2.6 Center of mass and gravity

For every system and at every instant in time, there is a unique location in space that is the average position of the system’s mass. This place is called the center of mass, commonly designated by cm, c.o.m., COM, G, c.g., or \( \mathbf{O} \).

One of the routine but important tasks of many real engineers is to find the center of mass of a complex machine\(^{1}\). Just knowing the location of the center of mass of a car, for example, is enough to estimate whether it can be tipped over by maneuvers on level ground. The center of mass of a boat must be low enough for the boat to be stable. Any propulsive force on a space craft must be directed towards the center of mass in order to not induce rotations. Tracking the trajectory of the center of mass of an exploding plane can determine whether or not it was hit by a massive object. Any rotating piece of machinery must have its center of mass on the axis of rotation if it is not to cause much vibration.

Also, many calculations in mechanics are greatly simplified by making use of a system’s center of mass. In particular, the whole complicated distribution of near-earth gravity forces on a body is equivalent to a single force at the body’s center of mass. Many of the important quantities in dynamics are similarly simplified using the center of mass.

The center of mass of a system is the point at the position \( \mathbf{r}_{\text{cm}} \) defined by

\[
\mathbf{r}_{\text{cm}} = \frac{\sum \mathbf{r}_i m_i}{m_{\text{tot}}} \quad \text{for discrete systems} \tag{2.30}
\]

\[
= \frac{\int \mathbf{r} \, dm}{m_{\text{tot}}} \quad \text{for continuous systems}
\]

where \( m_{\text{tot}} = \sum m_i \) for discrete systems and \( m_{\text{tot}} = \int dm \) for continuous systems.

See boxes 2.5 and ?? for a discussion of the \( \sum \) and \( \int \) sum notations.

Often it is convenient to remember the rearranged definition of center of mass as

\[
m_{\text{tot}} \mathbf{r}_{\text{cm}} = \sum m_i \mathbf{r}_i \quad \text{or} \quad m_{\text{tot}} \mathbf{r}_{\text{cm}} = \int \mathbf{r} \, dm.
\]

For theoretical purposes we rarely need to evaluate these sums and integrals, and for simple problems there are sometimes shortcuts that reduce the calculation to a matter of observation. For complex machines one or both of the formulas ?? must be evaluated in detail.

**Example: System of two point masses**

Intuitively, the center of mass of the two masses shown in figure ?? is between the two masses and closer to the larger one. Referring to equation ??,

\[
\mathbf{r}_{\text{cm}} = \frac{\sum \mathbf{r}_i m_i}{m_{\text{tot}}} = \frac{\mathbf{r}_1 m_1 + \mathbf{r}_2 m_2}{m_1 + m_2} = \frac{\mathbf{r}_1 (m_1 + m_2) - \mathbf{r}_1 m_2 + \mathbf{r}_2 m_2}{m_1 + m_2}
\]
2.6 Center of mass and gravity

\[ \vec{r}_1 + \left( \frac{m_2}{m_1 + m_2} \right) \left( \vec{r}_2 - \vec{r}_1 \right). \]

the fraction of the distance that the cm is from \( \vec{r}_1 \) to \( \vec{r}_2 \)

the vector from \( \vec{r}_1 \) to \( \vec{r}_2 \)

so that the math agrees with common sense — the center of mass is on the line connecting the masses. If \( m_2 \gg m_1 \), then the center of mass is near \( m_2 \). If \( m_1 \gg m_2 \), then the center of mass is near \( m_1 \). If \( m_1 = m_2 \), the center of mass is right in the middle at \((\vec{r}_1 + \vec{r}_2)/2\).

Continuous systems

How do we evaluate integrals like \( \int (\text{something}) \, dm \)? In center of mass calculations, (something) is position, but we will evaluate similar integrals where (something) is some other scalar or vector function of position. Most often we label the material by its spatial position, and evaluate \( dm \) in terms of increments of position. For 3D solids \( dm = \rho dV \) where \( \rho \) is density (mass per unit volume). So \( \int (\text{something}) \, dm \)

2.9 \( \int \) means add

As discussed in box 2.5 on page 70 we often add things up in mechanics. For example, the total mass of some particles is

\[ m_{\text{tot}} = m_1 + m_2 + m_3 + \ldots = \sum m_i \]

or more specifically the mass of 137 particles is, say, \( m_{\text{tot}} = \sum_{i=1}^{137} m_i \).

And the total mass of a bicycle is:

\[ m_{\text{bike}} = \sum_{i=1}^{100,000,000,000,000,000,000,000} m_i \]

where \( m_i \) are the masses of each of the \( 10^{23} \) (or so) atoms of metal, rubber, plastic, cotton, and paint. But atoms are so small and there are so many of them. Instead we often think of a bike as built of macroscopic parts. The total mass of the bike is then the sum of the masses of the tires, the tubes, the wheel rims, the spokes and nipples, the ball bearings, the chain pins, and so on. And we would write:

\[ m_{\text{bike}} = \sum_{i=1}^{2,000} m_i \]

where now the \( m_i \) are the masses of the 2,000 or so bike parts. This sum is more manageable but still too detailed in concept for some purposes.

An approach that avoids attending to atoms or ball bearings, is to think of sending the bike to a big shredding machine that cuts it up into very small bits. Now we write

\[ m_{\text{bike}} = \sum m_i \]

where the \( m_i \) are the masses of the very small bits. We don’t fuss over whether one bit is a piece of ball bearing or fragment of cotton from the tire walls. We just chop the bike into bits and add up the contribution of each bit. If you take the letter S, as in SUM, and distort it ( \( \mathcal{S} \sum \int \mathcal{S} \) ) and you get a big old fashioned German ‘S’ as in \( \int \mathcal{U}.\mathcal{M} \) (sum). So we write

\[ m_{\text{bike}} = \int dm \]

to mean the \( \int \)um of all the teeny bits of mass. More formally we mean the value of that sum in the limit that all the bits are infinitesimal (not minding the technical fine point that its hard to chop atoms into infinitesimal pieces).

The mass is one of many things we would like to add up, though many of the others also involve mass. In center of mass calculations, for example, we add up the positions ‘weighted’ by mass.

\[ \int \vec{r} \, dm \text{ which means } \lim_{m_i \to 0} \sum \vec{r}_i m_i. \]

That is, you take your object of interest and chop it into a billion pieces and then re-assemble it. For each piece you make the vector which is the position vector of the piece multiplied by (‘weighted by’) its mass and then add up the billion vectors. Well really you chop the thing into a trillion trillion . . . pieces, but a billion gives the idea.
turns into a standard volume integral \( \int_V (\text{something}) \rho \, dV \). For thin flat things like metal sheets we often take \( \rho \) to mean mass per unit area \( A \) so then \( dm = \rho \, dA \) and \( \int (\text{something}) \, dm = \int_A (\text{something}) \rho \, dA \). For mass distributed along a line or curve we take \( \rho \) to be the mass per unit length or arc length \( s \) and so \( dm = \rho \, ds \) and \( \int (\text{something}) \, dm = \int_{\text{curve}} (\text{something}) \rho \, ds \).

**Example.** The center of mass of a uniform rod is naturally in the middle, as the calculations here show (see fig. ??a). Assume the rod has length \( L = 3 \, \text{m} \) and mass \( m = 7 \, \text{kg} \).

\[
\mathbf{r}_{\text{cm}} = \frac{\int x \, dm}{m_{\text{tot}}} = \frac{\int_0^L x \, \rho \, dx}{\int_0^L \rho \, dx} = \frac{\rho(L^2/2)}{L \rho} \mathbf{i} = (L/2) \mathbf{i}
\]

So \( \mathbf{r}_{\text{cm}} = (L/2) \mathbf{i} \), or by dotting with \( \mathbf{i} \) (taking the x component) we get that the center of mass is on the rod a distance \( d = L/2 = 1.5 \, \text{m} \) from the end. \( \Box \)

The center of mass calculation is *objective*. It describes something about the object that does not depend on the coordinate system. In different coordinate systems the center of mass for the rod above will have different coordinates, but it will always be at the middle of the rod.

**Example.** Find the center of mass using the coordinate system with \( s \) & \( \lambda \) in fig. ??b:

\[
\mathbf{r}_{\text{cm}} = \frac{\int \mathbf{r} \, dm}{m_{\text{tot}}} = \frac{\int_0^L \mathbf{s} \lambda \, \rho \, ds}{\int_0^L \rho \, ds} = \frac{\rho(L^2/2)}{L \rho} \mathbf{\hat{\lambda}} = (L/2) \mathbf{\hat{\lambda}},
\]

again showing that the center of mass is in the middle. \( \Box \)

Note, one can treat the center of mass vector calculations as separate scalar equations, one for each component. For example:

\[
\mathbf{i} \cdot \left\{ \mathbf{r}_{\text{cm}} = \frac{\int \mathbf{r} \, dm}{m_{\text{tot}}} \right\} \Rightarrow r_{x_{\text{cm}}} = x_{\text{cm}} = \frac{\int x \, dm}{m_{\text{tot}}}.
\]

Finally, there is no law that says you have to use the best coordinate system. One is free to make trouble for oneself and use an inconvenient coordinate system.

**Example.** Use the \( xy \) coordinates of fig. ??c to find the center of mass of the rod.

\[
x_{\text{cm}} = \frac{\int x \, dm}{m_{\text{tot}}} = \frac{\int_{\ell_1}^{\ell_2} s \cos \theta \, \rho \, ds}{\int_{\ell_1}^{\ell_2} \rho \, ds} = \frac{\rho \cos \theta (\ell_2^2 - \ell_1^2)}{\rho (\ell_2 - \ell_1)^2} = \frac{\cos \theta (\ell_2 - \ell_1)}{2}
\]

Similarly \( y_{\text{cm}} = \sin \theta (\ell_2 - \ell_1)/2 \) so

\[
\mathbf{r}_{\text{cm}} = \frac{\ell_2 - \ell_1}{2} (\cos \theta \mathbf{i} + \sin \theta \mathbf{j})
\]

which still describes the point at the middle of the rod. \( \Box \)

The most commonly needed center of mass that can be found analytically but not directly from symmetry is that of a triangle (see box ?? on page ??). You can find more examples using integration to find the center of mass (or centroid) in your calculus text.
Center of mass and centroid

For objects with uniform material density we have

\[ \vec{r}_{cm} = \frac{\int \vec{r} \, dm}{m_{tot}} = \frac{\int_V \vec{r} \, \rho \, dV}{\int_V \rho \, dV} = \frac{\rho \int_V \vec{r} \, dV}{\rho \int_V dV} = \frac{\int_V \vec{r} \, dV}{V} \]

where the last expression is just the formula for geometric centroid. Analogous calculations hold for 2D and 1D geometric objects. Thus for objects with density that does not vary from point to point, the geometric centroid and the center of mass coincide.

Center of mass and symmetry

The center of mass respects any symmetry in the mass distribution of a system. If the word ‘middle’ has unambiguous meaning in English then that is the location of the center of mass, as for the rod of fig. ?? and the other examples in fig. ??.

Systems of systems and composite objects

Another way of interpreting the formula

\[ \vec{r}_{cm} = \frac{\vec{r}_1 m_1 + \vec{r}_2 m_2 + \cdots}{m_1 + m_2 + \cdots} \]

is that the \( m \)'s are the masses of subsystems, not just points, and that the \( \vec{r}_i \) are the positions of the centers of mass of these systems. This subdivision is justified in box ?? on page ??: The center of mass of a single complex shaped object can be found by treating it as an assembly of simpler objects.
Example: Two rods

The center of mass of two rods shown in figure 2.71 can be found as

\[
\vec{r}_{cm} = \frac{\vec{r}_1 m_1 + \vec{r}_2 m_2}{m_1 + m_2}
\]

where \(\vec{r}_1\) and \(\vec{r}_2\) are the positions of the centers of mass of each rod and \(m_1\) and \(m_2\) are the masses.

Example: ‘L’ shaped plate

Consider the plate with uniform mass per unit area \(\rho\).

\[
\vec{r}_G = \frac{\vec{r}_I m_I + \vec{r}_II m_{II}}{m_I + m_{II}}
\]

\[
= \frac{(\frac{2}{3}a \hat{i} + a \hat{j})(2\rho a^2) + (\frac{3}{2}a \hat{i} + \frac{3}{2}a \hat{j})(\rho a^2)}{(2\rho a^2) + (\rho a^2)}
\]

\[
= \frac{5}{6}a(\hat{i} + \hat{j}).
\]

Composite objects using subtraction

It is sometimes useful to think of an object as composed of pieces, some of which have negative mass.

Example: ‘L’ shaped plate, again

Reconsider the plate from the previous example.

\[
\vec{r}_G = \frac{\vec{r}_I m_I + \vec{r}_II m_{II}}{m_I + m_{II}}
\]

\[
= \frac{(a \hat{i} + a \hat{j})(\rho(2a)^2) + (\frac{3}{2}a \hat{i} + \frac{3}{2}a \hat{j})(-\rho a^2)}{(\rho(2a)^2) + (\rho a^2)(-\rho a^2)}
\]

\[
= \frac{5}{6}a(\hat{i} + \hat{j}).
\]

Center of gravity

The force of gravity on each little bit of an object is \(gm_i\) where \(g\) is the local gravitational ‘constant’ and \(m_i\) is the mass of the bit. For objects that are small compared to the radius of the earth (a reasonable assumption for all but a few special engineering calculations) the gravity constant is indeed constant from one point on the object to another (see box A.1 on page A.1 for a discussion of the meaning and history of \(g\).)
2.6 Center of mass and gravity

Not only that, all the gravity forces point in the same direction, down. (For engineering purposes, the two intersecting lines that go from your two hands to the center of the earth are parallel.) Let’s call this the $-\hat{k}$ direction. So the net force of gravity on an object is:

$$\vec{F}_{\text{net}} = \sum \vec{F}_i = \sum m_i g (-\hat{k}) = -mg \hat{k}$$

for discrete systems, and

$$\int d\vec{F} = \int -g \hat{k} dm = -mg \hat{k}$$

for continuous systems.

That’s easy, the billions of gravity forces on an objects microscopic constituents add up to $mg$ pointed down. What about the net moment of the gravity forces? The answer turns out to be simple. The top line of the calculation below poses the question, the last line gives the lucky answer.\footnote{We do the calculation here using the $\int$ notation for sums. But it could be done just as well using $\sum$.}

$$\vec{M}_C = \int \vec{r} \times d\vec{F}$$

The net moment with respect to C.

$$= \int \vec{r}_C \times (-g \hat{k} dm)$$

A force bit is gravity acting on a mass bit.

$$= \left( \int \vec{r}_C dm \right) \times (-g \hat{k})$$

Cross product distributive law ($g$, $\hat{k}$ are constants).

$$= (\vec{r}_{\text{cm}}/cm) \times (-g \hat{k})$$

Definition of center of mass.

$$= \vec{r}_{\text{cm}} \times (-mg \hat{k})$$

Re-arranging terms.

2.10 THEORY

Why can subsystems be treated like particles when finding the center of mass?

Let’s look at the collection of 47 particles above and then think of it as a set of three subsystems: I, II, and III with 2, 14, and 31 particles respectively. We treat masses 1 and 2 as subsystem I with center of mass $\vec{r}_1$ and total mass $m_1$. Similarly, we call subsystem II masses $m_3$ to $m_{16}$, and subsystem III, masses $m_{17}$ to $m_{47}$. We can calculate the center of mass of the system by treating it as 47 particles, or we can re-arrange the sum as follows:

$$\vec{r}_{\text{cm}} = \frac{\vec{r}_1 m_1 + \vec{r}_2 m_2 + \cdots + \vec{r}_{46} m_{46} + \vec{r}_{47} m_{47}}{m_1 + m_2 + \cdots + m_{47}}$$

We do the calculation here using the $\int$ notation for sums. But it could be done just as well using $\sum$.

The formula for the center of mass of the whole system reduces to one that looks like a sum over three (aggregate) particles.

This idea is easily generalized to the integral formulae as well like this.

$$\vec{r}_{\text{cm}} = \frac{\int \vec{r} dm}{\int dm} = \frac{\int_{\text{region 1}} \vec{r} dm + \int_{\text{region 2}} \vec{r} dm + \int_{\text{region 3}} \vec{r} dm + \cdots}{\int_{\text{region 1}} dm + \int_{\text{region 2}} dm + \int_{\text{region 1}} dm + \cdots} = \frac{\vec{r}_1 m_1 + \vec{r}_2 m_2 + \cdots}{m_1 + m_2 + \cdots}$$

The general idea of the calculations above is that center of mass calculations are basically big sums (addition), and addition is ‘associative.’
\[ \vec{r}_{\text{cm/C}} \times \vec{F}_{\text{net}} \]

Express in terms of net gravity force.

Thus the net moment is the same as for the total gravity force acting at the center of mass.

The near-earth gravity forces acting on a system are equivalent to a single force, \( mg \), acting at the system’s center of mass.

For the purposes of calculating the net force and moment from near-earth (constant \( g \)) gravity forces, a system can be replaced by a point mass at the center of gravity. The words ‘center of mass’ and ‘center of gravity’ both describe the same point in space.

Although the result we have just found seems plain enough, here are two things to ponder about gravity when viewed as an inverse square law (and thus not constant like we have assumed) that may make the result above seem less obvious.

- The net gravity force on a sphere is indeed equivalent to the force of a point mass at the center of the sphere. It took the genius Isaac Newton 3 years to deduce this result and the reasoning involved is too advanced for this book.
- The net gravity force on systems that are not spheres is generally not equivalent to a force acting at the center of mass (this is important for the understanding of tides as well as the orientational stability of satellites).

**A recipe for finding the center of mass of a complex system**

You find the center of mass of a complex system by knowing the masses and mass centers of its components. You find each of these centers of mass by

- Treating it as a point mass, or
- Treating it as a symmetric body and locating the center of mass in the middle, or
- Using integration, or
- Using the result of an experiment (which we will discuss in statics), or
- Treating the component as a complex system itself and applying this very recipe.

The recipe is just an application of the basic definition of center of mass (eqn. ??) but with our accumulated wisdom that the locations and masses in that sum can be the centers of mass and total masses of complex subsystems.

One way to arrange one’s data is in a table or spreadsheet, like below. The first four columns are the basic data. They are the \( x \), \( y \), and \( z \) coordinates of the subsystem center of mass (relative to some clear reference point), and the masses of the subsystems, one row for each of the \( N \) subsystems.

<table>
<thead>
<tr>
<th>Subsys#</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subsys 1</td>
<td>( x_1 )</td>
<td>( y_1 )</td>
<td>( z_1 )</td>
<td>( m_1 )</td>
<td>( m_1 x_1 )</td>
<td>( m_1 y_1 )</td>
<td>( m_1 z_1 )</td>
</tr>
<tr>
<td>Subsys 2</td>
<td>( x_2 )</td>
<td>( y_2 )</td>
<td>( z_2 )</td>
<td>( m_2 )</td>
<td>( m_2 x_2 )</td>
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<tr>
<td>Subsys N</td>
<td>( x_N )</td>
<td>( y_N )</td>
<td>( z_N )</td>
<td>( m_N )</td>
<td>( m_N x_N )</td>
<td>( m_N y_N )</td>
<td>( m_N z_N )</td>
</tr>
<tr>
<td>Row N+1</td>
<td>sums</td>
<td>( m_{\text{tot}} = \sum m_i )</td>
<td>( \sum m_i x_i )</td>
<td>( \sum m_i y_i )</td>
<td>( \sum m_i z_i )</td>
<td></td>
<td></td>
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</tbody>
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<tr>
<th>Subsys#</th>
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<tr>
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<td>( m_1 )</td>
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<tr>
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<td>( z_2 )</td>
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<td>Subsys N</td>
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<tr>
<td>Row N+1</td>
<td>sums</td>
<td>( m_{\text{tot}} = \sum m_i )</td>
<td>( \sum m_i x_i )</td>
<td>( \sum m_i y_i )</td>
<td>( \sum m_i z_i )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ x_{\text{cm}} = \frac{\sum m_i x_i}{m_{\text{tot}}} \quad y_{\text{cm}} = \frac{\sum m_i y_i}{m_{\text{tot}}} \quad z_{\text{cm}} = \frac{\sum m_i z_i}{m_{\text{tot}}} \]
One next calculates three new columns (5, 6, and 7) which come from each coordinate multiplied by its mass. For example the entry in the 6th row and 7th column is the \( z \) component of the 6th subsystem’s center of mass multiplied by the mass of the 6th subsystem. Then one sums columns 4 through 7. The sum of column 4 is the total mass, the sums of columns 5 through 7 are the total mass-weighted positions. Finally the result, the system center of mass coordinates, are found by dividing columns 5-7 of row N+1 by column 4 of row N+1.

Of course, there are multiple ways of systematically representing the data. The spreadsheet-like calculation above is just one way to organize the calculation.

**Summary of center of mass**

All discussions in mechanics make frequent reference to the concept of center of mass because

\[
\vec{r}_{cm} = \frac{\sum \vec{r}_i m_i}{m_{tot}} \quad \text{for discrete systems or systems of systems}
\]
\[
= \frac{\int \vec{r} \, dm}{m_{tot}} \quad \text{for continuous systems}
\]
\[
m_{tot} = \sum m_i \quad \text{for discrete systems or systems of systems}
\]
\[
= \int dm \quad \text{for continuous systems.}
\]

*For systems with distributed mass, the expressions for gravitational moment, linear momentum, angular momentum, and energy are all simplified by using the center of mass.*
2.11 The center of mass of a uniform triangle is a third of the way up from the base

The center of mass of a 2D uniform triangular region is the centroid of the area.

First we consider a right triangle with perpendicular sides \( b \) and \( h \)

and find the \( x \) coordinate of the centroid as

\[
x_{\text{cm}} = \frac{2h}{3},
\]

a third of the way to the left of the vertical base on the right. By similar reasoning, in the \( y \) direction, the centroid is a third of the way up from the base.

The center of mass of an arbitrary triangle can be found by treating it as the sum of two right triangles

so the centroid is a third of the way up from the base of any triangle. Finally, the result holds for all three bases. Summarizing, the centroid of a triangle is at the point one third up from each of the bases.

Non-calculus approach

Consider the line segment from \( A \) to the midpoint \( M \) of side \( BC \).

We can divide triangle \( ABC \) into equal width strips that are parallel to \( AM \). We can group these strips into pairs, each a distance \( s \) from \( AM \). Because \( M \) is the midpoint of \( BC \), by proportions each of these strips has the same length \( \ell \). Now in trying to find the distance of the center of mass from the line \( AM \) we notice that all contributions to the sum come in canceling pairs because the strips are of equal area and equal distance from \( AM \) but on opposite sides. Thus the centroid is on \( AM \). Likewise for all three sides. Thus the centroid is at the point of intersection of the three side bisectors.

That the three side bisectors intersect a third of the way up from the three bases can be reasoned by looking at the 6 triangles formed by the side bisectors.

The two triangles marked \( a \) and \( a \) have the same area (let’s call it \( a \) ) because they have the same height and bases of equal length (\( BM \) and \( CM \)). Similar reasoning with the other side bisectors shows that the pairs marked \( b \) have equal area and so have the pairs marked \( c \). But the triangle \( ABM \) has the same base and height and thus the same area as the triangle \( ACM \). So \( a + b + b = a + c + c \). Thus \( b = c \) and by similar reasoning \( a = b \) and all six little triangles have the same area. Thus the area of big triangle \( ABC \) is 3 times the area of \( GBC \). Because \( ABC \) and \( GBC \) share the base \( BC \), and point \( G \) is thus a third of the way up from the base.

Where is the middle of a triangle?

We have shown that the centroid of a triangle is at the point that is at the intersection of: the three side bisectors; the three area bisectors (which are the side bisectors); and the three lines one third of the way up from the three bases.

If the triangle only had three equal point masses on its vertices the center of mass lands on the same place. Thus the ‘middle’ of a triangle seems pretty well defined. But, there is some ambiguity. If the triangle were made of bars along each edge, each with equal cross sections, the center of mass would be in a different location for all but equilateral triangles. Also, the three angle bisectors of a triangle do not intersect at the centroid. Unless we define middle to mean centroid, the “middle” of a triangle is not well defined.
SAMPLE 2.39 Center of mass in 1-D: Three particles (point masses) of mass 2 kg, 3 kg, and 3 kg, are welded to a straight massless rod as shown in the figure. Find the location of the center of mass of the assembly.

Solution Let us select the first mass, \( m_1 = 2 \text{ kg} \), to be at the origin of our co-ordinate system with the \( x \)-axis along the rod. Since all the three masses lie on the \( x \)-axis, the center of mass will also lie on this axis. Let the center of mass be located at \( x_{cm} \) on the \( x \)-axis. Then,

\[
\begin{align*}
  m_{tot} x_{cm} &= \sum_{i=1}^{3} m_i x_i = m_1 x_1 + m_2 x_2 + m_3 x_3 \\
                 &= m_1(0) + m_2(\ell) + m_3(2\ell) \\
  \Rightarrow x_{cm} &= \frac{m_2 \ell + m_3 2\ell}{m_1 + m_2 + m_3} \\
                 &= \frac{3 \text{ kg} \cdot 0.2 \text{ m} + 3 \text{ kg} \cdot 0.4 \text{ m}}{2 + 3 + 3} \\
                 &= \frac{1.8 \text{ m}}{8} = 0.225 \text{ m}.
\end{align*}
\]

Therefore, \( x_{cm} = 0.225 \text{ m} \).

Alternatively, we could find the center of mass by first replacing the two 3 kg masses with a single 6 kg mass located in the middle of the two masses (the center of mass of the two equal masses) and then calculate the value of \( x_{cm} \) for a two particle system consisting of the 2 kg mass and the 6 kg mass (see Fig. ??):

\[
\begin{align*}
  x_{cm} &= \frac{6 \text{ kg} \cdot 0.3 \text{ m}}{8 \text{ kg}} = \frac{1.8 \text{ m}}{8} = 0.225 \text{ m}.
\end{align*}
\]

SAMPLE 2.40 Center of mass in 2-D: Two particles of mass \( m_1 = 1 \text{ kg} \) and \( m_2 = 2 \text{ kg} \) are located at coordinates \((1 \text{ m}, 2 \text{ m})\) and \((-2 \text{ m}, 5 \text{ m})\), respectively, in the \( xy \)-plane. Find the location of their center of mass.

Solution Let \( \vec{r}_{cm} \) be the position vector of the center of mass. Then,

\[
\begin{align*}
  m_{tot} \vec{r}_{cm} &= m_1 \vec{r}_1 + m_2 \vec{r}_2 \\
  \Rightarrow \vec{r}_{cm} &= \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \\
                 &= \frac{1 \text{ kg}(1 \text{ m} \hat{i} + 2 \text{ m} \hat{j}) + 2 \text{ kg}(-2 \text{ m} \hat{i} + 5 \text{ m} \hat{j})}{3 \text{ kg}} \\
                 &= \frac{(1 \text{ m} - 4 \text{ m}) \hat{i} + (2 \text{ m} + 10 \text{ m}) \hat{j}}{3} \\
                 &= -1 \text{ m} \hat{i} + 4 \text{ m} \hat{j}.
\end{align*}
\]

Thus the center of mass is located at the coordinates \((-1 \text{ m}, 4 \text{ m})\).

Geometrically, this is just a 1-D problem like the previous sample. The center of mass has to be located on the straight line joining the two masses. Since the center of mass is a point about which the distribution of mass is balanced, it is easy to see (see Fig. ??) that the center of mass must lie one-third way from \( m_2 \) on the line joining the two masses so that \( 2 \text{ kg} \cdot (d/3) = 1 \text{ kg} \cdot (2d/3) \).
SAMPLE 2.41  Location of the center of mass. A structure is made up of three point masses, $m_1 = 1 \text{ kg}$, $m_2 = 2 \text{ kg}$ and $m_3 = 3 \text{ kg}$, connected rigidly by massless rods. At the moment of interest, the coordinates of the three masses are (1.25 m, 3 m), (2 m, 2 m), and (0.75 m, 0.5 m), respectively. At the same instant, the velocities of the three masses are $2 \text{ m/s} \hat{i}$, $2 \text{ m/s} (\hat{i} - 1.5 \hat{j})$ and $1 \text{ m/s} \hat{j}$, respectively. Find the coordinates of the center of mass of the structure.

Solution Just for fun, let us do this problem two ways — first using scalar equations for the coordinates of the center of mass, and second, using vector equations for the position of the center of mass.

(a) Scalar calculations: Let $(x_{cm}, y_{cm})$ be the coordinates of the mass-center. Then from the definition of mass-center,

$$x_{cm} = \frac{\sum m_i x_i}{\sum m_i} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = \frac{1 \text{ kg} \cdot 1.25 \text{ m} + 2 \text{ kg} \cdot 2 \text{ m} + 3 \text{ kg} \cdot 0.75 \text{ m}}{1 \text{ kg} + 2 \text{ kg} + 3 \text{ kg}} = \frac{7.5 \text{ kg} \cdot \text{ m}}{6 \text{ kg}} = 1.25 \text{ m}.$$  

Similarly,

$$y_{cm} = \frac{\sum m_i y_i}{\sum m_i} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} = \frac{1 \text{ kg} \cdot 3 \text{ m} + 2 \text{ kg} \cdot 2 \text{ m} + 3 \text{ kg} \cdot 0.5 \text{ m}}{1 \text{ kg} + 2 \text{ kg} + 3 \text{ kg}} = \frac{8.5 \text{ kg} \cdot \text{ m}}{6 \text{ kg}} = 1.42 \text{ m}.$$  

Thus the center of mass is located at the coordinates (1.25 m, 1.42 m). 

(b) Vector calculations: Let $\vec{r}_{cm}$ be the position vector of the mass-center. Then,

$$m_{tot} \vec{r}_{cm} = \sum_{i=1}^{3} m_i \vec{r}_i = m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 \Rightarrow \vec{r}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3}{m_1 + m_2 + m_3}.$$  

Substituting the values of $m_1$, $m_2$, and $m_3$, and $\vec{r}_1 = 1.25 \hat{m} + 3 \hat{j}$, $\vec{r}_2 = 2 \hat{m} + 2 \hat{j}$, and $\vec{r}_3 = 0.75 \hat{m} + 0.5 \hat{j}$, we get,

$$\vec{r}_{cm} = \frac{1 \text{ kg} \cdot (1.25 \hat{m} + 3 \hat{j}) \text{ m} + 2 \text{ kg} \cdot (2 \hat{m} + 2 \hat{j}) \text{ m} + 3 \text{ kg} \cdot (0.75 \hat{m} + 0.5 \hat{j}) \text{ m}}{(1 + 2 + 3) \text{ kg}} = \frac{(7.5 \hat{m} + 8.5 \hat{j}) \text{ kg} \cdot \text{ m}}{6 \text{ kg}} = 1.25 \hat{m} + 1.42 \hat{j}.$$  

which, of course, gives the same location of the mass-center as above.

$$\vec{r}_{cm} = 1.25 \hat{m} + 1.42 \hat{j}$$
SAMPLE 2.42 Center of mass of a bent bar: A uniform bar of mass 4 kg is bent in the shape of an asymmetric ‘Z’ as shown in the figure. Locate the center of mass of the bar.

Solution Since the bar is uniform along its length, we can divide it into three straight segments and use their individual mass-centers (located at the geometric centers of each segment) to locate the center of mass of the entire bar. The mass of each segment is proportional to its length. Therefore, if we let \( m_2 = m_3 = m \), then \( m_1 = 2m \); and \( m_1 + m_2 + m_3 = 4m = 4 \text{ kg} \) which gives \( m = 1 \text{ kg} \). Now, from Fig. ??,

\[
\bar{r}_1 = \ell \hat{i} + \ell \hat{j}
\]
\[
\bar{r}_2 = 2\ell \hat{i} + \frac{\ell}{2} \hat{j}
\]
\[
\bar{r}_3 = (2\ell + \frac{\ell}{2}) \hat{i} = \frac{5\ell}{2} \hat{i}
\]

So,

\[
\bar{r}_{cm} = \frac{m_1 \bar{r}_1 + m_2 \bar{r}_2 + m_3 \bar{r}_3}{m_{tot}}
\]
\[
= \frac{2m(\ell \hat{i} + \ell \hat{j}) + m(2\ell \hat{i} + \frac{\ell}{2} \hat{j}) + m(\frac{5\ell}{2} \hat{i})}{4m}
\]
\[
= \frac{m(\ell \hat{i} + 2 \hat{j} + 2 \hat{i} + \frac{\ell}{2} \hat{j} + \frac{5\ell}{2} \hat{i})}{4m}
\]
\[
= \frac{\ell}{8} (13 \hat{i} + 5 \hat{j})
\]
\[
= \frac{0.5m}{8} (13 \hat{i} + 5 \hat{j})
\]
\[
= 0.812m \hat{i} + 0.312m \hat{j}.
\]

Geometrically, we could find the center of mass by considering two masses at a time, connecting them by a line and locating their mass-center on that line, and then repeating the process as shown in Fig. ???. The center of mass of \( m_2 \) and \( m_3 \) (each of mass \( m \)) is at the mid-point of the line connecting the two masses. Now, we replace these two masses with a single mass \( 2m \) at their mass-center. Next, we connect this mass-center and \( m_1 \) with a line and find their combined mass-center at the mid-point of this line. The mass-center just found is the center of mass of the entire bar.
SAMPLE 2.43  Shift of mass-center due to cut-outs: A 2 m \times 2 m uniform square plate has mass \( m = 4 \text{ kg} \). A circular section of radius 250 mm is cut out from the plate as shown in the figure. Find the center of mass of the plate.

Solution  Let us use an \( xy \)-coordinate system with its origin at the geometric center of the plate and the \( x \)-axis passing through the center of the cut-out. Since the plate and the cut-out are symmetric about the \( x \)-axis, the new center of mass must lie somewhere on the \( x \)-axis. Thus, we only need to find \( x_{cm} \) (since \( y_{cm} = 0 \)). Let \( m_1 \) be the mass of the plate with the hole, and \( m_2 \) be the mass of the circular cut-out. Clearly, \( m_1 + m_2 = m = 4 \text{ kg} \). The center of mass of the circular cut-out is at A, the center of the circle. The center of mass of the intact square plate (without the cut-out) must be at O, the middle of the square. Then,

\[
    m_1 x_{cm} + m_2 x_A = mx_O = 0
\]

\[
    \Rightarrow x_{cm} = -\frac{m_2}{m_1} x_A
\]

Now, since the plate is uniform, the masses \( m_1 \) and \( m_2 \) are proportional to the surface areas of the geometric objects they represent, \( \text{i.e.,} \)

\[
    \frac{m_2}{m_1} = \frac{\pi r^2}{\ell^2 - \pi r^2} = \frac{\pi}{(\frac{\ell}{r})^2 - \pi}
\]

Therefore,

\[
    x_{cm} = -\frac{m_2}{m_1} d = -\frac{\pi}{\left(\frac{\ell}{r}\right)^2 - \pi} d
\]

\[
    = -\frac{\pi}{\left(\frac{2 \text{ m}}{2.5 \text{ m}}\right)^2 - \pi} \cdot 0.5 \text{ m}
\]

\[
    = -25.81 \times 10^{-3} \text{ m} = -25.81 \text{ mm}
\]

Thus the center of mass shifts to the left by about 26 mm because of the circular cut-out of the given size.

\[
    x_{cm} = -25.81 \text{ mm}
\]

Comments: The advantage of finding the expression for \( x_{cm} \) in terms of \( r \) and \( \ell \) as in eqn. (2.4) is that you can easily find the center of mass of any size circular cut-out located at any distance \( d \) on the \( x \)-axis. This is useful in design where you like to select the size or location of the cut-out to have the center of mass at a particular location.
SAMPLE 2.44 Center of mass of two objects: A square block of side 0.1 m and mass 2 kg sits on the side of a triangular wedge of mass 6 kg as shown in the figure. Locate the center of mass of the combined system.

Solution The center of mass of the triangular wedge is located at \( h/3 \) above the base and \( \ell/3 \) to the right of the vertical side. Let \( m_1 \) be the mass of the wedge and \( \vec{r}_1 \) be the position vector of its mass-center. Then, referring to Fig. ??,

\[
\vec{r}_1 = \frac{\ell}{3} \hat{i} + \frac{h}{3} \hat{j}.
\]

The center of mass of the square block is located at its geometric center \( C_2 \). From geometry, we can see that the line \( AE \) that passes through \( C_2 \) is horizontal since \( \angle OAB = 45^\circ \) (\( h = \ell = 0.3 \) m) and \( \angle DAE = 45^\circ \). Therefore, the coordinates of \( C_2 \) are \((d/\sqrt{2}, h)\). Let \( m_2 \) and \( \vec{r}_2 \) be the mass and the position vector of the mass-center of the block, respectively. Then,

\[
\vec{r}_2 = \frac{d}{\sqrt{2}} \hat{i} + h \hat{j}.
\]

Now, noting that \( m_1 = 3m_2 \) or \( m_1 = 3m \), and \( m_2 = m \) where \( m = 2 \) kg, we find the center of mass of the combined system:

\[
\vec{r}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} = \frac{3m(\frac{\ell}{3} \hat{i} + \frac{h}{3} \hat{j}) + m(\frac{d}{\sqrt{2}} \hat{i} + h \hat{j})}{3m + m} = \frac{3m(\ell + \frac{d}{\sqrt{2}}) \hat{i} + 2h \hat{j}}{4m} = \frac{1}{4} \left( \frac{d}{\sqrt{2}} + \ell \right) \hat{i} + \frac{h}{2} \hat{j} = \frac{1}{4} \left( \frac{0.1 \text{ m}}{\sqrt{2}} + 0.3 \text{ m} \right) \hat{i} + \frac{0.3 \text{ m}}{2} \hat{j} = 0.093 \text{ m} \hat{i} + 0.150 \text{ m} \hat{j}.
\]

Thus, the center of mass of the wedge and the block together is slightly closer to the side OA and higher up from the bottom OB than \( C_1(0.1 \text{ m}, 0.1 \text{ m}) \). This is what we should expect from the placement of the square block.

Note that we could have, again, used a 1-D calculation by placing a point mass \( 3m \) at \( C_1 \) and \( m \) at \( C_2 \), connected the two points by a straight line, and located the center of mass \( C \) on that line such that \( CC_2 = 3CC_1 \). You can verify that the distance from \( C_1(0.1 \text{ m}, 0.1 \text{ m}) \) to \( C(0.093 \text{ m}, 0.15 \text{ m}) \) is one third the distance from \( C \) to \( C_2(0.071 \text{ m}, 0.3 \text{ m}) \).