow \qquad \cos\theta = \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

$$\text{and} \qquad \sin\theta = \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}.$$

Substituting these expressions in to the equations for $\ddot{x}$ and $\ddot{y}$ we get

$$\boxed{\ddot{x} = -\tfrac{k}{m}\dot{x}\sqrt{\dot{x}^2 + \dot{y}^2}, \quad \ddot{y} = -\tfrac{k}{m}\dot{y}\sqrt{\dot{x}^2 + \dot{y}^2} - g}$$

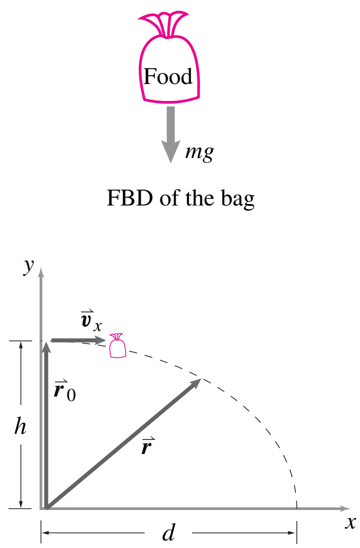

Figure 10.19: Free-body diagram of the bag and the geometry of its motion.

Filename:sfig6-1-2a

SAMPLE 10.6 Trajectory of a food-bag. In a flood hit area relief supplies are dropped in a 20 kg bag from a helicopter. The helicopter is flying parallel to the ground at 200 km/h and is 80 m above the ground when the package is dropped. How much horizontal distance does the bag travel before it hits the ground? Take the value of g, the gravitational acceleration, to be $10\,\mathrm{m/s^2}$. Ignore air drag.

Solution You must have solved such problems in elementary physics courses. Usually, in all projectile motion problems the equations of motion are written separately in the x and y directions, realizing that there is no force in the x direction, and then the equations are solved. Here we show you how to write and keep the equations in vector form all the way through.

The free-body diagram of the bag during its free flight is shown in Fig. 10.19. The only force acting on the bag is its weight. Therefore, from the linear momentum balance for the bag we get

$$m\vec{a} = -mg\hat{\jmath}.$$

Let us choose the origin of our coordinate system on the ground exactly below the point at which the bag is dropped from the helicopter. Then, the initial position of the bag $\vec{r}(0) = h\hat{\jmath} = 80\,\mathrm{m}\hat{\jmath}$. The fact that the bag is dropped from a helicopter flying horizontally gives us the initial velocity of the bag:

$$\vec{v}(0) \equiv \dot{\vec{r}}(0) = v_x\hat{\imath} = 200\,\mathrm{km/h}\hat{\imath}.$$

So now we have a 2nd order differential equation (from linear momentum balance):

$$\ddot{\vec{r}} = -g\hat{\jmath}$$

with two initial conditions:

$$\vec{r}(0) = h\hat{\jmath} \quad \text{and} \quad \dot{\vec{r}}(0) = v_x\hat{\imath}$$

which we can solve to get the position vector of the bag at any time. Since the basis vectors $\hat{\imath}$ and $\hat{\jmath}$ do not change with time, solving the differential equation is a matter of simple integration:

$$\begin{aligned} \ddot{\vec{r}} &\equiv \frac{d\dot{\vec{r}}}{dt} = -g\hat{\jmath} \\ \int d\dot{\vec{r}} &= -\hat{\jmath}\int g\,dt \\ \text{or} \qquad \dot{\vec{r}} &= -gt\hat{\jmath} + \vec{c}_1 \end{aligned} \tag{10.6}$$

and integrating once again, we get

$$\begin{aligned} \vec{r} &= \int(-gt\hat{\jmath} + \vec{c}_1)\,dt \\ &= -\frac{1}{2}gt^2\hat{\jmath} + \vec{c}_1 t + \vec{c}_2 \end{aligned} \tag{10.7}$$

where $\vec{c}_1$ and $\vec{c}_2$ are constants of integration and are vector quantities. Now substituting the initial conditions in eqn. (10.6) and eqn. (10.7) we get

$$\begin{aligned} \dot{\vec{r}}(0) &= v_x\hat{\imath} = \vec{c}_1, \quad \text{and} \\ \vec{r}(0) &= h\hat{\jmath} = \vec{c}_2. \end{aligned}$$

Therefore, the solution is

$$\begin{aligned} \vec{r}(t) &= -\frac{1}{2}gt^2\hat{\jmath} + v_x t\hat{\imath} + h\hat{\jmath} \\ &= v_x t\hat{\imath} + (h - \frac{1}{2}gt^2)\hat{\jmath}. \end{aligned}$$

So how do we find the horizontal distance traveled by the bag from our solution? The distance we are interested in is the x-component of $\vec{r}$, *i.e.*, $v_x t$. But we do not know t. However, when the bag hits the ground, its position vector has no y-component, *i.e.*, we can write $\vec{r} = d\hat{\imath} + 0\hat{\jmath}$ where d is the distance we are interested in. Now equating the components of $\vec{r}$ with the obtained solution, we get

$$d = v_x t \quad \text{and} \quad 0 = h - \frac{1}{2}gt^2.$$

Solving for t from the second equation and substituting in the first equation we get

$$d = v_x\sqrt{\frac{2h}{g}} = \frac{200\,\text{km}}{3600\,\text{s}} \cdot \sqrt{\frac{2 \cdot 80\,\text{m}}{10\,\text{m/s}^2}} = \frac{2}{9}\,\text{km} \approx 222\,\text{m}.$$

$$\boxed{d = 222\,\text{m}}$$

Comments: Here we have tried to show you that solving for position from the given acceleration in vector form is not really any different than solving in scalar form provided the unit vectors involved are fixed in time. As long as the right hand side of the differential equation is integrable, the solution can be obtained. If the method shown above seems too "*mathy*" or intimidating to you then follow the usual scalar way of doing this problem.

The scalar method:

From the linear momentum balance, $-mg\hat{\jmath} = m\vec{a}$, writing the acceleration as $\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath}$ and equating the x and y components from both sides, we get

$$a_x = 0 \quad \text{and} \quad a_y = -g.$$

Now using the formula for distance under uniform acceleration from Chapter 3, $x = x_0 + v_0 t + \frac{1}{2}at^2$, in both x and y directions, we get

$$\begin{aligned} d &= \overbrace{x_0}^{0} + v_x t + \frac{1}{2}\overbrace{a_x}^{0}t^2 \\ &= v_x t \\ 0 &= \overbrace{y_0}^{h} + \overbrace{v_y}^{0}t + \frac{1}{2}\overbrace{a_y}^{-g}t^2 \\ &= h - \frac{1}{2}gt^2 \\ \Rightarrow \quad t &= \sqrt{\frac{2h}{g}}. \end{aligned}$$

Substituting for t in the equation for d we get

$$d = v_x\sqrt{\frac{2h}{g}} = \frac{200\,\text{km}}{3600\,\text{s}} \cdot \sqrt{\frac{2 \cdot 80\,\text{m}}{10\,\text{m/s}^2}} = \frac{2}{9}\,\text{km} \approx 222\,\text{m}.$$

as above.

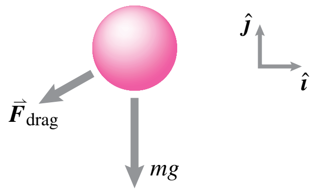

Figure 10.20:
Filename:sfig3-5-cart-cannon

* To be precise, if the launch speed is much faster than the 'terminal velocity' of the falling ball.

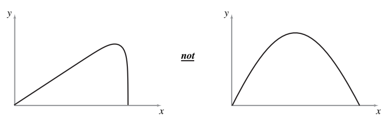

Figure 10.21:
Filename:sfig3-5-cart-cannon-graph

SAMPLE 10.7 Cartoon mechanics: The cannon. It is sometimes claimed that students have trouble with dynamics because they built their intuition by watching cartoons. This claim could be rebutted on many grounds.

1) Students don't have trouble with dynamics! They love the subject.
2) Nowadays many cartoons are made using 'correct' mechanics, and
3) the cartoons are sometimes more accurate than the pedagogues anyway.

Problem: What is the path of a cannon ball? In the cartoon world the cannon ball goes in a straight line out the cannon then comes to a stop and then starts falling. Of course a good physicist knows the path is a parabola. Or is it?

Solution The drag force on a cannon ball moving through air is approximately proportional to the speed squared and resists motion. Gravity is approximately constant. Then

$$\begin{aligned}
\vec{F}_{\text{drag}} &= cv^2 \cdot (\text{unit vector opposing motion}) \\
&= cv^2 \cdot \left(\frac{-\vec{v}}{|\vec{v}|}\right) \\
&= -c|\vec{v}|\vec{v} \\
&= -c\sqrt{\dot{x}^2 + \dot{y}^2}\,(\dot{x}\hat{\imath} + \dot{y}\hat{\jmath})
\end{aligned}$$

So the linear momentum balance gives

$$\sum \vec{F} = \dot{\vec{L}}$$
$$\left\{-mg\hat{\jmath} - c\sqrt{\dot{x}^2 + \dot{y}^2}(\dot{x}\hat{\imath} + \dot{y}\hat{\jmath}) = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath})\right\}$$

$$\begin{aligned}
\{\}\cdot\hat{\imath} \quad &\Rightarrow \quad \ddot{x} = \left[-c\sqrt{\dot{x}^2 + \dot{y}^2}\,\dot{x}/m\right] \\
\{\}\cdot\hat{\jmath} \quad &\Rightarrow \quad \ddot{y} = -c\sqrt{\dot{x}^2 + \dot{y}^2}\,\dot{y}/m - g
\end{aligned}$$

Solving these equations numerically with reasonable values * of $\dot{x}_0, \dot{y}_0, m$ and c gives

which is closer to a cartoon's triangle than to a naive physicist's parabola.

10.2 Linear momentum, angular momentum, work and energy

If you know a particle's starting position andvelocity, and you know the force on it as it moves, then you can use $\vec{F} = m\vec{a}$ to predict its path. That is the central idea of the previous section. We had no need for ideas related to momentum, angular momentum, work and energy. For one particle the one equation $\vec{F} = m\vec{a}$ tells the whole story. So, before we go on to discuss them further, let us be clear:

> The concepts of linear and angular momentum, work and energy are not needed to study particle mechanics. $\vec{F} = m\vec{a}$ is enough.

So, why do we bother to devote a section to these topics? Because

- These concepts will sometimes be needed when we discuss more complex systems;
- These concepts sometimes provide a shorter route for answering some dynamics questions;
- The simplest place to introduce the concepts is in the context of one particle;
- The concepts give a way to check the consistency of solutions of $\vec{F} = m\vec{a}$; and
- The concepts can be an aid to physical intuition.

For more complex systems, principles of momentum and energy transcend $\vec{F} = m\vec{a}$ and can generally not be derived from $\vec{F} = m\vec{a}$. But for a single particle, all of these are derived concepts, as worked out in box 10.4 on page 574. Note that

> All of the facts and theorems below apply to any motion of a particle that is consistent with $\vec{F} = m\vec{a}$.

Example: **Simple ballistics solution.**
Consider a ball thrown up at 45°: $\vec{F} = -mg\hat{\jmath}$, $\vec{r}(0) = \vec{0}$ and $\vec{v}(0) = v_0\hat{\imath} + v_0\hat{\jmath}$. We claimed (page 556) that a solution is

$$\vec{r} = v_0 t\hat{\imath} + (v_0 t - gt^2/2)\hat{\jmath} \qquad \text{and} \qquad \vec{v} = v_0\hat{\imath} + (v_0 - gt)\hat{\jmath}.$$

This solution is plotted various ways in *Fig*. 10.22. These functions of time are consistent with the initial conditions. Further they are consistent with the governing equations, the so called 'equations of motion', $\dot{\vec{v}} = \vec{F}/m$ and $\dot{\vec{r}} = \vec{v}$. All of the momentum and energy principles below must therefore apply.

Some of the ideas apply even if $\vec{F} \neq m\vec{a}$. For example, the work of a force is defined for imagined motions that might never occur.

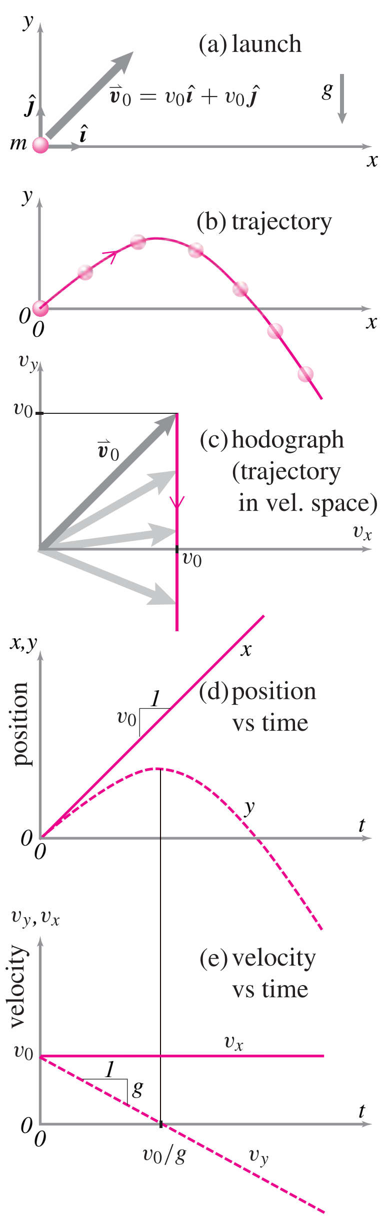

Figure 10.22: **(a) A simple ballistics problem.** (b) The well known parabolic trajectory. (c) The velocity trajectory, called a *hodograph*, shows the sequence of positions of the tip of the velocity vector. The velocity starts with equal x and y components and then the y component decreases. (d) Plot of x and y components vs time. The y component has a parabolic shape like the shape of the trajectory of the particle because x and t are proportional in this problem. (b) is a 'cross plot' of the two plots in (d) where the sequence of x and y values from (d) are plotted against each other. (e) shows how the velocity components vary in time. (c) is a cross plot of the plots in (e).

Filename:tfig10-ballistics-soln

Linear momentum

Linear momentum for a particle is defined as $\vec{L} = m\vec{v}$. The particle momentum-balance theorems (facts) are

$$\vec{F} = \frac{d}{dt}\vec{L} \qquad \text{and} \qquad \underbrace{\int_{t_1}^{t_2} \vec{F}\,dt}_{\text{Linear impulse}} = \vec{L}_2 - \vec{L}_1$$

These are so trivially related to $\vec{F} = m\vec{a}$ that it is hard to see any content in them. And, indeed, if we were only studying the mechanics of single particles we probably would not have introduced the concept of linear momentum. Nonetheless, the general result does apply:

> The net force $\vec{F}$ on a particle is the rate of change of its linear momentum, $\dot{\vec{L}}$.

A special important case is when there is no force and linear momentum is conserved (doesn't change). For a single particle momentum conservation means constant velocity motion.

Example: **Linear momentum check.**
For the simple ballistics solution above we evaluate the left side of the momentum balance equation

$$\int_0^t \vec{F}\,dt' = -mgt\,\hat{\jmath}.$$

Then evaluate the right side:

$$\Delta\vec{L} = \vec{L}_2 - \vec{L}_1 = [m(v_0\hat{\imath} + (v_0 - gt)\hat{\jmath})] - [m(v_0\hat{\imath} + v_0] = -mgt\,\hat{\jmath})$$

and check for equality: $-mgt\,\hat{\jmath} = -mgt\,\hat{\jmath}$. This force and momentum is plotted in $Fig.$ 10.23. The solution *is* consistent with linear momentum balance. Note that in this example there is no change in the component of linear momentum in the $\hat{\imath}$ direction; there is no force in the x direction so L_x is conserved.

Angular momentum

Angular momentum relative to point C is $\vec{H}_{/C} = \vec{r}_{/C} \times (m\vec{v})$, where $\vec{r}_{/C}$ is the position of the particle relative to fixed point C and $\vec{v}$ is the velocity of the particle. Angular momentum can be calculated relative to any point C. Which point you pick affects the value of the angular momentum. Sometimes $\vec{H}_{/C}$ it is written without the "/" as $\vec{H}_C$ The angular momentum theorems (facts) are:

$$\underbrace{\vec{r}_{/C} \times \vec{F}}_{\vec{M}_C} = \dot{\vec{H}}_{/C} \qquad \text{and} \qquad \underbrace{\int_{t_1}^{t_2} \vec{r}_{/C} \times \vec{F}\,dt}_{\text{Angular impulse}} = (\vec{H}_{/C})_2 - (\vec{H}_{/C})_1.$$

The torque $\vec{M}_C$ of *all* the external forces acting on a particle about point C is the rate of change of its angular momentum $\dot{\vec{H}}_{/C}$ about point C.

The intuitive notion is that angular momentum represents how much a particle is 'going around' point C. A particle gets more credit for going faster, for being more massive, and for being farther away*.

If the force on the particle is zero or passes through the point C, the torque (moment) of the force is zero and its angular momentum is conserved.

Example: **Angular momentum check.**]
Using the same ballistics example we check the solution for consistency with angular momentum balance. For no good reason lets use the origin for the angular momentum reference point. We could use any point. Again we compare the left and right sides and check for equality.

$$\underbrace{\int_0^t \vec{r}_{/0} \times \vec{F}\, dt}_{\text{Left side}} = \int_0^t \left(v_0 t'\hat{\imath} + (v_0 - g t'^2/2)\hat{\jmath}\right) \times (-mg\,\hat{\jmath})\, dt'$$

$$= \int_0^t -v_0 m g t' \hat{k}\, dt' \quad = \quad -v_0 m g \hat{k} t^2/2$$

$$\underbrace{\vec{H}_{/02} - \vec{H}_{/01}}_{\text{Right side}} = \vec{r}(t) \times (m\vec{v}(t)) - \vec{r}(0) \times (m\vec{v}(0))$$

$$= m\left(v_0 t\hat{\imath} + (v_0 t - g t^2/2)\hat{\jmath}\right) \times (v_0\hat{\imath} + (v_0 - g t)\hat{\jmath}) - m\left(\vec{0} \times \vec{v}(0)\right)$$

$$= \left(v_0 t - g t^2 - v_0 t + g t^2/2\right) m v_0 t \hat{k} \quad = \quad -v_0 m g t^2 \hat{k}/2.$$

This torque and angular momentum are plotted in *Fig*. 10.24. The 'left side' agrees with the 'right side' and angular momentum balance is satisfied, as it must be. In 2D problems like this there is only one non-trivial component to angular momentum, that in the out-of-plane (z) direction. In this case the angular momentum H_z is not conserved because the gravity force does have a torque about the z axis.

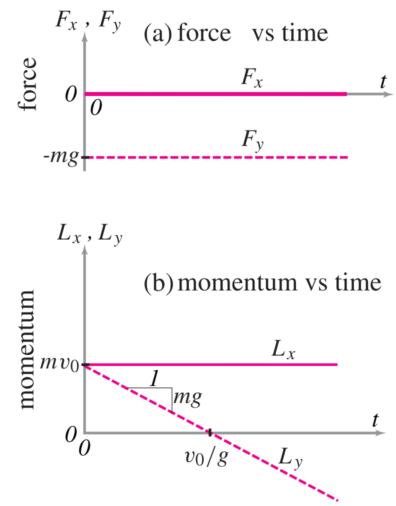

Figure 10.23: The force and linear momentum from the problem in *Fig*. 10.22. Note that the momentum curves are the integrals of the corresponding force curves.

Filename:tfig10-force-momentum.pdf

* **Intuition and angular momentum.** The notion that angular momentum is bigger if a given mass is further away might be counterintuitive. If one kilogram is going around your head once per second at a distance of a meter it has the same angular momentum, about your head, as a second equal mass going around at a radius of 10 meters and only going around once every 100 seconds. The second mass has 10 times the radius and one tenth the speed. Even though the second mass certainly doesn't make its rotation feel so present its angular momentum is as big. Intuitive or not, this is how angular momentum is defined. Its a useful concept so its worth adjusting your intuition to match it.

Power balance

The power P of a force $\vec{F}$ on a particle with velocity $\vec{v}$ is $P = \vec{F} \cdot \vec{v}$. The kinetic energy of a particle is $E_K = mv^2/2$. The power balance equations are

$$P = \dot{E}_K \qquad \text{and} \qquad \int_{t_1}^{t_2} P\, dt = E_{K2} - E_{K1}.$$

The power P of *all* the external forces acting on a particle is the rate of change of its kinetic energy $\dot{E}_K$. Or, integrating in time, the power added up over time is the net change in kinetic energy.

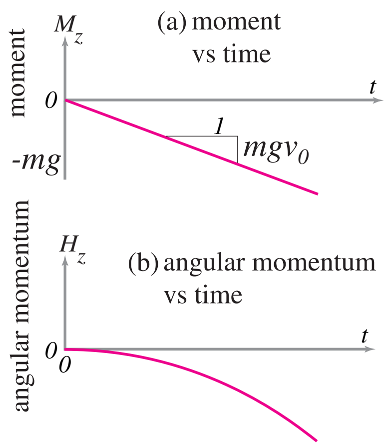

Figure 10.24: The torque and angular momentum about the origin for the problem in *Fig.* 10.22. For this 2D problem only the z component is non-trivial. Note that the angular momentum is the integral of the torque.

Filename:tfig10-torque-angmomentum

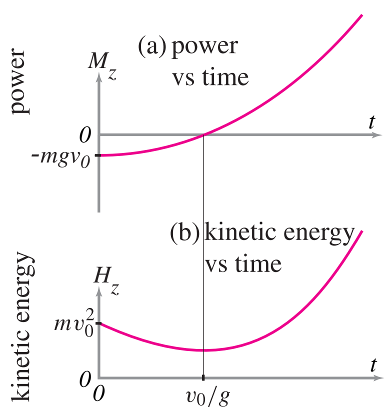

Figure 10.25: The power of the gravitational force and the total kinetic energy are plotted vs time for the ballistics problem of *Fig.* 10.22. Note that the kinetic energy is the integral of the power. The gravitational power starts out negative, goes to zero when the particle reaches the apex of its trajectory and then becomes positive evermore as the downwards gravitational force acts on a downwards moving particle. The kinetic energy starts at mv_0^2 and drops to $mv_0^2/2$ at the particle apex when v_y goes to zero. Then the kinetic energy increases forever more as the gravitational force does more and more work.

Filename:tfig10-power-energy

The situation is similar to that for 1D motion (section 9.2).

Power and work

The integral of power with respect to time can be replaced with a path integral for the work of a force. The key idea is in the differential expressions for an increment of work:

$$dW = P\,dt = \vec{F}\cdot\vec{v}\,dt = \vec{F}\cdot\frac{d\vec{r}}{dt}dt = \vec{F}\cdot d\vec{r}.$$

So

$$\begin{aligned} W_{12} &= W_{12} \\ \text{Power integrated in time} &= \text{Sum of work increments} \\ \int_{t_1}^{t_2} P\,dt &= \int_1^2 dW \\ \int_{t_1}^{t_2} \vec{F}\cdot\vec{v}\,dt &= \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r} \end{aligned} \tag{10.8}$$

Thus the power balance equation, integrated in time is equivalent to the "work energy" equation:

$$\begin{aligned} \text{Work on particle} &= \text{Change in its kinetic energy} \\ W_{12} &= \Delta E_{\mathrm{K}} \\ &= E_{\mathrm{K2}} - E_{\mathrm{K1}} \end{aligned} \tag{10.9}$$

Example: **Energy check.**
We can check the simple ballistics solution for consistency with energy balance. First lets compare the work to the change of kinetic energy.

$$\begin{aligned} \underbrace{\int_0^t \vec{F}\cdot\vec{v}\,dt}_{\text{Work}} &= \int_0^t (-mg\,\hat{\jmath})\cdot(v_0\hat{\imath} + (v_0 - gt')\hat{\jmath})\,dt' \\ &= \int_0^t -mg(v_0 - gt')\,dt' = -mg\,v_0 t + mg^2t^2/2 \\ \underbrace{E_{\mathrm{K2}} - E_{\mathrm{K1}}}_{\text{Change in kinetic energy}} &= m(|\vec{v}|^2 - |\vec{v}_0|^2)/2 \\ &= m\left((v_0^2 + (v_0 - gt)^2 - 2v_0^2\right)/2 = \underbrace{-mg\,v_0 t + mg^2t^2/2}_{\text{agrees}} \end{aligned}$$

This power and kinetic energy are plotted in *Fig.* 10.25. In this check we have not taken advantage of the fact that this particular force is conservative.

The work of a force $\vec{F}$: W_{12}

Previously in Physics, and more recently in one dimensional dynamics here, you learned that

Work is force times distance.

This is actually a special case of the formula

$$P = \vec{F} \cdot \vec{v}.$$

How is that? If $\vec{F}$ is constant and parallel to the displacement $\Delta\vec{x}$, then

$$\begin{aligned} W_{12} &= \int \dot{W}\, dt = \int P\, dt = \int \vec{F} \cdot \underbrace{\vec{v}\, dt}_{d\vec{x}} = \int \vec{F} \cdot d\vec{x} = \vec{F} \cdot \int d\vec{x} \\ &= \vec{F} \cdot \Delta\vec{x} = F\Delta x = \text{Force} \cdot \text{distance}. \end{aligned}$$

In 2 and 3 dimensions there are subtleties involved with the concept of work because of its dependence on which path in space the force works on. These 'path dependence' subtleties are often covered in some detail in calculus courses in the sections on vector calculus, path integrals, gradient and curl. We discuss the relevant highlights below.

Potential energy of a force

Some forces (read force fields $\vec{F} = \vec{F}(\vec{r})$) have the property that the work they do is independent of the path followed by the material point as the force acts. If the work of a force is path independent in this way (see box 10.3 on page 573, then a potential energy can be defined so that the work done by the force is the decrease in the Potential Energy

$$-\Delta E_{\text{P}} = W_{12} = E_{\text{P}1} - E_{\text{P}2}$$

The common examples are listed below:

- **linear spring:** $E_{\text{P}} = (1/2)k(\text{stretch})^2$.
- **gravity near earth's surface:** $E_{\text{P}} = mgh$
- **gravity between spheres or points:** $E_{\text{P}} = -MmG/r$
- **constant force $\vec{F}$ acting on a point:** $E_{\text{P}} = -\vec{F} \cdot \vec{r}$

In all cases a constant could be added to the potential energy and it would still be a legitimate potential energy for the force.

In the cases of the spring and gravity between spheres, the change in potential energy is the net work done by the spring or gravity on the pair of objects between which the force acts. If both ends of a spring are moving, the net work of the spring on the two objects to which it is connected is the decrease in potential energy of the spring.

There is a possible source of confusion in our using the same symbol E_P to represent the potential work of an external force and for internal potential energy. In practice, however, they are used identically, so we use the same symbol for both. The potential energy in a stretched spring is the same whether it is the cause of force on a system or it is internal to the system.

Example: **Checking conservation of energy**
Because the gravity force is conservative we can also check our simple ballistics solution for consistency with conservation of energy. Taking the potential energy as $E_\text{P} = mgh = mgy$ we find. as expected, that the solution does have the property that

$$\begin{aligned} E_{tot1} &\stackrel{?}{=} E_{tot2} \\ E_{\text{K}1} + E_{\text{P}1} &\stackrel{?}{=} E_{\text{K}2} + E_{\text{P}2} \\ mv_0^2 + 0 &\stackrel{?}{=} \left(v_0^2 + (v_0 - gt)^2\right) m/2 + mg\,\overbrace{(v_0 t - gt^2/2)}^{y} \\ mv_0^2 &\stackrel{\checkmark}{=} mv_0^2 \quad \text{(Checks)} \end{aligned}$$

Using momentum and energy as a check of a numerical solution

You obtain a numerical solution to $\vec{F} = m\vec{a}$ by setting up the set of first order differential equations 10.2 on page 555. In turn, these can be written in explicit scalar form as $eqn.$ (10.3).

While you solve these equations you can add further first order equations that you can use in your energy and momentum checks. These evaluate the integrals for linear impulse, angular impulse and work.

$$\begin{aligned} \frac{d}{dt}(\text{linear impulse}) &= \vec{F}, \\ \frac{d}{dt}(\text{angular impulse}) &= \vec{r}_{/C} \times \vec{F}, \text{and} \\ \dot{W} &= \vec{F} \cdot \vec{v}. \end{aligned}$$

The first two equations are short hand for 2 (or 3) first order scalar equations for motion in 2 (or 3) spatial dimensions. If these are added to the system of ordinary differential equations that you solve, they can be used to check the solution.

Summary on using energy and momentum to check a solution

Because the momentum and energy facts and theorems apply to any motion consistent with $\vec{F} = m\vec{a}$ they can be used as a consistency check on any solutions you find to the differential equations of motion ($\vec{F} = m\vec{a}$).

Here is the general situation. You are given $\vec{F}(t, \vec{r}, \vec{v})$. You are given initial conditions $\vec{r}_0$ and $\vec{v}_0$ at, say, $t = 0$. Using computer integration or pencil and paper methods, you solve the differential equation $\vec{a} = \vec{F}/m$ to get $\vec{r}(t)$ and $\vec{v}(t)$. Now your solution can be checked for consistency with the energy and momentum theorems. In particular, your solution, if it is correct, must satisfy

- Linear momentum balance: $\int_0^t \vec{F}\,dt = \vec{L}_2 - \vec{L}_1$;
- Angular momentum balance: $\int_0^t \vec{r} \times \vec{F}\,dt = \vec{H}_2 - \vec{H}_1$;
- Work-energy: $\int_0^t \vec{F} \cdot \vec{v}\,dt = E_{K2} - E_{K1}$.

These have been used in the simple ballistics example above. That linear momentum balance, angular momentum balance and energy balance, all are consistent to an assumed solution lends credence to its correctness. For simple problems with such simple analytical solutions, using this consistency is not the most efficient way of checking a candidate solution's veracity. We would be better off just plugging the proposed solution back into the differential equation to see if it was satisfied. But in more complex problems and in numerical solutions, checks like those here are sometimes simpler to make. Some more comments about these checks:

- The angular momentum check can be used relative to any fixed point you choose. If you can find a point where, say, the applied force has no moment, then the change of angular momentum should be zero about that point.

10.3 *THEORY*
Conservative forces and non-conservative forces

Imagine that the force $\vec{F}$ on a particle is known to depend on the position $\vec{r}$ of a particle as it moves. This dependence of $\vec{F}$ on $\vec{r}$ is called a *force field*:

$$\vec{F} = \vec{F}(\vec{r}).$$

As the particle moves from one point $\vec{r}_1$ to another $\vec{r}_2$ we can evaluate the work of this force field as

$$W_{12} = \int_{t_1}^{t_2} P\,dt = \int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\,dt = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}(\vec{r}) \cdot d\vec{r}.$$

But what if the particle moves between the same two points but along a different path, is the work W_{12} the same? If that is true then the work going from $\vec{r}_1$ to $\vec{r}_2$ and back would be zero. Which means the work of the force when the particle moves on any closed path would be zero. Here is an example of a force field $\vec{F}(\vec{r})$ in the xy plane where the work in going on a closed path, from $\vec{r}_1$ to $\vec{r}_1$, from home to home, is not zero:

$$\vec{F} = C[\hat{k} \times \vec{r}] = C[-y\hat{\imath} + x\hat{\jmath}]$$

where C is a constant. This force pushes the particle around in circles. So, if the particle moves on the circular path

$$\vec{r} = r_0[\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}] \qquad (0 \le \theta \le 2\pi)$$

then the work is the force magnitude times the arc-length (the force is parallel to the velocity for this path) and so,

$$\int_{\theta_1}^{\theta_2} \vec{F} \cdot d\vec{r} = 2\pi C r_0 \neq 0.$$

This force field gives non-zero work for some closed paths, thus is path dependent for open paths and therefor is *non-conservative.* How can you tell if a force field is conservative or not. This, you learn in vector calculus, holds if the curl of $\vec{F}$ is zero, $\vec{\nabla} \times \vec{F} = \vec{0}$, everywhere.

Forces from any combination of springs and gravity are always conservative.

10.4 *THEORY*

Derivation of momentum, angualr momentum and energy theorems for a point mass

For a point-mass particle the principles of linear momentum, angular momentum and energy are theorems that can be derived simply from

$$\vec{F} = m\vec{a}$$

as follows.

Linear momentum

Define linear momentum as $\vec{L} = m\vec{v}$ then differentiating we have the equation $\vec{F} = m\vec{a}$. It is not so much a derivation but a restatement to write:

$$\vec{F} = \dot{\vec{L}}.$$

Integrating both sides in time we get

$$\underbrace{\int_{t_1}^{t_2} \vec{F}\, dt}_{\text{impulse}} = \underbrace{\vec{L}_2 - \vec{L}_1}_{\text{change of momentum}}.$$

This is the principle of impulse and momentum.

Angular momentum

Start with $\vec{F} = m\vec{a}$ and take the cross product of both sides with the position relative to a fixed point C and you get

$$\vec{r}_{/C} \times \vec{F} = \vec{r}_{/C} \times (m\vec{a}).$$

Now if we define $\vec{H}_{/\mathrm{C}} = \vec{r}_{/C} \times (m\vec{v})$ we can differentiate to find that, writing out all details,

$$\begin{aligned}
\dot{\vec{H}}_{/\mathrm{C}} &= \frac{d}{dt}\left(\vec{r}_{/\mathrm{C}} \times (m\vec{v})\right) \\
&= \frac{d}{dt}\left((\vec{r} - \vec{r}_{\mathrm{C}}) \times m\vec{v}\right) \\
&= m\left(\dot{\vec{r}} - \dot{\vec{r}}_{\mathrm{C}}\right) \times \vec{v} + m\left(\vec{r} - \vec{r}_{\mathrm{C}}\right) \times \dot{\vec{v}} \\
&= m\left(\vec{v} - \vec{0}\right) \times \vec{v} + m\left(\vec{r} - \vec{r}_{\mathrm{C}}\right) \times \dot{\vec{v}} \\
&= m\underbrace{\vec{v} \times \vec{v}}_{\vec{0}} + m\left(\vec{r} - \vec{r}_{\mathrm{C}}\right) \times \dot{\vec{v}} \\
&= m\vec{r}_{/\mathrm{C}} \times (m\dot{\vec{v}}) \\
&= \vec{r}_{/\mathrm{C}} \times (m\vec{a})
\end{aligned}$$

Putting these together we have

$$\vec{r}_{/C} \times \vec{F} = \dot{\vec{H}}_{/\mathrm{C}}.$$

Integrating both sides with respect to time we get that the net angular impulse is the change in angular momentum.

$$\int_{t_1}^{t_2} \vec{r}_{/C} \times \vec{F}\, dt = (\vec{H}_{/\mathrm{C}})_2 - (\vec{H}_{/\mathrm{C}})_1.$$

Power and kinetic energy

The power equation is found with a shade more difficulty. We take the equation $\vec{F} = m\vec{a}$ and dot both sides with the velocity $\vec{v}$ of the particle:

$$\vec{F} \cdot \vec{v} = m\vec{a} \cdot \vec{v}. \tag{10.10}$$

Evaluating $\vec{v} \cdot \vec{a}$ is most easily done with the benefit of hindsight. So we cheat and look at the time derivative of the speed squared:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}v^2\right) &= \frac{1}{2}\frac{d}{dt}(\vec{v} \cdot \vec{v}) \\
&= \frac{1}{2}\left(\dot{\vec{v}} \cdot \vec{v} + \vec{v} \cdot \dot{\vec{v}}\right) \\
&= \vec{v} \cdot \dot{\vec{v}} \\
&= \vec{v} \cdot \vec{a}
\end{aligned}$$

Applying this result to $eqn.$ (10.10) we get

$$\underbrace{\vec{F} \cdot \vec{v}}_{P} = \frac{d}{dt}\underbrace{\left(\frac{1}{2}mv^2\right)}_{E_{\mathrm{K}}},$$

the energy (or power balance) equation for a particle.

Integrating in time we get

$$\underbrace{\int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\, dt}_{\text{Work}} = \underbrace{E_{\mathrm{K}2} - E_{\mathrm{K}1}}_{\text{Change in kinetic energy}},$$

Power and work and energy

Because $\vec{F} \cdot \vec{v}\, dt = \vec{F} \cdot d\vec{r}$ the time integral of power can be replaced with a path integral, the standard work integral:

$$\int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\, dt = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}.$$

If $\vec{F}$ is a conservative force field, meaning a function of position, then $E_{\mathrm{P}}(\vec{r})$ exists, so that

$$-\vec{\nabla} E_{\mathrm{P}} = \vec{F} \quad \text{then} \quad \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = E_{\mathrm{P}}(\vec{r}_2) - E_{\mathrm{P}}(\vec{r}_1)$$

and the work-energy equation becomes

$$E_2 = E_1$$

Where $E = E_{\mathrm{K}} + E_{\mathrm{P}}$ is defined as the total energy.

- If the applied force is conservative, the work integral can be replaced by the change in potential energy and the work-energy check is a check of the conservation of energy.
- If you try to make the checks with pencil and paper the checks can sometimes be harder to implement than it was to find the original solution.
- These checks are often *very useful*, and this is perhaps an understatement, for checking the validity of numerical solutions of dynamics equations. Basically you shouldn't trust yours or any body else's code unless such checks have been made. It is hard to write correct code without making such checks. And such checks are a strong sign of code reliability because an error in computer code will usually lead to an error in momentum balance, angular momentum balance or energy balance.

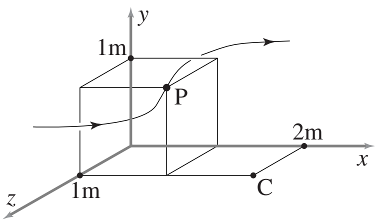

Figure 10.26:

Filename:sfig1-1-DH1

SAMPLE 10.8 **Basic calculations:** Find $\vec{L}$, $\dot{\vec{L}}$, $\vec{H}_{/C}$, $\dot{\vec{H}}_{/C}$, E_K, $\dot{E}_K$ for a given particle P with mass $m_P = 1\,\text{kg}$, given position, velocity, acceleration, and a point C. Specifically, we are given $\vec{r}_P = (\hat{\imath} + \hat{\jmath} + \hat{k})\,\text{m}$, $\vec{v}_P = 3\,\text{m/s}(\hat{\imath}+\hat{\jmath})$, $\vec{a}_P = 2\,\text{m/s}^2(\hat{\imath}-\hat{\jmath}-\hat{k})$, and $\vec{r}_C = (2\hat{\imath}+\hat{k})\,\text{m}$.

Solution Since $\vec{r}_P = (\hat{\imath} + \hat{\jmath} + \hat{k})\,\text{m}$ and $\vec{r}_C = (2\hat{\imath} + \hat{k})\,\text{m}$,

$$\vec{r}_{P/C} = \vec{r}_P - \vec{r}_C = (-\hat{\imath} + \hat{\jmath})\,\text{m}.$$

So we have the motion quantities

$$\begin{aligned}
\vec{L} &= m\vec{v}_P \\
&= (1\,\text{kg})\cdot[(3\,\text{m/s})(\hat{\imath} + \hat{\jmath})] \\
&= 3(\hat{\imath} + \hat{\jmath})\frac{\text{kg}\cdot\text{m}}{\text{s}} \\
&= 3\,\text{N}\cdot\text{s}(\hat{\imath} + \hat{\jmath})
\end{aligned}$$

$$\begin{aligned}
\dot{\vec{L}} &= m\vec{a}_P \\
&= (1\,\text{kg})[(2\,\text{m/s}^2)(\hat{\imath} - \hat{\jmath} - \hat{k})] \\
&= 2(\hat{\imath} - \hat{\jmath} - \hat{k})\frac{\text{kg}\cdot\text{m}}{\text{s}^2} \\
&= 2\,\text{N}(\hat{\imath} - \hat{\jmath} - \hat{k})
\end{aligned}$$

$$\begin{aligned}
\vec{H}_{/C} &= \vec{r}_{P/C} \times m\vec{v}_P \\
&= [(-\hat{\imath} + \hat{\jmath})\,\text{m}] \times [(1\,\text{kg})3\,\text{m/s}(\hat{\imath} + \hat{\jmath})] \\
&= -(6\,\text{kg}\cdot\text{m}^2/\,\text{s})\hat{k} \qquad (10.11)
\end{aligned}$$

$$\begin{aligned}
\dot{\vec{H}}_{/C} &= \vec{r}_{P/C} \times m\vec{a} \\
&= [(-\hat{\imath} + \hat{\jmath})m] \times [(1\,\text{kg})2\,\text{m/s}^2(\hat{\imath} - \hat{\jmath} - \hat{k})] \\
&= -(2\,\text{kg}\cdot\text{m}^2/\,\text{s}^2)(\hat{\imath} + \hat{\jmath})
\end{aligned}$$

$$\begin{aligned}
E_K &= \frac{1}{2}m|\vec{v}_P|^2 \\
&= \frac{1}{2}(1\,\text{kg})(3\sqrt{2}\,\text{m/s})^2 \\
&= 9\frac{\text{kg}\cdot\text{m}^2}{\text{s}^2} \\
&= 9\,\text{N}\cdot\text{m}
\end{aligned}$$

$$\begin{aligned}
\dot{E}_K &= \frac{d}{dt}(\frac{m}{2}\vec{v}_P\cdot\vec{v}_P) \\
&= \frac{m}{2}[\vec{v}_P\cdot\dot{\vec{v}}_P + \dot{\vec{v}}_P\cdot\vec{v}_P] \\
&= m\vec{v}_P\cdot\vec{a}_P \\
&= 1\,\text{kg}[(3\,\text{m/s})(\hat{\imath} + \hat{\jmath})]\cdot[(2\,\text{m/s}^2)(\hat{\imath} - \hat{\jmath} - \hat{k})] \\
&= 0.
\end{aligned}$$

Note: $\frac{d}{dt}(\frac{1}{2}v^2) \neq |\vec{v}||\vec{a}|$.

SAMPLE 10.9 Direct application of the formulas: A 2 kg block is moving with a velocity $\vec{v}(t) = u_0 e^{-ct}\hat{\imath} + v_0\hat{\jmath}$, where $u_0 = 5\,\text{m/s}$, $v_0 = 10\,\text{m/s}$, and $c = 0.5/\,\text{s}$. Consider the time interval between $t_1 = 1\,\text{s}$ to $t_2 = 3\,\text{s}$.

1. Find the net change in the linear momentum of the block, $\Delta\vec{L} = \vec{L}(t_2) - \vec{L}(t_1)$.
2. Find the force $\vec{F}(t)$ on the block and compute the impulse $\int_{t_1}^{t_2}\vec{F}\,dt$ and show that it is the same as $\Delta\vec{L}$ computed above.
3. Find the change in kinetic energy from direct computation of energy and compare with work done by computing $\int_{t_1}^{t_2} P\,dt$.

Solution

1. For the given block we have, $\vec{L} = m\vec{v} = m(u_0 e^{-ct}\hat{\imath} + v_0\hat{\jmath})$. Therefore,

$$\Delta L = \vec{L}(t_2) - \vec{L}(t_1) = m u_0\left(e^{-ct_2} - e^{-ct_1}\right)\hat{\imath}.$$

Substituting the given values, $m = 2\,\text{kg}$, $u_0 = 5\,\text{m/s}$, $v_0 = 10\,\text{m/s}$, $c = 0.5/\,\text{s}$, $t_1 = 1\,\text{s}$, and $t_2 = 3\,\text{s}$ we get

$$\Delta\vec{L} = 2\,\text{kg}\cdot 5\,\text{m/s}(e^{-0.5/\,\text{s}\cdot 3\,\text{s}} - e^{-0.5/\,\text{s}\cdot 1\,\text{s}})\hat{\imath} = -(3.83\,\text{kg}\cdot\text{m/s})\hat{\imath}.$$

$$\boxed{\Delta\vec{L} = -(3.83\,\text{kg}\cdot\text{m/s})\hat{\imath}}$$

2. To calculate the impulse, $\int\vec{F}\,dt$, we need to find the force first. SInce $\vec{F} = m\vec{a} = m\dot{\vec{v}}$, we get

$$\vec{F}(t) = m\frac{d}{dt}(u_0 e^{-ct}\hat{\imath} + v_0\hat{\jmath}) = -mcu_0 e^{-ct}\hat{\imath}.$$

Hence, the impulse is

$$\begin{aligned}\int_{t_1}^{t_2}\vec{F}\,dt &= -\int_{t_1}^{t_2} mcu_0 e^{-ct}\,dt\hat{\imath} = mu_0(e^{-ct_2} - e^{-ct_1})\\ &= 2\,\text{kg}\cdot 5\,\text{m/s}(e^{-0.5/\,\text{s}\cdot 3\,\text{s}} - e^{-0.5/\,\text{s}\cdot 1\,\text{s}})\hat{\imath}\\ &= -3.83\ \text{kg}\cdot\text{m/s}\hat{\imath}\end{aligned}$$

which is, expectedly, the same answer as obtained above for ΔL.

3. To find the kinetic energy, we need the speed of the particle, $v = |\vec{v}| = \sqrt{v_x^2 + v_y^2}$. Now, the change in kinetic energy is

$$\begin{aligned}\Delta E_\text{K} &= E_{\text{K}2} - E_{\text{K}1} = \frac{1}{2}m\left(v_2^2 - v_1^2\right)\\ &= \frac{1}{2}m\left(u_0^2 e^{-2ct_2} + v_0^2 - u_0^2 e^{-2ct_1} - v_0^2\right)\\ &= \frac{1}{2}mu_0^2\left(e^{-2ct_2} - e^{-2ct_1}\right) = -7.95\,\text{N}\cdot\text{m}.\end{aligned}$$

Now, we can compare this value by computing the work done $\int P\,dt$, since $\Delta E_\text{K} = \int P\,dt$. To compute the power $P = \vec{F}\cdot\vec{v}$, we need to find the dot product between the force and the velocity. Since $\vec{F} = -mcu_0 e^{-ct}\hat{\imath}$, and $\vec{v} = u_0 e^{-ct}\hat{\imath} + v_0\hat{\jmath}$, we get, $\vec{F}\cdot\vec{v} = -mcu_0^2 e^{-2ct}$. Therefore, the work done is,

$$\begin{aligned}W &= \int_{t_1}^{t_2} P\,dt = \int_{t_1}^{t_2} -mcu_0^2 e^{-2ct}\,dt\\ &= -mcu_0^2(e^{-2ct_2} - e^{-2ct_1}) = -7.95\,\text{N}\cdot\text{m}.\end{aligned}$$

$$\boxed{\Delta E_\text{K} = -7.95\,\text{N}\cdot\text{m}, \quad W = -7.95\,\text{N}\cdot\text{m}}$$

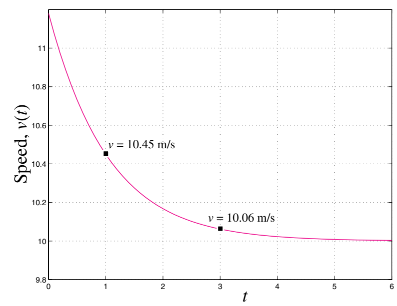

Figure 10.27: The plot of speed $v = |\vec{v}| = \sqrt{v_x^2 + v_y^2}$ vs time. The speeds at t_1 and t_2 are of interest for computing the kinetic energy at the two instants.

Filename:sfig10-2-speed

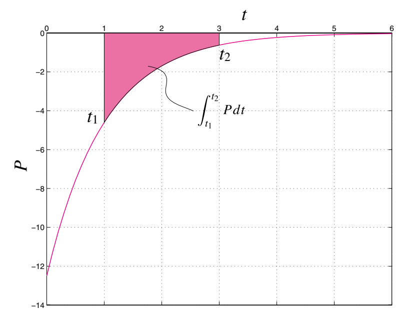

Figure 10.28: The area under the $P(t)$ curve between t_1 and t_2 is the work done $W = \int_{t_1}^{t_2} P\,dt$.

Filename:sfig10-2-power

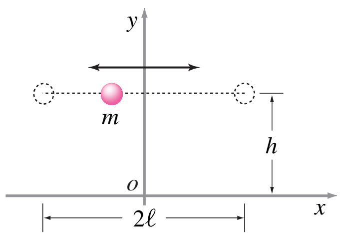

Figure 10.29:

Filename:sfig3-2-direct-appl

SAMPLE 10.10 Angular momentum: direct application of the formula. The position of a particle of mass $m = 0.5\,\text{kg}$ is $\vec{\boldsymbol{r}}(t) = \ell\,\sin(\lambda t)\hat{\imath} + h\,\hat{\jmath}$; where $\lambda = \pi/2\,\text{rad/s}, h = 2\,\text{m}, \ell = 2\,\text{m}$, and $\vec{\boldsymbol{r}}$ is measured from the origin.

1. Find the net change in linear momentum $\Delta\vec{\boldsymbol{L}}$ of the particle between $t = 1\,\text{s}$ and $t = 3\,\text{s}$.
2. Find the net change in angular momentum $\Delta\vec{\boldsymbol{H}}_{/\text{O}}$ of the particle about the origin between $t = 1\,\text{s}$ and $t = 3\,\text{s}$.
3. Find the angular impulse $\int_{1\,\text{s}}^{3\,\text{s}} \boldsymbol{M}\,dt$ about the origin and compare the result with $\Delta\vec{\boldsymbol{H}}_{/\text{O}}$ found above.

Solution

1. Linear momentum: Let the two instants of interest be $t_1(= 1\,\text{s})$ and $t_2(= 3\,\text{s})$. The net change in linear momentum, $\Delta\vec{\boldsymbol{L}} = \vec{\boldsymbol{L}}_2 - \vec{\boldsymbol{L}}_1 = m(\vec{\boldsymbol{v}}_2 - \vec{\boldsymbol{v}}_1)$. Since $\vec{\boldsymbol{v}} = \dot{\vec{\boldsymbol{r}}} == \ell\,\lambda\,\cos(\lambda t)\hat{\imath}$, we get

$$
\begin{aligned}
\Delta\vec{\boldsymbol{L}} &= m(\vec{\boldsymbol{v}}_2 - \vec{\boldsymbol{v}}_1) = m\;\ell\;\lambda(\cos\lambda t_2 - \cos\lambda t_1)\hat{\imath} \\
&= (0.5\,\text{kg})(2\,\text{m})\left(\frac{\pi}{2}\,\text{rad/s}\right)\left(\cos\frac{\pi}{2} - \cos\frac{3\pi}{2}\right)\hat{\imath} \\
&= \vec{0}.
\end{aligned}
$$

The answer makes sense because both $\vec{\boldsymbol{v}}_1 = \vec{0}$ and $\vec{\boldsymbol{v}}_2 = \vec{0}$. In fact, finding the velocity at $t_1 = 1\,\text{s}$ and $t_2 = 3\,\text{s}$ would have made the calculation much simpler.

$$\boxed{\Delta\vec{\boldsymbol{L}} = \vec{0}}$$

2. Angular momentum: The net change in angular momentum between t_1 and t_2 is,

$$\Delta\vec{\boldsymbol{H}}_{/\text{O}} = (\vec{\boldsymbol{H}}_{/\text{O}})_2 - (\vec{\boldsymbol{H}}_{/\text{O}})_1 = \vec{\boldsymbol{r}}_{/\text{O}} \times m\underbrace{\vec{\boldsymbol{v}}_2}_{\vec{0}} - \vec{\boldsymbol{r}}_{/\text{O}} \times m\underbrace{\vec{\boldsymbol{v}}_1}_{\vec{0}} = \vec{0}.$$

$$\boxed{\Delta\vec{\boldsymbol{H}}_{/\text{O}} = \vec{0}}$$

Note that it so happens that velocities at the two instants are zero and hence, both $(\vec{\boldsymbol{H}}_{/\text{O}})_1$ and $(\vec{\boldsymbol{H}}_{/\text{O}})_2$ are zero, making $\Delta\vec{\boldsymbol{H}}_{/\text{O}}$ also zero. It is, however, possible that we could get $\Delta\vec{\boldsymbol{H}}_{/\text{O}}$ to be zero even if the $(\vec{\boldsymbol{H}}_{/\text{O}})_1$ and $(\vec{\boldsymbol{H}}_{/\text{O}})_2$ were non-zero (when they are equal).

3. Moment impulse: Now, let us find the impulse due to the moment, $\int M\,dt$ between the two given time instants and see if that matches with the net zero change in angular momentum. We first need to compute the moment $\vec{\boldsymbol{M}}_\text{O} = \vec{\boldsymbol{r}}_{/\text{O}} \times \vec{\boldsymbol{F}} = \vec{\boldsymbol{r}}_{/\text{O}} \times m\,\vec{\boldsymbol{a}}$:

$$\vec{\boldsymbol{M}}_{/\text{o}}(t) = \vec{\boldsymbol{r}}_{/\text{O}} \times \times m\vec{\boldsymbol{a}} = (\ell\;\sin(\lambda t)\hat{\imath} + h\,\hat{\jmath}) \times m(-\ell\,\lambda^2\sin\lambda t\,\hat{\imath}) = m\,\ell\,h\,\lambda^2\sin\lambda t\,\hat{\boldsymbol{k}}.$$

Therefore, the impulse due to this moment is

$$
\begin{aligned}
\int_{t_1}^{t_2}\vec{\boldsymbol{M}}_{/\text{O}}dt &= \int_{1\,\text{s}}^{3\,\text{s}}(m\,\ell\,h\,\lambda^2\sin\lambda t\,\hat{\boldsymbol{k}})dt = m\,\ell\,h\,\lambda^2\hat{\boldsymbol{k}}\int_{1\,\text{s}}^{3\,\text{s}}\sin(\lambda t)dt \\
&= m\,\ell\,h\,\lambda^2\hat{\boldsymbol{k}}\left[-\frac{\cos\lambda t}{\lambda}\right]_{1\,\text{s}}^{3\,\text{s}} = m\,\ell\,h\,\lambda\hat{\boldsymbol{k}}\left[\cos\frac{3\pi}{2} - \cos\frac{\pi}{2}\right] \\
&= \vec{0}
\end{aligned}
$$

as expected. It can also be seen from a plot of $|\vec{M}_{/O}|$ vs t, as shown in *Fig.* 10.30, that the net area under the moment between t_1 and t_2 is zero, giving a zero moment impulse.

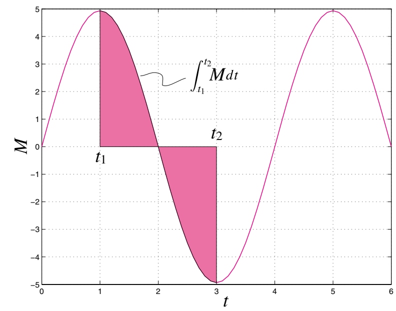

Figure 10.30: Plot of $M(t)$ where $\vec{M}_{/O}(t) = M(t)\hat{k}$. The area under the moment curve between t_1 and t_2 is the magnitude of the moment impulse.

Filename:sfig10-2-moment

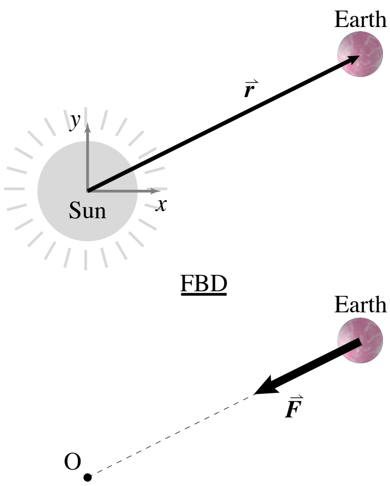

Figure 10.31: The earth moving around a fixed sun. The attraction force $\vec{F}$ is directed "centrally" towards the sun and has magnitude proportional to both masses and inversely proportional to the distance squared.

Filename:tfigure-earthfixedsun

10.3 Central-force motion and celestial mechanics

One of Isaac Newton's greatest achievements was the explanation of Kepler's laws of planetary motion. Kepler, using the meticulous observations of Tycho Brahe characterized the orbits of the planets about the sun with his 3 famous laws:

- Each planet travels on an ellipse with the sun at one focus.
- Each planet goes faster when it is close to the sun and slower when it is further. It speeds and slows so that the line segment connecting the planet to the sun sweeps out area at a constant rate.
- Planets that are further from the sun take longer to go around. More exactly, the periods are proportional to the lengths of the ellipses to the 3/2 power.

Newton, using his equation $\vec{F} = m\vec{a}$ and his law of universal gravitational attraction, was able to formulate a differential equation governing planetary motion. He was also able to solve this equation and found that it exactly predicts all three of Kepler's laws.

The Newtonian description of planetary motion is the most historically significant example of *central-force motion* where,

- the only force acting on a particle is directed towards the origin of a given coordinate system, and
- the magnitude of the force depends only on distance between attracting points.

If we define the position of the particle as $\vec{r}$ with magnitude r, linear momentum balance for central-force motion is

$$\begin{aligned} \sum \vec{F}_i &= \dot{\vec{L}} \\ \Rightarrow \quad \vec{F} &= m\vec{a} \\ \Rightarrow \quad F(r)\left(\frac{-\vec{r}}{r}\right) &= m\ddot{\vec{r}}, \end{aligned} \tag{10.12}$$

where $-\vec{r}/r$ is a unit vector pointed toward the origin and $F(r)$ is the magnitude of the origin-attracting force.

For the rest of this section we consider some of the consequences of *eqn.* (10.12). We start with the most historically important example.

Motion of the earth around a fixed sun

For simplicity let's assume that the sun does not move and that the motion of the earth lies in a plane. Newton's law of gravitation says that the attractive force of the sun on the earth is proportional to the masses of the sun and earth and inversely proportional to the distance between them squared (*Fig.* 10.31). Thus we have *

$$F = \frac{G m_e m_s}{r^2}$$

where m_e and m_s are the masses of the earth and sun, r is the distance between the earth and sun. 'Big G' is a universal constant $G \approx 6.67 \cdot 10^{-17}\,\mathrm{N\,m^2/kg^2}$. What is the vector-valued force on the earth? It is its magnitude times a unit vector in the appropriate direction.

$$\begin{aligned}
\vec{F} &= \left(\frac{G m_e m_s}{r^2}\right)\left(\frac{-\vec{r}}{|\vec{r}|}\right) \\
\Rightarrow \quad \vec{F} &= -G m_e m_s \left(\frac{\vec{r}}{r^3}\right) \\
\Rightarrow \quad \vec{F} &= -G m_e m_s \left(\frac{x\hat{\imath} + y\hat{\jmath}}{(x^2+y^2)^{3/2}}\right) \qquad (10.13)
\end{aligned}$$

where we have used that $\vec{r} = x\hat{\imath} + y\hat{\jmath}$, $r = |\vec{r}| = \sqrt{x^2+y^2}$, and $\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$. Now we can write the linear momentum balance equation for the earth in great detail.

$$\vec{F} = m\vec{a} \quad \Rightarrow \quad -G m_e m_s \left(\frac{x\hat{\imath} + y\hat{\jmath}}{(x^2+y^2)^{3/2}}\right) = m_e(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}) \qquad (10.14)$$

Taking the dot product of equation 10.14 with $\hat{\imath}$ and $\hat{\jmath}$ successively (*i.e.*, taking x and y components) gives two scalar second order ordinary differential equations:

$$\ddot{x} = \frac{-G m_s x}{(x^2+y^2)^{3/2}} \quad \text{and} \quad \ddot{y} = \frac{-G m_s y}{(x^2+y^2)^{3/2}}. \qquad (10.15)$$

This pair of coupled second order differential equations describes the motion of the earth.* Pencil and paper solution is possible, Newton did it, but is a little too hard for this book. So we resort to computer solution. To set this up we put equations *eqn.* (10.15) in the form of a set of coupled first order ordinary differential equations. If we define $z_1 = x$, $z_2 = \dot{x}$, $z_3 = y$, and $z_4 = \dot{y}$. We can now write equations 10.15 as

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= -G m_s z_1/(z_1^2 + z_3^2)^{3/2} \\
\dot{z}_3 &= z_4 \\
\dot{z}_4 &= -G m_s z_3/(z_1^2 + z_3^2)^{3/2}. \qquad (10.16)
\end{aligned}$$

To actually solve these numerically we need a value for $G m_s$ and initial conditions. The solutions of these equations on the computer are all, within numerical error, consistent with Kepler's laws.

Without a full solution, there are some things we can figure out relatively easily.

* Soon after Newton, Cavendish found G in his lab by delicately measuring the *small* attractive force between two balls. The gravitational attraction between two 1 kg balls a meter apart is about a ten-millionth of a billionth of a Newton (a Newton is about a fifth of a pound).

* Note that G appears in the product $G m_s$. Newton didn't know the value of big G, but he could do a lot of figuring without it. All he needed was the product $G m_s$ which he could find from the period and radius of the earth's orbit. The entanglement of G with the mass of the sun is why some people call Cavendish's measurement of big G, "weighing the sun". From Newton's calculation of $G m_s$ and Cavendish's measurement of G you can find m_s. Naturally, the real history is a bit more complicated. Cavendish presented his result as weighing the earth.

Circular orbits

We generally think of the motions of the planets as being roughly circular orbits. In fact, for any attractive central force one of the possible motions is a circular orbit. Rather than trying to derive this, let's assume a circular solution and see if it solves the equations of motion. A constant speed circular orbit with angular frequency ω and radius r_o obeys the parametric equation

$$\begin{aligned} \vec{\boldsymbol{r}} &= r_o\,(\cos(\omega t)\hat{\imath} + \sin(\omega t)\hat{\jmath}) \\ \text{differentiating twice} \quad \Rightarrow \quad \ddot{\vec{\boldsymbol{r}}} &= -\omega^2 r_o\,(\cos(\omega t)\hat{\imath} + \sin(\omega t)\hat{\jmath}) \\ &= -\omega^2 \vec{\boldsymbol{r}}. \end{aligned} \tag{10.17}$$

Comparing *eqn.* (10.17) with *eqn.* (10.12) we see we have an identity (a solution to the equation) if

$$\omega^2 = \frac{F(r)}{mr}.$$

In the case of gravitational attraction where $m = m_e$ we have $F(r) = Gm_s m_e / r^2$ so we get circular motion with

$$\omega^2 = \frac{Gm_s}{r^3} \qquad \Rightarrow \qquad T = 2\pi\sqrt{\frac{}{Gm_s}} \quad r^{\frac{3}{2}} \tag{10.18}$$

because angular frequency is inversely proportional to the period ($\omega = 2\pi/T$). We have, for the special case of circular orbits, derived Kepler's third law. The orbital period is proportional to the orbital size to the 3/2 power.

Conservation of energy

Any force of the form

$$\vec{\boldsymbol{F}} = -F(r)\frac{\vec{\boldsymbol{r}}}{r}$$

is conservative and is associated with a potential energy given by the indefinite integral

$$E_{\mathrm{P}} = \int_{-\infty}^{r} F(r)dr.$$

For the case of gravitational attraction, the potential energy is

$$E_{\mathrm{P}} = \frac{-Gm_s m_e}{r}$$

where we could add an arbitrary constant. Thus, one of the features of planetary motion is that for a given orbit the energy is constant in time:

$$\begin{aligned} \text{Constant} &= E_{\mathrm{K}} + E_{\mathrm{P}} \\ &= = \frac{1}{2}mv^2 + \frac{-Gm_s m_e}{r} \\ &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{-Gm_s m_e}{\sqrt{x^2 + y^2}}. \end{aligned} \tag{10.19}$$

If that constant is bigger than zero than the orbit has enough energy to have positive kinetic energy even when infinitely far from the sun. Such orbits are said to have more than "escape velocity" and they do indeed have open hyperbola-shaped orbits, and only pass close to the sun at most once.

Motion of rockets and artificial satellites

Rockets and the like move around the earth much like planets, comets and asteroids move around the sun. All of the equations for planetary motion apply. But you need to substitute the mass of the earth for m_s and the mass of the satellite for m_e. Thus we can write the governing equation eqn. (10.14) as

$$-GMm\left(\frac{x\hat{\imath}+y\hat{\jmath}}{(x^2+y^2)^{3/2}}\right) = m\,(\ddot{x}\hat{\imath}+\ddot{y}\hat{\jmath}) \tag{10.20}$$

where now M is the mass of the earth and m is the mass of the satellite. At the surface of the earth $r = R$, the earth's radius, and $GM/R^2 = g$ so we can rewrite the governing equation for rockets and the like as

$$-gR^2\left(\frac{x\hat{\imath}+y\hat{\jmath}}{(x^2+y^2)^{3/2}}\right) = (\ddot{x}\hat{\imath}+\ddot{y}\hat{\jmath})\,. \tag{10.21}$$

Another central-force example: force proportional to radius

A less famous, but also useful, example of central force is where the attraction force is proportional to the radius. In this case the governing equations are:

$$\begin{aligned} \vec{F} &= m\vec{a} \\ -k\vec{r} &= m\ddot{\vec{r}} \\ -k(x\hat{\imath}+y\hat{\jmath}) &= m\,(\ddot{x}\hat{\imath}+\ddot{y}\hat{\jmath})\,. \end{aligned} \tag{10.22}$$

Dotting both sides with $\hat{\imath}$ and $\hat{\jmath}$ we get two uncoupled linear homogeneous constant coefficient differential equations:

$$\ddot{x}+\frac{k}{m}x=0 \quad \text{and} \quad \ddot{y}+\frac{k}{m}y=0.$$

These you recognize as the harmonic oscillator equations so we can pick off the general solutions immediately as:

$$x = A\cos(\lambda t)+B\sin(\lambda t) \quad \text{and} \quad y = C\cos(\lambda t)+D\sin(\lambda t) \tag{10.23}$$

where A, B, C, and D are arbitrary constants which are determined by initial conditions. For all A, B, C, and D eqn. (10.23) describes an ellipse (or a special case of an ellipse, like a circle or a straight line). In the case of planetary motion we also had ellipses. In this case, however, the center of attraction is at the center of the ellipse and not at one of the foci.

Conservation of angular momentum and Kepler's second law

If we take the linear momentum balance equation *eqn.* (10.12) and take the cross product of both sides with $\vec{\boldsymbol{r}}$ we get the following.

$$\begin{aligned}
\vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \\
\Rightarrow \quad F(r)\left(\frac{-\vec{\boldsymbol{r}}}{r}\right) &= m\ddot{\vec{\boldsymbol{r}}} \\
\Rightarrow \quad \vec{\boldsymbol{r}} \times \left(F(r)\left(\frac{-\vec{\boldsymbol{r}}}{r}\right)\right) &= \vec{\boldsymbol{r}} \times \left(m\ddot{\vec{\boldsymbol{r}}}\right) \\
\Rightarrow \quad \vec{0} &= \frac{d}{dt}\left(m\vec{\boldsymbol{r}} \times \dot{\vec{\boldsymbol{r}}}\right) \qquad (\text{because} \dot{\vec{\boldsymbol{r}}} \times \dot{\vec{\boldsymbol{r}}} = \vec{0}) \\
\Rightarrow \quad \text{constant} &= m\vec{\boldsymbol{r}} \times \dot{\vec{\boldsymbol{r}}}. \qquad (10.24)
\end{aligned}$$

But this last quantity is exactly the rate at which area is swept out by a moving particle. Thus Kepler's third law has been derived for all central-force motions (not just inverse square attractions). The last quantity is also the angular momentum of the particle. Thus for a particle in central force motion we have derived conservation of angular momentum from $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$.

SAMPLE 10.11 Circular orbits of planets: Refer to *eqn*. (10.15) in the text that governs the motion of planets around a fixed sun.

1. Let $x = A\cos(\lambda t)$ and $y = A\sin(\lambda t)$. Show that x and y satisfy the equations of planetary motion and that they describe a circular orbit.
2. Show that the solution assumed in (a) satisfies Kepler's third law by showing that the orbital period $T = 2\pi/\lambda$ is proportional to the 3/2 power of the size of the orbit (which can be characterized by its radius).

Solution

1. The governing equation of planetary motion can be written as

$$\begin{aligned} \frac{\ddot{x}}{x} = \frac{-Gm_s}{(x^2+y^2)^{3/2}} &= \frac{\ddot{y}}{y} \\ \Rightarrow \quad \ddot{x}y - \ddot{y}x &= 0 \end{aligned} \tag{10.25}$$

Now,

$$\begin{aligned} x &= A\cos(\lambda t) \quad \Rightarrow \quad \ddot{x} = -\lambda^2 A\cos(\lambda t) \\ y &= A\sin(\lambda t) \quad \Rightarrow \quad \ddot{y} = -\lambda^2 A\cos(\lambda t) \end{aligned}$$

Substituting these values in *eqn*. (10.25), we get

$$-\lambda^2 A^2 \cos(\lambda t)\cdot\sin(\lambda t) + \lambda^2 A\sin(\lambda t)\cdot\cos(\lambda t) \stackrel{\checkmark}{=} 0$$

Thus the assumed form of x and y satisfy the governing equations of planetary motion, *i.e.*, $x(t) = A\cos(\lambda t)$ and $y(t) = A\sin(\lambda t)$ form a solution of planetary motion. Now, it is easy to show that

$$x^2 + y^2 = A^2\cos^2(\lambda t) + A^2\sin^2(\lambda t) = A^2,$$

i.e., x and y satisfy the equation of a circle with radius A. Thus, the assumed solution gives a circular orbit.

2. Substituting $x = A\cos(\lambda t)$ in *eqn*. (10.15), and noting that square of the radius of the orbit is $r^2 = x^2 + y^2 = A^2$, we get

$$\begin{aligned} -\lambda^2 A\cos(\lambda t) &= -Gm_s\frac{A\cos(\lambda t)}{r^3} \\ \Rightarrow \quad \lambda^2 &= \frac{Gm_s}{A^3} \\ \text{or} \quad \left(\frac{2\pi}{T}\right)^2 &= \frac{Gm_s}{A^3} \\ \Rightarrow \quad T^2 &= \frac{4\pi^2}{Gm_s}A^3 \\ \text{or} \quad T &= KA^{3/2} \end{aligned}$$

where $K = 2\pi/\sqrt{Gm_s}$ is a constant. Thus the orbital period T is proportional to the 3/2 power of the radius, or the size, of the circular orbit.

Of, course, the same holds true for elliptic orbits too, but it is harder to show that analytically using cartesian coordinates, x and y.

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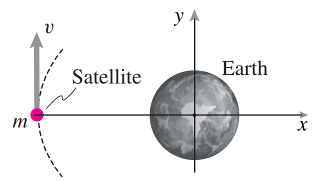

Figure 10.32:

Filename:sfig5-9-satorbit

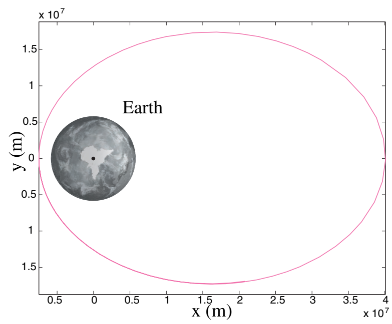

Figure 10.33: The elliptical orbit of the satellite, obtained from numerical integration of the equations of motion.

Filename:sfig5-9-satorbit-a

SAMPLE 10.12 Numerical computation of satellite orbits: The following data is known for an earth satellite: mass = 2000 kg, the distance to the closest point, the perigee, on its orbit from the earth's surface = 1100 km, and its velocity at perigee, which is purely tangential, is 9500 m/s. The radius of the earth is 6400 km and the acceleration due to gravity $g = 9.81\,\mathrm{m/s^2}$.

1. Solve the equations of motion of the satellite numerically with the given data and show that the orbit of the satellite is elliptical. Find the apogee of the orbit and the speed of the satellite at the apogee.
2. From the data at apogee and perigee show that the angular momentum and the energy of the satellite are conserved.
3. Find the orbital period of the satellite and show that it satisfies Kepler's third law (in equality form).

Solution

1. The equations of motion of a satellite around a fixed earth are

$$\ddot{x} = \frac{-gR^2 x}{(x^2+y^2)^{3/2}} \quad \text{and} \quad \ddot{y} = \frac{-gR^2 y}{(x^2+y^2)^{3/2}}.$$

where g is the acceleration due to gravity and R is the radius of the earth (see eqn. (10.20) in the text). From the given data at perigee, the initial conditions are

$$x(0) = -7500\,\mathrm{km}, \quad \dot{x}(0) = 0, \quad y(0) = 0, \quad \dot{y}(0) = 9500\,\mathrm{m/s}.$$

In order to solve the equations of motion by numerical integration, we first rewrite these equations as four first order equations:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -gR^2 z_1/(z_1^2+z_3^2)^{3/2} \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= -gR^2 z_3/(z_1^2+z_3^2)^{3/2}. \end{aligned}$$

Now the given initial conditions in terms of the new variables are

$$z1(0) = -7.5\times 10^6\,\mathrm{m}, \quad z2(0) = 0, \quad z3(0) = 0, \quad z4(0) = 9500\,\mathrm{m/s}.$$

We are now ready to go to a computer. We implement the following pseudocode on the computer to solve the problem.

```
ODEs ={z1dot=z2,  z2dot=-g*R^2*z_1/(z_1^2+z_3^2)^{3/2},
   z3dot=z4,  z4dot=-g*R^2*z_3/(z_1^2+z_3^2)^{3/2}}
IC   ={z1(0)=-7.5E06, z2(0)=0, z3(0)=0, z4(0)=9500}
Set  g = 9.81, R = 6.4E06
Solve ODEs with IC for t=0 to t=4E04
Plot z1 vs z3
```

Results obtained from implementing the code above with a Runge-Kutta method based integrator is shown in Fig. 10.33 where we have also plotted the earth centered at the origin to put the orbit in perspective. The orbit is clearly elliptical. From the computer output, we find the following data for the apogee.

$$x = 4.0049\times 10^7\,\mathrm{m}, \quad \dot{x} = 0, \quad y = 0, \quad \dot{y} = -1.7791\times 10^3\,\mathrm{m/s}$$

2. The expressions for energy E and angular momentum H for a satellite are,

$$E = E_K + E_P = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{GMm}{r}$$
$$\vec{\boldsymbol{H}}_O = \vec{\boldsymbol{r}} \times m\vec{\boldsymbol{v}} = (x\hat{\imath} + y\hat{\jmath}) \times m(\dot{x}\hat{\imath} + \dot{y}\hat{\jmath}) = m(x\dot{y} - y\dot{x})\hat{\boldsymbol{k}}$$

At both apogee and perigee, $y = 0$ and the velocity (which is tangential) is in the y direction, *i.e.*, $\dot{x} = 0$. Therefore, the expressions for energy and angular momentum become simpler:

$$E = \frac{1}{2}m\dot{y}^2 - \frac{GMm}{r} = \frac{1}{2}m\dot{y}^2 - \frac{gR^2m}{|x|},$$
$$\text{and} \quad H = mx\dot{y}.$$

Let E_1 and H_1 be the energy and the angular momentum of the satellite at the perigee, respectively, and E_2 and H_2 be the respective quantities at the apogee. Then, from the given data,

$$E_1 = \frac{1}{2}m\dot{y}_1^2 - \frac{gR^2m}{|x_1|} = \frac{1}{2}2000\,\text{kg}\cdot(9500\,\text{m/s})^2 - \frac{9.81\,\text{m/s}^2\cdot(6.4\times10^6\,\text{m})^2}{7.5\times10^6\,\text{m}}$$
$$= -1.6901\times10^{10}\ \text{Joules}$$
$$H_1 = mx_1\dot{y}_1 = 2000\,\text{kg}\cdot(-7.5\times10^6\,\text{m})\cdot(9500\,\text{m/s})$$
$$= -1.4250\times10^{14}\,\text{N·m}\cdot\text{s}$$
$$E_2 = \frac{1}{2}m\dot{y}_2^2 - \frac{gR^2m}{|x_2|} = \frac{1}{2}2000\,\text{kg}\cdot(-1779\,\text{m/s})^2 - \frac{9.81\,\text{m/s}^2\cdot(6.4\times10^6\,\text{m})^2}{4.0049\times10^7\,\text{m}}$$
$$= -1.6901\times10^{10}\,\text{Joules}$$
$$H_2 = mx_2\dot{y}_2 = 2000\,\text{kg}\cdot(4.0049\times10^7\,\text{m})\cdot(-1779\,\text{m/s})$$
$$= -1.4250\times10^{14}\,\text{N·m}\cdot\text{s}$$

Clearly, the energy and the angular momentum are conserved.

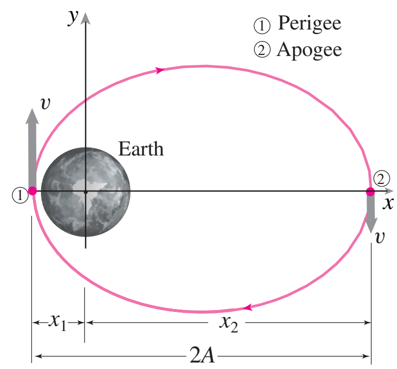

Figure 10.34: The elliptical orbit of the satellite. The perigee and apogee are marked as points 1 and 2 on the orbit.

Filename:sfig5-9-satorbit-b

3. From the computer output, we find the time at which the satellite returns to the perigee for the first time. This is the orbital period. From the output data, we get the orbital period to be 3.6335×10^4 s = 10.09 hrs. Now let us compare this result with the analytical value of the orbital period.

   Let A be the semimajor axis of the elliptic orbit. Then the square of the orbital time period T is given by

$$T^2 = \frac{4\pi^2A^3}{gR^2}.$$

For the orbit we have obtained by numerical integration,

$$2A = |x_1| + |x_2|$$
$$= 7.5\times10^6\,\text{m} + 4.0049\times10^7\,\text{m}$$
$$= 4.7549\times10^7\,\text{m}$$
$$\Rightarrow \quad A = 2.3774\times10^7\,\text{m}$$

Hence,

$$T = \sqrt{\frac{4\pi^2\cdot(2.3774\times10^7\,\text{m})^3}{9.81\,\text{m/s}^2\cdot(6.4\times10^6\,\text{m})^2}}$$
$$= 3.6335\times10^4\,\text{s}.$$

which is the same value as obtained from numerical solution.

$$\boxed{T = 3.6335\times10^4\,\text{s} = 10.09\ \text{hrs}}$$

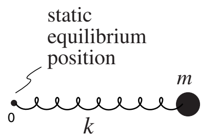

Figure 10.35:

Filename:sfig5-9-zerospring

* No spring can have zero relaxed length, however, a spring can be configured in various ways to make it behave as if it has zero relaxed length. See box 6.1 on page 329

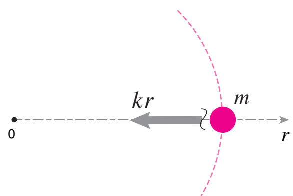

Figure 10.36: Free-body diagram of the mass.

Filename:sfig5-9-zerospring-a

SAMPLE 10.13 Zero-length spring and central force motion: A zero-length spring * (the relaxed length is zero) is tied to a mass $m = 1\,\mathrm{kg}$ on one end and fixed on the other end. The spring stiffness is $k = 1\,\mathrm{N}/m$.

1. Find appropriate initial conditions for the mass so that its trajectory is a straight line along the y-axis.
2. Find appropriate initial conditions for the mass so that its trajectory is a circle.
3. Can you find any condition on initial conditions that guarantees elliptic orbits of the mass?
4. Let $\vec{r}(0) = 0.5\,\mathrm{m}\hat{\imath}$ and $\dot{\vec{r}}(0) = (0.5\hat{\imath} + 0.6\hat{\jmath})\,\mathrm{m/s}$. Describe the motion of the mass by plotting its trajectory for 12 s.

Solution Let the position of the mass be $\vec{r}$ at some instant t. Since the relaxed length of the spring is zero, the stretch in the spring is $|\vec{r}|$ and the spring force on the mass is $-k\vec{r}$. Then the equation of motion of the mass is

$$\begin{aligned} -k\vec{r} &= m\ddot{\vec{r}} \\ -k(x\hat{\imath} + y\hat{\jmath}) &= m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}) \\ \Rightarrow \quad \ddot{x} + \frac{k}{m}x &= 0 \\ \text{and} \quad \ddot{y} + \frac{k}{m}y &= 0. \end{aligned}$$

Thus the equations of motion are decoupled in the x and y directions. The solutions, as discussed in the text (see eqn. (10.23)), are

$$\begin{aligned} x &= A\cos(\lambda t) + B\sin(\lambda t) \\ \text{and} \quad y &= C\cos(\lambda t) + D\sin(\lambda t) \end{aligned} \tag{10.26}$$

where the constants A, B, C, D are determined from initial conditions. Let us take the most general initial conditions $x(0) = x_0, \dot{x}(0) = \dot{x}_0, y(0) = y_0$, and $\dot{y}(0) = \dot{y}_0$. By substituting these values in x and y equations above and their derivatives, we get

$$A = x_0, \quad B = \dot{x}_0/\lambda, \quad C = y_0, \quad D = \dot{y}_0/\lambda.$$

Substituting these values we get

$$\begin{aligned} x &= x_0\cos(\lambda t) + \dot{x}_0/\lambda\sin(\lambda t) \\ \text{and} \quad y &= y_0\cos(\lambda t) + \dot{y}_0/\lambda\sin(\lambda t). \end{aligned} \tag{10.27}$$

1. For a straight line motion along the y-axis, we should have the x-component of motion identically zero. We can, therefore, set $x_0 = 0, \dot{x}_0 = 0$ and take any value for y_0 and $\dot{y}_0$ to give

$$\begin{aligned} x(t) &= 0 \\ \text{and} \quad y(t) &= y_0\cos(\lambda t) + \dot{y}_0/\lambda\sin(\lambda t). \end{aligned}$$

2. For a circular trajectory, we must pick initial conditions such that we get $x^2 + y^2 = (\text{a constant})^2$. We can easily achieve this by choosing, say, $x(0) = x_0, \dot{x}(0) = 0, y(0) = 0$, and $\dot{y}(0) = x_0\lambda$. Substituting these values in eqn. (10.27), we get

$$x^2 + y^2 = x_0^2\cos^2(\lambda t) + \left(\frac{x_0\lambda}{\lambda}\right)^2\sin^2(\lambda t) = x_0^2$$

which is a circular orbit of radius x_0. Note that the initial position of the mass for this orbit is $\vec{r}(0) = x_0\hat{\imath})$, and the initial velocity is $(\vec{v}(0) = x_0\lambda\hat{\jmath})$, *i.e.*, the velocity is normal to the position vector $(\vec{r}\cdot\vec{v} = 0)$, and the magnitude of the velocity is dependent on the magnitude of the position vector, in fact , it must be exactly equal to the product of the distance from the center and the orbital frequency λ.

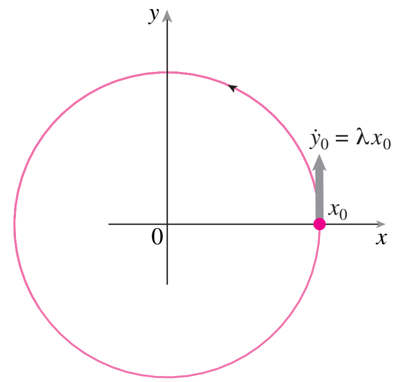

Figure 10.37: Circular trajectory of the mass.

Filename:sfig5-9-zerospring-b

3. In order to have elliptic orbits, the initial conditions should be selected such that x and y satisfy the equation of an ellipse. By examining the solutions in *eqn*. (10.27), we see that if we set $\dot{x}_0 = 0$ and $y_0 = 0$ and let the other two initial conditions have any arbitrary value, x_0 and $\dot{y}_0$, we get

$$\begin{aligned} x(t) &= x_0\cos(\lambda t), \\ \text{and} \quad y(t) &= (\dot{y}_0/\lambda)\sin(\lambda t), \\ \Rightarrow \quad \frac{x^2}{x_0^2} + \frac{y^2}{(\dot{y}_0/\lambda)^2} &= \cos^2(\lambda t) + \sin^2(\lambda t) \\ &= 1 \end{aligned}$$

which is the equation of an ellipse with semimajor axis x_0 and semiminor axis $\dot{y}/\lambda$. Of course, the symmetry of the equations implies that we could also get elliptic orbits by setting $x_0 = 0$ and $\dot{y}_0 = 0$, and letting the other two initial conditions be arbitrary. Thus the condition for elliptic orbits is to have the initial velocity normal to the position vector, *e.g.*,

$$\begin{aligned} \vec{r}(0) = x_0\hat{\imath} \quad &\text{and} \quad \dot{\vec{r}}(0) = \dot{y}_0\hat{\jmath}, \\ \text{or} \quad \vec{r}(0) = y_0\hat{\jmath} \quad &\text{and} \quad \dot{\vec{r}}(0) = \dot{x}_0\hat{\imath}, \end{aligned}$$

or, more generally,

$$\vec{r}(0) = r_0\hat{\lambda} \quad \text{and} \quad \dot{\vec{r}}(0) = v\hat{\boldsymbol{n}},$$

where $\hat{\lambda}$ is a unit vector along the position vector of the mass and $\hat{\boldsymbol{n}}$ is normal to $\hat{\lambda}$.

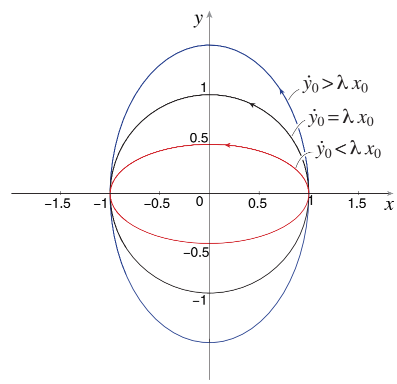

Figure 10.38: Elliptic orbits of the mass obtained from the initial conditions $x_0 = 1\,\text{m}, \dot{x}_0 = 0, y_0 = 0$, and various values of $\dot{y}_0$.

Filename:sfig5-9-zerospring-c

Note that the condition obtained in (b) for circular orbits is just a special case of the condition for elliptic orbits (well, a circle is just a special case of an ellipse). Therefore, if we keep x_0 fixed and vary $\dot{y}_0$ we can get different elliptic orbits, including a circular one, based on the same major axis. Taking $x_0 = 1\,\text{m}$, we show different orbits obtained for the mass by varying $\dot{y}_0$ in *Fig*. 10.38.

4. By substituting the given initial values $x_0 = 0.5\,\text{m}, \dot{x}(0) = 0.5\,\text{m/s}, y(0) = 0$ and $\dot{y} = 0.6\,\text{m/s}$ in *eqn*. (10.27) and and noting that $\lambda \equiv \sqrt{k/m} = \sqrt{(1\,\text{N/m})/(1\,\text{kg})} = (1/\,\text{s})$, we get

$$\begin{aligned} x(t) &= (0.5\,\text{m})\cdot\cos\left(\frac{1}{s}\cdot t\right) + \left(\frac{0.5\,\text{m/s}}{s}\right)\cdot\sin\left(\frac{1}{s}\cdot t\right) \\ y(t) &= \left(\frac{0.6\,\text{m/s}}{s}\right)\cdot\sin\left(\frac{1}{s}\cdot t\right) \end{aligned}$$

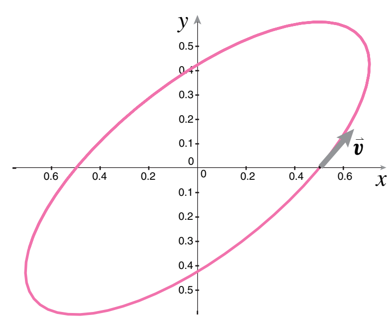

Figure 10.39: The orbit of the mass obtained from the initial conditions $x_0 = 0.5\,\text{m}, \dot{x}_0 = 0.5\,\text{m/s}, y_0 = 0$, and $\dot{y}_0 = 0.6\,\text{m/s}$.

Filename:sfig5-9-zerospring-d

The functions $x(t)$ and $y(t)$ do not seem to describe any simple geometric path immediately. We could, perhaps, do some mathematical manipulations and try to get a relationship between x and y that we can recognize. In stead, let us plot the orbit on a computer to see the path that the mass takes during its motion with these initial conditions. To plot this orbit, we evaluate x and y at, say, 100 values of t between 0 and 10 s and then plot x vs y.

```
t = [0 0.1 0.2 ... 9.9 10]
x = 0.5 * cos(t) + 0.6 * sin(t)
y = 0.6 * sin(t)
plot x vs y
```

The plot obtained by performing these operations on a computer is shown in *Fig*. 10.39.

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Problems for Chapter 10

Particle dynamics in space

10.1 Dynamics of a particle in space

Preparatory Problems

10.1 Given $\vec{r}(t) = A\sin(\omega t)\hat{\imath} + Bt\hat{\jmath} + C\hat{k}$, find

1. $\vec{v}(t)$
2. $\vec{a}(t)$
3. $\vec{r}(t) \times \vec{a}(t)$.

10.2 A particle of mass $=3\,\text{kg}$ travels in space with its position known as a function of time, $\vec{r} = (\sin\frac{t}{\text{s}})\,\text{m}\hat{\imath} + (\cos\frac{t}{\text{s}})\,\text{m}\hat{\jmath} + te^{\frac{t}{s}}\,\text{m/s}\hat{k}$. At $t = 3\,\text{s}$, find the particle's

a) velocity and

b) acceleration.

10.3 A particle of mass $m = 2\,\text{kg}$ travels in the xy-plane with its position known as a function of time, $\vec{r} = 3t^2\,\text{m/s}^2\hat{\imath} + 4t^3\frac{\text{m}}{s^3}\hat{\jmath}$. At $t = 5\,\text{s}$, find the particle's

a) velocity and

b) acceleration, and

c) draw the three vectors.

10.4 The velocity of a particle of mass m on a frictionless surface is given as $\vec{v} = (0.5\,\text{m/s})\hat{\imath} - (1.5\,\text{m/s})\hat{\jmath}$. If the displacement is given by $\Delta\vec{r} = \vec{v}t$, find (a) the distance traveled by the mass in 2 seconds and (b) a unit vector along the displacement.

10.5 If $\dot{\vec{r}} = (u_0 \sin \Omega t)\hat{\imath} + v_0\hat{\jmath}$ and $\vec{r}(0) = x_0\hat{\imath} + y_0\hat{\jmath}$, with u_0, v_0, and Ω as constants, find $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$.

10.6 For $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$ and $\vec{a} = 2\,\text{m/s}^2\hat{\imath} - (3\,\text{m/s}^2 - 1\,\text{m/s}^3\ t)\hat{\jmath} - 5\,\text{m/s}^4\ t^2\hat{k}$, write the vector equation $\vec{v} = \int \vec{a}\,dt$ as three scalar equations (*i.e.*, find $v_x(t)$, $v_y(t)$, and $v_z(t)$).

10.7 Find $\vec{r}(5\,\text{s})$ given that $\dot{\vec{r}} = v_1\sin(ct)\hat{\imath} + v_2\hat{\jmath}$ and $\vec{r}(0) = 2\,\text{m}\hat{\imath} + 3\,\text{m}\hat{\jmath}$, and that v_1 is a constant $4\,\text{m/s}$, v_2 is a constant $5\,\text{m/s}$, and c is a constant $4\,\text{s}^{-1}$.

10.8 Let $\dot{\vec{r}} = v_0\cos\alpha\hat{\imath} + v_0\sin\alpha\hat{\jmath} + (v_0\tan\theta - gt)\hat{k}$, where v_0, α, θ, and g are constants. If $\vec{r}(0) = \vec{0}$, find $\vec{r}(t)$.

10.9 On a smooth circular helical path the velocity of a particle is $\dot{\vec{r}} = -R\sin t\hat{\imath} + R\cos t\hat{\jmath} + gt\hat{k}$. If $\vec{r}(0) = R\hat{\imath}$, find $\vec{r}((\pi/3)\,\text{s})$.

More-Involved Problems

10.10 A particle travels on a path in the xy-plane given by $y(x) = \sin^2(\frac{x}{\text{m}})\,\text{m}$. where $x(t) = t^3(\frac{\text{m}}{s^3})$. What are the velocity and acceleration of the particle in cartesian coordinates when $t = (\pi)^{\frac{1}{3}}\,\text{s}$?

10.11 The position of a particle is given by $\vec{r}(t) = (t^2\,\text{m/s}^2\hat{\imath} + e^{\frac{t}{\text{s}}}\,\text{m}\hat{\jmath})$. What are the velocity and acceleration of the particle as functions of time? Draw the path of the particle and show the vectors $\vec{v}$ and $\vec{a}$ at $t = 1\,\text{s}$.

10.12 A particle travels on an elliptical path given by $y^2 = b^2(1-\frac{x^2}{a^2})$ with constant speed v. Find the velocity of the particle when $x = a/2$ and $y > 0$ in terms of a, b, and v.

10.13 A particle travels on a path in the xy-plane given by $y(x) = (1 - e^{-\frac{x}{\text{m}}})\,\text{m}$. Make a plot of the path. It is known that the x coordinate of the particle is given by $x(t) = t^2\,\text{m/s}^2$. What is the rate of change of speed of the particle? What angle does the velocity vector make with the positive x axis when $t = 3\,\text{s}$?

10.14 A particle starts at the origin in the xy-plane, $(x_0 = 0, y_0 = 0)$ and travels only in the positive xy quadrant. Its speed and x coordinate are known to be $v(t) = \sqrt{1 + (\frac{4}{s^2})t^2}\ \text{m/s}$ and $x(t) = t\ \text{m/s}$, respectively. What is $\vec{r}(t)$ in cartesian coordinates? What are the velocity, acceleration, and rate of change of speed of the particle as functions of time? What kind of path is the particle on? What are the distance of the particle from the origin and its velocity and acceleration when $x = 3\,\text{m}$?

10.15 For a particle, $\sum\vec{F} = m\vec{a}$. Two forces $\vec{F}_1$ and $\vec{F}_2$ act on a mass P as shown in the figure. P has mass 2 lbm. The acceleration of the mass is somehow measured to be $\vec{a} = -2\,\text{ft/s}^2\hat{\imath} + 5\,\text{ft/s}^2\hat{\jmath}$.

a) Write the equation

$$\sum\vec{F} = m\vec{a}$$

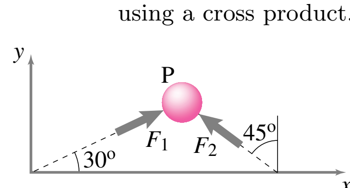
in vector form (evaluating each side as much as possible).

b) Write the equation in scalar form (use any method you like to get two scalar equations in the two unknowns F_1 and F_2).

c) Write the equation in matrix form.

d) Find $F_1 = |\vec{F}_1|$ and $F_2 = |\vec{F}_2|$ by the following methods:

1. from the scalar equations using hand algebra,
2. from the matrix equation using a computer, and
3. from the vector equation using a cross product.



Filename:h1-63a

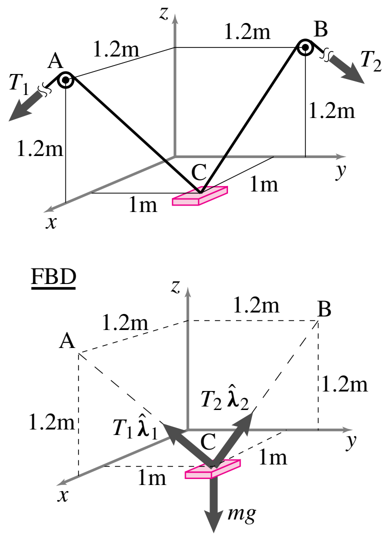
10.16 Three forces, $\vec{F}_1 = 20\,\text{N}\hat{\imath} - 5\,\text{N}\hat{\jmath}$, $\vec{F}_2 = F_{2x}\hat{\imath} + F_{2z}\hat{\boldsymbol{k}}$, and $\vec{F}_3 = F_3\hat{\lambda}$, where $\hat{\lambda} = \frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath}$, act on a body with mass 2 kg. The acceleration of the body is $\vec{a} = -0.2\,\text{m/s}^2\hat{\imath} + 2.2\,\text{m/s}^2\hat{\jmath} + 1.7\,\text{m/s}^2\hat{\boldsymbol{k}}$. Write the equation $\sum \vec{F} = m\vec{a}$ as scalar equations and solve them (most conveniently on a computer) for F_{2x}, F_{2z}, and F_3.

10.17 In three-dimensional space with no gravity a particle with $m = 3$ kg at A is pulled by three strings which pass through points B, C, and D respectively. The acceleration is known to be $\vec{a} = (1\hat{\imath} + 2\hat{\jmath} + 3\hat{\boldsymbol{k}})\ \text{m/s}^2$. The position vectors of B, C, and D relative to A are given in the first few lines of code below. Complete the pseudo-code to find the three tensions. The last line should read `T = ...` with `T` being assigned to be a 3-element column vector with the three tensions in Newtons. [Hint: If `x`, `y`, and `z` are three column vectors then `A=[x y z]` is a matrix with `x`, `y`, and `z` as columns.]

```
% Incomplete PSEUDO-CODE file
m   = 3;
a   = [ 1 2 3]$'$;
~
rAB = [ 2 3 5]$'$;
rAC = [-3 4 2]$'$;
rAD = [ 1 1 1]$'$;
uAB = rAB/(magnitude of rAB);
      .
      .
      .
T   =
```

10.18 The rate of change of linear momentum of a particle is known in two directions: $\dot{L}_x = 20\,\text{kg}\,\text{m/s}^2$, $\dot{L}_y = -18\,\text{kg}\,\text{m/s}^2$ and unknown in the z direction. The forces acting on the particle are $\vec{F}_1 = 25\,\text{N}\hat{\imath} + 32\,\text{N}\hat{\jmath} + 75\,\text{N}\hat{\boldsymbol{k}}$, $\vec{F}_2 = F_{2x}\hat{\imath} + F_{2y}\hat{\jmath}$ and $\vec{F}_3 = -F_3\hat{\boldsymbol{k}}$. Using $\sum \vec{F} = \dot{\vec{L}}$, separate the vector equation into scalar equations in the x, y, and z directions. Solve these equations (maybe with the help of a computer) to find F_{2x}, F_{2y}, and F_3.

10.19 A block of mass 100 kg is pulled with two strings AC and BC. Given that the tensions $T_1 = 1200$ N and $T_2 = 1500$ N, find the magnitude and direction of the acceleration of the block. [$\sum \vec{F} = m\vec{a}$]



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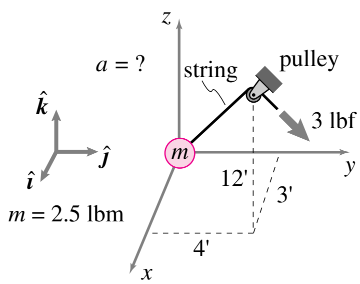
10.20 Neglecting gravity, the only force acting on the mass shown in the figure is from the string. Find the acceleration of the mass. Use the dimensions and quantities given. Recall that lbf is a pound force, lbm is a pound mass, and lbf/lbm $= g$. Use $g = 32\,\text{ft/s}^2$. Note also that $3^2 + 4^2 + 12^2 = 13^2$.



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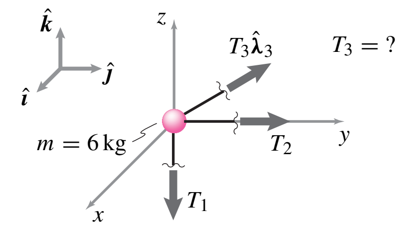
10.21 Three strings are tied to the mass shown with the directions indicated in the figure. They have unknown tensions T_1, T_2, and T_3. There is no gravity. The acceleration of the mass is given as $\vec{a} = (-0.5\hat{\imath} + 2.5\hat{\jmath} + \frac{1}{3}\hat{k})\,\text{m/s}^2$.

a) Given the free body diagram in the figure, write the equations of linear momentum balance for the mass.

b) Find the tension T_3.

$$\hat{\boldsymbol{\lambda}}_3 = \tfrac{1}{13}(3\hat{\boldsymbol{\imath}} + 12\hat{\boldsymbol{\jmath}} + 4\hat{\boldsymbol{k}})$$



$$\vec{\boldsymbol{a}} = (\text{-}.5\hat{\boldsymbol{\imath}} + 2.5\hat{\boldsymbol{\jmath}} + \tfrac{1}{3}\hat{\boldsymbol{k}})\tfrac{m}{s^2}$$

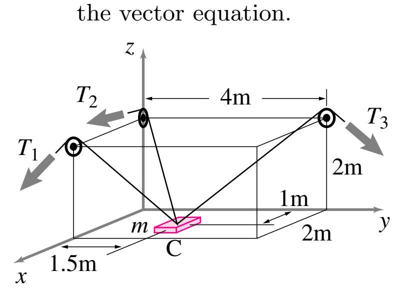
Filename:pfigure-blue-1-2

10.22 An object C of mass 2 kg is pulled by three strings as shown. The acceleration of the object at the position shown is $\boldsymbol{a} = \left(-0.6\hat{\imath} - 0.2\hat{\jmath} + 2.0\hat{\boldsymbol{k}}\right)\text{m/s}^2$.

a) Draw a free body diagram of the mass.

b) Write the equation of linear momentum balance for the mass. Use λ's as unit vectors along the strings.

c) Find the three tensions T_1, T_2, and T_3 at the instant shown. You may find these tensions by using hand algebra with the scalar equations, using a computer with the matrix equation, or by using a cross product on the vector equation.



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10.23 Particle moves on a strange path. Given that a particle moves in the xy plane for 1.77 s obeying

$$\begin{aligned} \boldsymbol{r} = &\quad (5\,\text{m})\cos^2(t^2/\,\text{s}^2)\hat{\imath} \\ &+ (5\,\text{m})\sin(t^2/\,\text{s}^2)\cos(t^2/\,\text{s}^2)\hat{\jmath} \end{aligned}$$

where x and y are the horizontal distance in meters and t is measured in seconds.

a) Accurately plot the trajectory of the particle.

b) Mark on your plot where the particle is going fast and where it is going slow. Explain how you know these points are the fast and slow places.

10.24 *Computer question:* **What's the plot? What's the mechanics question?** Shown are shown some pseudo computer commands that are not commented adequately, unfortunately, and no computer is available at the moment.

a) Draw as accurately as you can, assigning numbers etc, the plot that results from running these commands.

b) See if you can guess a mechanical situation that is described by this program. Sketch the system and define the variables to make the script file agree with the problem stated.

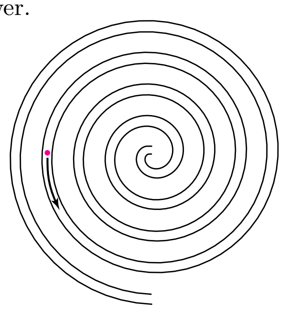
```
ODEs = {z1dot = z2
        z2dot = 0}
ICs = {z1 =1, z2=1}

Solve ODEs with ICs from t=0 to t=5

plot z2 and z1 vs t on the same plot
```

Filename:pfigure-s94f1p3

10.25 A particle is blown out through the uniform spiral tube shown, which lies flat on a horizontal frictionless table. Draw the particle's path after it is expelled from the tube. Defend your answer.



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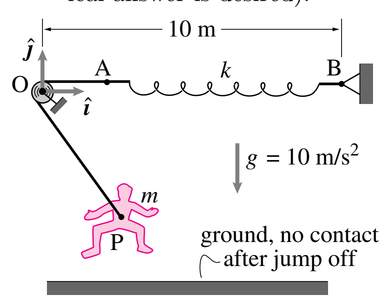
10.26 Bungy Jumping. In a new safer bungy jumping system, people jump up from the ground while suspended from a rope that runs over a pulley at O and is connected to a stretched spring anchored at B. The pulley has negligible size, mass, and friction. For the situation shown the spring AB has rest length $\ell_0 = 2\,\mathrm{m}$ and a stiffness of $k = 200\,\mathrm{N/m}$. The inextensible massless rope from A to P has length $\ell_r = 8\,\mathrm{m}$, the person has a mass of 100 kg. Take O to be the origin of an xy coordinate system aligned with the unit vectors $\hat{\imath}$ and $\hat{\jmath}$

a) Assume you are given the position of the person $\boldsymbol{r} = x\hat{\imath} + y\hat{\jmath}$ and the velocity of the person $\boldsymbol{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$. Find her acceleration in terms of some or all of her position, her velocity, and the other parameters given. Use the numbers given, where supplied, in your final answer.

b) Given that bungy jumper's initial position and velocity are $\boldsymbol{r}_0 = 1\,\mathrm{m}\hat{\imath} - 5\,\mathrm{m}\hat{\jmath}$ and $\boldsymbol{v}_0 = 0$ write computer commands to find her position at $t = \pi/\sqrt{2}\,\mathrm{s}$.

c) Find the answer to part (b) with pencil and paper (a final numerical answer is desired).



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10.27 A softball pitcher releases a ball of mass m upwards from her hand with speed v_0 and angle θ_0 from the horizontal. The only external force acting on the ball after its release is gravity.

a) What is the equation of motion for the ball after its release?

b) What are the position, velocity, and acceleration of the ball?

c) What is its maximum height?

d) At what distance does the ball return to the elevation of release?

e) What kind of path does the ball follow and what is its equation y as a function of x?

10.28 Find the trajectory of a not-vertically-fired cannon ball assuming the air drag is proportional to the speed. Assume the mass is 10 kg, $g = 10\,\mathrm{m/s}$, the drag proportionality constant is $C = 5\,\mathrm{N/(m/s)}$. The cannon ball is launched at 100 m/s at a 45 degree angle.

- Draw a free body diagram of the mass.
- Write linear momentum balance in vector form.
- Solve the equations on the computer and plot the trajectory.
- Solve the equations by hand and then use the computer to plot your solution.
- Compare the two plots and comment on the differences, if any.

10.29 A baseball pitching machine releases a baseball of mass m from its barrel with speed v_0 and angle θ_0 from the horizontal. The only external forces acting on the ball after its release are gravity and air resistance. The speed of the ball is given by $v^2 = \dot{x}^2 + \dot{y}^2$. Taking into account air resistance on the ball proportional to its speed squared, $\vec{\boldsymbol{F}}_d = -bv^2\hat{\boldsymbol{e}}_t$, find the equation of motion for the ball, after its release, in cartesian coordinates.

10.30 The equations of motion from problem 10.29 are nonlinear and cannot be solved in closed form for the position of the baseball. Instead, solve the equations numerically. Make a computer simulation of the flight of the baseball, as follows.

a) Convert the equation of motion into a system of first order differential equations.

b) Pick values for the gravitational constant g, the coefficient of resistance b, and initial speed v_0, solve for the x and y coordinates of the ball and make a plots its trajectory for various initial angles θ_0.

c) Use Euler's, Runge-Kutta, or other suitable method to numerically integrate the system of equations.

d) Use your simulation to find the initial angle that maximizes the distance of travel for ball, with and without air resistance.

e) If the air resistance is very high, what is a qualitative description for the curve described by the path of the ball?

10.31 A particle of mass m moves in a viscous fluid which resists motion with a force of magnitude $F = c|\vec{v}|$, where $\vec{v}$ is the velocity. Do not neglect gravity.

a) (easy) In terms of some or all of g, m, and c, what is the particle's terminal (steady-state) falling speed?

b) Starting with a free body diagram and linear momentum balance, find two second order scalar differential equations that describe the two-dimensional motion of the particle.

c) (Challenge, long calculation) Assume the particle is thrown from $\vec{r} = \vec{0}$ with $\vec{v} = v_{x0}\hat{\imath} + v_{y0}\hat{\jmath}$ at a vertical wall a distance d away. Find the height h along the wall where the particle hits. (Answer in terms of some or all of v_{x0}, v_{y0}, m, g, c, and d.) [Hint: i) find $x(t)$ and $y(t)$, ii) eliminate t, iii) substitute $x = d$. The answer is not tidy. In the limit $d \rightarrow 0$ the answer reduces to a sensible dependence on d (The limit $c \rightarrow 0$ is also sensible.).]

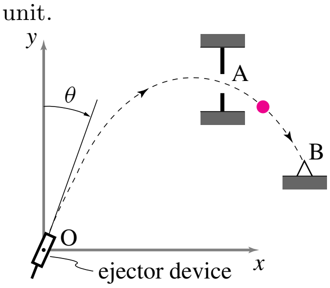
d) (Challenge, computer simulation). Do a computer simulation of the problem and find the solution in your simulation. Choose non-trivial numbers for all constants. To get an accurate solution you need an accurate interpolation to find at what time the particle hits the wall.

10.32 Someone in a violent part of the world shot a projectile at someone else. The basic facts:
Launched from the origin.
Projectile mass = 1 kg.
Launch angle 30° above horizontal.
Launch speed 172 m/s.
Drag force $\propto cv^2$ with $c = .01\,\text{kg}/\text{m}$.
Gravity $g = 10\,\text{m/s}$.

a) Write and execute computer code to find the height at $t = 1\,\text{s}$. [Hints: sketch of problem, FBD, write drag force in vector form, LMB, 1st order equations, numerical setup, find height at 1 s].

b) Estimate the height at $t = 1\,\text{s}$ using pencil and paper. An answer in meters is desired. [Hints: Assume g is negligible. Good calculus skills are needed but no involved arithmetic is needed. $1 + 1.72 = 2.72 \approx e$. After you have found a solution check that the force of gravity is a small fraction of the drag force throughout the duration of one second of your solution.]

10.33 In the arcade game shown, the object of the game is to propel the small ball from the ejector device at O in such a way that is passes through the small aperture at A and strikes the contact point at B. The player controls the angle θ at which the ball is ejected and the initial velocity v_0. The trajectory is confined to the frictionless xy-plane, which may or may not be vertical. Find the value of θ that gives success. The coordinates of A and B are $(2\ell, 2\ell)$ and $(3\ell, \ell)$, respectively, where ℓ is your favorite length unit.



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10.2 Momentum and energy for particle motion

Preparatory Problems

10.34 What symbols do we use for the following quantities? What are the definitions of these quantities? Which are vectors and which are scalars? What are the SI and US standard units for the following quantities?

a) linear momentum

b) rate of change of linear momentum

c) angular momentum

d) rate of change of angular momentum

e) kinetic energy

f) rate of change of kinetic energy

g) moment

h) work

i) power

10.35 Does angular momentum depend on reference point? (Assume that all candidate points are fixed in the same Newtonian reference frame.)

10.36 Does kinetic energy depend on reference point? (Assume that all candidate points are fixed in the same Newtonian reference frame.)

10.37 What is the relation between the dynamics 'Linear Momentum Balance' equation and the statics 'Force Balance' equation?

10.38 What is the relation between the dynamics 'Angular Momentum Balance' equation and the statics 'Moment Balance' equation?

10.39 A ball of mass $m = 0.1\,\text{kg}$ is thrown from a height of $h = 10\,\text{m}$ above the ground with velocity $\vec{v} = 120\,\text{km/h}\hat{\imath} - 120\,\text{km/h}\hat{\jmath}$. What is the kinetic energy of the ball at its release?

10.40 A ball of mass $m = 0.2\,\text{kg}$ is thrown from a height of $h = 20\,\text{m}$ above the ground with velocity $\vec{v} = 120\,\text{km/h}\hat{\imath} - 120\,\text{km/h}\hat{\jmath} - 10\,\text{km/h}\hat{\boldsymbol{k}}$. What is the kinetic energy of the ball at its release?

10.41 How do you calculate P, the power of all external forces acting on a particle, from the forces $\vec{F}_i$ and the velocity $\vec{v}$ of the particle?

10.42 A particle A has velocity $\vec{v}_A$ and mass m_A. A particle B has velocity $\vec{v}_B = 2\ \vec{v}_A$ and mass equal to the other $m_B = m_A$. What is the relationship between:

a) $\vec{L}_\text{A}$ and $\vec{L}_\text{B}$,

b) $\vec{H}_\text{A/C}$ and $\vec{H}_\text{B/C}$, and

c) E_KA and E_KB?

10.43 A bullet of mass 50 g travels with a velocity $\vec{v} = 0.8\,\text{km/s}\hat{\imath} + 0.6\,\text{km/s}\hat{\jmath}$. (a) What is the linear momentum of the bullet? (Answer in consistent units.)

10.44 A particle has position $\vec{r} = 4\,\text{m}\hat{\imath} + 7\,\text{m}\hat{\jmath}$, velocity $\vec{v} = 6\,\text{m/s}\hat{\imath} - 3\,\text{m/s}\hat{\jmath}$, and acceleration $\vec{a} = -2\,\text{m/s}^2\hat{\imath} + 9\,\text{m/s}^2\hat{\jmath}$. For each position of a point P defined below, find $\vec{H}_P$, the angular momentum of the particle with respect to the point P.

a) $\vec{r}_P = 4\,\text{m}\hat{\imath} + 7\,\text{m}\hat{\jmath}$,

b) $\vec{r}_P = -2\,\text{m}\hat{\imath} + 7\,\text{m}\hat{\jmath}$, and

c) $\vec{r}_P = 0\,\text{m}\hat{\imath} + 7\,\text{m}\hat{\jmath}$,

d) $\vec{r}_P = \vec{0}$

10.45 The position vector of a particle of mass 1 kg at an instant t is $\vec{r} = 2\,\text{m}\hat{\imath} - 0.5\,\text{m}\hat{\jmath}$. If the velocity of the particle at this instant is $\vec{v} = -4\,\text{m/s}\hat{\imath} + 3\,\text{m/s}\hat{\jmath}$, compute (a) the linear momentum $\vec{L} = m\vec{v}$ and (b) the angular momentum ($\vec{H}_{/\text{O}} = \vec{r}_{/\text{O}} \times (m\vec{v})$).

10.46 The position of a particle of mass $m = 0.5\,\text{kg}$ is $\vec{r}(t) = \ell\,\sin(\omega t)\hat{\imath} + h\hat{\jmath}$; where $\omega = 2\,\text{rad/s}, h = 2\,\text{m}, \ell = 2\,\text{m}$, and $\vec{r}$ is measured from the origin.

a) Find the kinetic energy of the particle at $t = 0\,\text{s}$ and $t = 5\,\text{s}$.

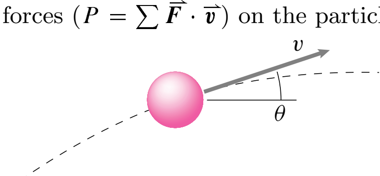
b) Find the rate of change of kinetic energy at $t = 0\,\text{s}$ and $t = 5\,\text{s}$.

10.47 For a particle

$$E_\text{K} = \frac{1}{2}\ m\ v^2.$$

Why does it follow that $\dot{E}_\text{K} = m\ \vec{v}\cdot\vec{a}$? [hint: write v^2 as $\vec{v}\cdot\vec{v}$ and then use the product rule of differentiation.]

10.48 Consider a projectile of mass m at some instant in time t during its flight. Let $\vec{v}$ be the velocity of the projectile at this instant (see the figure). In addition to the force of gravity, a drag force acts on the projectile. The drag force is proportional to the square of the speed (speed$\equiv |\vec{v}| = v$) and acts in the opposite direction. Find an expression for the net power of these forces ($P = \sum \vec{F}\cdot\vec{v}$) on the particle.



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10.49 A 10 gm wad of paper is tossed into the air. At a particular instant of interest, the position, velocity, and acceleration of its center of mass are $\vec{r} = 3\,\text{m}\hat{\imath} + 3\,\text{m}\hat{\jmath} + 6\,\text{m}\hat{k}$, $\vec{v} = -9\,\text{m/s}\hat{\imath} + 24\,\text{m/s}\hat{\jmath} + 30\,\text{m/s}\hat{k}$, and $\vec{a} = -10\,\text{m/s}^2\hat{\imath} + 24\,\text{m/s}^2\hat{\jmath} + 32\,\text{m/s}^2\hat{k}$, respectively. What is the translational kinetic energy of the wad at the instant of interest?

10.50 A 2 kg particle moves so that its position $\vec{r}$ is given by

$$\vec{r}(t) = [5\sin(at)\hat{\imath} + bt^2\hat{\jmath} + ct\hat{k}]\,\text{meters}$$

where $\text{a} = \pi/\sec, b = .25/\sec^2, c = 2/\sec$.

a) What is the linear momentum of the particle at $t = 1\sec$?

b) What is the force acting on the particle at $t = 1\sec$?

10.51 A particle A has mass m_A and velocity $\vec{v}_\text{A}$. A particle B at the same location has mass $m_B = 2\ m_A$ and velocity equal to the other $\vec{v}_\text{B} = \vec{v}_\text{A}$. Point C is a reference point. What is the relationship between:

a) $\vec{L}_\text{A}$ and $\vec{L}_\text{B}$,

b) $\vec{H}_\text{A/C}$ and $\vec{H}_\text{B/C}$, and

c) E_KA and E_KB?

10.52 A particle of mass $m = 3\,\text{kg}$ moves in space. Its position, velocity, and acceleration at a particular instant in time are $\vec{r} = 2\,\text{m}\hat{\imath} + 3\,\text{m}\hat{\jmath} + 5\,\text{m}\hat{k}$, $\vec{v} = -3\,\text{m/s}\hat{\imath} + 8\,\text{m/s}\hat{\jmath} + 10\,\text{m/s}\hat{k}$, and $\vec{a} = -5\,\text{m/s}^2\hat{\imath} + 12\,\text{m/s}^2\hat{\jmath} + 16\,\text{m/s}^2\hat{k}$, respectively. For this particle at the instant of interest, find its:

a) linear momentum $\vec{L}$,

b) rate of change of linear momentum $\dot{\vec{L}}$,

c) angular momentum about the origin $\vec{H}_{/\text{O}}$,

d) rate of change of angular momentum about the origin $\dot{\vec{H}}_{/\text{O}}$,

e) kinetic energy E_K, and

f) rate of change of kinetic energy $\dot{E}_\text{K}$.

10.53 A particle has position $\vec{r} = 3\,\text{m}\hat{\imath} - 2\,\text{m}\hat{\jmath} + 4\,\text{m}\hat{k}$, velocity $\vec{v} = 2\,\text{m/s}\hat{\imath} - 3\,\text{m/s}\hat{\jmath} + 7\,\text{m/s}\hat{k}$, and acceleration $\vec{a} = 1\,\text{m/s}^2\hat{\imath} - 8\,\text{m/s}^2\hat{\jmath} + 3\,\text{m/s}^2\hat{k}$. For each position of a point P defined below, find the rate of change of angular momentum, $\dot{\vec{H}}_P$, of the particle with respect to the point P.

a) $\vec{r}_P = 3\,\text{m}\hat{\imath} - 2\,\text{m}\hat{\jmath} + 4\,\text{m}\hat{k}$,

b) $\vec{r}_P = 6\,\text{m}\hat{\imath} - 4\,\text{m}\hat{\jmath} + 8\,\text{m}\hat{k}$,

c) $\vec{r}_P = -9\,\text{m}\hat{\imath} + 6\,\text{m}\hat{\jmath} - 12\,\text{m}\hat{k}$, and

d) $\vec{r}_P = \vec{0}$

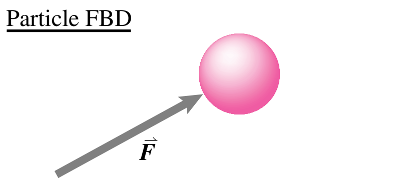
More-Involved Problems

10.54 A particle of mass $m = 6\,\mathrm{kg}$ is moving in space. Its position, velocity, and acceleration at a particular instant in time are $\vec{r} = 1\,\mathrm{m}\hat{\imath} - 2\,\mathrm{m}\hat{\jmath} + 4\,\mathrm{m}\hat{k}$, $\vec{v} = 3\,\mathrm{m/s}\hat{\imath} + 4\,\mathrm{m/s}\hat{\jmath} - 7\,\mathrm{m/s}\hat{k}$, and $\vec{a} = 5\,\mathrm{m/s^2}\hat{\imath} + 11\,\mathrm{m/s^2}\hat{\jmath} - 9\,\mathrm{m/s^2}\hat{k}$, respectively. For this particle at the instant of interest, find its:

a) the net force $\sum \vec{F}$ on the particle,

b) the net moment on the particle about the origin $\sum \vec{M}_O$ due to the applied forces, and

c) the power P of the applied forces.



FBD of the particle

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10.55 At a particular instant of interest, a particle of mass $m_1 = 5\,\mathrm{kg}$ has position, velocity, and acceleration $\vec{r}_1 = 3\,\mathrm{m}\hat{\imath}$, $\vec{v}_1 = -4\,\mathrm{m/s}\hat{\jmath}$, and $\vec{a}_1 = 6\,\mathrm{m/s^2}\hat{\jmath}$, respectively, and a particle of mass $m_2 = 5\,\mathrm{kg}$ has position, velocity, and acceleration $\vec{r}_2 = -6\,\mathrm{m}\hat{\imath}$, $\vec{v}_2 = 5\,\mathrm{m/s}\hat{\jmath}$, and $\vec{a}_2 = -4\,\mathrm{m/s^2}\hat{\jmath}$, respectively. For the system of particles, find its

a) linear momentum $\vec{L}$,

b) rate of change of linear momentum $\dot{\vec{L}}$

c) angular momentum about the origin $\vec{H}_{/O}$,

d) rate of change of angular momentum about the origin $\dot{\vec{H}}_{/O}$,

e) kinetic energy E_K, and

f) rate of change of kinetic energy $\dot{E}_K$.

10.56 A particle of mass $m = 250$ gm is shot straight up (parallel to the y-axis) from the x-axis at a distance $d = 2\,\mathrm{m}$ from the origin. The velocity of the particle is given by $\vec{v} = v\hat{\jmath}$ where $v^2 = v_0^2 - 2ah$, $v_0 = 100\,\mathrm{m/s}$, $a = 10\,\mathrm{m/s^2}$ and h is the height of the particle from the x-axis.

a) Find the linear momentum of the particle at the outset of motion ($h = 0$).

b) Find the angular momentum of the particle about the origin at the outset of motion ($h = 0$).

c) Find the linear momentum of the particle when the particle is 20 m above the x-axis.

d) Find the angular momentum of the particle about the origin when the particle is 20 m above the x-axis.

10.3 Central force motion

Experts note that none of these problems use polar coordinates or other fancy coordinate systems. Such descriptions come later in the text. At this point we want to lay out the basic equations and the qualitative features that can be found by numerical integration of the equations.

Preparatory Problems

10.57 What exactly is meant by "central force motion"?

10.58 Under what circumstances is the angular momentum of a system, calculated relative to a point C which is fixed in a Newtonian frame, conserved?

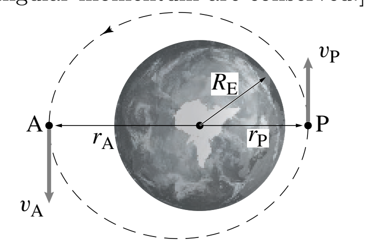
10.59 The mass of the earth is M, the mass of a satellite orbiting the earth is m, the radius of the earth is R, the force of gravity at the earth's surface is mg, the universal gravitational constant is G.

a) If the sattelite is at distance r what is the force of the earth's gravity in terms of r, M, m and G?

b) If the sattelite is at distance r what is the force of the earth's gravity in terms of r, R, m and g? (hint: evaluate the formula from the first part at $r = R$).

More-Involved Problems

10.60 A satellite is put into an elliptical orbit around the earth and has a speed v_P at position P. Find an expression for the speed v_A at position A (in terms of R_E, r_P, r_A, g, and v_P. The radii to A and P are, respectively, r_A and r_P. [Hint: both total energy and angular momentum are conserved.]



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10.61 The mechanics of nuclear war. A missile, modelled as a point, is launched on a ballistic trajectory from the surface of the earth. The force on the missile from the earth's gravity is $F = mgR^2/r^2$ and is directed towards the center of the earth. When it is launched from the equator it has speed v_0 and in the direction shown, 45° from horizontal. For the purposes of this calculation ignore the earth's rotation. That is, you can think of this problem as two-dimensional in the plane shown. If you need numbers, use the following values:

$m = 1000\,\mathrm{kg}$ is the mass of the missile,

$g = 10\,\mathrm{m/s^2}$ is earth's gravitational constant at the earth's surface,

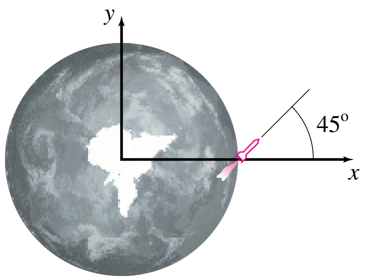
$R = 6,400,000\,\mathrm{m}$ is the radius of the earth, and

$v_0 = 9000\,\mathrm{m/s}$

$r(t)$ is the distance of the missile from the center of the earth.

a) Draw a free body diagram of the missile. Write the linear momentum balance equation. Break this equation into x and y components. Rewrite these equations as a system of 4 first order ODE's suitable for computer solution. Write appropriate initial conditions for the ODE's.

b) Using the computer (or any other means) plot the trajectory of the rocket after it is launched for a time of 6670 seconds. [Use a much shorter time when debugging your program.] On the same plot draw a (round) circle for the earth.



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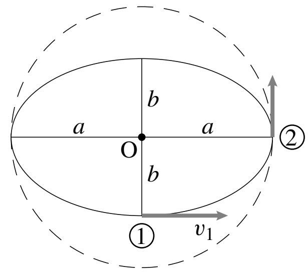
10.62 A particle of mass 2 kg moves in the horizontal xy-plane under the influence of a central force $\vec{F} = -k\vec{r}$ (attraction force proportional to distance from the origin), where $k = 200\,\mathrm{N/m}$ and $\vec{r}$ is the position of the particle relative to the force center. Neglect all other forces.

a) Show that circular trajectories are possible, and determine the relation between speed v and circular radius r_o which must hold on a circular trajectory. [hint: Write $\vec{F} = m\vec{a}$, break into x and y components, solve the separate scalar equations, pick fortuitous values for the free constants in your solutions.]

b) It turns out that trajectories are in general elliptical, as depicted in the diagram.

For a particular elliptical trajectory with $a = 1\,\mathrm{m}$ and $b = 0.8\,\mathrm{m}$, the velocity of the particle at point 1 is observed to be perpendicular to the radial direction, with magnitude v_1, as shown. When the particle reaches point 2, its velocity is again perpendicular to the radial direction.

Determine the speed increment Δv which would have to be added (instantaneously) to the particle's speed at point 2 to transfer it to the circular trajectory through point 2 (the dotted curve).



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10.63 Circular motion. Generally when people talk about central force motion they not only mean that the only force is directed at the origin but that the magnitude of the force only depends on the distance from the origin. Thus in 2D

$$\vec{F} = -\underbrace{\frac{x\hat{\imath} + y\hat{\jmath}}{\sqrt{x^2+y^2}}}_{\text{Unit vector}} \cdot \underbrace{F\left(\sqrt{x^2+y^2}\right)}_{\text{Magnitude of force}}$$

where the scalar function $F(r)$ expresses the dependence of the central attractive force on distance $r = \sqrt{x^2+y^2}$. Consider a particle with mass m on a candidate circular orbit

$$\vec{r} = R\cos\lambda t\,\hat{\imath} + R\sin\lambda t\,\hat{\jmath}$$

with constant speed $v = |\vec{v}| = |\dot{\vec{r}}| = R\lambda$. For each of the cases below find the speed v for circular motion at radius R. Find this by plugging the circular motion equation into $\vec{F} = m\vec{a}$ using the form of $F(r)$ given. Answer in terms of other constants given (e.g., k, m, M, G)

a) $F(r) = kr$ (zero-rest-length attractive spring)

b) $F(r) = -GMm/r^2$ (inverse-square gravitational attraction)

c) $F(r) = r^n$ (arbitrary power law attraction)

d) $F(r) = F(r)$ (arbitrary function). In this case you need to find how the speed depends on F in general.

e) Can you find a function $F(r)$ for which there are two or more circular orbits at the same speed v?

10.64 Circular motion, numerical solution. For each of the cases in problem 10.63 pick values for the physical constants. Then pick initial conditions which, according to theory, should give circular orbits. Then numerically solve the 4 coupled first order ODEs that describe planar motion, make a plot, and show that you do indeed get circular orbits. How big is the disrepancy between your numerical solution and an exact circle?

10.65 Two equal mass satellites have circular orbits at two different radii. The one that is closer to the earth has smaller potential energy and bigger kinetic energy. Which satellite has bigger total energy?

10.66 Find initial conditions corresponding to circular motion for a central force problem and simulate this motion on the computer. Use any central force attraction law you like (e.g., zero-length spring, inverse square,...) Check that you get closed circular orbits by plotting several revolutions. Now, in your simulation, apply a slight drag force opposing motion $\vec{F} = -c\,\vec{v}$. Pick a value for c so that the orbit slowly spirals in (say, less than 10% per orbit).

a) Make a plot of the spiraling orbit.

b) Plot the speed $|\vec{v}|$ vs time as it spirals in.

c) How is it that a drag force causes the satellite to speed up? Is that numerical error? An approximation in our formulation of the governing equations? A relativistic effect? What?

10.67 Circular motion, numerical solution. For each of the cases in problem 10.63 pick values for the physical constants.

a) Pick initial conditions which, according to theory, should give circular orbits.

b) Numerically solve the 4 coupled first order ODEs that describe planar motion, make a plot, and show that you do indeed get circular orbits.

c) How big is the disrepancy between your numerical solution and an exact circle? Between the theoretically predicted period and the actual period?

10.68 Conic sections, numerical solution. Newton discovered that with $\vec{F} = m\vec{a}$ and a central attractive force of $F = C/r^2$ that all motions were conic sections. In particular, consider this problem, all in consistent units: $m = 1, C = 1, x_0 = 1, y_0 = 0, \dot{x}_0 = 0, \dot{y}_0 = v_0$. Newton claimed that there is a special values for v_0, lets call them v_0^c and v_0^p with $v_0^c < v_0^p$ so that

- for $v_0 < v_0^c$ all orbits are ellipses with maximum distance from the origin being 1;
- for $v_0 = v_0^c$ the orbit is a circle of radius 1;
- for $v_0^c < v_0 < v_0^p$ the orbit is an ellipse with minimum distance from the origin being 1;
- for $v_0 = v_0^p$ the orbit is a left-opening parabola with base at the initial condition. (to draw the complete parabola you need also to use al the same initial conditions but with $\dot{y}_0 = -v_0$).
- for $v_0^p < v_0$ the orbit is a left-opening hyperbola, asymptoting to a straight line. (again you need to use $\dot{y}_0 = -v_0$ to draw the complete hyperbola.

a) By a sequence of more or less systematic numerical guesses find as accurately, as you can, v_0^c and v_0^p.

b) Numerically solve the 4 coupled first order ODEs for initial conditions that correspond to each of the 5 cases above.

c) Plot all 5 cases on one plot showing the 5 shapes clearly.